Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(7): 23149–23172.
DOI: 10.3934/math.2026933
Received: 02 June 2026
Revised: 17 July 2026
Accepted: 24 July 2026
Published: 30 July 2026

Research article

Nonuniform Berry–Esseen bound for character ratios

Natthapol Dejtrakulwongse[1], **Suporn Jongpreechaharn**[1,2] **and Kritsana Neammanee**[1,2,*]

[1] Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Pathumwan 10330, Bangkok, Thailand

[2] Center of Excellence in Mathematics, Ministry of Higher Education, Science, Research and Innovation, Ratchathewi 10400, Bangkok, Thailand

* **Correspondence:** Email: kritsana.n@chula.ac.th.

Abstract: The first uniform bound in the normal approximation for character ratios was obtained by Fulman (2005) with the rate of convergence $O(n^{-1/4})$, and later improved to $O(n^{-1/2})$ by Shao and Su (2006). In this work, we use Stein's method and the Jack measure to derive a nonuniform Berry–Esseen-type bound for character ratios. Our result yields a sharper constant compared to previously known estimates. We also discuss an application of this approximation, particularly its role in random walk on the symmetric group S_n.

Keywords: character ratio; normal approximation; uniform and nonuniform bounds; Stein's method and Jack measure
Mathematics Subject Classification: 60F05, 20C30, 60J10

1. Character ratio

Let n be a natural number and S_n be the symmetric group on $\{1, 2, 3, \ldots, n\}$, that is, the set of all permutations of $\{1, 2, 3, \ldots, n\}$. The character ratio serves as a bridge between the representation theory of S_n and the study of random walks on S_n. In particular, character ratios determine the eigenvalues of the transition matrix. In this paper, we establish an approximation of a normalized character ratio by the standard normal distribution. Before introducing the definition of the character ratio, we first review the definitions and key properties of representations of S_n, namely partitions, representations and irreducible representations, and matrix representations.

A partition λ of a natural number n is a tuple $(\lambda_1, \lambda_2, \ldots, \lambda_p)$, where $\lambda_1, \lambda_2, \ldots, \lambda_p \in \mathbb{N}$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_p = n$, and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$. For example, the set of all partitions of 4 is $\{(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1)\}$. For any vector space V, we define $\text{End}(V) = \{T : V \to V \mid T \text{ is a linear map}\}$. For a vector space V over $\mathbb{C}$, we say that (V, ρ) is a representation of S_n if $\rho : S_n \to$

End(V) is a homomorphism; that is, $\rho(\pi_1\pi_2) = \rho(\pi_1) \circ \rho(\pi_2)$ for all $\pi_1, \pi_2 \in S_n$. Now, suppose that U is a subspace of V. We say that $(U, \rho|_U)$ is a subrepresentation of a representation (V, ρ) if $(U, \rho|_U)$ is also a representation of S_n. To verify that $(U, \rho|_U)$ is a representation, it suffices to show that $\rho(\pi)(u) \in U$ for any $\pi \in S_n$ and $u \in U$. Moreover, we say that (V, ρ) is an irreducible representation of S_n if its only subrepresentations are (V, ρ) and $(\{\vec{0}\}, \rho|_{\{\vec{0}\}})$.

Example 1.1. *(1) Let $\rho_1 : S_n \to End(\mathbb{C})$ be defined by $\rho_1(\pi)(x) = x$ for any $\pi \in S_n$ and $x \in \mathbb{C}$. Then, $(\mathbb{C}, \rho_1)$ is a trivial representation of S_n.*

(2) Let $\rho_2 : S_n \to End(\mathbb{C})$ be defined by

$$\rho_2(\pi)(x) = \begin{cases} x, & \text{if } \pi \text{ is an even permutation,} \\ -x, & \text{if } \pi \text{ is an odd permutation} \end{cases}$$

for any $\pi \in S_n$ and $x \in \mathbb{C}$. Then, $(\mathbb{C}, \rho_2)$ is the sign representation.

(3) Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{C}^3$ over $\mathbb{C}$; that is, $e_1 = (1, 0, 0)^T, e_2 = (0, 1, 0)^T$, and $e_3 = (0, 0, 1)^T$ and $\rho_3 : S_3 \to End(\mathbb{C}^3)$ are defined in the following. For $\pi \in S_3$, let

$$\rho_3(\pi)(e_i) = e_{\pi(i)},$$

and for $v \in \mathbb{C}^3$, which can be written as a linear combination of the standard basis as $v = k_1e_1 + k_2e_2 + k_3e_3$ for some unique $k_1, k_2, k_3 \in \mathbb{C}$, we define

$$\rho_3(\pi)(v) = k_1\rho_3(\pi)(e_1) + k_2\rho_3(\pi)(e_2) + k_3\rho_3(\pi)(e_3).$$

Then, $(\mathbb{C}^3, \rho_3)$ is a representation of S_3. Let

$$U = \left\{(x_1, x_2, x_3)^T \in \mathbb{C}^3 \;\middle|\; x_1 + x_2 + x_3 = 0\right\}.$$

For any $(x_1, x_2, x_3)^T \in U$, we have $x = (x_1, x_2, x_3)^T = x_1e_1 + x_2e_2 + x_3e_3$ and $x_1 + x_2 + x_3 = 0$. For $\pi \in S_3$, we obtain that $x_{\pi(1)} + x_{\pi(2)} + x_{\pi(3)} = x_1 + x_2 + x_3 = 0$, which implies that $\rho_3(\pi)(x) \in U$. Hence, $(U, \rho_3|_U)$ is a subrepresentation of $(\mathbb{C}^3, \rho_3)$.

(4) For any representation (V, ρ) with $\dim V = 1$, any subspace of V must be either V or $\{\vec{0}\}$. It follows that the representation is irreducible, which can be seen in Example 1.1(1),(2).

(5) Observe that $\{(12), (123)\}$ is a generating set of S_3; that is, for any $\pi \in S_3$, $\pi = (12)^\alpha(123)^\beta$ for some $\alpha, \beta \in \mathbb{N} \cup \{0\}$. Let $\rho_4 : S_3 \to End(\mathbb{C}^2)$ be defined by

$$\rho_4((12))(x) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x, \quad \rho_4((123))(x) = \begin{bmatrix} \cos 120^\circ & -\sin 120^\circ \\ \sin 120^\circ & \cos 120^\circ \end{bmatrix} x$$

for any $x \in \mathbb{C}^2$, and

$$\rho_4(\pi) = \rho_4^\alpha((12)) \circ \rho_4^\beta((123)) \text{ for any } \pi = (12)^\alpha(123)^\beta \in S_3.$$

Then, $(\mathbb{C}^2, \rho_4)$ is a representation of S_3. Let U be a subspace of $\mathbb{C}^2$. Suppose that $(U, \rho_4|_U)$ is a subrepresentation of $(\mathbb{C}^2, \rho_4)$ and that U is neither $\mathbb{C}^2$ nor $\{\vec{0}\}$. Then, $\dim U = 1$. There are three possible cases to consider.

Case 1. *$U = span\{(1, 0)\}$. Then, $\rho_4((123))((1, 0)^T) = (\cos 120^\circ, \sin 120^\circ)^T \notin U$.*

Case 2. *$U = span\{(0,1)\}$. Then, $\rho_4((123))((0,1)^T) = (-\sin 120°, \cos 120°)^T \notin U$.*
Case 3. *$U = span\{(a,b)\}$ for some $a, b \in \mathbb{R} \setminus \{0\}$. Then, $\rho_4((12))((a,b)^T) = (a,-b)^T \notin U$ because $b \neq 0$.*
Hence, $(\mathbb{C}^2, \rho_4)$ is an irreducible representation of S_3.

It is known that every irreducible representation of S_n is uniquely determined, up to isomorphism, by a partition of n. That is, there is a one-to-one correspondence between irreducible representations of S_n and partitions of n [1, p. 58].

Let $\mathfrak{B} = \langle v_1, v_2, \ldots, v_m \rangle$ be an ordered basis of V, and let $v \in V$. The projection of v with respect to $\mathfrak{B}$ is defined by $[v]_{\mathfrak{B}} = (k_1, k_2, \ldots, k_m)^T$, where $v = k_1v_1 + k_2v_2 + \cdots + k_mv_m$. Then, for any linear map $T : V \to V$, there exists a unique $m \times m$ matrix $\mathbf{A}$ with respect to $\mathfrak{B}$ such that $[T(v)]_{\mathfrak{B}} = \mathbf{A}[v]_{\mathfrak{B}}$, which is described by $\mathbf{A} = [[T(v_1)]_{\mathfrak{B}} \quad [T(v_2)]_{\mathfrak{B}} \ \cdots \ [T(v_m)]_{\mathfrak{B}}]$ [2, p. 69]. The converse is trivially true; specifically, the matrix multiplication map $[T(v)]_{\mathfrak{B}} = \mathbf{A}[v]_{\mathfrak{B}}$ is linear. Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be ordered bases of a vector space V, and let $T \in \mathrm{End}(V)$. Suppose that $\mathbf{A}_{\mathfrak{B}_1}$ and $\mathbf{A}_{\mathfrak{B}_2}$ are the matrix representations of T with respect to $\mathfrak{B}_1$ and $\mathfrak{B}_2$, respectively. Then, $\mathrm{tr}(\mathbf{A}_{\mathfrak{B}_1}) = \mathrm{tr}(\mathbf{A}_{\mathfrak{B}_2})$, where $\mathrm{tr}(\cdot)$ is a trace function [2, p. 327]. This allows us to define the character of a representation as follows.

Let λ be a partition of n, and let (V, ρ) be an irreducible representation of S_n corresponding to λ, with $\dim(V) = k$. Then, there exists a $k \times k$ matrix $\mathbf{A}_\pi$ such that $\rho(\pi)(v) = \mathbf{A}_\pi v$ for each $\pi \in S_n$ and any $v \in V$. The character $\chi^\lambda : S_n \to \mathbb{R}$ is defined by the formula

$$\chi^\lambda(\pi) = \mathrm{tr}(\mathbf{A}_\pi) \text{ for any } \pi \in S_n.$$

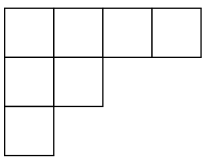
Note that any irreducible representation which is induced by the same partition has identical characters and are integers [3, p. 43]. The dimension of λ is also defined to be $\dim(\lambda) = \dim(V)$. We also have the fact that

$$\sum_{\lambda} \dim^2(\lambda) = n! \tag{1.1}$$

([4, p. 74]), where the summation runs over all partitions of n. The Plancherel measure on the set of all partitions λ of n is defined by

$$\mathbb{P}(\lambda) = \frac{\dim^2(\lambda)}{n!} \text{ for any partition } \lambda \text{ of } n. \tag{1.2}$$

Next, we introduce the formula for calculating the character ratio of a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ of n. The Young diagram of λ is an array of boxes, where the ith row contains λ_i boxes, starting from the left. For example, consider a partition $\lambda = (4, 2, 1)$ of $n = 7$. The Young diagram of λ is



We also say that a partition τ of $n + 1$ is above λ if τ can be obtained by adding a single box to λ, and we write $\lambda \nearrow \tau$. For examples, $(4, 2, 2) \nearrow (4, 3, 2)$ and $(4, 2, 2) \nearrow (4, 2, 2, 1)$. Next, we define the hooklength for each box in the Young diagram. Let x be a box in the Young diagram. The hooklength of x is defined as

$$h(x) = 1 + a(x) + l(x), \tag{1.3}$$

where $a(x)$ is the number of boxes to the right of x, and $l(x)$ is the number of boxes below x. For example, consider the Young diagram corresponding to the partition $\lambda = (4,2,1)$ of $n = 7$. The hooklength of the box x at the top-left is $1+3+2=6$ because there are three boxes in the first row to the right of x and two boxes in the first column and below x. The hooklength of the remaining boxes can be calculated in the same way. We can then fill the hooklength of each box in the Young diagram as

6	4	2	1
3	1		
1			

For an irreducible representation (V,ρ) that is induced by a certain partition λ, $\dim(\lambda)$ can be calculated by the hooklength formula as

$$\dim(\lambda) = \frac{n!}{\prod_x h(x)}, \tag{1.4}$$

where the product is taken over all boxes x in the Young diagram of λ [5, p. 559]. For example, from the Young diagram which is filled with all hooklengths above, we obtain for $\lambda = (4,2,1)$ that

$$\dim(\lambda) = \frac{7!}{6\cdot 4\cdot 2\cdot 1\cdot 3\cdot 1\cdot 1} = 35.$$

Notice that if $\pi_1, \pi_2 \in S_n$ have the same length, then $\chi^\lambda(\pi_1) = \chi^\lambda(\pi_2)$ for any partition λ of n [4, p. 31]. In this work, we focus on the character evaluated at a transposition, as it is the key to analyzing the random walk on a symmetric group [6], which is described later. By choosing a partition λ of n by the Plancherel measure, the character ratio is defined by $\chi^\lambda(12)/\dim(\lambda)$, namely, the ratio between character evaluated at a transposition (12) and the dimension of λ. An advantage of defining the character ratio at the transposition (12) is that we can work with symmetric group S_n for any $n \geq 2$.

For a partition $\lambda = (\lambda_1, \dots, \lambda_p)$ of n, we let

$$n(\lambda) = \sum_{i=1}^{p} (i-1)\lambda_i,$$

and we define $\lambda' = (\lambda'_1, \dots, \lambda'_q)$, where λ'_j is the number of boxes of the jth column of Young diagram of λ. For instance, let $\lambda = (2,1,1)$. Then, $\lambda' = (3,1)$, $n(\lambda) = (1-1)2 + (2-1)1 + (3-1)1 = 3$, and $n(\lambda') = (1-1)3 + (2-1)1 = 1$. Consequently, we have the fact that

$$\dim\lambda = \dim\lambda'. \tag{1.5}$$

The formula for calculating the character ratio for $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p)$ is then given by Frobenius [7] as

$$\frac{\chi^\lambda(12)}{\dim(\lambda)} = \frac{n(\lambda') - n(\lambda)}{\binom{n}{2}}. \tag{1.6}$$

In this work, we consider the normalized character ratio

$$W(\lambda) = \frac{(n-1)}{\sqrt{2}} \frac{\chi^\lambda(12)}{\dim(\lambda)} \tag{1.7}$$

for any partition λ of n. Fulman [5] showed that

$$\mathbb{E}W = 0, \tag{1.8}$$

and

$$\mathbb{E}W^2 = 1 - \frac{1}{n} \tag{1.9}$$

([5], pp. 563-564). By applying (1.6) to (1.7), we obtain

$$W(\lambda) = \frac{\sqrt{2}}{n}(n(\lambda') - n(\lambda)), \tag{1.10}$$

a form that will be used extensively. We also observe that $(\lambda')' = \lambda$. This implies that

$$W(\lambda) = -W(\lambda'). \tag{1.11}$$

2. Literature review and main result

In 1993, Kerov [8] outlined the proof of the asymptotic normality of the normalized character ratio W, defined in (1.10). Later, Ivanov and Olshanski [9] provided the complete proof of this result in 2002. Fulman [5] presented an error bound for the character ratio by constructing an exchangeable pair and applying Stein's method in 2005 to achieve the normal approximation with the rate of $O(n^{-1/4})$ as

$$\sup_{z\in\mathbb{R}} |\mathbb{P}(W \leq z) - \Phi(z)| \leq 40.1n^{-\frac{1}{4}}, \tag{2.1}$$

where $\Phi(z) = \left(1/\sqrt{2\pi}\right)\int\limits_{-\infty}^{z} e^{-\frac{x^2}{2}}\,dx$ is the standard normal distribution function. Fulman [5] also conjectured that the rate of convergence can be improved to $O(n^{-1/2})$. Later in 2006, (2.1) was improved to the rate of $O(n^{-1/2})$ by Shao and Su [10] as

$$\sup_{z\in\mathbb{R}} |\mathbb{P}(W \leq z) - \Phi(z)| \leq An^{-\frac{1}{2}}, \tag{2.2}$$

where A can be narrowed down to 150. Chen et al. [11] further improved this constant to 8.2 under certain conditions, as stated in the following theorem:

$$\sup_{z\in\mathbb{R}} |\mathbb{P}(W \leq z) - \Phi(z)| \leq \frac{8.2}{\sqrt{n}}. \tag{2.3}$$

We see that the constants in the error bounds of (2.1)–(2.3) do not depend on the value of z. This type of error bound is called a uniform bound. In contrast, a nonuniform bound includes an additional factor that decays as $|z|$ increases so that the bound becomes sharper for large $|z|$. In 2021, Chen et al. [11] obtained both uniform and nonuniform bounds for the Jack deformation of the character ratio using zero-bias coupling under certain conditions on α. In 2025, Thành and Tu [12] established a nonuniform bound for general unbounded exchangeable pairs, with the Jack deformation of the character ratio arising as one of its applications. The following is their result:

$$|\mathbb{P}(W \leq z) - \Phi(z)| \leq \frac{C(p)}{(1+|z|)^p\sqrt{n}}, \tag{2.4}$$

where the constant $C(p)$ depends on the order of p. In the case of $p = 3$, it turns out that the constant is very large; that is, $C(3) \geq 2.55 \times 10^{10}$ (see [12, Appendix A]).

In this paper, we improve a nonuniform bound for the normal approximation of the character ratio. Our key contribution is to give a sharper bound on the moments of W by utilizing the Jack measure and the Metropolis algorithm. Theorem 2.1 presents our main result.

Theorem 2.1. *For any $|z| \geq 35$ and $n \geq 51$, we have*

$$|\mathbb{P}(W \leq z) - \Phi(z)| \leq \frac{93.7}{\sqrt{n}(1+|z|)^3}. \tag{2.5}$$

We compare the coefficient 8.2 of $1/\sqrt{n}$ in the uniform bound (2.3) with that in our nonuniform bound in (2.5) for z varying from 35 to 100, as in Figure 1.

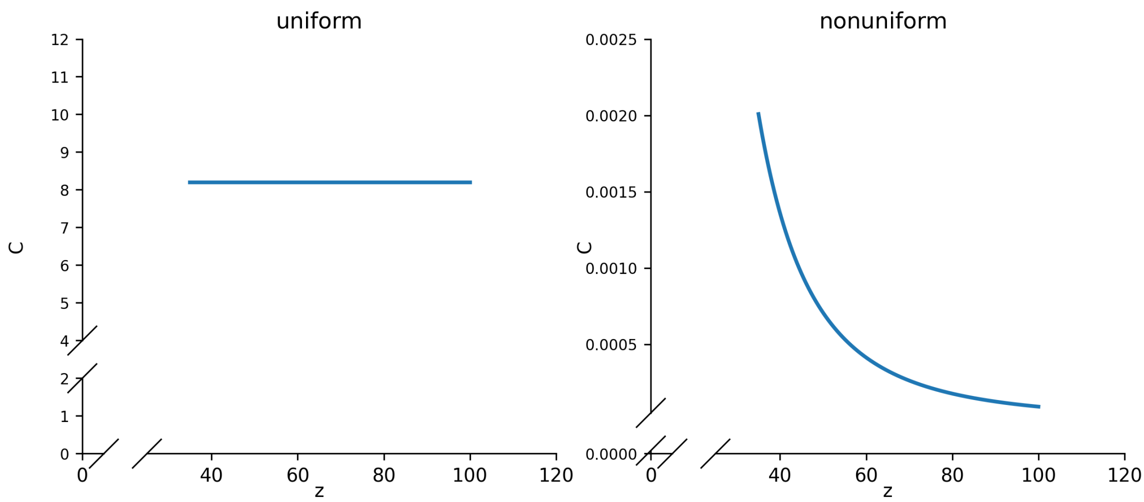


Figure 1. Constants C in the form of $C/\sqrt{n}$ on the bounds of (2.3) and (2.5).

Remark 1. *(1) For small values of z, specifically when $|z| < 35$, it suffices to apply the uniform bound.*

(2) By using the fact that $|W| \leq n/\sqrt{2}$, which will be proved below in Proposition 3.4, we have $\mathbb{P}(W \leq z) = 1$ for any $z \geq 35$ and $n \leq 50$. Thus, it is reasonable to assume in the main theorem that $n \geq 51$.

(3) For $|z| \geq 35$, we have

$$|\mathbb{P}(W \leq z) - \Phi(z)| \leq \frac{2.01 \times 10^{-3}}{\sqrt{n}},$$

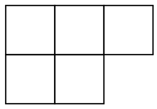
which provides a sharper constant than a uniform bound given in (2.3).

3. Moments of W

In order to make a sharper bound, we need to bound moments of W which are provided in this section. First, we note that the Jack$_\alpha$ measure on the set of all partitions of n is defined as

$$\mathbb{P}_\alpha(\lambda) = \frac{\alpha^n n!}{\prod_x (\alpha a(x) + l(x) + 1)(\alpha a(x) + l(x) + \alpha)},$$

where λ is a partition of n, and $a(\cdot)$ and $l(\cdot)$ are defined as in (1.3) (see [13] for more details). Hence, the Plancherel measure can be viewed as a special case of the Jack$_\alpha$ measure when $\alpha = 1$ and $W = \sqrt{(n-1)/n}W_1$. For example, the partition $(3, 2)$ of $n = 5$, which has the Young diagram



has Jack$_\alpha$ measure

$$\mathbb{P}_\alpha((3,2)) = \frac{30\alpha^2}{(\alpha+1)^2(3\alpha+1)(\alpha+2)(2\alpha+1)}.$$

Note that for any $\pi \in S_n$, π can be decomposed into a product of disjoint cycles with decreasing lengths [14, p. 31]. If $\pi \in S_n$ is the product of disjoint cycles of lengths $n_1, n_2, \ldots, n_r$ with $n_1 \geq n_2 \geq \cdots \geq n_r$ and $n_1 + \cdots + n_r = n$, then $(n_1, \ldots, n_r)$ is called the cycle type of π, and we define

$$c(\pi) = \text{ number of cycles of } \pi.$$

For a partition $\lambda = (n_1, \ldots, n_r)$ of n, we denote that $[\lambda] = \{\pi \in S_n \mid \lambda \text{ is the cycle type of } \pi.\}$. To be specific, for S_4, $[(2,1,1)] = \{(12),(13),(14),(23),(24),(34)\}$, $[(2,1^{n-2})] = \{\sigma \mid \sigma \text{ is a transposition in } S_n\}$, and $[(\underbrace{1,\ldots,1}_{n\text{ terms}})] = \{i_d\}$, where i_d is the identity permutation. Let $\pi_1, \pi_2 \in S_n$ and $\alpha \geq 1$. It should be noted that if $\pi_2 = \pi_1\sigma$ for some transposition σ, then σ can be uniquely determined as $\sigma = \pi_1^{-1}\pi_2$. The transition probability T_α on S_n is defined as

$$T_\alpha(\pi_1,\pi_2) = \begin{cases} \frac{(\alpha-1)n(\lambda')}{\alpha\binom{n}{2}}, & \text{if } \pi_2 = \pi_1 \text{ and } \pi_1 \in [\lambda], \\ \frac{1}{\binom{n}{2}}, & \text{if } \exists \text{a transposition } \sigma \in S_n \text{ s.t. } \pi_2 = \pi_1\sigma \text{ and } c(\pi_2) = c(\pi_1) - 1, \\ \frac{1}{\alpha\binom{n}{2}}, & \text{if } \exists \text{a transposition } \sigma \in S_n \text{ s.t. } \pi_2 = \pi_1\sigma \text{ and } c(\pi_2) = c(\pi_1) + 1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

In the case $\alpha = 1$, we observe that $T_1(\pi_1, \pi_2) = 0$ for all $\pi_1, \pi_2 \in S_n$ and that

$$T_1(\pi_1,\pi_2) = \begin{cases} \frac{1}{\binom{n}{2}}, & \text{if } \exists \text{a transposition } \sigma \in S_n \text{ s.t. } \pi_2 = \pi_1\sigma, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

For example, if we consider $\pi_1 = (12)$ and $\pi_2 = (123)$ in S_n, then $\pi_2 = \pi_1(23)$, which implies that $T_1(\pi_1, \pi_2) = 1/\binom{n}{2}$. Let λ_1 and λ_2 be partitions of n. Then, the Metropolis chain is defined by

$$K_\alpha(\lambda_1,\lambda_2) = \sum_{\pi_2 \in [\lambda_2]} T_\alpha(\pi_1,\pi_2), \tag{3.3}$$

where π_1 is a fixed permutation in $[\lambda_1]$, and π_2 ranges over all permutations in $[\lambda_2]$. Note that $K_\alpha(\lambda_1, \lambda_2)$ does not depend on how to choose π_1. We say that the probability that a partition λ_1 moves to λ_2 is $K_\alpha(\lambda_1, \lambda_2)$. For convenience, we denote the exponential notation of a partition (a^n, b^m) to mean the partition $(\underbrace{a,\ldots,a}_{n\text{ terms}},\underbrace{b,\ldots,b}_{m\text{ terms}})$ and use K and T for K_1 and T_1, respectively. To bound $\mathbb{E}W^4$ and $\mathbb{E}W^6$, we need the following $K(\lambda_1, \lambda_2)$ in Proposition 3.1.

Proposition 3.1. *For $n \geq 6$, we summarize the value of K in Table 1.*

Table 1. $K(\lambda_1, \lambda_2)$ for S_n.

$\lambda_1 \backslash \lambda_2$	(1^n)	$(2, 1^{n-2})$	$(2^2, 1^{n-4})$	$(3, 1^{n-3})$	$(2^3, 1^{n-6})$	$(3, 2, 1^{n-5})$	$(4, 1^{n-4})$
(1^n)	0	1	0	0	0	0	0
$(2, 1^{n-2})$	$\frac{1}{\binom{n}{2}}$	0	$\frac{\binom{n-2}{2}}{\binom{n}{2}}$	$\frac{2(n-2)}{\binom{n}{2}}$	0	0	0
$(2^2, 1^{n-4})$	0	$\frac{2}{\binom{n}{2}}$	0	0	$\frac{\binom{n-4}{2}}{\binom{n}{2}}$	$\frac{4(n-4)}{\binom{n}{2}}$	$\frac{4}{\binom{n}{2}}$
$(3, 1^{n-3})$	0	$\frac{3}{\binom{n}{2}}$	0	0	0	$\frac{\binom{n-3}{2}}{\binom{n}{2}}$	$\frac{3(n-3)}{\binom{n}{2}}$
$(2^3, 1^{n-6})$	0	0	$\frac{3}{\binom{n}{2}}$	0	0	x	x
$(3, 2, 1^{n-5})$	0	0	$\frac{3}{\binom{n}{2}}$	$\frac{1}{\binom{n}{2}}$	x	0	x
$(4, 1^{n-4})$	0	0	$\frac{2}{\binom{n}{2}}$	$\frac{4}{\binom{n}{2}}$	x	x	0

Remark 2. *Here,* x *indicates a probability that is not required for computing the moments of W.*

Proof. From (3.2), we know that $K(\lambda, \lambda) = 0$ for all partitions λ of n. To complete all entries in Table 1, we use the following procedure: For given λ_1 and λ_2, fix $\pi \in [\lambda_1]$. Then, we compute the size of the set A_{λ_1,λ_2}, where $A_{\lambda_1,\lambda_2} = \{\pi\sigma \in [\lambda_2] \mid \sigma \text{ is a transposition in } S_n.\}$. Thus,

$$K(\lambda_1, \lambda_2) = \frac{|A_{\lambda_1,\lambda_2}|}{\binom{n}{2}}.$$

It should be noted that K is well-defined. That is, it does not depend on the choice π in $[\lambda_1]$.

For $\lambda_1 = (1^n)$, we have that $[\lambda_1] = \{i_d\}$, and it follows that

$$A_{\lambda_1,\lambda_2} = \begin{cases} [(2, 1^{n-2})], & \text{if } \lambda_2 = (2, 1^{n-2}), \\ \emptyset, & \text{otherwise.} \end{cases}$$

Therefore,

$$K((1^n), (2, 1^{n-2})) = \frac{|A_{\lambda_1,(2,1^{n-2})}|}{\binom{n}{2}} = 1.$$

For $\lambda_1 = (2, 1^{n-2})$, we fix $\pi = (12) \in [(2, 1^{n-2})]$. Because $|A_{\lambda_1,(1^n)}| = |\{i_d\}| = 1$, we have

$$K((2, 1^{n-2}), (1^n)) = \frac{1}{\binom{n}{2}}.$$

Similarly, we have $|A_{\lambda_1,(2^2,1^{n-4})}| = \binom{n-2}{2}$ and $|A_{\lambda_1,(3,1^{n-3})}| = 2(n-2)$. Therefore,

$$K((2, 1^{n-2}), (2^2, 1^{n-4})) = \frac{\binom{n-2}{2}}{\binom{n}{2}}, \quad \text{and} \quad K((2, 1^{n-2}), (3, 1^{n-3})) = \frac{2(n-2)}{\binom{n}{2}}.$$

We remark that $\binom{n-2}{2} + 2(n-2) + 1 = \binom{n}{2}$, which implies that $K((2, 1^{n-2}), \lambda) = 0$ for $\lambda \notin \{(1^n), (2^2, 1^{n-4}), (3, 1^{n-3})\}$. For other cases, we can compute $K(\lambda_1, \lambda_2)$ with the same argument. $\square$

For convenience, let $K^{(k)}(\lambda_1, \lambda_2)$ denote the probability that λ_1 moves to λ_2 in k steps. In [15, p. 289], for $\alpha \geq 1$, Fulman defined

$$W_\alpha = \frac{\alpha n(\lambda') - n(\lambda)}{\sqrt{\alpha\binom{n}{2}}}$$

and showed that the rth moment of $W_\alpha / \sqrt{\alpha\binom{n}{2}}$ under the Jack$_\alpha$ measure is equal to the probability that $(\underbrace{1, \ldots, 1}_{n \text{ terms}})$ moves to itself in r steps under the Metropolis chain. Hence,

$$\mathbb{E}W^4 = \left(\frac{n-1}{n}\right)^2 \binom{n}{2}^2 K^{(4)}((1^n), (1^n)), \tag{3.4}$$

$$\text{and} \quad \mathbb{E}W^6 = \left(\frac{n-1}{n}\right)^3 \binom{n}{2}^3 K^{(6)}((1^n), (1^n)). \tag{3.5}$$

Proposition 3.2. *For $n \geq 4$, $\mathbb{E}W^4 \leq 3$.*

Proof. To prove the proposition, we will start from $K^{(4)}((1^n), (1^n))$. From Table 1, we illustrate how (1^n) can move to itself in four steps in the following form:

$$(1^n) \longrightarrow (2, 1^{n-2}) \underset{\text{2-steps}}{\cdots\cdots\cdots} (2, 1^{n-2}) \longrightarrow (1^n).$$

To compute $K^{(2)}((2, 1^{n-2}), (2, 1^{n-2}))$, we note from Table 1 that

$$\begin{tikzcd}
 & (1^n) \arrow[dr] & \\
(2, 1^{n-2}) \arrow[ur] \arrow[r] \arrow[dr] & (2^2, 1^{n-4}) \arrow[r] & (2, 1^{n-2}) \\
 & (3, 1^{n-3}) \arrow[ur] &
\end{tikzcd}$$

and hence,

$$\begin{aligned}
K^{(2)}((2, 1^{n-2}), (2, 1^{n-2})) &= K((2, 1^{n-2}), (1^n))K((1^n), (2, 1^{n-2})) \\
&\quad + K((2, 1^{n-2}), (2^2, 1^{n-4}))K((2^2, 1^{n-4}), (2, 1^{n-2})) \\
&\quad + K((2, 1^{n-2}), (3, 1^{n-3}))K((3, 1^{n-3}), (2, 1^{n-2})) \\
&= \frac{1}{\binom{n}{2}} + \frac{2\binom{n-2}{2}}{\binom{n}{2}^2} + \frac{6(n-2)}{\binom{n}{2}^2} = \frac{3n^2 + n - 12}{2\binom{n}{2}^2}.
\end{aligned} \tag{3.6}$$

Therefore,

$$\begin{aligned} K^{(4)}((1^n),(1^n)) &= K((1^n),(2,1^{n-2}))K^{(2)}((2,1^{n-2}),(2,1^{n-2}))K((2,1^{n-2}),(1^n)) \\ &= \frac{3n^2+n-12}{2\binom{n}{2}^3}. \end{aligned}$$

From this fact and (3.4),

$$\mathbb{E}W^4 = \frac{(3n^2+n-12)(n-1)}{n^3} \leq 3.$$

□

Proposition 3.3. *For $n \geq 6$, $\mathbb{E}W^6 \leq 15.4$.*

Proof. Note that the path from (1^n) to (1^n) in six steps must take the form of

$$(1^n) \longrightarrow (2,1^{n-2}) \underset{\text{4-steps}}{\cdots\cdots\cdots} (2,1^{n-2}) \longrightarrow (1^n).$$

From Table 1, we can separate the computation of $K^{(4)}((2,1^{n-2}),(2,1^{n-2}))$ into three cases as follows:

Case 1.

$$(2,1^{n-2}) \longrightarrow (1^n) \longrightarrow (2,1^{n-2}) \underset{\text{2-steps}}{\cdots\cdots\cdots} (2,1^{n-2})$$

Case 2.

$$\begin{tikzcd}
 & & (2,1^{n-2}) \arrow[r] \arrow[dr] \arrow[ddr] & (1^n) \arrow[dr] & \\
(2,1^{n-2}) \arrow[r] & (3,1^{n-3}) \arrow[ur] \arrow[r] \arrow[dr] & (3,2,1^{n-5}) \arrow[r] \arrow[dr] & (2^2,1^{n-4}) \arrow[r] & (2,1^{n-2}) \\
 & & (4,1^{n-4}) \arrow[ur] \arrow[r] & (3,1^{n-3}) \arrow[ur] &
\end{tikzcd}$$

Case 3.

$$\begin{tikzcd}
 & & (2,1^{n-2}) \arrow[r] \arrow[ddr] \arrow[ddddr] & (1^n) \arrow[ddr] & \\
 & & (3,2,1^{n-5}) \arrow[dr] \arrow[dddr] & & \\
(2,1^{n-2}) \arrow[r] & (2^2,1^{n-4}) \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & (2^2,1^{n-4}) \arrow[r] & (2,1^{n-2}) \\
 & & (2^3,1^{n-6}) \arrow[ur] & & \\
 & & (4,1^{n-4}) \arrow[uur] \arrow[r] & (3,1^{n-3}) \arrow[uur] &
\end{tikzcd}$$

Let E_1, E_2, and E_3 be events in Cases 1–3, respectively.

To compute $\mathbb{P}(E_1)$, by Table 1 and (3.6), we have that

$$\begin{aligned}\mathbb{P}(E_1) &= K((2,1^{n-2}),(1^n))K((1^n),(2,1^{n-2}))K^{(2)}((2,1^{n-2}),(2,1^{n-2}))\\ &= \frac{3n^2+n-12}{2\binom{n}{2}^3}.\end{aligned}$$

To compute $\mathbb{P}(E_2)$, observe that

$$\begin{aligned}K^{(2)}((3,1^{n-3}),(1^n)) &= K((3,1^{n-3}),(2,1^{n-2}))K((2,1^{n-2}),(1^n)) = \frac{3}{\binom{n}{2}^2},\\ K^{(2)}((3,1^{n-3}),(2^2,1^{n-4})) &= K((3,1^{n-3}),(2,1^{n-2}))K((2,1^{n-2}),(2^2,1^{n-4}))\\ &\quad + K((3,1^{n-3}),(3,2,1^{n-5}))K((3,2,1^{n-5}),(2^2,1^{n-4}))\\ &\quad + K((3,1^{n-3}),(4,1^{n-4}))K((4,1^{n-4}),(2^2,1^{n-4}))\\ &= \frac{3\binom{n-2}{2}}{\binom{n}{2}^2} + \frac{3\binom{n-3}{2}}{\binom{n}{2}^2} + \frac{6(n-3)}{\binom{n}{2}^2} = \frac{3(n-3)(n-1)}{\binom{n}{2}^2},\end{aligned}$$

and

$$K^{(2)}((3,1^{n-3}),(3,1^{n-3})) = \frac{n^2+29n-84}{2\binom{n}{2}^2}.$$

Therefore,

$$\begin{aligned}\mathbb{P}(E_2) &= K((2,1^{n-2}),(3,1^{n-3}))[K^{(2)}((3,1^{n-3}),(1^n))K((1^n),(2,1^{n-2}))\\ &\quad + K^{(2)}((3,1^{n-3}),(2^2,1^{n-4}))K((2^2,1^{n-4}),(2,1^{n-2}))\\ &\quad + K^{(2)}((3,1^{n-3}),(3,1^{n-3}))K((3,1^{n-3}),(2,1^{n-2}))]\\ &= \frac{18n^3-288n+432}{\binom{n}{2}^4}.\end{aligned}$$

With the same argument, we can show that

$$\mathbb{P}(E_3) = \frac{3n^4-10n^3-23n^2+110n-96}{\binom{n}{2}^4}.$$

Then,

$$\begin{aligned}&K^{(6)}((1^n),(1^n))\\ &= K((1^n),(2,1^{n-2}))\,(\mathbb{P}(E_1)+\mathbb{P}(E_2)+\mathbb{P}(E_3))\,K((2,1^{n-2}),(1^n))\\ &= 1\times\left(\frac{3n^2+n-12}{2\binom{n}{2}^3} + \frac{18n^3-288n+432}{\binom{n}{2}^4} + \frac{3n^4-10n^3-23n^2+110n-96}{\binom{n}{2}^4}\right)\times\frac{1}{\binom{n}{2}}\end{aligned}$$

$$= \frac{15n^4 + 30n^3 - 105n^2 - 700n + 1344}{4\binom{n}{2}^5}.$$

From this fact and (3.5),

$$\mathbb{E}W^6 = \frac{(n-1)(15n^4 + 30n^3 - 105n^2 - 700n + 1344)}{n^5} \leq 15.4.$$

□

Proposition 3.4. *For $n \geq 2$,*

(1) $|W| \leq (n-1)/\sqrt{2}$;
(2) $W \stackrel{d}{=} -W$, *where* $X \stackrel{d}{=} Y$ *means that random variables* X *and* Y *are equal in distribution;*
(3) $\mathbb{E}W^{2k-1} = 0$ *for any* $k \in \mathbb{N}$.

Proof. (1) Let λ be a partition of n. Suppose that $\lambda = (\lambda_1, \ldots, \lambda_p)$, and $\lambda' = (\lambda'_1, \ldots, \lambda'_q)$. By (1.10), all possible values of $W(\lambda)$ are given by

$$\begin{aligned} W(\lambda) &= \frac{\sqrt{2}}{n}[n(\lambda') - n(\lambda)] \\ &= \frac{\sqrt{2}}{n}\left[\sum_{j=1}^{q}\sum_{k=1}^{\lambda'_j}(j-1) - \sum_{i=1}^{p}\sum_{k=1}^{\lambda_i}(i-1)\right] \\ &= \frac{\sqrt{2}}{n}\left[\sum_{j=1}^{q}\sum_{k=1}^{\lambda'_j} j - \sum_{i=1}^{p}\sum_{k=1}^{\lambda_i} i\right]. \end{aligned}$$

We see that $\sum_{j=1}^{q}\sum_{k=1}^{\lambda'_j} j$ is maximal, and $\sum_{i=1}^{p}\sum_{k=1}^{\lambda_i} i$ is minimal when $\lambda = (n)$. Hence, $W(\lambda)$ is maximized when $\lambda = (n)$. In this case, $W(\lambda) = \left(\sqrt{2}/n\right)\sum_{j=1}^{n}(j-1) = (n-1)/\sqrt{2}$. In particular, $W(\lambda)$ attains its minimum at $\lambda' = (1^n)$. By (1.11), $W(\lambda') = -W(\lambda) = -(n-1)\sqrt{2}$. Hence, the proposition is proved.

(2) See [11, p. 464].

(3) By (2), we have $W^{2k-1} \stackrel{d}{=} -W^{2k-1}$. Hence, $\mathbb{E}W^{2k-1} = -\mathbb{E}W^{2k-1}$, which implies that $\mathbb{E}W^{2k-1} = 0$.

□

4. Proof of the main result

If a random variable V is discrete, and $V \stackrel{d}{=} -V$, then it is sufficient to make a bound for the normal approximation in case $z \geq 0$ (see [16] for more detail). Hence, by Proposition 3.4(2), we will prove Theorem 2.1 for $z \geq 0$. By Proposition 3.4(1), we know that $|W| \leq 49/\sqrt{2} < 35$ for $n \leq 50$. This implies that $\mathbb{P}(W \leq z) = 1$ for $n \leq 50$ and $z \geq 35$. Then, it suffices to investigate the theorem in case of

$n \geq 51$. Our main technique is Stein's method, which aims to make an error bound from a differential equation:

$$f'(w) - wf(w) = \mathbf{1}_{(-\infty,z]}(w) - \Phi(z), \tag{4.1}$$

where

$$\mathbf{1}_{(-\infty,z]}(w) = \begin{cases} 1, & \text{if } w \leq z, \\ 0, & \text{if } w > z. \end{cases}$$

The solution f_z for (4.1) is given in [17, p. 14] as

$$f_z(w) = \begin{cases} \sqrt{2\pi}\, e^{w^2/2}\Phi(w)[1 - \Phi(z)], & \text{if } w \leq z, \\ \sqrt{2\pi}\, e^{w^2/2}\Phi(z)[1 - \Phi(w)], & \text{if } w > z, \end{cases} \tag{4.2}$$

and from (4.1), we have

$$f_z'(w) = \begin{cases} wf_z(w) + 1 - \Phi(z), & \text{if } w \leq z, \\ wf_z(w) - \Phi(z), & \text{if } w > z. \end{cases} \tag{4.3}$$

The properties of f_z that will be used in the work are listed as follows:

$$|f_z'(u)| \leq 1 \text{ for } u \in \mathbb{R}, \tag{4.4}$$

$$|f_z'(u) - f_z'(v)| \leq 1 \text{ for } u, v \in \mathbb{R}, \tag{4.5}$$

and

$$wf_z(w) \text{ is an increasing function of } w. \tag{4.6}$$

([17], p. 16). If we replace w in (4.1) by W and take an expectation, then we have

$$|\mathbb{P}(W \leq z) - \Phi(z)| = |\mathbb{E}f_z'(W) - \mathbb{E}Wf_z(W)|. \tag{4.7}$$

To find a bound on the right-hand side of (4.7), a common approach is to apply the coupling method of exchangeable pairs which are used by [5, 10, 15]. For a pair of random variables U and U', we say that (U, U') is an exchangeable pair if the joint distribution of (U, U') is the same as of (U', U). In 2005, Fulman [5] first constructed an exchangeable pair (W, W') that satisfies

$$\mathbb{E}(W - W'|W) = \frac{2}{n+1}W. \tag{4.8}$$

An exchangeable pair that satisfies (4.8) has a consequence that

$$\mathbb{E}Wf(W) = \mathbb{E}\int_{-\infty}^{\infty} f'(W+t)K(t)dt, \tag{4.9}$$

where

$$K(t) = \frac{n+1}{4}(W' - W)\{\mathbf{1}(0 \leq t \leq W' - W) - \mathbf{1}(W' - W \leq t \leq 0\} \tag{4.10}$$

([17], p. 22). We denote the difference of an exchangeable pair as $\Delta := |W - W'|$. It can also be shown that

$$\Delta \leq 2\sqrt{2}, \tag{4.11}$$

and
$$\mathbb{P}\left(\Delta > \frac{4e\sqrt{2}}{\sqrt{n}}\right) \le 2e^{-2e\sqrt{n}} \tag{4.12}$$

([5], pp. 565-566). By applying (4.9) to (4.7), it follows that

$$\begin{aligned}|\mathbb{P}(W \le z) - \Phi(z)| &= \left|\mathbb{E}f_z'(W) - \mathbb{E}\int_{-\infty}^{\infty} f_z'(W+t)K(t)\,dt\right| \\ &\le \left|\mathbb{E}f_z'(W) - \mathbb{E}f_z'(W)\int_{-\infty}^{\infty} K(t)\,dt\right| \\ &\quad + \left|\mathbb{E}\int_{-\infty}^{\infty}\{f_z'(W) - f_z'(W+t)\}K(t)\,dt\right| \\ &= A + B,\end{aligned}$$

where

$$A = \left|\mathbb{E}f_z'(W) - \mathbb{E}f_z'(W)\int_{-\infty}^{\infty} K(t)\,dt\right|,$$

and

$$B = \left|\mathbb{E}\int_{-\infty}^{\infty}\{f_z'(W) - f_z'(W+t)\}K(t)\,dt\right|.$$

From this fact and the fact that

$$\left|f_z'(w) - f_z'(w+t) - \int_t^0 h(w+u)\,du\right| \le \mathbf{1}(z - \max(0,t) < w < z - \min(0,t)), \tag{4.13}$$

where

$$h(w) = (wf_z(w))' = \begin{cases}(\sqrt{2\pi}(1+w^2)e^{w^2/2}\Phi(w) + w)(1-\Phi(z)), & \text{if } w \le z,\\ (\sqrt{2\pi}(1+w^2)e^{w^2/2}(1-\Phi(w)) - w)\Phi(z), & \text{if } w > z\end{cases} \tag{4.14}$$

([18], p. 250), we have

$$|\mathbb{P}(W \le z) - \Phi(z)| \le A + B_1 + B_2 + B_3,$$

where

$$B_1 = \mathbb{E}\mathbf{1}\left(\Delta > \frac{4e\sqrt{2}}{\sqrt{n}}\right)\int_{-\infty}^{\infty}\left|\left(f_z'(W) - f_z'(W+t)\right)\right| K(t)\,dt, \tag{4.15}$$

$$B_2 = \mathbb{E}\mathbf{1}\left(\Delta \le \frac{4e\sqrt{2}}{\sqrt{n}}\right)\int_{-\infty}^{\infty}\mathbf{1}\,(z - \max(0,t) < W < z + \min(0,t))\,K(t)\,dt, \tag{4.16}$$

and

$$B_3 = \mathbb{E}\mathbf{1}\left(\Delta \le \frac{4e\sqrt{2}}{\sqrt{n}}\right)\int_{-\infty}^{\infty}\int_t^0 h(W+u)K(t)\,du\,dt. \tag{4.17}$$

Hence, we will divide the proof into four steps.

Step 1. Bounding A**.** By using (1.9), Propositions 3.2, 3.3, and 3.4(3), we have

$$\begin{aligned}\mathbb{P}(W > 0.9z) &\le \frac{\mathbb{E}(W+0.9)^6}{(0.9z+0.9)^6} \\ &= \left(\frac{10/9}{z+1}\right)^6 (\mathbb{E}W^6 + 5.4\mathbb{E}W^5 + 12.15\mathbb{E}W^4 \\ &\quad + 14.58\mathbb{E}W^3 + 9.8415\mathbb{E}W^2 + 3.54294\mathbb{E}W + 0.531441) \\ &\le \frac{117.084}{(z+1)^6}. \end{aligned} \tag{4.18}$$

From the Gaussian tail, $1 - \Phi(w) \le e^{-w^2/2}/\left(\sqrt{2\pi}w\right)$ for $w > 0$ ([17], p. 38) and (4.2), we have

$$0 \ge wf_z(w) = \sqrt{2\pi}we^{-w^2/2}(1-\Phi(-w))(1-\Phi(z)) \ge -(1-\Phi(z))$$

for $w < 0$. From this fact, (4.3), and (4.6), we have $0 \le f_z'(w) \le 0.9zf_z(0.9z) + 1 - \Phi(z)$ for $w \le 0.9z$. By applying the Gaussian tail for $w \le 0.9z$ and $z \ge 35$, we have

$$\begin{aligned}0 \le f_z'(w) &\le \left(0.9z\sqrt{2\pi}e^{81z^2/200} + 1\right)(1-\Phi(z)) \\ &\le \left(0.9z\sqrt{2\pi}e^{81z^2/200} + 1\right)\frac{1}{\sqrt{2\pi}ze^{z^2/2}} \\ &\le \frac{1}{e^{19z^2/200}}.\end{aligned}$$

By using the fact that $e^{19z^2/200}/(z+1)^6 \ge 1.59 \times 10^{41}$ for $z \ge 35$, we have $f_z'(w) \le 6.27 \times 10^{-42}/(z+1)^6$, which implies that

$$\mathbb{E}|f_z'(W)|\mathbf{1}(W \le 0.9z) \le \frac{6.27 \times 10^{-42}}{(z+1)^6}. \tag{4.19}$$

From (4.4), (4.18), and (4.19), it follows that

$$\begin{aligned}\mathbb{E}|f_z'(W)|^2 &\le \mathbb{E}|f_z'(W)| \\ &= \mathbb{E}|f_z'(W)|\mathbf{1}(W \le 0.9z) + \mathbb{E}|f_z'(W)|\mathbf{1}(W > 0.9z) \\ &\le \frac{6.27 \times 10^{-42}}{(z+1)^6} + \mathbb{P}(W > 0.9z) \\ &\le \frac{117.085}{(z+1)^6}.\end{aligned} \tag{4.20}$$

Shao and Su [10, p. 2155] also showed that

$$\sqrt{\mathbb{E}\left(1 - \frac{n+1}{4}\mathbb{E}^W\Delta^2\right)^2} \le \sqrt{\frac{3}{4n}}. \tag{4.21}$$

By applying (4.20), (4.21), and the fact that

$$\int_{-\infty}^{\infty} K(t)\,dt = \frac{n+1}{4}\Delta^2 \tag{4.22}$$

([17], p. 22) to term A, we have

$$
\begin{aligned}
A &= \left|\mathbb{E}f_z'(W)\left(1-\int_{-\infty}^{\infty}K(t)\,dt\right)\right| \\
&= \left|\mathbb{E}f_z'(W)\left(1-\frac{n+1}{4}\mathbb{E}^W\Delta^2\right)\right| \\
&\le \sqrt{\mathbb{E}|f_z'(W)|^2}\sqrt{\mathbb{E}\left(1-\frac{n+1}{4}\mathbb{E}^W\Delta^2\right)^2} \\
&= \frac{9.371}{\sqrt{n}(z+1)^3}.
\end{aligned}
\tag{4.23}
$$

Step 2. Bounding B_1. From (4.11), (4.12), (4.5), and (4.22),

$$
\begin{aligned}
B_1 &= \mathbb{E}\mathbf{1}\left(\Delta>\frac{4e\sqrt{2}}{\sqrt{n}}\right)\int_{-\infty}^{\infty}|f_z'(W)-f_z'(W+t)|K(t)\,dt \\
&\le \mathbb{E}\mathbf{1}\left(\Delta>\frac{4e\sqrt{2}}{\sqrt{n}}\right)\int_{-\infty}^{\infty}K(t)\,dt \\
&\le \frac{n+1}{4}\mathbb{E}\Delta^2\mathbf{1}\left(\Delta>\frac{4e\sqrt{2}}{\sqrt{n}}\right) \\
&\le 2(n+1)\mathbb{P}\left(\Delta>\frac{4e\sqrt{2}}{\sqrt{n}}\right) \\
&\le \frac{4(n+1)}{e^{2e\sqrt{n}}}.
\end{aligned}
\tag{4.24}
$$

Because $0<z<n/\sqrt{2}$, and $n\ge 51$,

$$
\frac{n+1}{e^{2e\sqrt{n}}}=\frac{n+1}{e^{e\sqrt{n}}e^{e\sqrt{n}}}\le\frac{n+1}{(n+1)\sqrt{n}e^{e\sqrt{n}}}\le\frac{1}{\sqrt{n}e^{e\sqrt{z\sqrt{2}}}}\le\frac{2.31\times10^{-4}}{\sqrt{n}(z+1)^3}.
$$

Hence,

$$
B_1\le\frac{9.24\times10^{-4}}{\sqrt{n}(1+z)^3}.
\tag{4.25}
$$

Step 3. Bounding B_2. Let $\delta=4e\sqrt{2}/\sqrt{n}$, and define $f_\delta:\mathbb{R}\to\mathbb{R}$ as

$$
f_\delta(t)=\begin{cases}0, & \text{if } t<z-2\delta,\\ t^5(t-z+2\delta), & \text{if } z-2\delta\le t\le z+2\delta,\\ 4\delta t^5, & \text{if } t>z+2\delta.\end{cases}
$$

Then, for $t\in(z-2\delta,z+2\delta)$,

$$
f_\delta'(t)=6t^5-5t^4z+10t^4\delta=6t^5-5t^4(z-2\delta)\ge 6t^5-5t^5=t^5.
$$

Therefore,

$$f_\delta'(t) \geq \begin{cases} t^5, & \text{if } t \in (z-2\delta, z+2\delta), \\ 0, & \text{if } t \notin [z-2\delta, z+2\delta]. \end{cases} \tag{4.26}$$

If $|t| \leq \Delta \leq \delta$, and $W > z-\delta$, then $W+t \geq W-\Delta \geq z-2\Delta \geq z-2\delta > 0$. That is, $(W+t)/(z-2\delta) \geq 1$. We also see that f_δ is increasing. For these reasons, together with (4.9) and (4.26), we have

$$\begin{aligned} 0 \leq B_2 &\leq \frac{1}{(z-2\delta)^5}\mathbb{E}\mathbf{1}\left(\Delta \leq \frac{4e\sqrt{2}}{\sqrt{n}}\right)\int_{-\infty}^{\infty}(W+t)^5\mathbf{1}(z-\Delta < W < z+\Delta)K(t)\,dt \\ &\leq \frac{1}{(z-2\delta)^5}\mathbb{E}\int_{-\infty}^{\infty} f_\delta'(W+t)K(t)\,dt \\ &\leq \frac{1}{(z-2\delta)^5}\mathbb{E}|W|f_\delta(W) \\ &\leq \frac{4}{(z-2\delta)^5}\mathbb{E}W^6\delta \\ &\leq \frac{246.4e\sqrt{2}}{\sqrt{n}(z-2\delta)^5}. \end{aligned}$$

From the fact that $(z-2\delta)^5 \geq 583.89(z+1)^3$ for $z \geq 35$ and $n \geq 51$, it follows that

$$B_2 \leq \frac{1.63}{\sqrt{n}(z+1)^3}. \tag{4.27}$$

Step 4. Bounding B_3. We separate B_3 into two terms as

$$B_3 = B_{31} + B_{32},$$

where

$$B_{31} = \mathbb{E}\mathbf{1}\left(\Delta \leq \frac{4e\sqrt{2}}{\sqrt{n}}\right)\int_{-\infty}^{\infty}\int_t^0 h(W+u)K(t)\mathbf{1}\left(W+u \leq \frac{99z}{100}\right)du\,dt, \tag{4.28}$$

and

$$B_{32} = \mathbb{E}\mathbf{1}\left(\Delta \leq \frac{4e\sqrt{2}}{\sqrt{n}}\right)\int_{-\infty}^{\infty}\int_t^0 h(W+u)K(t)\mathbf{1}\left(W+u > \frac{99z}{100}\right)du\,dt. \tag{4.29}$$

From (4.14), if $w \leq 99z/100$, then

$$\begin{aligned} 0 \leq h(w) &\leq \left[\sqrt{2\pi}\left(1+\frac{9801z^2}{10000}\right)e^{9801z^2/20000} + z\right]\frac{e^{-z^2/2}}{\sqrt{2\pi}z} \\ &= \left(\frac{1}{z}+\frac{9801z}{10000}\right)e^{-199z^2/20000} + \frac{e^{-z^2/2}}{\sqrt{2\pi}} \\ &\leq \frac{8.154}{(1+z)^3}, \end{aligned} \tag{4.30}$$

where the last inequality follows from the fact that $e^{199z^2/20000}/\left[z(z+1)^3\right] \geq 0.1203$, $e^{z^2/2}/(z+1)^3 \geq 2.17 \times 10^{261}$, and $ze^{199z^2/20000}/(z+1)^3 \geq 147.4$ for $z \geq 35$.

By using $f(t) = t$ in (4.9) and (4.22), we have $\mathbb{E}W^2 = (n+1)\,\mathbb{E}\Delta^2/4$. By (1.9), it follows that

$$\mathbb{E}\Delta^2 \leq \frac{4}{n+1}.$$

Hence,

$$\begin{aligned}
\mathbb{E}\mathbf{1}\left(\Delta \leq \frac{4e\sqrt{2}}{\sqrt{n}}\right)\int_{-\infty}^{\infty}\int_{t}^{0} K(t)\,du\,dt &\leq \mathbb{E}\mathbf{1}\left(\Delta \leq \frac{4e\sqrt{2}}{\sqrt{n}}\right)\int_{-\infty}^{\infty} |t|K(t)\,dt \\
&= \frac{n+1}{8}\mathbb{E}\mathbf{1}\left(\Delta \leq \frac{4e\sqrt{2}}{\sqrt{n}}\right)\Delta^3 \\
&\leq \frac{e(n+1)}{\sqrt{2n}}\mathbb{E}\Delta^2 \\
&= \frac{2e\sqrt{2}}{\sqrt{n}}.
\end{aligned} \tag{4.31}$$

From (4.28), (4.30), and (4.31), we have

$$B_{31} \leq \frac{8.154}{(1+z)^3}\mathbb{E}\mathbf{1}\left(\Delta \leq \frac{4e\sqrt{2}}{\sqrt{n}}\right)\int_{-\infty}^{\infty}\int_{t}^{0} K(t)\,du\,dt \leq \frac{62.692}{\sqrt{n}(z+1)^3}. \tag{4.32}$$

Chuntee and Neammanee [19] also showed that $0 \leq h(w) \leq 1.001(1+z)$ for $w > 0.99z$. By using the fact that $(z-2.1758)^6/(z+1)^6 \geq 0.5745$ for $z \geq 35$ and Proposition 3.3, we have

$$\mathbb{P}(W > 0.99z - 2.154) \leq \frac{\mathbb{E}W^6}{(0.99z-2.154)^6} \leq \frac{16.358}{(z-2.1758)^6} \leq \frac{28.48}{(z+1)^6} \leq \frac{0.02198}{(z+1)^4}. \tag{4.33}$$

Note that $4e\sqrt{2}/\sqrt{n} \leq 2.154$ for $n \geq 51$. Hence,

$$\begin{aligned}
B_{32} &\leq 1.001(1+z)\mathbb{E}\mathbf{1}\left(\Delta \leq \frac{4e\sqrt{2}}{\sqrt{n}}\right)\int_{-\infty}^{\infty}\int_{t}^{0} K(t)\mathbf{1}(W+u > 0.99z)\,du\,dt \\
&\leq 1.001(1+z)\mathbb{E}\mathbf{1}\left(\Delta \leq \frac{4e\sqrt{2}}{\sqrt{n}}\right)\int_{-\infty}^{\infty}\int_{t}^{0} K(t)\mathbf{1}(W+\Delta > 0.99z)\,du\,dt \\
&\leq 1.001(1+z)\mathbb{E}\mathbf{1}\left(\Delta \leq \frac{4e\sqrt{2}}{\sqrt{n}}\right)\int_{-\infty}^{\infty}\int_{t}^{0} K(t)\mathbf{1}\left(W > 0.99z - \frac{4e\sqrt{2}}{\sqrt{n}}\right)du\,dt \\
&\leq 1.001(1+z)\frac{n+1}{8}\mathbb{E}\mathbf{1}(W > 0.99z - 2.154)\mathbf{1}\left(\Delta \leq \frac{4e\sqrt{2}}{\sqrt{n}}\right)\Delta^3 \\
&\leq 1.001(1+z)\frac{1}{4}\frac{(4e\sqrt{2})^3}{\sqrt{n}}\mathbb{P}(W > 0.99z - 2.154) \\
&\leq \frac{20}{\sqrt{n}(1+z)^3}.
\end{aligned} \tag{4.34}$$

From (4.23), (4.25), (4.27), (4.32), and (4.34), the main theorem is proved.

5. Application

One important application of character ratios arises in the study of random walks on the symmetric group S_n. A fundamental challenge in this area is analyzing the convergence rate, or mixing time, of such Markov chains. Using Fourier analysis on finite groups, Diaconis and Shahshahani [6] showed that the eigenvalues of the transition matrix are given precisely by character ratios, providing the essential spectral information needed to bound the convergence rate. For illustration, we consider a probability measure T on the symmetric group S_n:

$$T(\pi) = \begin{cases} \dfrac{1}{n}, & \text{if } \pi \text{ is the identity,} \\ \dfrac{2}{n^2}, & \text{if } \pi \text{ is a transposition,} \\ 0, & \text{otherwise,} \end{cases}$$

where $\pi \in S_n$. Let $\pi_i, \pi_j \in S_n$. Because $\pi_j = \pi_j \pi_i^{-1} \pi_i$, we define a transition probability that a permutation π_i moves to π_j as

$$\mathbf{M}(\pi_i, \pi_j) = T(\pi_j \pi_i^{-1}).$$

For example, in S_3, we have $\mathbf{M}((12),(123)) = T((123)(12)^{-1}) = T((23)) = 2/9$ and $\mathbf{M}((13),(12)) = T((12)(13)^{-1}) = T((132)) = 0$. For a random walk on S_n, the transition matrix $\mathbf{M}$ is an $n! \times n!$ matrix, where the (i, j)-th entry is $\mathbf{M}(\pi_i, \pi_j)$. The following is the transition matrix of S_3.

Table 2. Transition matrix of S_3.

π_i \ π_j	e	(12)	(13)	(23)	(123)	(132)
e	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	0	0
(12)	$\frac{2}{9}$	$\frac{1}{3}$	0	0	$\frac{2}{9}$	$\frac{2}{9}$
(13)	$\frac{2}{9}$	0	$\frac{1}{3}$	0	$\frac{2}{9}$	$\frac{2}{9}$
(23)	$\frac{2}{9}$	0	0	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{2}{9}$
(123)	0	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{1}{3}$	0
(132)	0	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	0	$\frac{1}{3}$

In 1981, Diaconis and Shahshahani [6] showed that the eigenvalues of the transition matrix M are

$$\Sigma(\lambda) = \frac{1}{n} + \frac{n-1}{n} \frac{\chi^\lambda(12)}{\dim(\lambda)}, \tag{5.1}$$

where λ is a partition of n with multiplicity $\dim^2(\lambda)$. By applying Theorem 2.1 to (5.1), it follows that

$$\begin{aligned}\left|\mathbb{P}(\Sigma \le z) - \Phi\left(\frac{nz-1}{\sqrt{2}}\right)\right| &= \left|\mathbb{P}\left(\frac{\sqrt{2}W+1}{n} \le z\right) - \Phi\left(\frac{nz-1}{\sqrt{2}}\right)\right| \\ &= \left|\mathbb{P}\left(W \le \frac{nz-1}{\sqrt{2}}\right) - \Phi\left(\frac{nz-1}{\sqrt{2}}\right)\right| \\ &\le \frac{93.7}{\sqrt{n}\left(1+\left|\frac{nz-1}{\sqrt{2}}\right|\right)^3},\end{aligned}$$

provided that $|nz-1|/\sqrt{2} \ge 35$, and $n \ge 51$.

6. Conclusions and discussion

In this paper, we give a nonuniform bound in the normal approximation for the character ratio of the form $93.7/\left[\sqrt{n}(1+|z|)^3\right]$. In fact, one can apply Theorem 2 from [12] to obtain a bound, but the constant in our bound, 93.7, is much smaller than the corresponding constant obtained from [12], which is at least 2.55×10^{10}. The key tool is that our bound for $\mathbb{E}W^4$ and $\mathbb{E}W^6$ is much smaller than that obtained in [12, Appendix A], as we use the Jack measure and the Metropolis algorithm, whereas Thành and Tu [12] derived their bound using the exchangeable pairs method, which can be applied to (4.9).

In general, one can extend our idea to any order of nonuniform bound. That is, we can make a bound in the form of $C/\left[(1+|z|)^r\sqrt{n}\right]$ by using the method of the Metropolis chain to find the explicit form of $\mathbb{E}W^{2r}$.

Author contributions

Natthapol Dejtrakulwongse: Methodology, validation, visualization, writing–original draft, writing–review and editing; Suporn Jongpreechaharn: Conceptualization, methodology, supervision, validation, writing–original draft, writing–review and editing; Kritsana Neammanee: Conceptualization, methodology, supervision, validation, writing–original draft, writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

No artificial intelligence (AI) tools were used in the writing, analysis, or preparation of this manuscript. The authors take full responsibility for the content, interpretation, and conclusions of this work.

Acknowledgments

The authors would like to thank the referees for their valuable comments and suggestions, which have helped improve the quality of the manuscript.

Conflict of interest

All authors declare that they have no conflicts of interest.

References

1. P. Etingof, O. Golberg, S. Hensel, T. Liu, A. Schwendner, D. Vaintrob, et al., *Introduction to representation theory*, American Mathematical Society, 2011. https://doi.org/10.1090/stml/059
2. S. Axler, *Linear algebra done right*, 4 Eds., Springer Cham, 2024. https://doi.org/10.1007/978-3-031-41026-0
3. J. Serre, *Linear representations of finite groups*, Graduate Texts in Mathematics, Vol. 42, New York: Springer, 1977. https://doi.org/10.1007/978-1-4684-9458-7
4. B. Sagan, *The symmetric group: representations, combinatorial algorithms, and symmetric functions*, Graduate Texts in Mathematics, Vol. 203, New York: Springer, 2010. https://doi.org/10.1007/978-1-4757-6804-6
5. J. Fulman, Stein's method and Plancherel measure of the symmetric group, *Trans. Amer. Math. Soc.*, **357** (2005), 555–570.
6. P. Diaconis, M. Shahshahani, Generating a random permutation with random transpositions, *Z. Wahrscheinlichkeitstheor Verw. Gebiete*, **57** (1981), 159–179. https://doi.org/10.1007/BF00535487
7. F. G. Frobenius, Über die Charaktere der symmetrischen Gruppe, *Sitz. König. Preuß. Akad. Wissen.*, 1900, 516–534.
8. S. Kerov, Gaussian limit for the Plancherel measure of the symmetric group, *C. R. Acad. Sci. Paris*, **316** (1993), 303–308.
9. V. Ivanov, G. Olshanski, Kerov's central limit theorem for the Plancherel measure on Young diagrams, In: S. Fomin, *Symmetric functions 2001: surveys of developments and perspectives*, NATO Science Series, Springer, Dordrecht, **74** (2002), 93–151. https://doi.org/10.1007/978-94-010-0524-1_3
10. Q. M. Shao, Z. Su, The Berry-Esseen bound for character ratios, *Proc. Amer. Math. Soc.*, **134** (2006), 2153–2159.
11. L. H. Y. Chen, M. Raič, L. Thành, On the error bound in the normal approximation for Jack measures, *Bernoulli*, **27** (2021), 442–468. https://doi.org/10.3150/20-BEJ1245
12. L. Thành, N. Tu, Non-uniform Berry–Esseen bounds for exchangeable pairs with applications to the mean-field classical N-vector models and Jack measures, *C. R. Math.*, **363** (2025), 757–776. https://doi.org/10.5802/crmath.711
13. A. Okounkov, The uses of random partitions, *XIVth International Congress on Mathematical Physics*, 2006, 379–403. https://doi.org/10.1142/9789812704016_0037
14. D. S. Dummit, R. M. Foote, *Abstract algebra*, 3 Eds., Hoboken, 2004.
15. J. Fulman, Stein's method, Jack measure, and the Metropolis algorithm, *J. Comb. Theory, Ser. A*, **108** (2004), 275–296. https://doi.org/10.1016/j.jcta.2004.07.003

16. N. Dejtrakulwongse, K. Neammanee, S. Jongpreechaharn, Non-uniform bounds in normal approximation for descent and inversion, *AIMS Math.*, **11** (2026), 3903–3919. https://doi.org/10.3934/math.2026157
17. L. H. Y. Chen, L. Goldstein, Q. Shao, *Normal approximation by Stein's method*, Springer Berlin, 2011. https://doi.org/10.1007/978-3-642-15007-4
18. L. H. Y. Chen, Q. Shao, A non-uniform Berry–Esseen bound via Stein's method, *Probab. Theory Relat. Fields*, **120** (2001), 236–254. https://doi.org/10.1007/PL00008782
19. W. Chuntee, K. Neammanee, Exponential bounds for normal approximation of the number of descents and inversions, *Commun. Stat.-Theory Methods*, **46** (2017), 1218–1229. https://doi.org/10.1080/03610926.2015.1014109

Appendix

Lower bound for $C(3)$

In [12], Thành and Tu gave a nonuniform bound for exchangeable pairs stated in Theorem A.1.

Theorem A.1. *Let $p \geq 1$ and $a > 0$. Let (W, W') be an exchangeable pair satisfying $\mathbb{E}(W'|W) = (1-\lambda)W + R$ for some random variable R, and let $\Delta = W - W'$. Then, for every $z \in \mathbb{R}$, we have*

$$\begin{aligned}&|\mathbb{P}(W \leq z) - \Phi(z)|\\ &\leq \frac{C(p)(1+\mathbb{E}|W|^{2p})}{(1+|z|)^p}\left(\sqrt{\mathbb{E}\left(1 - \frac{1}{2\lambda}\mathbb{E}(\Delta^2 \mid W)\right)^2} + \frac{\sqrt{\mathbb{E}R^2}}{\lambda} + a + \frac{a^3}{\lambda} + \frac{\sqrt{\mathbb{E}\Delta^4 \mathbf{1}(|\Delta| > a)}}{\lambda}\right),\end{aligned}$$

where $C(p)$ is a constant depending only on p.

Because the character ratio W which is defined in (1.7) has an exchangeable pair W' that satisfies $\mathbb{E}(W'|W) = [1 - 2/(n+1)]\,W$, the components can be applied using Theorem A.1. In this appendix, we show that the constant in the case of $p = 3$ for a character ratio is greater than 2.55×10^{10}.

In the proof of Theorem A.1, the authors made a bound such that

$$|\mathbb{P}(W < z) - \Phi(z)| \leq T_1 + T_2 + \left(\frac{n+1}{4}\right)(T_{3,1} + T_{3,2,1} + T_{3,2,2}), \tag{A.1}$$

where

$$\begin{aligned}T_1 &= \mathbb{E}\left[f_z'(W)\left(1 - \frac{n+1}{4}(W - W')^2\right)\right],\\ T_2 &= \frac{n+1}{2}\mathbb{E}\left(f_z(W)R\right),\\ T_{3,1} &= \mathbb{E}\left(\Delta \mathbf{1}(|\Delta| > a)\left(f_z(W) - f_z(W - \Delta) - \Delta f_z'(W)\right)\right),\\ T_{3,2,1} &= \mathbb{E}\left(\Delta \mathbf{1}(|\Delta| \leq a)\int_{-\Delta}^{0}((W+t)f_z(W+t) - Wf_z(W))\,dt\right),\\ T_{3,2,2} &= \mathbb{E}\left(\Delta \mathbf{1}(|\Delta| \leq a)\int_{-\Delta}^{0}(\mathbf{1}(W + t \leq z) - \mathbf{1}(W \leq z))\,dt\right),\end{aligned}$$

f_z is the solution of (4.1), and

$$a = \frac{2e\sqrt{2(2+e)^2}}{\sqrt{n-1}} \leq \frac{36.277}{\sqrt{n-1}} \text{ for } n \geq 3. \tag{A.2}$$

To illustrate how large the constant in Theorem A.1 is, we focus on the constant of the term $T_{3,2,1}$ only. From p. 762 of [12], we have

$$|T_{3,2,1}| \leq \frac{an}{n+1} \int_0^a \int_{-t}^t \mathbb{E}h(W+u)\, du\, dt, \tag{A.3}$$

where h is defined by (4.14). We also make a bound of $\mathbb{E}h(W+u)$ as follows. First, we note that

$$(a+b)^n \leq 2^{n-1}(a^n + b^n) \tag{A.4}$$

for any $a, b > 0$. Equation (86) of [12] has

$$\mathbb{E}h^2(W+a) \leq \frac{\pi}{4} + 2\mathbb{E}(W+a)^2 \leq 2.786 + 2a^2 \leq 2.786 + \frac{2633}{n-1}. \tag{A.5}$$

By Hölder's inequality, Markov's inequality, (A.4), and (A.5), we obtain

$$\begin{aligned}
\mathbb{E}h(W+a)\mathbf{1}\left(W+a > \frac{z}{2}\right) &\leq \sqrt{\mathbb{E}h^2(W+a)}\sqrt{\mathbb{E}\mathbf{1}\left(W+a > \frac{z}{2}\right)} \\
&\leq \frac{\sqrt{\mathbb{E}h^2(W+a)}\sqrt{\mathbb{E}(W+a+1)^6}}{\left(1+\frac{z}{2}\right)^3} \\
&\leq \frac{\sqrt{2.786 + \frac{2633}{n-1}}\sqrt{32(\mathbb{E}(W^6 + (a+1)^6))}}{\left(1+\frac{z}{2}\right)^3} \\
&\leq \frac{C_0\sqrt{1+\mathbb{E}W^6}}{(1+z)^3},
\end{aligned}$$

where $C_0 \geq 75.6$. Thành and Tu [12] also showed that

$$\mathbb{E}h(W+a)\mathbf{1}(W+a \leq 0) + \mathbb{E}h(W+a)\mathbf{1}\left(0 < W+a \leq \frac{z}{2}\right) \leq \frac{C_1\sqrt{1+\mathbb{E}W^6}}{(1+z)^3}$$

for some constant $C_1 \geq 0$. Then, it follows that

$$\begin{aligned}
\mathbb{E}h(W+a) &= \mathbb{E}h(W+a)\mathbf{1}(W+a \leq 0) + \mathbb{E}h(W+a)\mathbf{1}\left(0 < W+a \leq \frac{z}{2}\right) \\
&\quad + \mathbb{E}h(W+a)\mathbf{1}\left(W+a > \frac{z}{2}\right) \leq \frac{C_2\sqrt{1+\mathbb{E}W^6}}{(1+z)^3}
\end{aligned}$$

for some $C_2 \geq 75.6$. Therefore,

$$|T_{3,2,1}| \leq \frac{an}{n+1}\frac{C_2\sqrt{1+\mathbb{E}W^6}}{(1+z)^3}\int_0^a \int_{-t}^t du\, dt = \frac{a^3 n}{n+1}\frac{C_2\sqrt{1+\mathbb{E}W^6}}{(1+z)^3}. \tag{A.6}$$

From (A.6), it is left to bound $\mathbb{E}W^6$. Thành and Tu applied [12] $f(t) = t^5$ in (4.9) and (4.10) to show that

$$
\begin{aligned}
\mathbb{E}W^6 &= 5\mathbb{E}\int_{-\infty}^{\infty}(W+t)^4K(t)\,dt \\
&= \frac{n+1}{4}\mathbb{E}\Delta^6 + \frac{5(n+1)}{4}\mathbb{E}W\Delta^5 + \frac{5(n+1)}{2}\mathbb{E}W^2\Delta^4 \\
&\quad + \frac{5(n+1)}{2}\mathbb{E}W^3\Delta^3 + \frac{5(n+1)}{4}\mathbb{E}W^4\Delta^2 \\
&\le \frac{n+1}{4}\mathbb{E}\Delta^6 + \left(\frac{1}{384}\mathbb{E}W^6 + \frac{5^{2.2}(n+1)^{1.2}}{6\cdot 2^{1.2}}\mathbb{E}\Delta^6\right) + \left(\frac{1}{24}\mathbb{E}W^6 + \frac{10(n+1)^{1.5}}{3}\mathbb{E}\Delta^6\right) \\
&\quad + \left(\frac{1}{8}\mathbb{E}W^6 + \frac{5(n+1)^2}{2}\mathbb{E}\Delta^6\right) + \left(\frac{1}{3\sqrt{2}}\mathbb{E}W^6 + \frac{5(n+1)^3}{24}\mathbb{E}\Delta^6\right),
\end{aligned}
$$

where we use the fact that $|ab| \le |a|^p/p + |b|^q/q$ for $1/p + 1/q = 1$ and $p, q > 0$ in the last inequality. Therefore,

$$
\mathbb{E}W^6 \le 1.69\left(\frac{n+1}{4} + \frac{5^{2.2}(n+1)^{1.2}}{6\cdot 2^{1.2}} + \frac{10(n+1)^{1.5}}{3} + \frac{5(n+1)^2}{2} + \frac{5(n+1)^3}{24}\right)\mathbb{E}\Delta^6. \tag{A.7}
$$

Because

$$
\mathbb{P}(\Delta > a) \le \frac{1}{e^{2e(2+e)\log n}} = \frac{1}{n^{2e(2+e)}}, \tag{A.8}
$$

and

$$
|\Delta| \le 4 \tag{A.9}
$$

([12], p. 769), we have

$$
\mathbb{E}\Delta^6 \le a^6 + 4^6\mathbb{P}(\Delta > a) \le \frac{36.277^6}{(n-1)^3} + \frac{4^6}{n^{2e(2+e)}} \le \frac{C_3}{(n-1)^3} \tag{A.10}
$$

for some $C_3 \ge 2.27\times 10^9$. By (A.7) and (A.10), it follows that

$$
\mathbb{E}W^6 \le C_4\left(\frac{n+1}{n-1}\right)^3
$$

for some $C_4 \ge 7.99\times 10^8$. From this fact and (A.6), it follows that

$$
|T_{3,2,1}| \le \frac{n}{n+1}\,\frac{C_2\sqrt{1+\mathbb{E}W^6}}{(1+z)^2}\,\frac{36.277^3}{(n-1)^{\frac{3}{2}}} \le \frac{C_5}{(1+z)^3(n-1)^{\frac{3}{2}}}
$$

for some $C_5 \ge 1.02\times 10^{11}$. Hence, by (A.1), $C(3) \ge 2.55\times 10^{10}$.

AIMS Press