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Research article

Spherically symmetric sprays of isotropic curvature

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Abstract: The classification of sprays with isotropic curvature is a central theme in spray geometry. In this paper, we investigated a class of spherically symmetric sprays—those invariant under spherical rotations. First, we derived a necessary and sufficient condition for such a spray to have isotropic curvature; the condition was a system of differential equations. For the important subclass of projectively affine spherically symmetric sprays, we then obtained the corresponding equivalent system of partial differential equations (PDEs). Furthermore, we constructed several explicit sprays of isotropic curvature.

Keywords: spray geometry; isotropic curvature; spherically symmetric spray; projectively affine
Mathematics Subject Classification: 53B40, 53C60

1. Introduction

Spray geometry provides a natural framework for studying geodesic behavior independently of a metric. A spray on a manifold is a vector field on the tangent bundle whose integral curves are interpreted as geodesics. Every Finsler metric induces a natural spray, but the converse is not true: there exist sprays that are not metrizable by any Finsler metric, and such examples have been discovered in recent years [1–4]. In local coordinates (x^i, y^i), a spray G on an n-dimensional manifold M is a smooth vector field on $TM \setminus \{0\}$ expressed as

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i},$$

where the coefficients $G^i = G^i(x, y)$ are smooth and positively homogeneous of degree two in y.

Spherical symmetry (rotational symmetry) has long played a distinguished role in differential geometry. A Finsler metric F on a domain $\mathcal{U} \subset \mathbb{R}^n$ is called spherically symmetric if it is invariant under the orthogonal group $O(n)$, i.e., $F(Ux, Uy) = F(x, y)$ for all $U \in O(n)$. It was first studied by

Rutz [5]. Zhou [6, 7] derived the general form of such metrics,

$$F = |y|\,\phi\Big(|x|^2, \langle x, y\rangle/|y|\Big),$$

where $\phi(r, s)$ is a positive smooth function and $r = |x|^2$, $s = \langle x, y\rangle/|y|$. Here and throughout, $|\cdot|$ and $\langle\cdot,\cdot\rangle$ denote the classical Euclidean norm and inner product, respectively. Zhou [6] also classified the projectively flat spherically symmetric Finsler metrics of constant flag curvature. More general results on projectively flat Finsler metrics of constant flag curvature can be found in [8, 9]. Many interesting metrics in Riemannian and Finsler geometry are spherically symmetric, such as Riemannian metrics of constant sectional curvature in $\mathbb{R}^n$, the Funk metric, the Berwald's metric, the Bryant metric, etc. There are many results on spherically symmetric Finsler metrics in the past years; some of them can be found in [10–15]. These studies naturally motivate us to study sprays that are invariant under orthogonal transformations.

Recently, Gui and Li [16] proved that a spray G on $\mathcal{U} \subset \mathbb{R}^n$ is invariant under orthogonal transformations if, and only if, its coefficients are of the form

$$G^i = |y|Ay^i + |y|^2Bx^i, \tag{1.1}$$

where $r = |x|^2$, $s = \langle x, y\rangle/|y|$, and $A = A(r, s)$, $B = B(r, s)$ are smooth functions. Such sprays are called spherically symmetric sprays. Obviously, the geodesic coefficients (also called spray coefficients) of any spherically symmetric Finsler metric are of the above form (1.1).

A spray is said to have *isotropic curvature* if its Riemann curvature tensor $R^i_{\ k}$ satisfies

$$R^i_{\ k} = R\delta^i_k - \tau_k y^i,$$

where $R = R(x, y)$ and $\tau_k = \frac{1}{2}\frac{\partial R}{\partial y^k}$ are homogeneous functions in y of degree two and one, respectively. This generalizes the notion of constant flag curvature from Finsler geometry. Li and Shen [4] systematically studied sprays with isotropic curvature, establishing fundamental properties and constructing examples that are not metrizable. Lou and Li [17] introduced sprays of weakly isotropic curvature and provided two families of non-metrizable examples. In [18], Cheng, Cao and Qing determined the relationship between the χ-curvatures of two projectively related sprays and found an approach to construct sprays of isotropic curvature. Yang [1] further explored the connection between projectively flat sprays and isotropic curvature, and exhibited projectively flat sprays that cannot be induced by any Finsler metric.

A natural problem therefore arises:

What is the classification of spherically symmetric sprays of isotropic curvature?

In this paper we study this problem. Our first main result provides a necessary and sufficient condition for a spherically symmetric spray to have isotropic curvature, expressed as a system of partial differential equations for $A(r, s)$ and $B(r, s)$. Here and throughout, the subscript "s" denotes the partial derivative of a function with respect to s. For example, $A_s = \frac{\partial A}{\partial s}$, $A_{rs} = \frac{\partial^2 A}{\partial r\partial s}$, $A_{ss} = \frac{\partial^2 A}{\partial s^2}$, etc. The Einstein summation is used in the whole paper.

Theorem 1. *Let G be a spherically symmetric spray on an open subset $\Omega \subset \mathbb{R}^n$ with G^i expressed in* (1.1). *Then, G has isotropic curvature if, and only if, the following system of equations holds:*

$$\begin{cases} 2\left[(r - s^2)A_{ss} - sA_s + A\right]B - 2sA_{rs} - A_{ss} + 2A_r = 0, \\ \left[2(r - s^2)B - 1\right]B_{ss} - 2sB_{rs} - (r - s^2)B_s^2 - 2sBB_s + 4B_r + 4B^2 = 0. \end{cases} \tag{1.2}$$

In particular, the spray has zero curvature if, and only if, G^i satisfies (1.2) and the following equation holds:

$$\left[2(r-s^2)B-1\right]A_s - 2sA_r + 2B + 2sAB + A^2 = 0. \tag{1.3}$$

Though the equations for A and B are given by the above theorem for isotropic curvature, it is difficult to find the general solution of the corresponding system of partial differential equations (PDEs). Therefore, we turn our attention to projectively affine sprays, whose Douglas curvature vanishes. This is the spray analogue of Douglas metrics in Finsler geometry. Douglas curvature is a projectively invariant geometric quantity introduced by Douglas [19] in 1927. For this class, the spray geodesics can be expressed as

$$G^i = P(x,y)y^i + \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k.$$

Then, for spherically symmetric sprays of the form (1.1), the function $B(r,s)$ is quadratic in s, i.e., there exist smooth functions $f(r)$, $g(r)$, and $h(r)$ such that

$$B(r,s) = f(r)s^2 + g(r)s + h(r). \tag{1.4}$$

Our second main result characterizes those projectively affine spherically symmetric sprays that have isotropic curvature, and for the zero-curvature case we obtain a family of special solutions in the following corollary.

Theorem 2. *Let G be a spherically symmetric projectively affine spray, so that its geodesic coefficients have the form* (1.1) *with B given by* (1.4)*. Then, G has isotropic curvature if, and only if, $g(r) = 0$, $f(r)$, and $h(r)$ satisfy*

$$(2rh(r)-1)f(r) + 2h(r)^2 + 2h'(r) = 0, \tag{1.5}$$

and the following equation holds:

$$2\left[(r-s^2)A_{ss} - sA_s + A\right](f(r)s^2 + h(r)) - 2sA_{rs} - A_{ss} + 2A_r = 0. \tag{1.6}$$

In particular, the spray is of zero curvature if, and only if, $A(r,s)$ and $B(r,s)$ further satisfy (1.3).

It is known that projectively affine spray of scalar curvature is locally projectively flat. This means that there exist local coordinates $(\tilde{x})$ locally such that $\tilde{G}^i = P(\tilde{x},y)y^i$. However, in most cases it is not easy to find the corresponding coordinate transformation. Thus Theorem 2 tells us how to directly determine the sprays of isotropic curvature when $B(r,s)$ is given by (1.4).

Due to the high nonlinearity of (1.6), it is difficult to give its general solution. Nevertheless we can find some special solutions, as presented in the following corollary.

Corollary 1.1. *Let*

$$\begin{aligned} A(r,s) = \frac{2}{p(r) + (2rh(r)-1)^2} \Big\{ &\left[1 - p(r) - 2rh(r)\right] h(r)s \\ &+ \sqrt{p(r)h(r)(rh(r)-1)\left[4h(r)(rh(r)-1)(r-s^2) + p(r) + 1\right]} \Big\}, \end{aligned} \tag{1.7}$$

and $B(r,s)$ *satisfies*

$$(2rh(r)-1)B(r,s) = -2\left(h(r)^2 + h'(r)\right)s^2 + h(r)(2rh(r)-1), \tag{1.8}$$

where $p(r) = 0$ *or* $p(r) = \left[c\exp\left(\int \frac{h(r)^2+h'(r)}{q(r)}\,\mathrm{d}r\right) - 1\right]^{-1}$, $q(r) = h(r)(2rh(r)-1)(rh(r)-1)$, *c is a constant, and* $h(r)$ *is a* C^∞ *function of r. Then, the spray G defined by* $G^i = |y|A(r,s)y^i + |y|^2B(r,s)x^i$ *is of zero curvature.*

Using these explicit solutions, we construct families of spherically symmetric sprays with isotropic curvature in Section 5.

The paper is organized as follows. Section 2 recalls the necessary preliminaries on sprays, curvature, and spherical symmetry. Section 3 proves Theorem 1, giving the equivalent conditions for isotropic curvature. Section 4 treats the projectively affine case and proves Theorem 2, including the explicit zero-curvature solutions. Section 5 presents concrete sprays of isotropic curvature.

2. Preliminaries

In this section, we recall some definitions and necessary notions used in this paper.

Let M be an n-dimensional smooth manifold with $n \geq 2$. A spray on M is a smooth vector field on the tangent bundle $TM\backslash\{0\}$, which in standard local coordinates (x^i, y^i) is expressed as

$$G = y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i}, \tag{2.1}$$

where the coefficients $G^i = G^i(x,y)$ are smooth functions, positively homogeneous of degree two in y, i.e., $G^i(x,\lambda y) = \lambda^2 G^i(x,y)$ for all $\lambda > 0$; these functions are called spray coefficients.

A Finsler metric on M is a scalar function $F = F(x,y)$ on TM such that (i) F is positively homogeneous: $F(x,\lambda y) = \lambda F(x,y)$ for all $\lambda > 0$; (ii) $F(x,y)$ is C^∞ on the punctured tangent bundle $TM \setminus \{0\}$; (iii) for every nonzero y, the matrix $g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x,y)$ is positive definite. Every Finsler metric F on M induces a spray G whose coefficients are given by

$$G^i = \frac{1}{4}g^{ij}\left\{\frac{\partial^2 F^2}{\partial y^j \partial x^k}y^k - \frac{\partial F^2}{\partial x^j}\right\}.$$

The geodesic equation $x = x(t)$ of (M,F) then reads

$$\frac{d^2x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0,$$

and the coefficients G^i are also called the geodesic coefficients of (M,F).

Two sprays $\bar{G}$ and G are called projectively equivalent if there exists a function $P(x,y)$, positively homogeneous of degree one in y, such that

$$\bar{G}^i(x,y) = G^i(x,y) + P(x,y)y^i,$$

where the scalar function $P(x,y)$ is called the projective factor. If $G^i(x,y) = P(x,y)y^i$, then G is called a projectively flat spray. This notion is intimately related to Hilbert's fourth problem on characterizing metrics whose geodesics are straight lines [1,20,21].

Some projectively invariant geometric quantities play important roles in spray geometry and related Finsler geometry. The Douglas curvature is one such quantity, introduced by J. Douglas [19]. The Douglas curvature tensor D^i_{jkl} of a spray G is defined by

$$D^i_{jkl} = B^i_{jkl} - \frac{2}{n+1}\left(E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + \frac{\partial E_{jk}}{\partial y^l}y^i\right),$$

where $B^i_{jkl}(y) = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}(y)$ is the Berwald curvature tensor and $E_{ij}(y) = \frac{1}{2}B^m_{ijm}(y)$. A Finsler metric F is called a Berwald metric if its spray coefficients are quadratic in y.

A spray is projectively affine if $D^i_{jkl} = 0$; then from [22],

$$G^i(x,y) = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k + P(x,y)y^i,$$

where $\frac{1}{2}\Gamma^i_{jk}(x)y^j y^k$ is the quadratic homogeneous part of G^i with respect to y, and its coefficients $\Gamma^i_{jk}(x)$ depend only on the position x. A Finsler metric is called a Douglas metric if its spray is projectively affine. Related results on Douglas Finsler metrics can be found in [23–25].

The Riemann curvature tensor $R^i_{\ k}$ of a spray G^i is defined by

$$R^i_{\ k} = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k}y^j + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{2.2}$$

The Ricci curvature Ric is then defined as the trace of $R^i_{\ k}$, i.e.,

$$\mathrm{Ric} = R^m_{\ m}.$$

Let $P \subset T_xM$ be a 2-dimensional plane and $y \in T_xM \setminus \{0\}$ such that $P = \mathrm{span}\{y,u\}$. Here, g_y denotes the fundamental tensor at y, namely, for any $u, v \in T_xM$,

$$g_y(u,v) = g_{ij}(x,y)u^i v^j.$$

The flag curvature $K(P,y)$ with flagpole y is defined by

$$K(P,y) = \frac{g_y(u, R_y(u))}{g_y(y,y)g_y(u,u) - [g_y(y,u)]^2},$$

where the Riemann curvature operator is $R_y(u) = R^i_{\ k}(y)u^k\frac{\partial}{\partial x^i}$.

A spray G is said to be of scalar curvature if the Riemann curvature tensor has the form

$$R^i_{\ k} = R\,\delta^i_k - \tau_k y^i, \tag{2.3}$$

where $R = R(x,y)$ and $\tau_k = \tau_k(x,y)$ satisfy $R = \tau_k y^k$. In particular, if $\tau_k = \frac{1}{2}\frac{\partial R}{\partial y^k}$, the spray is said to be of isotropic curvature. Consequently, a direct computation shows that a spray has isotropic curvature if, and only if,

$$R^i_{\ k} - \left[\frac{1}{n-1}R^m_{\ m}\delta^i_k - \frac{1}{2(n-1)}(R^m_{\ m})_{y^k}y^i\right] = 0. \tag{2.4}$$

3. Spherically symmetric spray of isotropic curvature

In this section, we first give the Riemann curvature of spherically symmetric sprays in Proposition 3.1 and then derive the equivalent equations for Ricci-flat sprays and for sprays of scalar curvature, respectively. Then, we prove Theorem 1, which provides equivalent conditions for a spherically symmetric spray to have isotropic curvature. For simplicity, throughout this and the following sections, the subscripts y^k and x^k denote the partial derivatives with respect to the corresponding variables. For example, $P_{y^k} = \frac{\partial P}{\partial y^k}$, $P_{x^k} = \frac{\partial P}{\partial x^k}$, etc.

Proposition 3.1. *Let G be a spherically symmetric spray on an open subset $\Omega \subset \mathbb{R}^n$ with G^i expressed in* (1.1). *Then, the Riemann curvature of G is given by*

$$\begin{aligned}R^i{}_k = &\Big\{\Big[2(r-s^2)B-1\Big]A_s - 2sA_r + 2(1+sA)B + A^2\Big\}\Big(|y|^2\delta^i_k - y^k y^i\Big)\\ &+\Big\{\Big[2(r-s^2)B-1\Big]B_{ss} - 2sB_{rs} - (r-s^2)B_s^2 - 2sBB_s + 4B_r + 4B^2\Big\}\Big(|y|x^k - sy^k\Big)|y|x^i\\ &+\Big\{2\Big[(r-s^2)A_{ss} - sA_s + A\Big]B - 2sA_{rs} - A_{ss} + 4A_r - AA_s\\ &-\Big[(r-s^2)A_s + sA + 1\Big]B_s\Big\}\Big(|y|x^k - sy^k\Big)y^i. \end{aligned} \tag{3.1}$$

Proof. Set $P = |y|\,A(r,s)$ and $Q = |y|^2B(r,s)$. Differentiating P and Q with respect to x^k and y^k gives

$$\begin{cases} P_{x^k} = 2A_r|y|x^k + A_s y^k, \quad P_{y^k} = (A - sA_s)\frac{y^k}{|y|} + A_s x^k,\\ P_{x^j y^k} = A_s\delta_{jk} + \frac{2}{|y|}\Big[A_r y^k + A_{rs}(|y|x^k - sy^k)\Big]x^j + \frac{1}{|y|^2}A_{ss}(|y|x^k - sy^k)y^j,\\ P_{y^j y^k} = \frac{A - sA_s}{|y|^3}(|y|^2\delta_{jk} - y^j y^k) + \frac{A_{ss}}{|y|^3}(|y|x^j - sy^j)(|y|x^k - sy^k),\\ Q_{x^k} = 2B_r|y|^2x^k + B_s|y|y^k, \quad Q_{y^k} = (2B - sB_s)y^k + B_s|y|x^k,\\ Q_{x^j y^k} = B_s|y|\delta_{jk} + (2B_{rs}x^j + B_{ss}\frac{y^j}{|y|})(|y|x^k - sy^k) + (4B_r x^j + B_s\frac{y^j}{|y|})y^k,\\ Q_{y^j y^k} = (2B - sB_s)\delta_{jk} + \frac{B_{ss}}{|y|^2}(|y|x^j - sy^j)(|y|x^k - sy^k)\\ \qquad\qquad + \frac{B_s}{|y|^2}\Big[|y|(x^j y^k + x^k y^j) - sy^j y^k\Big]. \end{cases} \tag{3.2}$$

Then

$$\begin{cases} G^i_{x^k} = P_{x^k}y^i + Q_{x^k}x^i + Q\delta^i_k,\\ G^i_{y^k} = P_{y^k}y^i + P\delta^i_k + Q_{y^k}x^i,\\ G^i_{x^j y^k} = P_{x^j y^k}y^i + P_{x^j}\delta^i_k + Q_{x^j y^k}x^i + Q_{y^k}\delta^i_j,\\ G^i_{y^j y^k} = P_{y^j y^k}y^i + P_{y^k}\delta^i_j + P_{y^j}\delta^i_k + Q_{y^j y^k}x^i. \end{cases} \tag{3.3}$$

Substituting (3.2) and (3.3) into (2.2) yields (3.1). □

From the proposition above we obtain Ricci curvature $\mathrm{Ric} = R^m_m$ as

$$\begin{aligned}\mathrm{Ric} = R^m_m = &(r-s^2)|y|^2\Big\{\Big[2(r-s^2)B-1\Big]B_{ss} - 2sB_{rs} - (r-s^2)B_s^2 - 2sBB_s + 4B_r + 4B^2\Big\}\\ &+ (n-1)|y|^2\Big\{\Big[2(r-s^2)B-1\Big]A_s - 2sA_r + 2(1+sA)B + A^2\Big\}. \end{aligned} \tag{3.4}$$

Then, the following corollary is an immediate consequence.

Corollary 3.1. *Let G be a spherically symmetric spray on an open subset $\Omega \subset \mathbb{R}^n$ with G^i expressed in* (1.1)*. Then, G is Ricci flat (i.e.,* $\mathrm{Ric} = 0$*) if, and only if, the following equation holds,*

$$(r - s^2)\Big\{ \Big[2(r - s^2)B - 1\Big] B_{ss} - 2sB_{rs} - (r - s^2)B_s^2 - 2sBB_s + 4B_r + 4B^2\Big\} \\ + (n - 1)\Big\{ \Big[2(r - s^2)B - 1\Big] A_s - 2sA_r + 2(1 + sA)B + A^2\Big\} = 0.$$

From Proposition 3.1 and the definition of scalar curvature (2.3), the following corollary is a direct consequence.

Corollary 3.2. *Let G be a spherically symmetric spray on an open subset $\Omega \subset \mathbb{R}^n$ with G^i expressed in* (1.1)*. Then, G has scalar curvature if, and only if, the following equation holds,*

$$\Big[2(r - s^2)B - 1\Big] B_{ss} - 2sB_{rs} - (r - s^2)B_s^2 - 2sBB_s + 4B_r + 4B^2 = 0. \tag{3.5}$$

Due to the high nonlinearity, it is difficult to find the general solutions of (3.5). However, in the polynomial case we can derive the following corollary, which will be used in the next section to prove Theorem 2.

Corollary 3.3. *Let G be a spherically symmetric spray on an open subset $\Omega \subset \mathbb{R}^n$ with G^i expressed in* (1.1)*. If $B(r, s)$ is a polynomial in s, then G has scalar curvature if, and only if, $B(r, s) = f(r)s^2 + h(r)$, where $f(r)$ and $h(r)$ satisfy*

$$(2rh(r) - 1)f(r) + 2h(r)^2 + 2h'(r) = 0. \tag{3.6}$$

Proof. We first prove necessity. Let $B(r, s) = \displaystyle\sum_{k=0}^{n} a_k s^k$ $(n \geq 6)$, where $a_k = a_k(r)$ are smooth functions of r and k ranges over nonnegative integers. A direct computation yields

$$\begin{cases} B_r(r, s) = \sum_{k=0}^{n} a_k' s^k, \quad B_{rs}(r, s) = \sum_{k=1}^{n} k a_k' s^{k-1}; \\ B_s(r, s) = \sum_{k=1}^{n} k a_k s^{k-1}, \quad B_{ss}(r, s) = \sum_{k=2}^{n} k(k - 1)a_k s^{k-2}. \end{cases} \tag{3.7}$$

By Corollary 3.2, $B(r, s)$ must satisfy Eq (3.5); substituting (3.7) into Eq (3.5) gives

$$2r \sum_{k=0}^{n} a_k s^k \sum_{k=2}^{n} k(k - 1)a_k s^{k-2} - r\left(\sum_{k=1}^{n} k a_k s^{k-1}\right)^2 + \left(\sum_{k=1}^{n} k a_k s^k\right)^2 \\ - 2\sum_{k=0}^{n} a_k s^k \sum_{k=2}^{n} k(k - 1)a_k s^k + 2\sum_{k=0}^{n} a_k s^k \sum_{k=0}^{n}(2 - k)a_k s^k \\ - \sum_{k=2}^{n} k(k - 1)a_k s^{k-2} + 2\sum_{k=0}^{n}(2 - k)a_k' s^k = 0.$$

Since each $a_k = a_k(r)$ is a function of r only, the coefficients of s^{2n}, s^{2n-1}, s^{2n-2}, s^{2n-3}, s^{2n-4}, $\cdots$, s^4, s^3, s^2, s^1, s^0 must vanish. This yields

$$(4 - n^2)a_n^2 = 0, \quad -2(n^2 - n - 3)a_n a_{n-1} = 0,$$

$$(n^2-2n)ra_n^2+(3-n)(n+1)a_{n-1}^2-2(n^2-2n)a_na_{n-2}=0,$$

$$2(n-1)(n-2)ra_na_{n-1}-2(n^2-3n-1)a_{n-1}a_{n-2}-2(n^2-3n+5)a_na_{n-3}=0,$$

$$\begin{aligned}&(n-1)ra_{n-1}^2-2(n-1)(n-4)a_{n-2}^2+2r(n^2-4n+6)a_na_{n-2}-2(n^2-4n+12)a_na_{n-4}\\&-2(n-1)(n-3)a_{n-1}a_{n-3}=0,\end{aligned}$$

$$\cdots\cdots$$

$$30(2ra_0-1)a_6+30ra_1a_5+12(ra_2-2a_0)a_4+3a_3(ra_3-2a_1)-4a_4'=0,$$

$$20a_5(2ra_0-1)+16ra_1a_4+2a_3(2ra_2-5a_0)+2a_1a_2-2a_3'=0,$$

$$12a_4(2ra_0-1)+6ra_1a_3+3a_1^2=0,$$

$$6a_3(2ra_0-1)+6a_0a_1+2a_1'=0,$$

$$2a_2(2ra_0-1)+4a_0^2+4a_0'-ra_1^2=0.$$

From the above equations we conclude that $a_n(r)\equiv 0$, then $a_{n-1}(r)\equiv 0$, and, inductively, $a_{n-2}(r)$, $a_{n-3}(r),\cdots,a_4(r),a_3(r),a_1(r)$ all vanish identically. Consequently,

$$B(r,s)=a_2(r)s^2+a_0(r)=f(r)s^2+h(r). \tag{3.8}$$

Substituting this expression for $B(r,s)$ into (3.5) yields

$$(2rh(r)-1)f(r)+2h(r)^2+2h'(r)=0. \tag{3.9}$$

The sufficiency follows directly by substituting (3.8) and (3.9) back into (3.5). □

Now before we give the proof of Theorem 1, some computations are needed. Differentiating both sides of (3.4) with respect to y^k yields

$$\begin{aligned}(R^m_{\ m})_{y^k}=&\Big\{(r-s^2)\Big[2(r-s^2)B-1\Big]B_{sss}-2s(r-s^2)B_{rss}-2s\Big[5(r-s^2)B-1\Big]B_{ss}\\&+2(r+s^2)B_{rs}+2s(r-s^2)B_s^2+2(3r-s^2)BB_s-8s(B^2+B_r)\Big\}\Big(|y|x^k-sy^k\Big)\\&+2(r-s^2)\Big\{\Big[2(r-s^2)B-1\Big]B_{ss}-2sB_{rs}-(r-s^2)B_s^2-2sBB_s+4B_r+4B^2\Big\}y^k\\&+(n-1)\Big\{2\Big[(r-s^2)A_s+sA+1\Big]B_s+2\Big[(r-s^2)A_{ss}-sA_s+A\Big]B-2sA_{rs}\\&-A_{ss}-2A_r+2AA_s\Big\}\Big(|y|x^k-sy^k\Big)+2(n-1)\Big\{2\Big[(r-s^2)A_s+sA+1\Big]B\\&+A^2-2sA_r-A_s\Big\}y^k.\end{aligned} \tag{3.10}$$

Contracting $R^i_{\ k}$ with x^k yields

$$\begin{aligned}R^i_{\ k}x^k=&\Big\{\Big[2(r-s^2)B-1\Big]A_s-2sA_r+2(1+sA)B+A^2\Big\}|y|\Big(|y|x^i-sy^i\Big)\\&+(r-s^2)\Big\{\Big[2(r-s^2)B-1\Big]B_{ss}-2sB_{rs}-(r-s^2)B_s^2-2sBB_s+4B_r\\&+4B^2\Big\}|y|^2x^i+(r-s^2)\Big\{2\Big[(r-s^2)A_{ss}-sA_s+A\Big]B-2sA_{rs}-A_{ss}+4A_r\end{aligned}$$

$$-AA_s-\Big[(r-s^2)A_s+sA+1\Big]B_s\Big\}|y|y^i. \tag{3.11}$$

Let the left-hand side of (2.4) be $LR^i{}_k$; then (2.4) can be written as $LR^i{}_k=0$. Substituting (3.1), (3.4), and (3.10) into (2.4) yields

$$\begin{aligned}LR^i{}_k=&\Big\{2(r-s^2)BB_{ss}-B_{ss}-2sB_{rs}-(r-s^2)B_s^2-2sBB_s+4B_r+4B^2\Big\}\Big(|y|(|y|x^k-sy^k)x^i\\&+\frac{1}{n-1}\Big[(ry^k-s|y|x^k)y^i-(r-s^2)|y|^2\delta^i_k\Big]\Big)+\Big\{(r-s^2)\Big([2(r-s^2)B-1]B_{sss}\\&-2sB_{rss}-6sBB_{ss}+2B_{rs}+6BB_s\Big)+3(n-1)\Big(2\Big[(r-s^2)A_{ss}-sA_s+A\Big]B\\&-2sA_{rs}-A_{ss}+2A_r\Big)\Big\}\frac{(|y|x^k-sy^k)y^i}{2(n-1)}.\end{aligned} \tag{3.12}$$

Contracting $LR^i{}_k$ with x^k yields

$$\begin{aligned}LR^i{}_kx^k=&\frac{n-2}{n-1}(r-s^2)\Big\{2(r-s^2)BB_{ss}-B_{ss}-2sB_{rs}-(r-s^2)B_s^2-2sBB_s+4B_r\\&+4B^2\Big\}|y|^2x^i+\frac{(r-s^2)}{2(n-1)}\Big\{(r-s^2)\Big([2(r-s^2)B-1]B_{sss}-2sB_{rss}-6sBB_{ss}+2B_{rs}\\&+6BB_s\Big)+3(n-1)\Big(2\Big[(r-s^2)A_{ss}-sA_s+A\Big]B-2sA_{rs}-A_{ss}+2A_r\Big)\Big\}|y|y^i.\end{aligned} \tag{3.13}$$

Proof of Theorem 1. The proof is divided into two cases: Case (i) the spray has isotropic curvature; Case (ii) the spray has zero curvature.

Case (i). We first establish the necessity of (1.2) for isotropic curvature.

Since $LR^i{}_k=0$, we have $LR^i{}_kx^k=0$. If y is parallel to x, then $r-s^2=0$; by the continuity of $A(r,s)$ and $B(r,s)$, $LR^i{}_kx^k=0$ holds trivially. If y is not parallel to x, then $r-s^2\neq 0$. By (3.13), and by the linear independence of x^i and y^i, the coefficients of x^i and y^i must both vanish. Thus, (3.13) reduces to

$$2(r-s^2)BB_{ss}-B_{ss}-2sB_{rs}-(r-s^2)B_s^2-2sBB_s+4B_r+4B^2=0, \tag{3.14}$$

$$\begin{aligned}&(r-s^2)\Big\{[2(r-s^2)B-1]B_{sss}-2sB_{rss}-6sBB_{ss}+2B_{rs}+6BB_s\Big\}\\&\quad+3(n-1)\Big\{2\Big[(r-s^2)A_{ss}-sA_s+A\Big]B-2sA_{rs}-A_{ss}+2A_r\Big\}=0.\end{aligned} \tag{3.15}$$

Differentiating both sides of (3.14) with respect to s yields

$$\Big[2(r-s^2)B-1\Big]B_{sss}-2sB_{rss}-6sBB_{ss}+2B_{rs}+6BB_s=0. \tag{3.16}$$

Then, (3.15)$-(r-s^2)\times$ (3.16) gives, after simplification,

$$2\Big[(r-s^2)A_{ss}-sA_s+A\Big]B-2sA_{rs}-A_{ss}+2A_r=0. \tag{3.17}$$

Thus, Eqs (3.14) and (3.17) together constitute (1.2).

We now prove sufficiency. Assume that $A(r,s)$ and $B(r,s)$ satisfy (1.2). Differentiating the second equation of (1.2) with respect to s and then adding the first equation of (1.2) yields (3.15). Substituting the first equation of (1.2) and (3.15) into (3.12) gives $LR^i{}_k=0$. Thus, whenever $A(r,s)$, $B(r,s)$ satisfy (1.2), we have $LR^i{}_k=0$, and the curvature is isotropic.

Case (ii). We first prove necessity. Assume the curvature is zero; then $R^i_{\ k} = 0$. From (3.1), the coefficient of δ^i_k must vanish, i.e.,

$$\Big[2(r - s^2)B - 1\Big] A_s - 2sA_r + 2(1 + sA)B + A^2 = 0,$$

which is exactly (1.3). Since zero curvature implies isotropic curvature, by Case (i), we have that (1.2) holds.

We now prove sufficiency. Assume that $A(r, s)$ and $B(r, s)$ satisfy (1.2) and (1.3). Differentiating both sides of (1.3) with respect to s yields

$$2\Big[(r - s^2)A_{ss} - sA_s + A\Big] B - 2sA_{rs} - A_{ss} + 2\Big[(r - s^2)A_s + sA + 1\Big] B_s + 2AA_s - 2A_r = 0.$$

Subtracting the first equation of (1.2) from this expression gives

$$\Big[(r - s^2)A_s + sA + 1\Big] B_s + AA_s - 2A_r = 0. \tag{3.18}$$

Then, substituting (1.2), (1.3), and (3.18) into (3.1) yields $R^i_{\ k} = 0$. □

4. Projectively affine sprays of isotropic curvature

In this section we prove Theorem 2, which provides equivalent conditions for a spherically symmetric spray to be projectively affine and, hence, of the form satisfying (1.5) and (1.6).

Let G be a projectively affine spray. Then, by Corollary 3.3, $B(r, s) = f(r)s^2 + h(r)$; hence

$$G^i = |y|A(r, s)y^i + |y|^2 \Big(f(r)s^2 + h(r)\Big) x^i.$$

Proof of Theorem 2. By Corollary 3.3, any projectively affine spray $G = |y|A(r, s)y^i + |y|^2B(r, s)x^i$ has scalar curvature if, and only if,

$$B(r, s) = f(r)s^2 + h(r), \tag{4.1}$$

where $(2rh(r) - 1)f(r) + 2h(r)^2 + 2h'(r) = 0$, i.e., equation (3.6). This implies $g(r) = 0$ in (1.4) and, hence, (1.5) holds. Furthermore, by Theorem 1, the spray has isotropic curvature if, and only if, (1.2) holds. In fact, by Corollary 3.2, $B(r, s)$ in (4.1) automatically satisfies the second equation of (1.2). Thus, we only need to consider the first equation of (1.2). Substituting (4.1) into (1.2) yields

$$2\Big[(r - s^2)A_{ss} - sA_s + A\Big]\Big(f(r)s^2 + h(r)\Big) - 2sA_{rs} - A_{ss} + 2A_r = 0, \tag{4.2}$$

which is exactly (1.6). Thus, the first part of the theorem is proved.

Now for the case of zero curvature, the result follows from Theorem 1. In fact, by Theorem 1 G has zero curvature if, and only if, $A(r, s)$ and $B(r, s)$ further satisfy Eq (1.3), i.e.,

$$\Big[2(r - s^2)(f(r)s^2 + h(r)) - 1\Big] A_s - 2sA_r + A^2 + 2\Big(f(r)s^2 + h(r)\Big)(sA + 1) = 0. \tag{4.3}$$

□

To construct sprays of isotropic curvature, we need to solve (1.6). Due to its nonlinearity, it is difficult to give a general solution. Nevertheless we can find some special solutions; see Example 5.1–5.3.

For zero curvature, the key is to solve (1.6) and (4.3). There is no general way to solve them together. However, we can go a step further and obtain a family of special solutions in Corollary 1.1.

Proof of Corollary 1.1. A direct, though lengthy, verification shows that the given $A(r, s)$ and $B(r, s)$ satisfy (1.3), (1.5), and (1.6). Hence, G has zero curvature.

Alternatively, we give a more efficient derivation below to show how the form of $A(r, s)$ in (1.7) is discovered. From the proof of Theorem 2, we need to consider Eqs (1.6) and (4.3). Substituting the expression (3.6) for $f(r)$ into equations $(2rh(r) - 1)\times(1.6)$ and $(2rh(r) - 1)\times(4.3)$, respectively, yields

$$\begin{aligned}&4\left[(r-s^2)A_{ss}-sA_s+A\right]\left[(r-s^2)h(r)^2-s^2h'(r)\right]+2\left[(s^2-2r)A_{ss}-2rsA_{rs}+sA_s\right.\\&\left.+2rA_r-A\right]h(r)+2sA_{rs}+A_{ss}-2A_r=0,\end{aligned} \tag{4.4}$$

$$\begin{aligned}&4\left[(r-s^2)A_s+sA+1\right]\left[(r-s^2)h(r)^2-s^2h'(r)\right]+2\left[(s^2-2r)A_s-2rsA_r+(rA-s)A\right.\\&\left.-1\right]h(r)-A^2+2sA_r+A_s=0.\end{aligned} \tag{4.5}$$

Differentiating both sides of (4.5) with respect to s gives

$$\begin{aligned}&4\left[(r-s^2)^2A_{ss}-3s(r-s^2)A_s+(r-3s^2)A-2s\right]h^2(r)+2\left[2r(AA_s-sA_{rs}-A_r)\right.\\&\left.+sA_s-A+(s^2-2r)A_{ss}\right]h(r)-4s\left[s(r-s^2)A_{ss}+(2r-3s^2)A_s+3sA+2\right]h'(r)\\&-2(AA_s-sA_{rs}-A_r)+A_{ss}=0.\end{aligned} \tag{4.6}$$

Subtracting Eq (4.4) from Eq (4.6), we obtain

$$4s\left[(r-s^2)A_s+sA+1\right]\left[h^2(r)+h'(r)\right]+(AA_s-2A_r)(1-2rh(r))=0. \tag{4.7}$$

Eliminating A_r from Eqs (4.5) and (4.7) gives

$$\left[2h(r)(r-s^2)-sA-1\right]A_s+2h(r)(sA+1)+A^2=0. \tag{4.8}$$

A direct substitution shows that the following expression satisfies (4.8) identically for any smooth function $p(r)$:

$$\begin{aligned}A(r,s)=&\frac{2}{p(r)+(2rh(r)-1)^2}\Big\{\left[1-p(r)-2rh(r)\right]h(r)s\\&+\sqrt{p(r)h(r)(rh(r)-1)\left[4h(r)(rh(r)-1)(r-s^2)+p(r)+1\right]}\Big\}.\end{aligned}$$

Moreover, a direct verification follows that it also satisfies (4.7).

Besides, this $A(r, s)$ should simultaneously satisfy Eqs (4.4) and (4.5). Direct substitution yields

$$4h(r)(rh(r)-1)(r-s^2)X(r)=0,$$

$$2sX(r)\Big[-4h(r)(rh(r)-1)(4r^2h(r)^2-4rh(r)-p(r)+1)s^2+(2rh(r)-1)^2(4r^2h(r)^2-4rh(r)$$

$$+p(r)+1)-4\sqrt{p(r)h(r)(rh(r)-1)\left[4h(r)(rh(r)-1)(r-s^2)+p(r)+1\right](2rh(r)-1)s}\Big]=0,$$

where

$$X(r)=h(r)\,(2rh(r)-1)\,(rh(r)-1)p'(r)+(h^2(r)+h'(r))p^2(r)+(h^2(r)+h'(r))p(r). \qquad (4.9)$$

It can be seen that both equations hold simultaneously if $X(r)=0$, i.e., (4.9) vanishes identically. In fact, Eq (4.9) is a Bernoulli equation; solving directly yields that $p(r)$ satisfies the following form:

$$p(r)=0, \qquad p(r)=\frac{1}{c\exp\left(\int\frac{h(r)^2+h'(r)}{q(r)}\mathrm{d}r\right)-1},$$

where

$$q(r)=h(r)(2rh(r)-1)(rh(r)-1),$$

c is a constant, and $h(r)$ is a C^∞ function of r. Hence, $A(r,s)$ in (1.7) with $p(r)$ and $q(r)$, as above satifies Eq (1.6) and (4.3). Therefore, by Theorem 2, the corresponding spray G defined by $G^i = |y|A(r,s)y^i+|y|^2B(r,s)x^i$ has zero curvature. □

5. Examples

In this section we present explicit sprays that satisfy Theorems 1 and 2.

Let G be a projectively affine spherically symmetric spray. By Corollary 3.2, $B(r,s)=f(r)s^2+h(r)$. Then,

$$G^i=|y|A(r,s)y^i+|y|^2\left(f(r)s^2+h(r)\right)x^i.$$

We first present examples of nonzero isotropic curvature (Examples 5.1–5.3).

Example 5.1. *Let*

$$A(r,s)=0, \quad B(r,s)=\frac{1}{r+c_1},$$

where c_1 is a constant, and $r+c_1\neq 0$, i.e., $|x|^2+c_1\neq 0$. It can be directly verified that they satisfy (1.2) *in Theorem 1. In this case, $G^i=|y|^2x^i/(|x|^2+c_1)$. Thus, if $c_1>0$, then $|x|^2+c_1>0$ for all $x\in\mathbb{R}^n$; if $c_1<0$, then $|x|^2+c_1\neq 0$ requires $|x|^2<-c_1$; hence G^i is defined on the open ball $\mathbb{B}^n_{\sqrt{-c_1}}(0)$ of radius $\sqrt{-c_1}$ centered at the origin* 0*. This spray is projectively affine and has nonzero isotropic curvature.*

Example 5.2. *Let*

$$A(r,s)=sf(r)+\frac{s}{r+c_1}\int\frac{\eta(\gamma)}{s^2}\mathrm{d}s, \quad B(r,s)=\frac{1}{r+c_1},$$

where $\gamma=(s^2-r)/(r+c_1)^2$, $f(r)$ and $\eta(\gamma)$ are smooth functions of r and γ, respectively, and c_1 is a constant with $r+c_1\neq 0$. It can be directly verified that they satisfy (1.2) *in Theorem 1. In this case,*

$$G^i=\langle x,y\rangle\left[f(r)+\frac{1}{|x|^2+c_1}\int\frac{\eta(\gamma)}{s^2}\mathrm{d}s\right]y^i+\frac{|y|^2}{|x|^2+c_1}x^i.$$

Then the corresponding spray G is projectively affine and has nonzero isotropic curvature. The smoothness of the corresponding spray depends on the choice of $\eta(\gamma)$. By choosing $\eta(\gamma)$ suitably, one *can obtain a C^∞ spray on its domain. In particular, the choices in cases (i) and (ii) below yield smooth sprays. The following are two special cases:*

(i) Take $\eta(\gamma) = c_2$, where c_2 is a constant. Then,

$$A(r, s) = sf(r) - \frac{c_2}{r + c_1}, \qquad B(r, s) = \frac{1}{r + c_1}.$$

(ii) Take $\eta(\gamma) = \gamma^n$, where $n \in \mathbb{Z}^+$. Then,

$$A(r, s) = sf(r) + \frac{1}{(r + c_1)^{2n+1}} \sum_{k=0}^{n} \binom{n}{k} (-r)^k \frac{s^{2n-2k}}{2n - 2k - 1}, \qquad B(r, s) = \frac{1}{r + c_1}.$$

Now we consider a $B(r, s)$ different from those in Examples 5.1 and 5.2.

Example 5.3. *Let $h(r) = c_1$, where c_1 is a constant; then $(1.2)_2$ reduces to*

$$(2c_1 r - 1)f(r) + 2c_1^2 = 0. \tag{5.1}$$

Solving for $f(r)$ gives $f(r) = 2c_1^2/(1 - 2c_1 r)$; hence

$$B(r, s) = \frac{2c_1^2 s^2}{1 - 2c_1 r} + c_1.$$

Substituting this expression into $(1.2)_1$ yields

$$2\Big[(r - s^2)A_{ss} - sA_s + A\Big]\frac{2c_1^2(r - s^2) - c_1}{2c_1 r - 1} - 2sA_{rs} - A_{ss} + 2A_r = 0. \tag{5.2}$$

We now present the function $A(r, s)$ satisfying Eq (5.2):

$$A(r, s) = h(r)s + \frac{s}{2} \int \frac{f(u)\sqrt{-2\alpha/(c_1\beta)}}{s^2} \mathrm{d}s,$$

where $\alpha = 2c_1 r - 1$, $\beta = \alpha - 2c_1 s^2$, $u = (-r\beta - r + \frac{1}{2c_1})/\beta$, $f(u)$ is a function of u, $h(r)$ is a function of r and c_1 is a constant such that $2c_1 r - 1 \neq 0$. Then,

$$G^i = \langle x, y \rangle \Big[h(r) + \frac{1}{2} \int \frac{f(u)\sqrt{-2\alpha/(c_1\beta)}}{s^2} \mathrm{d}s\Big] y^i + |y|^2 \Big(\frac{\beta - \alpha}{\alpha} + c_1\Big) x^i,$$

which is projectively affine and has nonzero isotropic curvature. The smoothness of the corresponding spray depends essentially on the choice of $f(u)$. By choosing $f(u)$, and when necessary the constant c_1, suitably, one can obtain a C^∞ spray on its domain. For instance, in case (i) below, a smooth spray is obtained; while in case (ii), the spray may fail to be smooth for some values of c_1; by choosing c_1 appropriately, one can ensure its smoothness.

The following are two special cases:

(i) Take $f(u) = 0$. Then,

$$A(r, s) = \big(h(r) + c_2\big)s, \qquad B(r, s) = \frac{2c_1^2 s^2}{1 - 2c_1 r} + c_1.$$

(ii) Take $f(u) = 1$, $h(r) = 0$. Then,

$$A(r, s) = -\frac{\sqrt{-2(2c_1 r - 1)/c_1}\,\sqrt{2c_1(r - s^2) - 1}}{2(2c_1 r - 1)}, \qquad B(r, s) = \frac{2c_1^2 s^2}{1 - 2c_1 r} + c_1.$$

Now we turn to examples of zero curvature (Examples 5.4–5.6). Example 5.4 corresponds to a projectively flat spray; detailed computations are given in [16].

Example 5.4. *[16]*

$$A(r,s) = \frac{-s \pm \sqrt{s^2 - c - r}}{r + c}, \qquad B(r,s) = 0,$$

where $s^2 - r - c \geq 0$ *and* $r + c \neq 0$*, i.e.,* $|x|^2 + c \neq 0$ *and* $\langle x, y\rangle^2/|y|^2 - |x|^2 - c \geq 0$*. Thus, when* $c < 0$ *we have* $|x|^2 + c < 0$*, so* $|x|^2 < -c$*; hence, the spray coefficient* G^i *is defined on the open ball* $B_{\sqrt{-c}}(0)$ *of radius* $\sqrt{-c}$ *centered at the origin* 0*. In this case,*

$$G^i = \frac{-\langle x, y\rangle + \sqrt{\langle x, y\rangle^2 - |x|^2 - c}}{|x|^2 + c} y^i.$$

To obtain more nontrivial zero-curvature examples, we take nonzero $B(r,s)$ in Examples 5.5 and 5.6. Then we can find suitable $A(r,s)$ from (1.7) using Corollary 1.1.

Example 5.5. *Let* $B(r,s) = \frac{1}{r+c_1}$*. From* (1.7) *we obtain directly:*

Case $p(r) = 0$***:*** *Direct computation yields:*

$$A(r,s) = \frac{2s}{c_1 - r}.$$

In this case,

$$G^i = \frac{2\langle x, y\rangle}{c_1 - |x|^2} y^i + \frac{|y|^2}{|x|^2 + c_1} x^i,$$

where $c_1 \neq 0$ *is a constant, and* $|x| \neq \sqrt{|c_1|}$ *throughout the domain.*

Case $p(r) \neq 0$***:*** *Clearly,* $h(r) = \frac{1}{r+c_1}$*, then* $p(r)$ *is given by* $p(r) = \frac{1}{c-1}$*, and consequently*

$$\begin{aligned} A(r,s) = &\frac{2\sqrt{-cc_1^3 - \left[(4s^2 - 2r)c + 4(r - s^2)\right] c_1^2 + cc_1 r^2}}{(c_1 - r)^2 c + 4c_1 r} \\ &+ \frac{2c(c_1 s - rs) - 4c_1 s}{(c_1 - r)^2 c + 4c_1 r}, \end{aligned}$$

where $c \neq 1$ *and* $c_1 \neq 0$ *are constants, and* $|x| \neq \sqrt{|c_1|}$ *throughout the domain. Substituting* $A(r,s)$ *and* $B(r,s)$ *into Eq (1.3) yields an identity. Then, the corresponding spray G is projectively affine and has zero curvature.*

In the next example, by choosing $h(r) = c_1$ and, hence, $B(r,s) = \frac{2c_1^2 s^2}{1-2c_1 r} + c_1$ as in Example 5.3, we can construct corresponding $A(r,s)$ using Corollary 1.1.

Example 5.6. *Let* $h(r) = c_1$*, where* c_1 *is a constant; then by Corollary 1.1, we have*

$$p(r) = \frac{2c_1 r - 1}{c_1 r(c - 2) - c + 1}.$$

Direct computation yields

$$A(r,s) = \frac{2c_1(c-2)(1-2c_1r)\,s + 2\sqrt{c_1(2c_1r-1)\left[4c_1(r-s^2)(c_1(c-2)r-c+1)+c\right]}}{[2c_1(c-2)r-c]\,(2c_1r-1)},$$

$$B(r,s) = \frac{2c_1^2s^2}{1-2c_1r} + c_1,$$

where $c_1 < 0, c \geq 2$, *and* $|x|^2 = r$ *must satisfy*

$$2c_1|x|^2 - 1 < 0, \qquad 2c_1(c-2)|x|^2 - c < 0.$$

Hence $A(r,s)$ *and* $B(r,s)$ *are smooth on* $T\Omega \setminus \{0\}$. *Substituting* $A(r,s)$ *and* $B(r,s)$ *into Eq (1.3) yields an identity. Hence, the corresponding spray G is projectively affine and has zero curvature.*

6. Conclusions

We conclude by relating the present results to the special case of sprays generated by spherically symmetric Finsler metrics. Every spherically symmetric Finsler metric induces a spherically symmetric spray of the form

$$G^i = |y|A(r,s)y^i + |y|^2B(r,s)x^i.$$

Therefore, Theorem 1 gives necessary conditions for the geodesic spray of such a metric to have isotropic curvature. In this sense, the present work extends the study of isotropic curvature from spherically symmetric Finsler metrics to the broader setting of spherically symmetric sprays.

However, not every spray of the form above is induced by a Finsler metric. Thus, the classification obtained in this paper is formulated independently of the metrizability problem. For metric-induced sprays, the functions $A(r,s)$ and $B(r,s)$ are determined by the metric function $\phi(r,s)$, and the system in Theorem 1 can be further translated into equations for ϕ. It would be interesting to characterize which of the examples constructed here are metrizable by spherically symmetric Finsler metrics. This question will be considered in future work.

Author contributions

Benling Li: Conceptualization, funding acquisition, methodology, resources, supervision, validation, writing—review & editing; Zuhuan Jia: Data curation, investigation, writing-original draft. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. G. Yang, On sprays of scalar curvature and metrizability, *J. Geom. Anal.*, **33** (2023), 120. https://doi.org/10.1007/s12220-022-01165-x
2. S. G. Elgendi, Z. Muzsnay, Metrizability of holonomy invariant projective deformation of sprays, *Canad. Math. Bull.*, **66** (2023), 701–714. https://doi.org/10.4153/S0008439520000016
3. Y. Li, X. Mo, Y. Yu, Inverse problem of sprays with scalar curvature, *Int. J. Math.*, **30** (2019), 1950041. https://doi.org/10.1142/S0129167X19500411
4. B. Li, Z. Shen, Sprays of isotropic curvature, *Int. J. Math.*, **29** (2018), 1850003. https://doi.org/10.1142/S0129167X18500039
5. S. Rutz, Symmetry in Finsler spaces, *Contemp. Math.*, **196** (1996), 289–300.
6. L. Zhou, Projective spherically symmetric Finsler metrics with constant flag curvature in $\mathbb{R}^n$, *Geom. Dedicata*, **158** (2012), 353–364. https://doi.org/10.1007/s10711-011-9639-3
7. L. Zhou, Spherically symmetric Finsler metrics in $\mathbb{R}^n$, *Publ. Math. Debrecen*, **80** (2012), 67–77. https://doi.org/10.5486/PMD.2012.4870
8. Z. Shen, Projectively flat Finsler metrics of constant flag curvature, *Trans. Amer. Math. Soc.*, **355** (2003), 1713–1728. https://doi.org/10.1090/S0002-9947-02-03216-6
9. B. Li, On the classification of projectively flat Finsler metrics with constant flag curvature, *Adv. Math.*, **257** (2014), 266–284. https://doi.org/10.1016/j.aim.2014.02.022
10. R. L. Bryant, L. Huang, X. Mo, On Finsler surfaces of constant flag curvature with a Killing field, *J. Geom. Phys.*, **116** (2017), 345–357. https://doi.org/10.1016/j.geomphys.2017.02.012
11. A. Mezrag, Z. Muzsnay, The holonomy of spherically symmetric projective Finsler metrics of constant curvature, *J. Geom. Anal.*, **34** (2024), 257. https://doi.org/10.1007/s12220-024-01691-w
12. E. Sevim, Z. Shen, S. Ulgen, Spherically symmetric Finsler metrics with constant Ricci and flag curvature, *Publ. Math. Debrecen*, **87** (2015), 463–472. https://doi.org/10.5486/PMD.2015.7358
13. X. Mo, N. Solórzano, K. Tenenblat, On spherically symmetric Finsler metrics with vanishing Douglas curvature, *Differ. Geom. Appl.*, **31** (2013), 746–758. https://doi.org/10.1016/j.difgeo.2013.09.002
14. N. Solórzano, B. Leandro, On spherically symmetric Finsler metric with scalar and constant flag curvature, *Differ. Geom. Appl.*, **79** (2021), 101828. https://doi.org/10.1016/j.difgeo.2021.101828
15. H. Zhu, H. Zhang, Projective Ricci flat spherically symmetric Finsler metrics, *Int. J. Math.*, **29** (2018), 1850078. https://doi.org/10.1142/S0129167X18500787
16. Y. Gui, B. Li, On spherically symmetric sprays, 2026. https://doi.org/10.48550/arXiv.2604.11196
17. Y. Lou, B. Li, Sprays of weakly isotropic curvature, *Differ. Geom. Appl.*, **82** (2022), 101896. https://doi.org/10.1016/j.difgeo.2022.101896

18. X. Cheng, K. Cao, C. Qing, The characterizations and constructions of sprays of isotropic curvature, *Acta Math. Sin. English Ser.*, **38** (2022), 1612–1620. https://doi.org/10.1007/s10114-022-1120-8

19. J. Douglas, The general geometry of paths, *Ann. Math.*, **29** (1927), 143–168. https://doi.org/10.2307/1967989

20. S. Bácsó, Z. Szilasi, On the projective theory of sprays, *Acta Math. Acad. Paedagog. Nyházi.*, **26** (2010), 171–207.

21. G. Yang, Some classes of sprays in projective spray geometry, *Differ. Geom. Appl.*, **29** (2011), 606–614. https://doi.org/10.1016/j.difgeo.2011.04.041

22. Z. Shen, *Differential geometry of spray and Finsler spaces*, Springer Science & Business Media, 2001. https://doi.org/10.1007/978-94-015-9727-2

23. S. Bácsó, M. Matsumoto, On Finsler spaces of Douglas type II: Projectively flat spaces, *Publ. Math. Debrecen*, **53** (1998), 423–438. https://doi.org/10.5486/pmd.1998.2138

24. B. Li, Y. Shen, Z. Shen, On a class of Douglas metrics, *Stud. Sci. Math. Hung.*, **46** (2009), 355–365. https://doi.org/10.1556/SScMath.2009.1096

25. H. Liu, X. Mo, On a class of Finsler metrics of Douglas type, *Acta Math. Sin. English Ser.*, **41** (2025), 1491–1507. https://doi.org/10.1007/s10114-025-3309-0