AIMS Mathematics, 11(7): 23090–23131.
DOI:10.3934/math.2026931
Received: 16 April 2026
Revised: 03 July 2026
Accepted: 14 July 2026
Published: 30 July 2026

https://www.aimspress.com/journal/Math

Research article

Inference for the Lindley distribution under partially accelerated life tests with adaptive progressive hybrid censoring

Mohamed A. T. El-Shahat[1], Fardous Elsayed[1], Tmader Alballa[2,*], Doaa Basalamah[3] and Said G. Nassr[4]

[1] Department of Statistics and Insurance, Faculty of Commerce, Zagazig University, Egypt
[2] Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
[3] Mathematics Department, Faculty of Science, Umm Al-Qura University, P.O.Box 24231, Makka, Saudi Arabia
[4] Department of Statistics and Insurance, Faculty of Commerce, Arish University, Al-Arish, Egypt

* **Correspondence:** Email: tsalballa@pnu.edu.sa.

Abstract: The purpose of this research is to examine the statistical inference for the Lindley distribution using step-stress partially accelerated life testing (SS-PALT) and an adaptive Type I progressive hybrid censoring scheme (AT-I PHCS). To improve the analysis of lifetime observations, the suggested method combines adaptive censoring processes with acceleration mechanisms. Both maximum likelihood and Bayesian approaches are used to estimate the acceleration factor and unknown distribution parameters. The asymptotic properties of the maximum likelihood estimators are derived, upon which approximate confidence intervals are built. In the Bayesian context, estimation is performed under the squared error loss function, while posterior summaries and credible intervals are obtained using Markov chain Monte Carlo (MCMC) methods. An extensive simulation study is carried out to evaluate the performance of the suggested estimators across different censoring schemes and acceleration conditions. Additionally, optimal design considerations are taken into account to determine efficient sampling plans within the SS-PALT setting. The practical relevance of the developed methodology is demonstrated through an application to real data, confirming its effectiveness in fitting lifetime data collected from accelerated experiments. Overall, the findings indicate that the combination of adaptive censoring strategies with partially accelerated life testing can lead to improved estimation accuracy and greater experimental efficiency.

Keywords: Lindley distribution; partially accelerated life test; adaptive Type I progressive hybrid censoring scheme; maximum likelihood estimation; Bayesian method; MCMC method
Mathematics Subject Classification: 62F10, 62F12, 62F15, 62N01, 62N02, 62N05

1. Introduction

The reliability of products and materials evaluated under normal settings has made it more difficult to predict their lifetime, as a result of continual advancements in manufacturing design. Under these conditions, lifelong testing is both time-consuming and costly. In order to find out how long these materials last, we can run a sample of them through some more extreme tests than normal. Conditions such as heat, electricity, pressure, vibration, cycling speed, and load are among the many stresses that can induce these strains. In accelerated life testing (ALT), objects are put through extreme stress in order to get failure data more quickly. In order to make educated guesses about the unknown design-condition lifetime distribution characteristics, we use the high-stress life data. We can use one of three ALT approaches. The order of operation is as follows: Progressive stress, step-stress, and steady-state ALT. The three methods are presented independently. A lengthy exam is required in the absence of stress. A continual level of tension calls for the first tactic. The other two approaches can save time, money, and materials while reducing testing time, according to [1]. Furthermore, ordinary accelerated life testing (OALT) and partially accelerated life testing (PALT) are two varieties of accelerated life testing that are available. The mathematical model used to estimate the lifetime of the unit in relation to stress must either be known or be easy to assume, which is a fundamental assumption in OALT. However, such models may not exist or be difficult to assume. Therefore, the life test can be conducted using PALT. First and foremost, there is no known or acceptable mathematical model that has been utilized to determine the mean lifetime-stress connection in PALT. Tests can also be conducted using PALT in an accelerated state with the intention of extrapolating the results to the normal state. For these instances, the acceleration factor is best estimated using the PALT approach. This component is the ratio of the rate of accelerated danger to the condition of usual use ($\beta > 1$).

There are two main types of PALT. The first approach, known as step-stress PALT (SS-PALT), changes the test's mode from normal to accelerated at regular intervals. The second one is constant-stress PALT (CS-PALT). All items undergo either normal or accelerated settings in this type of testing. The topic of ALT has been extensively covered in the literature. Two studies [2, 3] examined accelerated life experiments. The degradation of power capacitors was evaluated using ALT in the study of [4]. However, Alrashidi et al. [5] estimated unknown parameters and an acceleration factor based on CS-PALT with unified hybrid censored data. According to [6], stress change is studied in multiple disciplines related to reliability. Particularly in stress and fracture mechanics, stress change is often cyclical and can lead to the gradual onset and propagation of cracks. Monte Carlo methods are commonly used to estimate the uncertainty in damage development. Also, according to Ismial [7], the acceleration factor and parameters of the second type of Pareto distribution were determined using Bayesian and maximum likelihood approaches. The problem of optimal design and the Gompertz distribution estimate in SS-PALT with Type I censored data were examined in the work of Ismial [8]. Numerous researchers have also taken an interest in it, as seen in studies such as [9–12].

Experiments that measure dependability and longevity often use censorship. There may be cost or time constraints that force the experimenter to end the test before all items fail. Finding out how long things last before they break down is what life testing is all about. It is from these experiments that the suppressed data arises. Reducing the testing time and costs is a major motivation for censorship: The total time spent conducting the experiment, the number of units used, and the statistical inference drawn from the results would all be best balanced by an ideal censoring approach. Many different kinds

of censorship exist. Type I and Type II censorship are the most common. A life testing experiment with Type I censoring ends at time T, and the number of failures out of a sample size n is random. The duration of the test is determined at random by Type II censoring, which ends the trial after the rth failure. It is not possible to remove units before the trial ends using conventional Type I and Type II censoring processes. This is a drawback of these methods.

An approach to censorship that combines Type I and Type II censoring algorithms was suggested by [13]. In reliability or life-testing tests, this hybrid censoring method is frequently used. A predefined period $\eta \in (0, \infty)$ and a stochastic moment $\eta^* = min(x_{m:m:n}, \eta)$ are used to end the operation in these trials. One limitation of these methods is that units cannot be removed from the test at any point other than the end. A broader censorship strategy, sometimes called gradual censoring or progressive Type II censoring, can be used. This issue is also addressed by using Type II hybrid censoring systems. With a predetermined number of items to be examined (m), a life test is conducted on n items in a progressive Type II censoring system. The first failure happens at $x_{1:m:n}$, and out of the $n-1$ units that are still alive, R_1 units are chosen at random. When the second failure also occurs, from the remaining $n-1-R_1$ units, $x_{2:m:n}$ and R_2 units are selected at random, and the process proceeds in the same manner. Failure occurs at the mth. $x_{m:m:n}$ instant. We discard all units that remain after $n-m-R_1-R_2-\ldots-R_{m-1}$ have been eliminated. Some researchers [14–16] have written extensive monographs that readers can consult for further information about Type II progressive censoring and the benefits it offers. The Type 1 progressive hybrid censoring scheme (T-I PHCS) was proposed by Kundu and Joarder [17] and involves arranging n items in a life test with a certain quantity η and a progressive censoring scheme $R_1, R_2, \ldots, R_m$. The experiment is then ended using a randomly generated time, denoted as $\eta^* = \min(x_{m:m:n}, \eta)$. The experiment will end at $x_{m:m:n}$ if its value is less than η. If the value of $x_{m:m:n}$ is greater than η, the experiment will end at time η. The advantage of this censoring technique is that the maximum period the experimenter can sustain determines the value of η. There is a time limit on the experiment. There are two options for the data format that will be accepted under the T-I PHCS.

Case I: $x_{1:m:n} < \ldots < x_{m:m:n}$ if $x_{m:m:n} \leq \eta$,

Case II: $x_{1:m:n} < \ldots < x_{J:m:n}$ if $x_{m:m:n} > \eta$,

where J represents the number of failures that occurred before time η. For more information, see [18]. Studies [19–22] indicate that several authors have drawn statistical conclusions from the SS-PALT, as per T-I PHCS. The statistical inference process is less efficient in T-I PHCS because the effective sample size is random and there is a small chance that failures will occur before the allotted period. The AT-II PHCS, or adaptive Type II progressive hybrid censoring system, was introduced in [23]. This can help with the issue that was brought up earlier. Due diligence on product dependability is essential given the short time allotted for creation. Many devices used in the field have stopped using failure censoring methods like Type II and AT-II PHCS. The AT-I PHCS was subsequently introduced in the work of Lin and Huang [24]. Studies such as [25–27] show that it has also caught the interest of several researchers.

Timelines for product development necessitate rapid reliability testing. Many product categories no longer suit failure-censored approaches. A result of this was the creation of an AT-I PHCS approach by the work of Lin and Huang [24]. This method produces more accurate estimates of efficiency and guarantees that the experiment will terminate at η. We can provide a concise explanation of this method. Consider an experiment where n units are used with the censoring procedures $R_1, R_2, \ldots, R_m$. Within the interval of $(0, \infty)$, the experiment comes to a predetermined finish at time η. The initial

failure causes the random deletion of some of the remaining units. A representation of these units is R_1. The remaining units with a value of R_2 that survive the second failure are also eliminated from the experiment at random. J is the number of units that failed prior to η. If the m-th failure happens before η, the process will continue to detect failures until η. When all leftover units of $R_J^* = n - J - \sum_{i=1}^{J} R_i$ are eliminated at η, the experiment is declared complete. $R_1, R_2, \ldots, R_J$ will be used as a progressive censoring scheme in the time η leading up to the m-th failure; otherwise, the process will not adhere to this design. The likelihood function in this case is given by:

$$L = C_J \prod_{i=1}^{J} f(x_{(i)})[1 - F(x_{(i)})]^{R_i}[1 - F(\eta)]^{R_J^*} \tag{1.1}$$

where $C_J = \prod_{i=1}^{J} \varphi$ with $\varphi = m - i + 1 + \sum_{j=1}^{m} R_i$ and $R_J^* = n - J - \sum_{i=1}^{J} R_i$.

- Type I and adaptive Type I progressive hybrid censoring schemes
  Consider a life test with n units and an expected number of failures m. The removal vector $R = (R_1, R_2, \ldots, R_m)$ specifies a progressive hybrid censoring scheme, where R_i represents the number of surviving units that are randomly removed at the moment of the i-th recorded failure. A fixed censoring time $\eta > 0$ is denoted by this.
- Type I progressive hybrid censoring scheme
  The experiment ends at time $\min(x_{m:m:n}, \eta)$ when Type-I PHCS is used. The given is the removal pattern and the observed failure data $\{x_{1:m:n}, x_{2:m:n}, \ldots, x_{D:m:n}\}$:
  - Case I: If $x_{m:m:n} \leq \eta$, then $D = m$. All m failures are observed before η, and the removal vector is $R = (R_1, R_2, \ldots, R_m)$ with progressive removals at each failure. The experiment terminates at $x_{m:m:n}$.
  - Case II: If $x_{m:m:n} > \eta$, then $D = J < m$. Only J failures are observed before η, and the removal vector is $(R_1, R_2, \ldots, R_J)$. The experiment terminates at the fixed time η, and the remaining $R_J^* = n - J - \sum_{i=1}^{J} R_i$ units are censored at η.
- Adaptive Type I progressive hybrid censoring scheme
  To ensure termination at time η, the AT-I PHCS is engineered for situations with stringent time limitations. The procedure following the m-th failure is where the main distinction is as follows.
  - Case I: If $x_{m:m:n} \leq \eta$, the experiment continues past the m-th failure. The removal vector is extended such that $R_m = R_{m+1} = \cdots = R_D = 0$, with $D > m$ being the total number of failures observed up to η. Failures $x_{m+1:n}, x_{m+2:n}, \ldots, x_{D:n}$ are recorded, while no additional removals are made after m (i.e., $R_i = 0$ for $i \geq m$). All remaining $R_D^* = n - D - \sum_{i=1}^{D} R_i$ units are censored at η.
  - Case II: If $x_{m:m:n} > \eta$, $D = J < m$ and the result matches Type I PHCS Case II: Only J failures are observed before η, the removals $(R_1, R_2, \ldots, R_J)$, and $R_J^* = n - J - \sum_{i=1}^{J} R_i$ is censored at η.

  Hence, for Type I APHCS Case I, the removal vector may be written as $(R_1, R_2, \ldots, R_{m-1}, R_m = 0, R_{m+1} = 0, \ldots, R_D = 0)$, explicitly reflecting that no units are removed after the m-th failure.
- Unified data structure
  The observed data and removals for likelihood construction are:
  - Type I PHCS Case I: $\{x_{1:m:n}, \ldots, x_{m:m:n}; R_1, \ldots, R_m\}$ $(D = m)$.

– Type I PHCS Case II: $\{x_{1:m:n}, \ldots, x_{J:m:n}; R_1, \ldots, R_J; R_J^*\}$ $(D = J < m)$.
– AT-I PHCS Case I: $\{x_{1:m:n}, \ldots, x_{m:m:n}, x_{m+1:n}, \ldots, x_{D:n}; R_1, \ldots, R_{m-1}, R_m = 0, \ldots, R_D = 0; R_D^*\}$ $(D > m)$.
– AT-I PHCS Case II: Identical to Type I PHCS Case II.

Figure 1 shows the process of generating order statistics from the SS-PALT under AT-I PHCS.

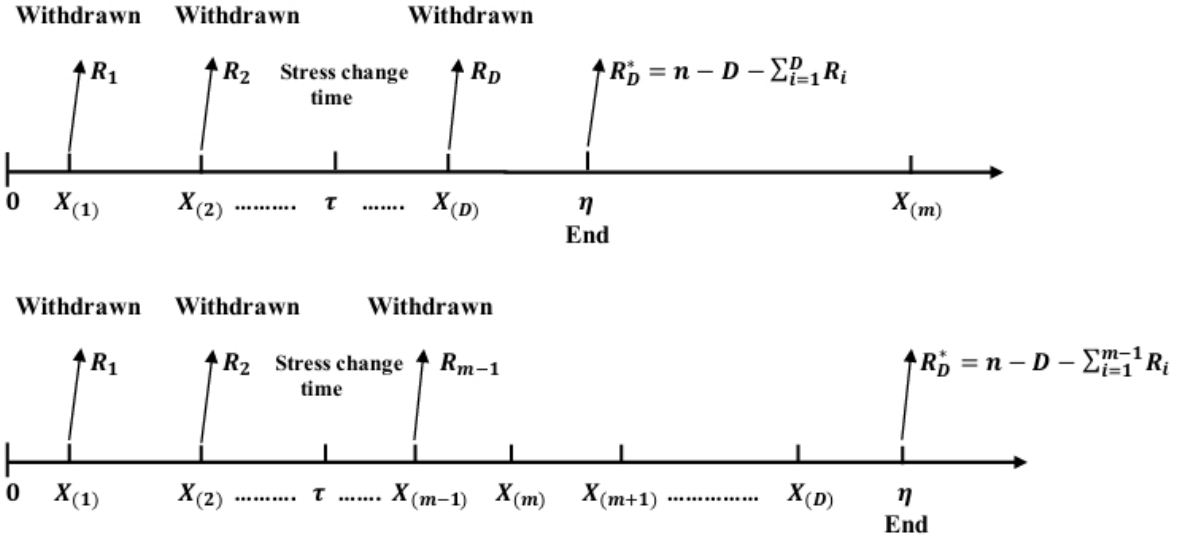

Figure 1. The process of generating order statistics from the PALT under AT-I PHCS.

In the subsequent section, we delineate the principal functions of the Lindley distribution. This research is motivated by three principal factors.

(1) The amalgamation of the AT-I PHCS with SS-PALTs: This combination facilitates earlier test termination while maintaining precise estimations of unknown parameters under both normal and accelerated settings. This is advantageous when failures are infrequent, as is customary for contemporary high-reliability products. The AT-I PHCS plan improves the efficiency of statistical inference by guaranteeing that a sufficient number of failures are recorded within a specified timeframe, thus addressing the shortcomings of previous censoring methods, including the T-I PHCS.
(2) The Lindley distribution's versatility in modeling lifetime data: It can depict constant, declining, and increasing rates. This adaptability mitigates the danger of model misspecification and facilitates more dependable extrapolation from standard usage to heightened stress conditions.
(3) There is a minor deficiency in the prior applications of the AT-I PHCS inside SS-PALT contexts. This deficiency underscores the originality of implementing the AT-I PHCS framework in conjunction with the Lindley model in the context of SS-PALTs.

These factors underscore the uniqueness and practical significance of our work. This research seeks to establish methodologies for estimating the parameters of the Lindley distribution within a cohesive inferential framework that integrates the SS-PALT model with the AT-I PHCS. This study's contributions are the following:

(1) Development of SS-PALTs utilizing the AT-I PHCS framework, and derivation of the comprehensive joint likelihood.
(2) The formulation of classical maximum likelihood estimates (MLEs) is warranted due to their consistency, asymptotic efficiency, and capacity to facilitate the direct production of confidence intervals from observed information, interval estimates for unknown parameters, and an acceleration factor.
(3) Establishment of a Bayesian estimating framework due to its capability to manage small samples and significant censoring, facilitate the integration of prior information, and deliver comprehensive posterior uncertainty via Bayesian credible intervals. Bayesian estimates are obtained by the squared error loss function and the Markov chain Monte Carlo (MCMC) method.
(4) A comparison of how well the methods work in a simulation study, along with proof that they are useful in the real world.

The following is the outline of the paper. The suggested model is detailed in Section 2. Section 3 introduces the acceleration factor and the unknown parameters together with the maximum likelihood estimator and asymptotic confidence interval. In Section 4, we cover topics such as Bayesian estimation with the squared error loss function, how to build credible intervals, and how to implement the MCMC technique in the Gibbs sampler. Sections 5 and 6 discuss optimal selection of the stress change time and prior sensitivity analysis across different scenarios. The results and methodology of the simulation analysis are detailed in Section 7. In Section 8, we see an example of real-world data in action. Finally, we provide concluding remarks in Section 9.

2. Model description

The Lindley distribution, proposed in [28], has emerged as a valuable model in reliability and survival research. It is a continuous probability distribution characterized by positive data, including the lifespan, waiting periods, and failure times. The distribution is extensively utilized due to its straightforward mathematical structure and practical adaptability. In contrast to the exponential distribution, the Lindley model can characterize scenarios in which the failure rate escalates over time. This feature renders it appropriate for various technical and industrial applications. Moreover, its probability density and cumulative distribution functions are expressed in closed form, facilitating estimation and statistical analysis. Numerous academics have created extensions of the Lindley distribution to enhance its capacity to accommodate various data types. These encompass weighted, inverse, and generalized Lindley models. The Lindley distribution has attracted significant interest in applied statistics because to its equilibrium of simplicity and efficacy. The related probability density function (PDF) is defined as

$$f(x,\theta) = \theta^2(\theta+1)^{-1}(1+x)\exp(-\theta x)\,; \quad x>0, \theta>0. \tag{2.1}$$

The survival function of the Lindley distribution and the hazard rate function are given by:

$$s(x,\theta) = (\theta+1+\theta x)(\theta+1)^{-1}\exp(-\theta x)\,; \quad x>0, \theta>0, \tag{2.2}$$

$$h(x,\theta) = [\theta^2(1+x)](\theta+1+\theta x)^{-1}\,; \quad x>0, \theta>0. \tag{2.3}$$

Figure 2 shows the PDF and cumulative distribution function (CDF) of the Lindley distribution for various values of θ.

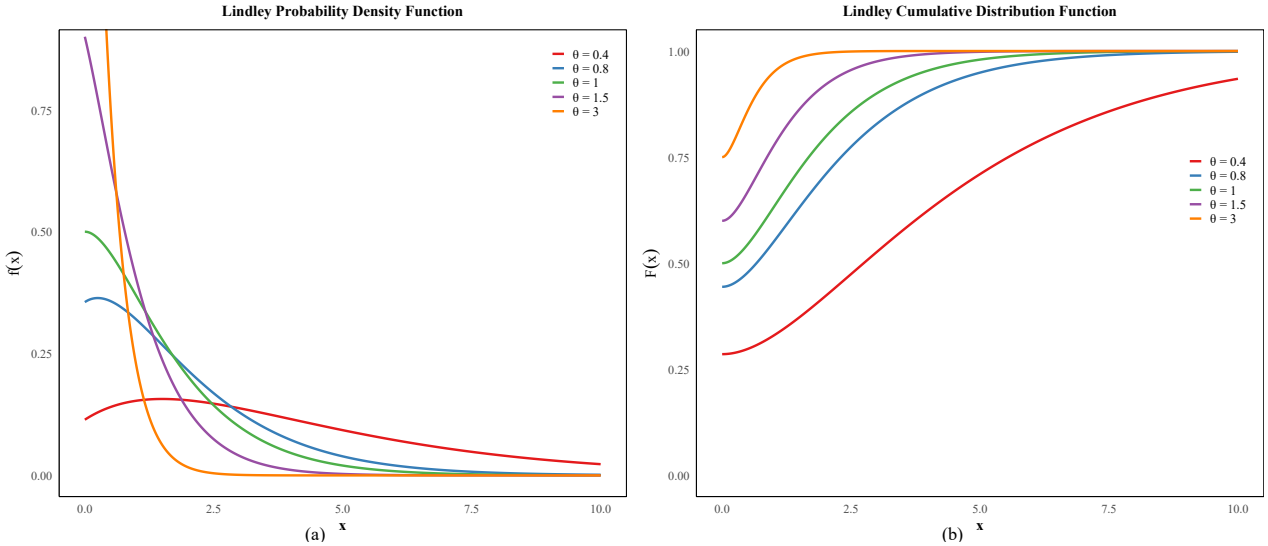

Figure 2. PDF and CDF of the Lindley distribution for different values of the parameter θ.

Under SS-PALT, the lifetime of the unit is given as follows:

$$X = \begin{cases} T & if \;\; T \leq \tau, \\ \tau + \frac{T-\tau}{\beta} & if \;\; T > \tau, \end{cases}$$

where τ is the stress change time, β value of more than one is the acceleration factor, and T is the unit's lifetime under typical operating conditions. Under the SS-PALT paradigm, the PDF of X can be found by

$$f(x) = \begin{cases} 0 & if \;\; x < 0, \\ f_1(x) \equiv f(x,\theta) & if \;\; 0 < x < \tau, \\ f_2(x) & if \;\; x > \tau, \end{cases}$$

where $f_2(x)$ is given by

$$f_2(x) \equiv f_2(x;\theta,\beta) = \theta^2(\theta+1)^{-1}\beta\{1+[\tau+\beta(x-\tau)]\}\exp\{-\theta[\tau+\beta(x-\tau)]\}, \tag{2.4}$$

and the corresponding survival function of the Lindley distribution is

$$s_2(x) \equiv s_2(x,\theta,\beta) = \{\theta+1+\theta[\tau+\beta(x-\tau)]\}(\theta+1)^{-1}\exp\{-\theta[\tau+\beta(x-\tau)]\}. \tag{2.5}$$

The one-parameter Lindley distribution has numerous appealing analytical aspects, as several significant reliability characteristics, including the survival function, hazard function, cumulative hazard function, and mean residual life, may be explicitly determined in closed forms. This property obviates the necessity for intricate numerical integration and enables dependable parameter estimation in both maximum likelihood and Bayesian inference, especially in the presence of censored observations. Despite the proposal of numerous extensions with supplementary parameters in the

literature, the original single-parameter Lindley model continues to be favored in many applications due to its optimal balance of simplicity, interpretability, and computational efficiency, particularly in the analysis of upper-hybrid censored data. Moreover, the distribution conforms to the concept of parsimony by characterizing lifetime behavior with a singular parameter, hence improving model stability and facilitating interpretation, as illustrated in previous studies [29–32].

Over the last few years, academics have demonstrated an increased interest in improving the traditional Lindley distribution by creating more versatile extensions that can accommodate a greater range of lifetime data properties. This work resulted in the introduction of several significant variations. Among them, the Q-Lindley distribution offers more flexibility through additional shape parameters, making it suited for simulating severely skewed observations; see [33]. The X-Lindley distribution was developed to depict a broader range of hazard rate patterns, whereas the truncated X-Lindley distribution has proven useful for assessing truncated lifetime data, which are frequently encountered in reliability and life-testing investigations; see [34]. Furthermore, the Weibull-Lindley distribution combines the strengths of the Weibull and Lindley families, providing greater capability for expressing both monotonic and non-monotonic hazard functions, see [35]. Because of their versatility and high empirical performance, these extensions have found broad use in domains such as reliability analyse, survival studies, and accelerated life-testing tests. For more details on other similar Lindley-type templates, see [36–40].

3. Statistical inference based on AT-I PHCS

This section uses AT-I PHCS with SS-PALTs data to explore the MLEs of θ and the acceleration factor β. Under typical usage circumstances, the MLEs of θ and β are given. Additionally, the asymptotic theory of MLEs is used to obtain the interval estimations of different parameters.

3.1. Point estimation

Suppose the observed progressive hybrid censored data consists of D failure times $x_1, x_2, \ldots, x_D$ up to a censoring time η, with progressive removals $\mathbf{R} = (R_1, \ldots, R_D)$ according to the scheme described above, and the remaining $R_D^* = n - D - \sum_{i=1}^{D} R_i$ are censored at η. Note that for clarity, the original order statistics are denoted as $X_{i:m:n}$, but for simplicity, we use X_i throughout the likelihood and inference developments, assuming that the sampling context is fixed.

Let n_u be the number of failures occurring prior to the stress-change time τ (i.e., $x_i \leq \tau$ for $i = 1, \ldots, n_u$), and let $n_a = D - n_u$ the number of failures after τ. The likelihood function of the Lindley SS-PALT model under either Type I PHCS or AT-I PHCS is:

$$L(\theta,\beta) \propto \left[\prod_{i=1}^{n_u} f_1(x_i;\theta) S_1(x_i;\theta)^{R_i}\right]\left[\prod_{i=n_{u+1}}^{D} f_2(x_i;\theta,\beta) S_2(x_i;\theta,\beta)^{R_i}\right] S_2(\eta;\theta,\beta)^{R_D^*}, \tag{3.1}$$

where the removal vector components $R_m, R_{m+1}, \ldots, R_D$ are zero in AT-I PHCS Case I for $D > m$. Substituting (2.1), (2.2), (2.4), and (2.5) in the likelihood function (3.1), we have

$$L(\theta,\beta) \quad \propto \quad \beta^{n_a}\left[\theta^2(\theta+1)^{-1}\right]^D \prod_{i=1}^{n_u}\left\{(1+x_i)\left[\{\theta+1+\theta(x_i)\}(\theta+1)^{-1}\right]^{R_i}\right\}$$

$$* \quad \prod_{i=n_{u+1}}^{D} \left\{ \{1 + W_i(\beta)\} \left[\frac{\theta + 1 + \theta[W_i(\beta)]}{\theta + 1} \right]^{R_i} \right\} \left\{ \frac{\theta + 1 + \theta[W^*(\eta)]}{\theta + 1} \right\}^{R_D^*}$$

$$* \quad \exp\left\{ -\theta \left[\sum_{i=1}^{n_u} (x_i)(1 + R_i) + \sum_{i=n_{u+1}}^{D} W_i(\beta)(1 + R_i) + R_D^* W^*(\eta) \right] \right\}, \tag{3.2}$$

where $x_{(i)} = x_i$, $W_i(\beta) = \tau + \beta(x_i - \tau)$, $W^*(\eta) = \tau + \beta(\eta - \tau)$, and η represent the ideal experiment termination time, m is the prespecified minimum number of failures required, D is the number of observed failures before termination, and J is an indicator defining the termination case.

Taking the natural logarithm likelihood function $l = \ln L$ in (3.2), we obtain

$$\begin{aligned} l &= D \ln\left[\theta^2(\theta + 1)^{-1}\right] + n_a \ln\beta + \sum_{i=1}^{n_u} \ln(1 + x_i) + \sum_{i=1}^{n_u} R_i \ln\left[\{\theta + 1 + \theta(x_i)\}(\theta + 1)^{-1}\right] \\ &+ \sum_{i=n_{u+1}}^{D} \ln\{1 + W_i(\beta)\} + \sum_{i=n_{u+1}}^{D} R_i \ln\left\{\frac{\theta + 1 + \theta[W_i(\beta)]}{\theta + 1}\right\} + R_D^* \ln\left\{\frac{\theta + 1 + \theta[W^*(\eta)]}{\theta + 1}\right\} \\ &- \theta\left\{\sum_{i=1}^{n_u} (x_i)(1 + R_i) + \sum_{i=n_{u+1}}^{D} W_i(\beta)(1 + R_i) + R_D^* W^*(\eta)\right\}. \end{aligned} \tag{3.3}$$

The first-order derivatives of (3.3) with respect to θ and β are given, respectively, by

$$\begin{aligned} \frac{\partial l}{\partial \theta} &= D\left[2\theta^{-1} - (\theta + 1)^{-1}\right] + \sum_{i=1}^{nu} R_i \left[\frac{x_i}{(\theta + 1 + \theta x_i)(\theta + 1)}\right] + \sum_{i=n_{u+1}}^{D} R_i \left\{\frac{W_i(\beta)}{\{\theta + 1 + \theta[W_i(\beta)]\}(\theta + 1)}\right\} \\ &+ \frac{R_D^* W^*(\eta)}{\{\theta + 1 + \theta[W^*(\eta)]\}(\theta + 1)} - \sum_{i=1}^{n_u} (x_i)(1 + R_i) - \sum_{i=n_{u+1}}^{D} W_i(\beta)(1 + R_i) - R_D^* W^*(\eta), \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \frac{\partial l}{\partial \beta} &= n_a \beta^{-1} + \sum_{i=n_{u+1}}^{D} (x_i - \tau)\{1 + W_i(\beta)\}^{-1} + \sum_{i=n_{u+1}}^{D} R_i \left\{[\theta(x_i - \tau)](\theta + 1 + \theta[W_i(\beta)])^{-1}\right\} \\ &+ R_D^* \left\{\frac{\theta(\eta - \tau)}{\theta + 1 + \theta[W^*(\eta)]}\right\} - \theta\left\{\sum_{i=n_{u+1}}^{D} (x_i - \tau)(1 + R_i) + R_D^*(\eta - \tau)\right\}. \end{aligned} \tag{3.5}$$

By equating Eqs (3.4) and (3.5) to zero, we derive two nonlinear equations. The system of these nonlinear equations is not amenable to analytical solutions. We can use numerical solutions through iterative strategies to obtain the MLEs.

3.2. Interval estimation

Using the asymptotic distributions of the MLEs for the parameters θ and β, we may approximate the confidence intervals of the parameters. Here is a formula that shows how the MLEs of θ and β tend to cluster as time goes on:

$$[(\hat{\theta} - \theta), (\hat{\beta} - \beta)] \longrightarrow N(0, I^{-1}(\theta, \beta)),$$

where $I^{-1}(\varphi)$ is the variance-covariance matrix of the parameters $\varphi = (\theta, \beta)$, and $I^{-1}(\theta, \beta)$ is the corresponding inequality. By inverting the Fisher information matrix, the asymptotic variance-covariance matrix can be obtained in the following way:

$$\boldsymbol{I}^{-1}(\hat{\varphi}) = \begin{pmatrix} -\frac{\partial L_1^2}{\partial \theta^2} & -\frac{\partial L_1^2}{\partial \theta \partial \beta} \\ -\frac{\partial L_1^2}{\partial \theta \partial \beta} & -\frac{\partial L_1^2}{\partial \beta^2} \end{pmatrix}^{-1}_{(\theta,\beta)=(\hat{\theta},\hat{\beta})} = \begin{pmatrix} var(\hat{\theta}) & cov(\hat{\theta},\hat{\beta}) \\ cov(\hat{\theta},\hat{\beta}) & var(\hat{\beta}) \end{pmatrix},$$

According to the Fisher information matrix, the negative second partial derivatives of Eq (3.3) with regard to θ and β are defined as

$$\begin{aligned}
\frac{\partial l^2}{\partial \theta^2} &= -2D\theta^{-2} + D(\theta+1)^{-2} - \sum_{i=1}^{n_u} R_i \frac{x_i(2\theta + 2\theta x_i + 2 + x_i)}{[(\theta+1+\theta x_i)(\theta+1)]^2} \\
&- \sum_{i=n_{u+1}}^{D} R_i \frac{W_i(\beta)\{\theta+1+\theta W_i(\beta) + (\theta+1)[1+W_i(\beta)]\}}{\{\{\theta+1+\theta W_i(\beta)\}(\theta+1)\}^2} \\
&- R_D^* \frac{W^*(\eta)\{\theta+1+\theta W^*(\eta) + (\theta+1)[1+W^*(\eta)]\}}{\{\{\theta+1+\theta W^*(\eta)\}(\theta+1)\}^2}, \\
\frac{\partial l^2}{\partial \beta^2} &= -n_a \beta^{-2} - \sum_{i=nu+1}^{D} (x_i - \tau)^2 \{1 + W_i(\beta)\}^{-2} \\
&- \theta^2 \left\{ \sum_{i=n_{u+1}}^{D} R_i \left[\frac{(x_i-\tau)^2}{\{\theta+1+\theta W_i(\beta)\}^2} \right] + R_D^* \left[\frac{(\eta-\tau)^2}{\{\theta+1+\theta W^*(\eta)\}^2} \right] \right\}, \\
&\textit{and} \\
\frac{\partial l^2}{\partial \beta \partial \theta} &= \sum_{i=n_{u+1}}^{D} R_i \left\{ \frac{\{\theta+1+\theta W_i(\beta)\}(x_i-\tau) - \theta(x_i-\tau)\{1+W_i(\beta)\}}{\{\theta+1+\theta W_i(\beta)\}^2} \right\} - R_D^*(\eta-\tau) \\
&+ R_D^* \left\{ \frac{\{\theta+1+\theta W^*(\eta)\}(\eta-\tau) - \theta(\eta-\tau)\{1+W^*(\eta)\}}{\{\theta+1+\theta W^*(\eta)\}^2} \right\} - \sum_{i=n_{u+1}}^{D} (x_i-\tau)(1+R_i).
\end{aligned}$$

Thus, the approximate confidence intervals of θ and β are, respectively, given by

$$\hat{\beta} \pm Z_{\alpha/2}\sqrt{var(\hat{\beta})} \quad and \quad \hat{\theta} \pm Z_{\alpha/2}\sqrt{var(\hat{\theta})}.$$

where $Z_{\alpha/2}$ is the upper $\alpha/2$ the percentile of a standard normal distribution.

4. Bayesian estimation

The Bayesian estimations of the parameters θ and β, as well as the generation of credible intervals, are examined in this section. There has been a lot of buzz about using the Bayesian estimation approach to failure time data. It takes both known parameters and easily available data into account. Using the maximum likelihood function, this study provides Bayesian estimations to produce point and interval estimates of the unknown parameters. Assuming that a and b are independent and have

independent gamma (G) priors, Bayesian estimators are generated using the square error (SE) loss function. For example, according to Ahmad et al. [41], several authors have argued that the acceleration factor follows a non informative prior and so provides little information on the parameter. An assumed one-parameter location-parameter G distribution is applied to the independent variable. Because of its adaptability and the wealth of information it yields about unknown features, the G distribution is useful. So, $\theta \sim G(c_1, b_1)$ and $\beta \sim G(c_2, b_2, 1)$. The joint prior distribution of the unknown parameters can be expressed as follows, given the assumptions made above:

$$\pi(\theta,\beta) \propto \theta^{(c_1-1)}(\beta-1)^{(c_2-1)} \exp\left[-b_1\theta - b_2(\beta-1)\right],\ \theta > 0,\ \beta > 1. \tag{4.1}$$

where, $c_k > 0$ and $b_k, k = 1, 2$ are the hyper-parameters.

The joint posterior distribution derived from the observed data provided by the likelihood function in (3.2) and the joint prior distribution articulated in (4.1) can be stated via the maximum likelihood technique as follows:

$$\begin{aligned}
\pi^*(\theta,\beta|x) &= \mathcal{H}(\beta-1)^{n_a+c_2-1}\left[\theta^2(\theta+1)^{-1}\right]^{D+c_1-1} e^{-b_2(\beta-1)} \prod_{i=1}^{n_u}\left\{(1+x_i)\left[\frac{\theta+1+\theta(x_i)}{\theta+1}\right]^{R_i}\right\} \\
&* \prod_{i=n_{u+1}}^{D}\left\{\{1+W_i(\beta)\}\left[\frac{\theta+1+\theta[W_i(\beta)]}{\theta+1}\right]^{R_i}\right\}\left\{\frac{\theta+1+\theta[W^*(\eta)]}{\theta+1}\right\}^{R_D^*} \\
&* \exp\left\{-\theta\left[b_1 + \sum_{i=1}^{n_u}(1+R_i)(x_i) + \sum_{i=n_{u+1}}^{D}(1+R_i)W_i(\beta) + R_D^* W^*(\eta)\right]\right\}
\end{aligned} \tag{4.2}$$

where $\mathcal{H}$ is the normalized constant. According to the SE loss function, the Bayesian estimator for any function of the unknown parameters, denoted as $\xi_{SEL}(\varphi)$, can be obtained as follows:

$$\hat{\xi}_{SEL}(\varphi) = \frac{\int_1^\infty \int_0^\infty \xi(\varphi)\pi^*(\theta,\beta|x)\ d\theta d\beta}{\int_1^\infty \int_0^\infty \pi^*(\theta,\beta|x)\ d\theta d\beta}. \tag{4.3}$$

It is evident from (4.3) that the Bayesian estimate of θ or β cannot be derived analytically, as the posterior expectation entails double integrals in both the numerator and the denominator, rendering the Bayesian expressions for θ or β difficult to get. The Bayesian MCMC procedure is used to generate samples from which the provided Bayesian estimates and credible intervals can be derived.

4.1. MCMC method

See previous studies [42–45] for a complete rundown of the benefits of MCMC and other Bayesian statistical computation methods. You may also find recent examples of MCMC being used to create and evaluate novel statistical models in [46]. Here, we take a look at Gibbs sampling, an MCMC algorithm that calculates Bayes estimates (BEs) and credible intervals under the SE loss function by taking a series of random samples from the posterior function. According to (4.2), the conditional posterior density functions for the parameters θ and β can be expressed in the following ways, respectively:

$$\pi^*(\theta|\beta;x) \propto \left[\theta^2(\theta+1)^{-1}\right]^{D+c_1-1} \prod_{i=1}^{n_u}\left\{[\theta+1+\theta(x_i)]^{R_i}(\theta+1)^{-R_i}\right\}$$

$$
\begin{aligned}
&* \prod_{i=n_{u+1}}^{D} \left[\{\theta+1+\theta[W_i(\beta)]\}^{R_i}(\theta+1)^{-R_i}\right]\left\{(\theta+1+\theta[W^*(\eta)])^{R_D^*}(\theta+1)^{-R_D^*}\right\} \\
&* \exp\left\{-\theta\left[b_1+\sum_{i=1}^{n_u}(1+R_i)(x_i)+\sum_{i=n_{u+1}}^{D}(1+R_i)W_i(\beta)+R_D^*W^*(\eta)\right]\right\} \qquad (4.4)
\end{aligned}
$$

and

$$
\begin{aligned}
\pi^*(\beta|\theta;x) \;\propto\; & (\beta-1)^{n_a+c_2-1}(\theta+1)^{-R_D^*}\{\theta+1+\theta[W^*(\eta)]\}^{R_D^*}\exp[-b_2(\beta-1)] \\
&* \prod_{i=n_{u+1}}^{D}\left\{\{1+W_i(\beta)\}(\theta+1+\theta[W_i(\beta)])^{R_i}(\theta+1)^{-R_i}\right\} \\
&* \exp\left\{-\theta\left[\sum_{i=n_{u+1}}^{D}(1+R_i)W_i(\beta)+R_D^*W^*(\eta)\right]\right\} \qquad (4.5)
\end{aligned}
$$

A known distribution based on the conditional posterior distributions of θ and β cannot be analytically reduced to Eqs (4.4) and (4.5). Therefore, we employ the Metropolis-Hastings algorithm, which generates random samples based on the normal proposal distribution, to address this issue.

4.2. Metropolis-Hastings within the Gibbs sampler

Every cycle of the Gibbs sampler incorporates a Metropolis-Hastings step, as described in [47, 48]. The steps that make up the algorithm are as follows:

(1) Starting values. For every parameter index $i = 1, 2$, initialize the chain by setting $\zeta_i^{(0)} = \hat{\zeta}_i$.
(2) Loop counter. Begin with $j = 1$.
(3) Generating proposals. For each $i = 1, 2$, denote the full conditional posterior distribution as

$$\pi_i^*(\zeta_i^{(j-1)} \mid \varphi_{-i}^{(j-1)}, \text{data}),$$

where $\varphi_{-i}^{(j-1)}$ refers to the other parameters, taken at either their most recently updated or earlier values. At the current iteration j, sample a candidate value θ_i^* from a normal proposal distribution $\mathcal{N}(\zeta_i^{(j-1)}, \hat{\sigma}_i^2)$, with $\hat{\sigma}_i^2$ representing the i-th diagonal entry of the inverse Fisher information matrix (refer to Subsection 3.2).
(4) Acceptance rule. For each $i = 1, 2$, calculate the acceptance probability

$$\vartheta_i = \min\left\{1, \frac{\pi_i^*(\zeta_i^* \mid \varphi_{-i}^{(j-1)}, \text{data})}{\pi_i^*(\zeta_i^{(j-1)} \mid \varphi_{-i}^{(j-1)}, \text{data})}\right\},$$

then draw $u_i \sim U(0, 1)$, and update the chain according to

$$\zeta_i^{(j)} = \begin{cases} \zeta_i^*, & \text{if } u_i < \vartheta_i, \\ \zeta_i^{(j-1)}, & \text{otherwise.} \end{cases}$$

(5) Advance counter. Increase j by one, i.e., set $j = j + 1$.
(6) Iteration. Continue executing Steps 3 through 5 for $j = 1, 2, \ldots, \mathcal{N}$.

In order to lessen reliance on the selected initial values, the first $\mathcal{M}$ samples drawn are eliminated during the burn-in phase. Given a sufficiently large total number of iterations $\mathcal{N}$, the remaining draws $\{\zeta_i^{(j)} : j = \mathcal{M}+1, \mathcal{M}+2, \ldots, \mathcal{N}\}$ for i from 1 to 2 serve as approximate samples from the posterior distribution. Let $\mathcal{N}_B = \mathcal{N} - \mathcal{M}$; based on the loss functions given in Eq (4.3), the Bayesian point estimates can be calculated as follows:

$$\hat{\zeta}_{i_{\mathrm{SEL}}} = \frac{1}{\mathcal{N}_B} \sum_{j=\mathcal{M}+1}^{\mathcal{N}} \zeta_i^{(j)}.$$

When using informative prior distributions within the Bayesian framework, choosing appropriate hyperparameters represents an essential and unavoidable task. This selection is typically achieved by combining expert opinion, results from earlier studies, and empirical evidence that captures relevant prior knowledge. Earlier work [49, 50] highlighted how this approach enhances the results in terms of both the models' robustness and their interpretability.

In the context of the SS-PALT framework, let W represent the overall count of whole datasets produced by the Lindley distribution. The MLEs are $\hat{\varphi}_i = (\hat{\theta}_i, \hat{\beta}_i)$ for every replication $i = 1, 2, \ldots, W$. After that, the empirical moments are determined as:

$$\bar{\zeta}_q = \frac{1}{W} \sum_{i=1}^{W} \hat{\zeta}_{q,i}, \quad \text{and} \quad \sigma^2_{\zeta_q} = \frac{1}{W-1} \sum_{i=1}^{W} \left(\hat{\zeta}_{q,i} - \bar{\zeta}_q\right)^2, \quad \text{for } q = 1, 2.$$

These empirical quantities are then equated to the theoretical moments of the prior distributions given in Eq (4.1), expressed as:

$$\bar{\zeta}_q = \begin{cases} \frac{c_q}{b_q}, & \text{for } q = 1 \\ c_q, & \text{for } q = 2, \end{cases} \quad \text{and} \quad \sigma^2_{\zeta_q} = \begin{cases} \frac{c_q}{b_q^2}, & \text{for } q = 1 \\ b_q, & \text{for } q = 2. \end{cases}$$

Solving these equations leads to the following closed-form estimators for the hyperparameters:

$$\hat{c}_q = \begin{cases} \frac{\bar{\zeta}_q^2}{\sigma^2_{\zeta_q}}, & \text{for } q = 1, \\ \bar{\zeta}_q, & \text{for } q = 2, \end{cases} \quad \text{and} \quad \hat{b}_q = \begin{cases} \frac{\bar{\zeta}_q}{\sigma^2_{\zeta_q}}, & \text{for } q = 1, \\ \sigma^2_{\zeta_q}, & \text{for } q = 2. \end{cases}$$

This moment-based matching approach anchors the prior distributions to the observed variability in the MLEs, thereby ensuring consistency between the data-driven estimates and the selected prior specifications.

4.3. Credible intervals

Each parameter ζ_i has its credible intervals (CRIs) built using the Gibbs sampler described in Subsection 4.2, which incorporates the Metropolis-Hastings steps. For each value of i from 1 to 2, arrange the post-burn-in drawings in a non-decreasing sequence following this formula: $\zeta_i^{(M+1)} \leq \zeta_i^{(M+2)} \leq \cdots \leq \zeta_i^{(N)}$.

We say that the shortest interval that contains a fraction $(1-\gamma)$ of these ordered samples is a two-sided CRI $100(1-\gamma)\%$. To be more specific, we identify the index j^* that reduces the statement to the minimum

$$\zeta_i^{(j+\lfloor(1-\gamma)(N-M)\rfloor)} - \zeta_i^{(j)}, \quad j = 1, 2, \ldots, \lceil\gamma(N-M)\rceil,$$

where $\lfloor \chi \rfloor$ denotes the greatest integer less than or equal to χ. The resulting CRI is then given by

$$(\zeta_{i_L}, \zeta_{i_U}) = \left(\zeta_i^{(j^*)}, \zeta_i^{(j^*+\lfloor(1-\gamma)(N-M)\rfloor)}\right).$$

5. Optimal selection of the stress change time τ

The stress change time τ determines the point at which the experiment switches from the normal stress level to the accelerated stress level. Since the amount of information collected before and after the stress change depends on τ, its selection has a direct effect on the precision of the estimators. Several criteria were considered when selecting τ. A simple reference value is the median lifetime under the baseline condition,

$$\tau_{\text{med}} = F^{-1}(0.5),$$

where $F(\cdot)$ denotes the baseline cumulative distribution function. In addition, empirical values obtained from preliminary simulation experiments were also examined. From an optimal design perspective, the D-optimal criterion selects the value of τ that maximizes the determinant of the Fisher information matrix,

$$\tau_D = \arg\max_{\tau} \ \det\left(\mathbf{I}(\theta, \beta; \tau)\right).$$

Similarly, the A-optimal criterion minimizes the trace of the inverse Fisher information matrix,

$$\tau_A = \arg\min_{\tau} \ \operatorname{tr}\left(\mathbf{I}^{-1}(\theta, \beta; \tau)\right).$$

To evaluate the relative contribution of the two parameters, the following information-balance criterion was also considered:

$$\tau_{\text{IB}} = \arg\min_{\tau} \ \left|\frac{I_{\beta\beta}(\tau)}{I_{\theta\theta}(\tau)} - 1\right|.$$

Here, $I_{\theta\theta}(\tau)$ and $I_{\beta\beta}(\tau)$ denote the diagonal elements of the Fisher information matrix. Another criterion is based on prediction accuracy and selects τ by minimizing the variance of the estimated survival function at a specified time point t_0

$$\tau_V = \arg\min_{\tau} \ \operatorname{Var}\left(\widehat{S}(t_0)\right).$$

Using the delta method, this variance can be approximated by

$$\operatorname{Var}\left(\widehat{S}(t_0)\right) \approx \nabla S(t_0)^{\top} \mathbf{I}^{-1}(\theta, \beta; \tau) \nabla S(t_0).$$

Since different criteria may lead to different recommendations, the final choice of τ was based on both the optimality measures and the simulation performance of the estimators. For the simulation study, the values $\tau = 0.90$, $\tau = 0.50$, and $\tau = 0.15$ were adopted for $\theta = 0.8$, 2.0, and 3.0, respectively. These values provided a reasonable compromise among estimation accuracy, interval performance, and computational efficiency.

5.1. Performance validation

The simulation results presented in Section 7 confirm that the selected values of τ produce satisfactory estimation results for both parameters. Across all scenarios, the estimators exhibit small bias, moderate mean squared errors, and coverage probabilities close to the nominal level. Overall, the selected stress change times provided stable performance over the considered parameter settings and were therefore adopted in all subsequent analyses.

6. Prior sensitivity analysis

To assess the impact of prior specifications, we compared 10 prior combinations across three scenarios ($\theta = 0.8, 2.0, 3.0$ with $\beta = 1.5$). Table 1 shows the mean, bias, mean square error (MSE), and coverage probability (CP). Also, summarizes prior sensitivity analysis across different scenarios.

Table 1. Prior sensitivity analysis across different scenarios.

Scenario	Prior	θ				β				DIC
		Mean	Bias	MSE	CP	Mean	Bias	MSE	CP	
$\theta = 0.8$	Both vague	0.881	0.081	0.027	0.93	1.050	-0.450	0.250	0.48	100.0
	Both weak	0.805	0.005	0.013	0.97	1.497	-0.003	0.114	0.98	99.1
	Both Infinite	0.861	0.061	0.015	0.95	1.444	-0.056	0.025	0.98	98.9
$\theta = 2.0$	Both vague	2.176	0.176	0.086	1.00	1.118	-0.382	0.177	0.30	35.2
	Both weak	1.937	-0.063	0.053	1.00	1.599	0.099	0.061	1.00	34.7
	Both Infinite	2.015	0.015	0.042	1.00	1.463	-0.037	0.012	1.00	34.1
$\theta = 3.0$	Both vague	3.616	0.616	0.582	0.90	1.172	-0.328	0.139	0.60	-8.8
	Both weak	2.666	-0.334	0.223	0.90	1.821	0.321	0.184	1.00	-9.8
	Both Infinite	2.869	-0.131	0.095	1.00	1.556	0.056	0.018	1.00	-10.2

The analysis reveals the following

(1) Vague priors are unsuitable for β: Coverage probabilities drop to 0.30-0.60, indicating convergence issues near the boundary $\beta = 1$.
(2) The Both Weak prior (Gamma(1,1)) is optimal: It provides the best balance across all scenarios with:
   - CPs ≥ 0.90 (typically 0.97-1.00)
   - Reasonable bias and MSE
   - Stable MCMC convergence
(3) Informative priors are less robust: Weak priors perform better at $\theta = 0.8$ (bias: 0.005 vs 0.061; CP: 0.97 vs 0.95).

Therefore, we adopted the Both Weak Gamma(1,1) prior for all Bayesian analyses reported in this paper.

7. Simulation study

In this part, we use SS-PALT to evaluate the MLEs and Bayesian estimators for the Lindley distribution and acceleration factor parameters using AT-I PHCS data in a Monte Carlo simulation. Using R version 4.1.2, 1000 AT-I PHCS samples were generated from the Lindley distribution for each simulation iteration. Following the optimal selection criteria discussed in Section 5, the stress change time τ was set to $\tau = 0.90, 0.50$, and 0.15 for $\theta = 0.8$, 2.0, and 3.0, respectively, with the corresponding censoring times $\eta = 2.5$, 1.0, and 0.8 to ensure balanced information across the normal and accelerated phases. For prior specification, and based on the prior sensitivity analysis presented in Section 6, independent gamma priors were assigned using the Both Weak specification: $\theta \sim$ Gamma$(1, 1)$ and $\beta - 1 \sim$ Gamma$(1, 1)$, which provide a prior mean of 1 for both parameters while maintaining weakly informative properties. The Metropolis-Hastings algorithm was executed for 10,000 iterations, with a 2000-iteration burn-in period.

The estimating procedure is conducted in several primary steps.

- Initialization. Specify (n, m, τ, η) and set the starting values (θ, β).
- The acceleration factor was set at $\beta = 1.5$ in all simulation scenarios, indicating a moderate level of stress acceleration. This decision permits the evaluation of the estimators' inferential performance without causing undue lifespan compression by establishing a meaningful and realistic split between the normal stress and accelerated stress phases. In addition, by setting β, the impacts of the sample size, censoring technique, and baseline parameter θ on the accuracy of estimation may be more clearly evaluated.
- Generation of samples. Generate n full lifetimes from the Lindley distribution under SS-PALT by sampling from the PDF $f_2(x)$ and the survival function s_2 as specified in Eqs (5) and (6), respectively, and thereafter implement the AT-I PHCS outlined in Section 2.
- Calculate the MLEs $\hat{\theta}$ and $\hat{\beta}$ at $\tau = 0.5$. Develop the corresponding 95% asymptotic confidence intervals (ACIs) outlined in section 3.
- Bayesian estimation. Utilize informative priors and establish their hyperparameters as delineated in Section 4 by producing $W = 1000$ complete random samples from the Lindley distribution, using the starting parameter values, with each sample including $n = 1000$ observations. Consequently, the hyperparameters are delineated as follows: $c_1 = 4, b_1 = 2, c_2 = 2$, and $b_2 = 4$. Subsequently, execute an MCMC sampler for $N = 10,000$ iterations, eliminating the initial $M = 2000$ as burn-in. Extract the BEs and 95% CRIs from the posterior draws.
- Reproduction. Reiterate Steps 1–4 for $B = 1000$ Monte Carlo replications to guarantee consistent inference.
- Performance indicators. For each point estimate of $\varphi = (\theta, \beta)$, calculate the following:

$$Mean(\varphi) = \frac{1}{N}\sum_{i=1}^{N}\hat{\varphi}^{(i)},\ MSE(\varphi) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\varphi}^{(i)} - \varphi)^2.\ \text{and}\ MRE(\varphi) = \frac{1}{N}\sum_{i=1}^{N}\left|\frac{\hat{\varphi}^{i} - \varphi}{\varphi}\right|.$$

Similarly, the evaluation of interval estimate is conducted utilizing the following metrics:

$$AL_{(1-\alpha)\%}(\varphi) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\varphi}_U^{(i)} - \hat{\varphi}_L^{(i)})\ ,\ \text{and}\ CP_{(1-\alpha)\%}(\varphi) = \frac{1}{N}\sum_{i=1}^{N} I_{\left(\hat{\varphi}_L^{(i)}, \hat{\varphi}_U^{(i)}\right)}(\varphi)\,.$$

Throughout all simulation runs, the mean is used to represent the average value of the estimator.

One way to evaluate the precision of parameter estimation is by looking at the MSE. The mean relative absolute bias (MRE) measures how far the estimator is on average from the actual parameter value. Whether they are confidence intervals or CRIs, average length (AL) shows the average width of the produced intervals. One measure of statistical significance is the CP, which indicates the percentage of intervals that adequately capture the actual parameter value.

Using the AT-I PHCS scheme, the SS-PALT model's results for all parameters are summarized in Tables 2–10. The measurements that are presented for each parameter correspond to the 95% ACIs or CRIs, and they include the mean, MSE, MRE, AL, and CP. In the tables, MLEs, Bayesian estimators, and estimates obtained under SE loss function. Results with a lower MSE, lower MRE, shorter interval lengths, and sufficient coverage probabilities show that point and interval estimates are both relevant. From these results, we can deduce a number of significant points.

For Type I PHCS (Figure 3, $n = 50, m = 35, S\,1$), The standard deviations of the MCMC chains are 0.1145 for θ and 0.3722 for β, with acceptance rates of 0.429 and 0.447, respectively. The effective sample sizes (ESSs) are 1050.5 for θ and 931.3 for β (both > 200). Geweke Z-scores are -1.247 for θ and 0.724 for β (both within ± 1.96). The Gelman-Rubin (R-hat) values are 1.003 for θ and 1.004 for β (both < 1.05). Figures 3 and 4 (MCMC diagnostics) confirm good mixing and convergence.

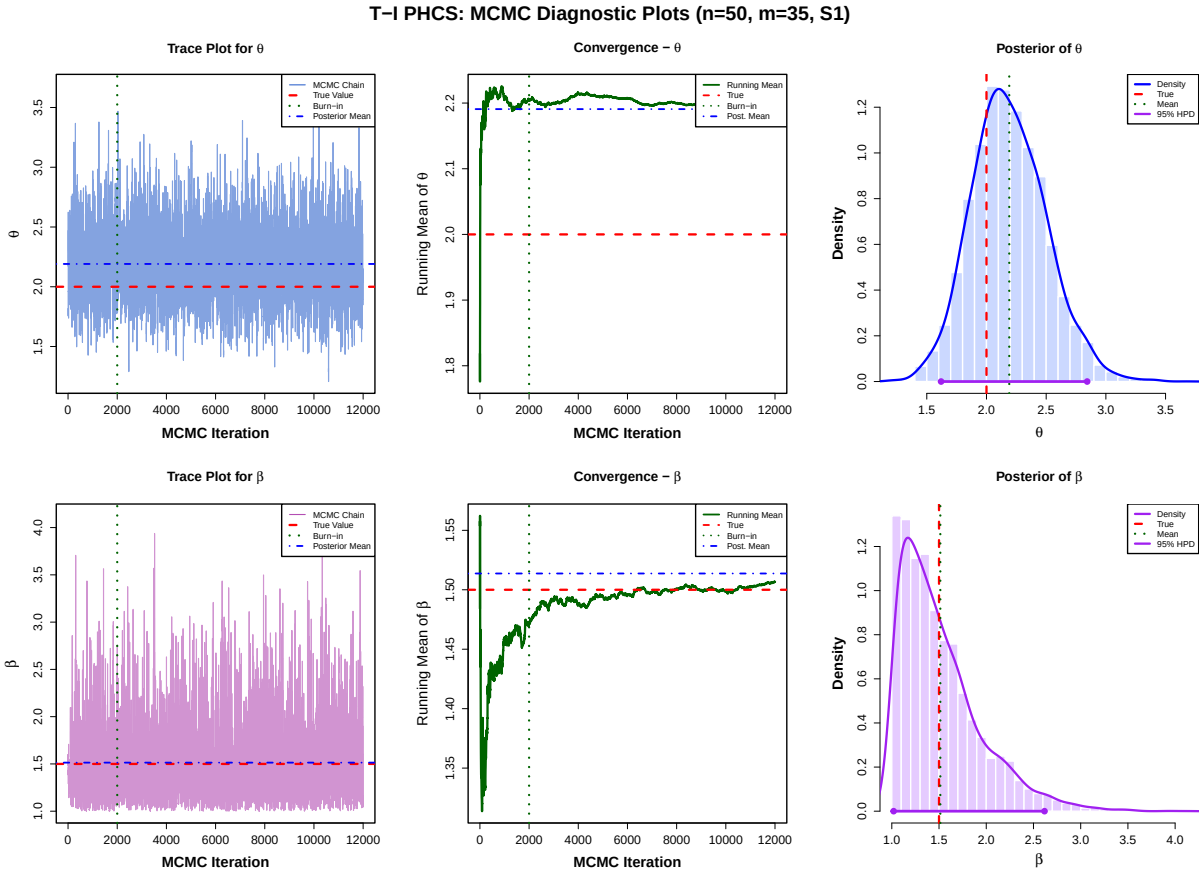

Figure 3. Graphical illustration of the Type I PHCS with $n = 50$, $m = 35$, and $S\,1$.

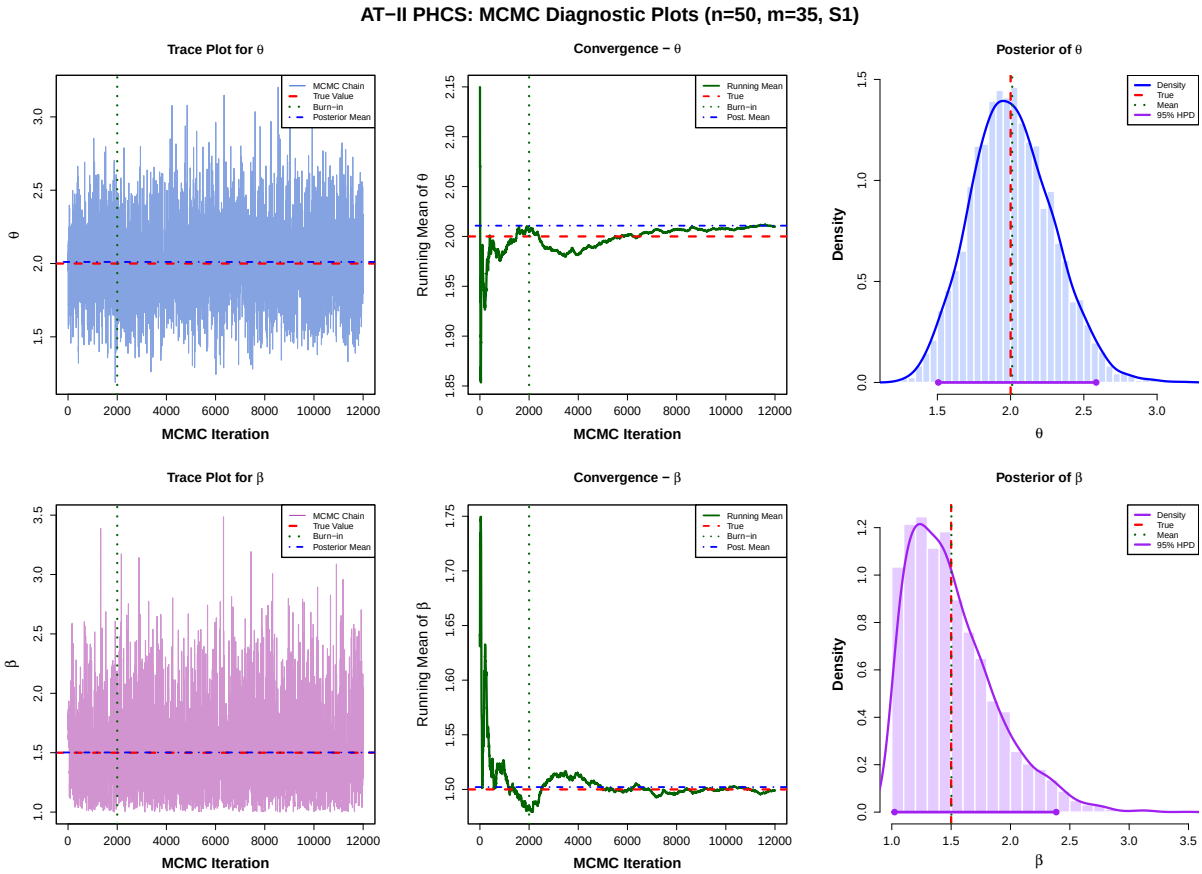

Figure 4. Graphical illustration of the AT-II PHCS with $n = 50, m = 35$, and S 1.

Algorithm 1: Simulation study for SS-PALT under Type I PHCS or Type I APHCS with MLE and Bayes MCMC estimation

Input: N_{sim} (number of simulations), n (sample size), θ, β (true parameters), τ (stress change time), T (termination time), m (pre-fixed number of failures, if needed), $R(\cdot)$ (removal pattern), hyperparameters c_1, b_1, c_2, b_2, MCMC parameters N (total iterations), M (burn-in), proposal variances, significance level γ

Output: Summary table: Average estimates, biases, MSEs for MLE and Bayesian estimates, and coverage probabilities of credible intervals

for $s \leftarrow 1$ **to** N_{sim} **do**

  Generate $u_i \sim U(0,1)$ for $i = 1, \ldots, n$;

  Solve T_i from $F_{\text{Lindley}}(T_i; \theta) = u_i$;

  $X_i \leftarrow \begin{cases} T_i & \text{if } T_i \le \tau \\ \tau + (T_i - \tau)/\beta & \text{otherwise} \end{cases}$;

  Apply Type I PHCS (or APHCS) to $(X_1, \ldots, X_n)$;

  Obtain observed data: x_{obs} (failure times), S_{obs} (censoring times), R^*, T^*, D (number of failures);

  Compute MLEs $(\hat{\theta}_{\text{MLE}}, \hat{\beta}_{\text{MLE}})$ by minimizing $-\ell(\theta, \beta \mid \text{data})$;

  Initialize $\theta^{(0)} = \hat{\theta}_{\text{MLE}}, \beta^{(0)} = \hat{\beta}_{\text{MLE}}$;

  Set initial proposal scales $\sigma_\theta = \text{se}(\hat{\theta}_{\text{MLE}})$ and $\sigma_\beta = \text{se}(\hat{\beta}_{\text{MLE}})$; target acceptance rate $\alpha_{\text{target}} = 0.44$; adaptation interval $L = 50$ iterations. **for** $d \leftarrow 1$ **to** N **do**

    Propose $\theta' \sim q_\theta(\cdot \mid \theta^{(d-1)})$; compute acceptance prob. $\alpha_\theta = \min\Big(1, \frac{\pi^*(\theta' \mid \beta^{(d-1)}, \text{data})}{\pi^*(\theta^{(d-1)} \mid \beta^{(d-1)}, \text{data})} \frac{q_\theta(\theta^{(d-1)} \mid \theta')}{q_\theta(\theta' \mid \theta^{(d-1)})}\Big)$; accept/reject to get $\theta^{(d)}$;

    Propose $\beta' \sim q_\beta(\cdot \mid \beta^{(d-1)})$; similarly compute α_β using $\pi^*(\beta' \mid \theta^{(d)}, \text{data})$; accept/reject to get $\beta^{(d)}$;

    Every L iterations during burn-in, update proposal scales: $\sigma_\theta \leftarrow \sigma_\theta \exp(\alpha_\theta - \alpha_{\text{target}})$ and $\sigma_\beta \leftarrow \sigma_\beta \exp(\alpha_\beta - \alpha_{\text{target}})$, then reset acceptance counters.

  Discard first M samples; keep $\{\theta^{(d)}, \beta^{(d)}\}_{d=M+1}^{N}$;

  Bayes point estimates (SE loss): $\hat{\theta}_{\text{SE}} = \frac{1}{N-M} \sum_{d=M+1}^{N} \theta^{(d)}, \hat{\beta}_{\text{SE}} = \frac{1}{N-M} \sum_{d=M+1}^{N} \beta^{(d)}$;

  $100(1-\gamma)\%$ credible intervals: $\Big(\theta_{[(\gamma/2)(N-M)]},\ \theta_{[(1-\gamma/2)(N-M)]}\Big)$ and $\Big(\beta_{[(\gamma/2)(N-M)]},\ \beta_{[(1-\gamma/2)(N-M)]}\Big)$, where $\theta_{[k]}$ denotes the k-th order statistic of the posterior samples;

  Store $\hat{\theta}_{\text{MLE}}, \hat{\beta}_{\text{MLE}}, \hat{\theta}_{\text{SE}}, \hat{\beta}_{\text{SE}}$, and indicators I_θ, I_β (whether true parameter lies in its credible interval);

Compute for each estimator (MLE and Bayes): Average estimate, bias, MSE;

Compute coverage probability for each parameter: $\frac{1}{N_{\text{sim}}} \sum_{s=1}^{N_{\text{sim}}} I_\theta^{(s)}$ (and similarly for β);

Return summary table;

- Tables 2–4 report point estimate summaries for $n = 50$. For each censoring scheme (S1, S2, S3), with selected values of m, the table gives the empirical mean of the estimators, the bias (mean minus true value), MSE, and MRE. the results are split into two blocks: *AT I PHCS* and *Type I PHCS* (Type I PHCS obtained via the conversion rule). Within each block, we present both MLEs and Bayesian point estimates. The table highlights how estimator bias and MSE respond to the removal pattern and to the truncation level m: In small-n settings, the compression of life-times induced by SS-PALT together with aggressive removals (e.g., S3) increases variance and bias for β, whereas θ remains relatively stable.
- Tables 5–7 are the analog of Table 2 for $n = 100$. Because the sample size doubles, both the MLE and Bayesian mean estimates for θ and β typically display a reduced MSE and MRE. The results

are organized identically (AT-I PHCS vs Type I PHCS; MLE vs Bayes), so a direct comparison with Tables 2–4 reveal asymptotic improvements: Narrower MSE columns, smaller biases, and more reliable coverage for Bayesian CRIs.

- Tables 8–10 summarize interval-estimate behavior at $n = 50, 100$. For each scheme and m, we report the AL of the 95% confidence (or credible) intervals CIs and CRIs and empirical CP for both MLE-based Wald intervals and Bayesian credible intervals, separately for AT-I PHCS and Type I PHCS. When AL is large but CP < 0.95, the intervals are conservative in length but still fail to cover nominal probability; conversely, a short AL with CP being near nominal indicates precision and correct uncertainty calibration. Table 3 shows the interplay between removal design and interval performance: Schemes concentrating removals late (S1) often provide a shorter AL for θ, while schemes dispersing removals (S3) increase AL for β. When $n = 100$, these tables show a shorter AL and CPs closer to or exceeding the nominal 0.95 in many configurations. The tables also make it evident when the Bayesian CRIs outperform the asymptotic Wald intervals for small to moderate sample sizes (particularly for β), a behavior attributable to the informative effect of the priors combined with the compression produced by SS-PALT.
- The simulation findings offer concrete proof of how the suggested estimators behave during convergence. With an increase in the sample size from 50 to 100, the MLEs and Bayesian estimators show better interval estimation performance, and decreased MSE and MRE values. These results show that the suggested inferential processes are consistent and asymptotically stable under AT-I PHCS. Under various censoring configurations, the Bayesian estimators show better numerical stability and less estimation variability, especially for the acceleration parameter.
- Definition of Schemes S1, S2, and S3 (notation consistent with the text). Suppose that AT-I PHCS is implemented on a sample of size n, and the progressive removal vector is $(R_1, R_2, \ldots, R_m)$, where m denotes the planned number of failures to be observed before the terminal rule. We define the three removal policies used in the simulations as follows.
  - S1: All removals after the last failure. For a given (n, m) we have

$$R^{(\mathrm{S1})} = (0, 0, \ldots, 0, n - m),$$

    i.e., $R_i = 0$ for $i = 1, \ldots, m - 1$ and $R_m = n - m$. This concentrates censoring at the final removal epoch.
  - S2: Two-block removals (first and last). If we let $T_{\mathrm{rem}} = n - m$, then

$$R_1^{(\mathrm{S2})} = \lfloor T_{\mathrm{rem}}/2 \rfloor, \quad R_m^{(\mathrm{S2})} = T_{\mathrm{rem}} - R_1^{(\mathrm{S2})},$$

    and $R_i^{(\mathrm{S2})} = 0$ for $i \notin \{1, m\}$. This places the removals primarily at the beginning and at the end of the observed failure sequence.
  - S3: Three-block removals (first, middle, last). Let $T_{\mathrm{rem}} = n - m$ and $j = (m/2)$ (the central index). Then

$$R_1^{(\mathrm{S3})} = \lfloor T_{\mathrm{rem}}/3 \rfloor, \quad R_j^{(\mathrm{S3})} = \lfloor T_{\mathrm{rem}}/3 \rfloor, \quad R_m^{(\mathrm{S3})} = T_{\mathrm{rem}} - R_1^{(\mathrm{S3})} - R_j^{(\mathrm{S3})},$$

    with other components zero. This spreads removals across three epochs, approximating equal partitioning while preserving integer counts.

Table 2. Point estimates: Mean, bias, MSE, and MRE at $\theta = 0.8, \beta = 1.5, \tau = 0.9, \eta = 2.5, n = 50$.

			AT-I PHCS								Type I PHCS							
			MLE				Bayes				MLE				Bayes			
Sch.	m	Par.	Mean	Bias	MSE	MRE	Mean	Bias	MSE	MRE	Mean	Bias	MSE	MRE	Mean	Bias	MSE	MRE
S1	30	θ	0.7997	-0.0003	0.0184	0.1340	0.8518	0.0518	0.0129	0.1089	0.7988	-0.0012	0.0191	0.1357	0.8016	0.0016	0.0146	0.1166
S1	30	β	1.6043	0.1043	0.2161	0.2366	1.4605	-0.0395	0.0274	0.0904	1.7197	0.2197	0.3686	0.2981	1.7174	0.2174	0.1818	0.2012
S2	30	θ	0.7949	-0.0051	0.0218	0.1447	0.8601	0.0601	0.0153	0.1190	0.7945	-0.0055	0.0226	0.1464	0.7977	-0.0023	0.0156	0.1197
S2	30	β	1.6565	0.1565	0.2856	0.2626	1.4668	-0.0332	0.0247	0.0852	1.7074	0.2074	0.3557	0.2861	1.7061	0.2061	0.1782	0.1965
S3	30	θ	0.7967	-0.0033	0.0219	0.1449	0.8584	0.0584	0.0159	0.1196	0.7961	-0.0039	0.0223	0.1453	0.7978	-0.0022	0.0160	0.1209
S3	30	β	1.6601	0.1601	0.2999	0.2650	1.4728	-0.0272	0.0258	0.0858	1.7001	0.2001	0.3542	0.2858	1.7016	0.2016	0.1790	0.1962
S1	35	θ	0.7992	-0.0008	0.0186	0.1340	0.8517	0.0517	0.0130	0.1091	0.7985	-0.0015	0.0191	0.1354	0.8000	-0.0000	0.0141	0.1138
S1	35	β	1.6068	0.1068	0.2158	0.2357	1.4613	-0.0387	0.0272	0.0900	1.6671	0.1671	0.2846	0.2652	1.6764	0.1764	0.1542	0.1869
S2	35	θ	0.7929	-0.0071	0.0214	0.1433	0.8599	0.0599	0.0143	0.1146	0.7927	-0.0073	0.0217	0.1438	0.7973	-0.0027	0.0145	0.1158
S2	35	β	1.6841	0.1841	0.2841	0.2561	1.4797	-0.0203	0.0251	0.0842	1.6989	0.1989	0.3129	0.2664	1.6953	0.1953	0.1670	0.1901
S3	35	θ	0.7964	-0.0036	0.0217	0.1443	0.8670	0.0670	0.0156	0.1200	0.7965	-0.0035	0.0219	0.1444	0.8030	0.0030	0.0148	0.1171
S3	35	β	1.7258	0.2258	0.3057	0.2680	1.4956	-0.0044	0.0247	0.0830	1.7260	0.2260	0.3075	0.2693	1.7156	0.2156	0.1702	0.1959
S1	40	θ	0.7970	-0.0030	0.0182	0.1327	0.8512	0.0512	0.0129	0.1087	0.7965	-0.0035	0.0183	0.1330	0.7977	-0.0023	0.0132	0.1112
S1	40	β	1.6156	0.1156	0.2144	0.2334	1.4641	-0.0359	0.0268	0.0889	1.6377	0.1377	0.2439	0.2460	1.6508	0.1508	0.1405	0.1785
S2	40	θ	0.7900	-0.0100	0.0206	0.1412	0.8664	0.0664	0.0148	0.1182	0.7901	-0.0099	0.0206	0.1412	0.8004	0.0004	0.0141	0.1158
S2	40	β	1.7548	0.2548	0.3064	0.2684	1.5079	0.0079	0.0264	0.0849	1.7538	0.2538	0.3058	0.2682	1.7333	0.2333	0.1791	0.2028
S3	40	θ	0.7922	-0.0078	0.0205	0.1396	0.8666	0.0666	0.0150	0.1182	0.7923	-0.0077	0.0204	0.1396	0.8022	0.0022	0.0141	0.1150
S3	40	β	1.7557	0.2557	0.3071	0.2693	1.5110	0.0110	0.0271	0.0860	1.7551	0.2551	0.3068	0.2690	1.7347	0.2347	0.1818	0.2031

Table 3. Point estimates: Mean, bias, MSE, and MRE at $\theta = 2, \beta = 1.5, \tau = 0.5, \eta = 1$, and $n = 50$.

			AT-I PHCS								Type I PHCS							
			MLE				Bayes				MLE				Bayes			
Sch.	m	Par.	Mean	Bias	MSE	MRE	Mean	Bias	MSE	MRE	Mean	Bias	MSE	MRE	Mean	Bias	MSE	MRE
S1	30	θ	2.0222	0.0222	0.0986	0.1225	2.0420	0.0420	0.0631	0.0971	2.0323	0.0323	0.1141	0.1286	1.9940	-0.0060	0.0953	0.1201
S1	30	β	1.5837	0.0837	0.2262	0.2470	1.5013	0.0013	0.0308	0.0913	2.1244	0.6244	2.5181	0.5579	1.9233	0.4233	0.3529	0.3005
S2	30	θ	2.0098	0.0098	0.1138	0.1330	2.0400	0.0400	0.0712	0.1030	2.0102	0.0102	0.1185	0.1341	1.9591	-0.0409	0.0920	0.1180
S2	30	β	1.6324	0.1324	0.3017	0.2750	1.5101	0.0101	0.0298	0.0869	1.7861	0.2861	0.6424	0.3536	1.7951	0.2951	0.2616	0.2399
S3	30	θ	2.0158	0.0158	0.1176	0.1330	2.0428	0.0428	0.0754	0.1053	2.0158	0.0158	0.1216	0.1337	1.9622	-0.0378	0.0946	0.1192
S3	30	β	1.6409	0.1409	0.3103	0.2818	1.5111	0.0111	0.0273	0.0848	1.7823	0.2823	0.6299	0.3599	1.7888	0.2888	0.2478	0.2389
S1	35	θ	2.0185	0.0185	0.0993	0.1231	2.0410	0.0410	0.0632	0.0973	2.0174	0.0174	0.1031	0.1235	1.9746	-0.0254	0.0832	0.1113
S1	35	β	1.5936	0.0936	0.2273	0.2472	1.5049	0.0049	0.0307	0.0908	1.8033	0.3033	0.7605	0.3659	1.7832	0.2832	0.2422	0.2357
S2	35	θ	2.0081	0.0081	0.1108	0.1315	2.0445	0.0445	0.0670	0.0993	2.0077	0.0077	0.1123	0.1317	1.9582	-0.0418	0.0832	0.1134
S2	35	β	1.6548	0.1548	0.2757	0.2634	1.5212	0.0212	0.0292	0.0859	1.7048	0.2048	0.3611	0.2934	1.7465	0.2465	0.2029	0.2141
S3	35	θ	2.0145	0.0145	0.1143	0.1316	2.0609	0.0609	0.0692	0.0989	2.0147	0.0147	0.1151	0.1317	1.9694	-0.0306	0.0819	0.1104
S3	35	β	1.7186	0.2186	0.3046	0.2705	1.5388	0.0388	0.0272	0.0809	1.7290	0.2290	0.3317	0.2779	1.7622	0.2622	0.1952	0.2085
S1	40	θ	2.0171	0.0171	0.0977	0.1221	2.0404	0.0404	0.0623	0.0964	2.0168	0.0168	0.1004	0.1223	1.9679	-0.0321	0.0772	0.1076
S1	40	β	1.5954	0.0954	0.2265	0.2463	1.5051	0.0051	0.0307	0.0908	1.6534	0.1534	0.3108	0.2775	1.7057	0.2057	0.1822	0.2027
S2	40	θ	2.0109	0.0109	0.1098	0.1301	2.0675	0.0675	0.0649	0.0971	2.0110	0.0110	0.1102	0.1302	1.9721	-0.0279	0.0776	0.1091
S2	40	β	1.7340	0.2340	0.2888	0.2686	1.5523	0.0523	0.0316	0.0888	1.7369	0.2369	0.2986	0.2722	1.7679	0.2679	0.1984	0.2158
S3	40	θ	2.0079	0.0079	0.1065	0.1265	2.0678	0.0678	0.0647	0.0972	2.0079	0.0079	0.1061	0.1264	1.9722	-0.0278	0.0757	0.1067
S3	40	β	1.7602	0.2602	0.3044	0.2753	1.5607	0.0607	0.0321	0.0892	1.7603	0.2603	0.3085	0.2769	1.7836	0.2836	0.2064	0.2213

Table 4. Point estimates: Mean, bias, MSE, and MRE at $\theta = 3, \beta = 1.5, \tau = 0.15, \eta = 0.8$, and $n = 50$.

			AT-I PHCS								Type I PHCS							
			MLE				Bayes				MLE				Bayes			
Sch.	m	Par.	Mean	Bias	MSE	MRE	Mean	Bias	MSE	MRE	Mean	Bias	MSE	MRE	Mean	Bias	MSE	MRE
S1	30	θ	3.0017	0.0017	0.4002	0.1664	2.9428	-0.0572	0.1375	0.0978	3.0008	0.0008	0.4183	0.1682	2.8064	-0.1936	0.2769	0.1411
S1	30	β	1.6390	0.1390	0.2913	0.2597	1.5619	0.0619	0.0381	0.0986	1.7440	0.2440	0.4834	0.3247	1.8499	0.3499	0.3116	0.2680
S2	30	θ	2.9745	-0.0255	0.4642	0.1810	2.9198	-0.0802	0.1519	0.1032	2.9726	-0.0274	0.4805	0.1828	2.7543	-0.2457	0.2988	0.1472
S2	30	β	1.6929	0.1929	0.3960	0.3004	1.5723	0.0723	0.0386	0.0991	1.7518	0.2518	0.5073	0.3328	1.8731	0.3731	0.3393	0.2827
S3	30	θ	2.9858	-0.0142	0.4620	0.1789	2.9219	-0.0781	0.1592	0.1049	2.9842	-0.0158	0.4770	0.1802	2.7700	-0.2300	0.2983	0.1459
S3	30	β	1.6876	0.1876	0.3930	0.2971	1.5686	0.0686	0.0381	0.0976	1.7381	0.2381	0.4966	0.3270	1.8542	0.3542	0.3220	0.2732
S1	35	θ	2.9998	-0.0002	0.3963	0.1658	2.9443	-0.0557	0.1360	0.0969	2.9966	-0.0034	0.4073	0.1668	2.8011	-0.1989	0.2634	0.1364
S1	35	β	1.6433	0.1433	0.2908	0.2584	1.5639	0.0639	0.0379	0.0982	1.7003	0.2003	0.3943	0.2946	1.8201	0.3201	0.2796	0.2531
S2	35	θ	2.9676	-0.0324	0.4416	0.1761	2.9205	-0.0795	0.1462	0.1013	2.9687	-0.0313	0.4596	0.1781	2.7602	-0.2398	0.2870	0.1437
S2	35	β	1.6905	0.1905	0.3825	0.2931	1.5742	0.0742	0.0403	0.1010	1.7246	0.2246	0.4612	0.3143	1.8480	0.3480	0.3164	0.2698
S3	35	θ	2.9759	-0.0241	0.4367	0.1734	2.9259	-0.0741	0.1545	0.1023	2.9741	-0.0259	0.4427	0.1740	2.7710	-0.2290	0.2813	0.1422
S3	35	β	1.6908	0.1908	0.3653	0.2906	1.5720	0.0720	0.0382	0.0998	1.7139	0.2139	0.4076	0.3060	1.8369	0.3369	0.2936	0.2651
S1	40	θ	2.9994	-0.0006	0.3971	0.1655	2.9434	-0.0566	0.1373	0.0975	2.9961	-0.0039	0.4050	0.1663	2.8012	-0.1988	0.2596	0.1351
S1	40	β	1.6417	0.1417	0.2894	0.2576	1.5631	0.0631	0.0378	0.0980	1.6773	0.1773	0.3545	0.2807	1.8030	0.3030	0.2659	0.2466
S2	40	θ	2.9770	-0.0230	0.4210	0.1722	2.9307	-0.0693	0.1400	0.0988	2.9766	-0.0234	0.4270	0.1727	2.7722	-0.2278	0.2696	0.1400
S2	40	β	1.6795	0.1795	0.3454	0.2816	1.5734	0.0734	0.0396	0.0999	1.6910	0.1910	0.3720	0.2908	1.8220	0.3220	0.2814	0.2561
S3	40	θ	2.9788	-0.0212	0.4181	0.1706	2.9327	-0.0673	0.1422	0.0994	2.9781	-0.0219	0.4169	0.1705	2.7789	-0.2211	0.2648	0.1377
S3	40	β	1.6786	0.1786	0.3402	0.2777	1.5713	0.0713	0.0389	0.0986	1.6833	0.1833	0.3557	0.2834	1.8115	0.3115	0.2708	0.2493

Table 5. Point estimates: Mean, bias, MSE, and MRE at $\theta = 0.8, \beta = 1.5, \tau = 0.9, \eta = 2.5$, and $n = 100$.

			AT-I PHCS								Type I PHCS							
			MLE				Bayes				MLE				Bayes			
Sch.	m	Par.	Mean	Bias	MSE	MRE	Mean	Bias	MSE	MRE	Mean	Bias	MSE	MRE	Mean	Bias	MSE	MRE
S1	65	θ	0.7987	-0.0013	0.0091	0.0935	0.8307	0.0307	0.0066	0.0788	0.7976	-0.0024	0.0091	0.0933	0.7999	-0.0001	0.0074	0.0839
S1	65	β	1.5515	0.0515	0.0956	0.1595	1.4668	-0.0332	0.0281	0.0911	1.5942	0.0942	0.1307	0.1871	1.6015	0.1015	0.0899	0.1505
S2	65	θ	0.7971	-0.0029	0.0104	0.0997	0.8350	0.0350	0.0074	0.0836	0.7964	-0.0036	0.0104	0.0998	0.7995	-0.0005	0.0080	0.0873
S2	65	β	1.5745	0.0745	0.1163	0.1730	1.4713	-0.0287	0.0278	0.0905	1.5933	0.0933	0.1295	0.1817	1.6002	0.1002	0.0905	0.1478
S3	65	θ	0.7973	-0.0027	0.0100	0.0982	0.8362	0.0362	0.0073	0.0832	0.7972	-0.0028	0.0100	0.0982	0.8011	0.0011	0.0078	0.0864
S3	65	β	1.5997	0.0997	0.1194	0.1753	1.4854	-0.0146	0.0268	0.0874	1.6059	0.1059	0.1236	0.1792	1.6087	0.1087	0.0874	0.1469
S1	75	θ	0.7979	-0.0021	0.0089	0.0927	0.8305	0.0305	0.0065	0.0785	0.7971	-0.0029	0.0090	0.0931	0.8002	0.0002	0.0073	0.0833
S1	75	β	1.5548	0.0548	0.0942	0.1578	1.4683	-0.0317	0.0277	0.0903	1.5774	0.0774	0.1092	0.1697	1.5830	0.0830	0.0794	0.1416
S2	75	θ	0.7942	-0.0058	0.0100	0.0988	0.8395	0.0395	0.0074	0.0849	0.7942	-0.0058	0.0100	0.0989	0.8013	0.0013	0.0079	0.0877
S2	75	β	1.6437	0.1437	0.1218	0.1748	1.5088	0.0088	0.0272	0.0866	1.6445	0.1445	0.1238	0.1760	1.6380	0.1380	0.0927	0.1500
S3	75	θ	0.7937	-0.0063	0.0101	0.0984	0.8418	0.0418	0.0078	0.0871	0.7937	-0.0063	0.0101	0.0984	0.8023	0.0023	0.0082	0.0889
S3	75	β	1.6892	0.1892	0.1382	0.1874	1.5347	0.0347	0.0290	0.0877	1.6886	0.1886	0.1379	0.1873	1.6747	0.1747	0.1050	0.1616
S1	85	θ	0.7977	-0.0023	0.0090	0.0927	0.8305	0.0305	0.0066	0.0787	0.7976	-0.0024	0.0090	0.0927	0.8005	0.0005	0.0073	0.0832
S1	85	β	1.5565	0.0565	0.0935	0.1570	1.4691	-0.0309	0.0274	0.0899	1.5577	0.0577	0.0946	0.1582	1.5652	0.0652	0.0698	0.1335
S2	85	θ	0.7927	-0.0073	0.0099	0.0972	0.8411	0.0411	0.0078	0.0872	0.7927	-0.0073	0.0099	0.0972	0.8015	0.0015	0.0082	0.0885
S2	85	β	1.6886	0.1886	0.1400	0.1882	1.5380	0.0380	0.0320	0.0924	1.6887	0.1887	0.1401	0.1882	1.6763	0.1763	0.1097	0.1649
S3	85	θ	0.7915	-0.0085	0.0098	0.0972	0.8416	0.0416	0.0080	0.0875	0.7915	-0.0085	0.0098	0.0972	0.8010	0.0010	0.0082	0.0894
S3	85	β	1.7175	0.2175	0.1531	0.1979	1.5559	0.0559	0.0348	0.0955	1.7175	0.2175	0.1531	0.1979	1.7014	0.2014	0.1206	0.1743

Table 6. Point estimates: Mean, bias, MSE, and MRE at $\theta = 2, \beta = 1.5, \tau = 0.5, \eta = 1$, and $n = 100$.

			AT-I PHCS								Type I PHCS							
			MLE				Bayes				MLE				Bayes			
Sch.	m	Par.	Mean	Bias	MSE	MRE	Mean	Bias	MSE	MRE	Mean	Bias	MSE	MRE	Mean	Bias	MSE	MRE
S1	65	θ	2.0039	0.0039	0.0479	0.0868	2.0224	0.0224	0.0339	0.0727	2.0024	0.0024	0.0478	0.0864	1.9799	-0.0201	0.0423	0.0816
S1	65	β	1.5519	0.0519	0.1034	0.1688	1.5013	0.0013	0.0294	0.0900	1.6884	0.1884	0.3086	0.2582	1.7033	0.2033	0.1610	0.1872
S2	65	θ	2.0002	0.0002	0.0552	0.0924	2.0225	0.0225	0.0379	0.0762	1.9990	-0.0010	0.0550	0.0923	1.9722	-0.0278	0.0449	0.0836
S2	65	β	1.5727	0.0727	0.1309	0.1866	1.5084	0.0084	0.0310	0.0903	1.6187	0.1187	0.1745	0.2112	1.6518	0.1518	0.1206	0.1687
S3	65	θ	2.0056	0.0056	0.0544	0.0918	2.0279	0.0279	0.0371	0.0751	2.0050	0.0050	0.0544	0.0918	1.9778	-0.0222	0.0436	0.0821
S3	65	β	1.5882	0.0882	0.1476	0.1948	1.5139	0.0139	0.0322	0.0914	1.6145	0.1145	0.1743	0.2106	1.6473	0.1473	0.1201	0.1670
S1	75	θ	2.0037	0.0037	0.0478	0.0865	2.0220	0.0220	0.0339	0.0725	2.0023	0.0023	0.0475	0.0863	1.9792	-0.0208	0.0399	0.0790
S1	75	β	1.5502	0.0502	0.1026	0.1683	1.5003	0.0003	0.0293	0.0899	1.5958	0.0958	0.1442	0.1955	1.6265	0.1265	0.1003	0.1572
S2	75	θ	1.9970	-0.0030	0.0518	0.0900	2.0307	0.0307	0.0342	0.0718	1.9968	-0.0032	0.0517	0.0900	1.9775	-0.0225	0.0414	0.0804
S2	75	β	1.6229	0.1229	0.1238	0.1786	1.5342	0.0342	0.0303	0.0876	1.6317	0.1317	0.1351	0.1844	1.6537	0.1537	0.1037	0.1574
S3	75	θ	2.0004	0.0004	0.0527	0.0907	2.0428	0.0428	0.0357	0.0738	2.0004	0.0004	0.0528	0.0907	1.9856	-0.0144	0.0425	0.0818
S3	75	β	1.6872	0.1872	0.1476	0.1959	1.5659	0.0659	0.0340	0.0928	1.6872	0.1872	0.1499	0.1966	1.6981	0.1981	0.1188	0.1722
S1	85	θ	1.9994	-0.0006	0.0473	0.0859	2.0202	0.0202	0.0340	0.0727	1.9992	-0.0008	0.0472	0.0858	1.9776	-0.0224	0.0397	0.0791
S1	85	β	1.5599	0.0599	0.1003	0.1654	1.5049	0.0049	0.0287	0.0885	1.5691	0.0691	0.1093	0.1712	1.5985	0.0985	0.0816	0.1434
S2	85	θ	1.9986	-0.0014	0.0520	0.0905	2.0487	0.0487	0.0372	0.0758	1.9986	-0.0014	0.0520	0.0905	1.9881	-0.0119	0.0434	0.0831
S2	85	β	1.7145	0.2145	0.1507	0.2011	1.5880	0.0880	0.0398	0.1014	1.7151	0.2151	0.1520	0.2015	1.7224	0.2224	0.1284	0.1833
S3	85	θ	1.9971	-0.0029	0.0504	0.0888	2.0533	0.0533	0.0380	0.0765	1.9971	-0.0029	0.0504	0.0888	1.9894	-0.0106	0.0429	0.0821
S3	85	β	1.7578	0.2578	0.1724	0.2166	1.6121	0.1121	0.0453	0.1091	1.7577	0.2577	0.1725	0.2166	1.7585	0.2585	0.1476	0.1987

Table 7. Point estimates: Mean, bias, MSE, and MRE at $\theta = 3, \beta = 1.5, \tau = 0.15, \eta = 0.8$, and $n = 100$.

			AT-I PHCS								Type I PHCS							
			MLE				Bayes				MLE				Bayes			
Sch.	m	Par.	Mean	Bias	MSE	MRE	Mean	Bias	MSE	MRE	Mean	Bias	MSE	MRE	Mean	Bias	MSE	MRE
S1	65	θ	2.9956	-0.0044	0.1915	0.1154	2.9683	-0.0317	0.0885	0.0778	2.9934	-0.0066	0.1895	0.1148	2.8841	-0.1159	0.1460	0.1019
S1	65	β	1.5699	0.0699	0.1185	0.1755	1.5473	0.0473	0.0356	0.0967	1.6015	0.1015	0.1607	0.2020	1.6845	0.1845	0.1390	0.1798
S2	65	θ	2.9921	-0.0079	0.2220	0.1237	2.9598	-0.0402	0.0948	0.0808	2.9898	-0.0102	0.2216	0.1236	2.8645	-0.1355	0.1605	0.1057
S2	65	β	1.5867	0.0867	0.1512	0.1939	1.5540	0.0540	0.0381	0.0985	1.6044	0.1044	0.1769	0.2072	1.6988	0.1988	0.1543	0.1875
S3	65	θ	2.9997	-0.0003	0.2134	0.1224	2.9649	-0.0351	0.0958	0.0820	2.9990	-0.0010	0.2137	0.1224	2.8775	-0.1225	0.1561	0.1053
S3	65	β	1.5838	0.0838	0.1497	0.1947	1.5499	0.0499	0.0381	0.0988	1.5961	0.0961	0.1622	0.2018	1.6866	0.1866	0.1416	0.1820
S1	75	θ	2.9975	-0.0025	0.1938	0.1160	2.9696	-0.0304	0.0888	0.0780	2.9962	-0.0038	0.1948	0.1161	2.8895	-0.1105	0.1457	0.1014
S1	75	β	1.5696	0.0696	0.1191	0.1757	1.5470	0.0470	0.0359	0.0969	1.5878	0.0878	0.1413	0.1885	1.6700	0.1700	0.1265	0.1717
S2	75	θ	2.9991	-0.0009	0.2174	0.1236	2.9691	-0.0309	0.0954	0.0812	2.9984	-0.0016	0.2180	0.1238	2.8801	-0.1199	0.1566	0.1056
S2	75	β	1.5812	0.0812	0.1372	0.1874	1.5518	0.0518	0.0370	0.0979	1.5905	0.0905	0.1471	0.1932	1.6809	0.1809	0.1337	0.1779
S3	75	θ	2.9959	-0.0041	0.2046	0.1200	2.9641	-0.0359	0.0930	0.0797	2.9954	-0.0046	0.2049	0.1200	2.8811	-0.1189	0.1507	0.1033
S3	75	β	1.5797	0.0797	0.1378	0.1884	1.5495	0.0495	0.0369	0.0986	1.5847	0.0847	0.1421	0.1908	1.6715	0.1715	0.1274	0.1759
S1	85	θ	3.0019	0.0019	0.1952	0.1163	2.9726	-0.0274	0.0897	0.0784	3.0009	0.0009	0.1952	0.1164	2.8948	-0.1052	0.1449	0.1013
S1	85	β	1.5665	0.0665	0.1180	0.1752	1.5453	0.0453	0.0355	0.0966	1.5757	0.0757	0.1264	0.1799	1.6581	0.1581	0.1161	0.1667
S2	85	θ	2.9940	-0.0060	0.2036	0.1194	2.9685	-0.0315	0.0900	0.0787	2.9939	-0.0061	0.2033	0.1194	2.8835	-0.1165	0.1489	0.1023
S2	85	β	1.5797	0.0797	0.1287	0.1818	1.5509	0.0509	0.0361	0.0969	1.5810	0.0810	0.1323	0.1834	1.6664	0.1664	0.1216	0.1712
S3	85	θ	2.9932	-0.0068	0.2072	0.1199	2.9762	-0.0238	0.0928	0.0800	2.9938	-0.0062	0.2071	0.1199	2.8884	-0.1116	0.1528	0.1038
S3	85	β	1.6016	0.1016	0.1346	0.1830	1.5631	0.0631	0.0377	0.0983	1.5987	0.0987	0.1344	0.1826	1.6813	0.1813	0.1271	0.1740

Table 8. Interval estimates: AL and CP at $\theta = 0.8, \beta = 1.5, \tau = 0.9, \eta = 2.5$, and $n = 50, 100$.

			AT-I PHCS				Type I PHCS			
			MLE		Bayes		MLE		Bayes	
Scheme	m	Par.	AL	CP	AL	CP	AL	CP	AL	CP
						$n = 50$				
S1	30	θ	0.5127	0.9313	0.4413	0.9501	0.4683	0.8398	0.4809	0.9563
S1	30	β	1.5914	0.8668	0.9547	0.9948	2.1117	0.8866	1.7422	0.9875
S2	30	θ	0.5681	0.9332	0.4831	0.9582	0.5186	0.8476	0.5182	0.9624
S2	30	β	1.8422	0.8674	1.0014	0.9990	2.0428	0.8904	1.7046	0.9833
S3	30	θ	0.5508	0.9268	0.4757	0.9571	0.5006	0.8400	0.5035	0.9592
S3	30	β	1.8570	0.8745	1.0132	0.9990	2.0189	0.8933	1.6827	0.9801
S1	35	θ	0.5136	0.9291	0.4413	0.9489	0.4705	0.8478	0.4754	0.9531
S1	35	β	1.6017	0.8728	0.9560	0.9948	1.8496	0.8905	1.5830	0.9875
S2	35	θ	0.5553	0.9301	0.4712	0.9604	0.5143	0.8665	0.5023	0.9604
S2	35	β	1.8526	0.9093	1.0100	0.9990	1.9115	0.9135	1.6187	0.9739
S3	35	θ	0.5504	0.9194	0.4726	0.9520	0.5162	0.8796	0.5010	0.9592
S3	35	β	1.9312	0.9286	1.0359	0.9990	1.9459	0.9316	1.6413	0.9786
S1	40	θ	0.5167	0.9335	0.4413	0.9515	0.4736	0.8576	0.4716	0.9578
S1	40	β	1.6401	0.8977	0.9602	0.9947	1.7084	0.8998	1.4861	0.9736
S2	40	θ	0.5497	0.9231	0.4670	0.9574	0.5268	0.8981	0.5002	0.9615
S2	40	β	1.9088	0.9501	1.0360	1.0000	1.9116	0.9532	1.6235	0.9699
S3	40	θ	0.5406	0.9268	0.4626	0.9557	0.5195	0.9021	0.4933	0.9567
S3	40	β	1.9019	0.9567	1.0371	0.9979	1.9035	0.9567	1.6191	0.9691
						$n = 100$				
S1	65	θ	0.3708	0.9300	0.3303	0.9557	0.3606	0.9073	0.3527	0.9537
S1	65	β	1.1408	0.9392	0.8286	0.9835	1.3540	0.9547	1.2161	0.9722
S2	65	θ	0.4071	0.9334	0.3575	0.9611	0.3953	0.9180	0.3813	0.9631
S2	65	β	1.2810	0.9498	0.8731	0.9846	1.3408	0.9549	1.2083	0.9693
S3	65	θ	0.3964	0.9356	0.3518	0.9571	0.3864	0.9274	0.3731	0.9601
S3	65	β	1.3215	0.9632	0.8939	0.9908	1.3456	0.9673	1.2144	0.9785
S1	75	θ	0.3717	0.9329	0.3305	0.9567	0.3649	0.9226	0.3527	0.9577
S1	75	β	1.1508	0.9484	0.8307	0.9835	1.2425	0.9649	1.1323	0.9649
S2	75	θ	0.3985	0.9368	0.3534	0.9582	0.3937	0.9348	0.3763	0.9613
S2	75	β	1.3063	0.9786	0.9000	0.9959	1.3118	0.9806	1.1971	0.9643
S3	75	θ	0.3912	0.9414	0.3507	0.9525	0.3891	0.9403	0.3722	0.9525
S3	75	β	1.3432	0.9808	0.9241	0.9960	1.3439	0.9808	1.2316	0.9656
S1	85	θ	0.3731	0.9349	0.3307	0.9566	0.3632	0.9204	0.3518	0.9555
S1	85	β	1.1633	0.9617	0.8318	0.9824	1.1685	0.9617	1.0750	0.9679
S2	85	θ	0.3885	0.9422	0.3488	0.9483	0.3859	0.9391	0.3699	0.9533
S2	85	β	1.2961	0.9716	0.9079	0.9899	1.2963	0.9716	1.1945	0.9533
S3	85	θ	0.3844	0.9364	0.3471	0.9485	0.3820	0.9343	0.3682	0.9475
S3	85	β	1.3135	0.9727	0.9249	0.9899	1.3135	0.9727	1.2175	0.9374

Table 9. Interval estimates: AL and CP at $\theta = 2, \beta = 1.5, \tau = 0.5, \eta = 1$, and $n = 50, 100$.

			AT-I PHCS				Type I PHCS			
			MLE		Bayes		MLE		Bayes	
Scheme	m	Par.	AL	CP	AL	CP	AL	CP	AL	CP
						$n = 100$				
S1	30	θ	1.1745	0.9355	1.0281	0.9570	1.0529	0.8055	1.1384	0.9427
S1	30	β	1.6029	0.8178	1.0388	0.9990	3.7972	0.7195	2.6579	0.9898
S2	30	θ	1.3023	0.9438	1.1253	0.9740	1.1270	0.7992	1.2004	0.9532
S2	30	β	1.8404	0.8106	1.0934	1.0000	2.4773	0.8470	2.0044	0.9823
S3	30	θ	1.2968	0.9375	1.1269	0.9677	1.0875	0.7781	1.1910	0.9490
S3	30	β	1.9183	0.8125	1.1092	1.0000	2.4702	0.8260	2.0013	0.9875
S1	35	θ	1.1741	0.9346	1.0277	0.9564	1.0003	0.7830	1.1015	0.9460
S1	35	β	1.6201	0.8235	1.0439	0.9990	2.5668	0.8328	1.9674	0.9865
S2	35	θ	1.2679	0.9415	1.0939	0.9708	1.1089	0.8246	1.1547	0.9551
S2	35	β	1.8652	0.8622	1.0973	1.0000	2.0638	0.8674	1.7872	0.9802
S3	35	θ	1.2838	0.9337	1.1092	0.9741	1.1800	0.8694	1.1644	0.9606
S3	35	β	2.0750	0.9150	1.1413	0.9990	2.1484	0.9202	1.8174	0.9824
S1	40	θ	1.1758	0.9358	1.0279	0.9565	1.0091	0.8021	1.0851	0.9492
S1	40	β	1.6352	0.8332	1.0445	0.9990	1.8565	0.8446	1.6573	0.9731
S2	40	θ	1.2580	0.9354	1.0825	0.9713	1.1731	0.8862	1.1413	0.9641
S2	40	β	1.9653	0.9323	1.1264	1.0000	1.9846	0.9292	1.7300	0.9764
S3	40	θ	1.2453	0.9369	1.0778	0.9710	1.1747	0.8987	1.1343	0.9586
S3	40	β	2.0179	0.9390	1.1459	0.9990	2.0296	0.9400	1.7561	0.9731
						$n = 100$				
S1	65	θ	0.8387	0.9378	0.7618	0.9606	0.7708	0.8589	0.8053	0.9419
S1	65	β	1.1934	0.9098	0.8995	0.9938	1.9308	0.9035	1.6144	0.9803
S2	65	θ	0.9176	0.9375	0.8260	0.9677	0.8513	0.8719	0.8673	0.9573
S2	65	β	1.3216	0.9042	0.9500	0.9938	1.5026	0.9135	1.3739	0.9750
S3	65	θ	0.9165	0.9365	0.8279	0.9703	0.8522	0.8710	0.8653	0.9570
S3	65	β	1.3912	0.9058	0.9733	0.9959	1.5010	0.9171	1.3610	0.9724
S1	75	θ	0.8396	0.9374	0.7616	0.9600	0.7833	0.8737	0.8004	0.9487
S1	75	β	1.1981	0.9158	0.8986	0.9938	1.4208	0.9179	1.2957	0.9805
S2	75	θ	0.9001	0.9378	0.8106	0.9710	0.8753	0.9201	0.8505	0.9595
S2	75	β	1.3652	0.9554	0.9643	0.9959	1.4009	0.9554	1.2846	0.9699
S3	75	θ	0.8982	0.9320	0.8125	0.9706	0.8870	0.9289	0.8499	0.9523
S3	75	β	1.4574	0.9746	1.0084	0.9949	1.4657	0.9756	1.3367	0.9736
S1	85	θ	0.8422	0.9374	0.7616	0.9603	0.8027	0.8967	0.7985	0.9509
S1	85	β	1.2281	0.9405	0.9042	0.9937	1.2604	0.9415	1.1714	0.9729
S2	85	θ	0.8811	0.9364	0.7996	0.9616	0.8726	0.9303	0.8383	0.9515
S2	85	β	1.3875	0.9727	0.9947	0.9939	1.3889	0.9727	1.2932	0.9505
S3	85	θ	0.8784	0.9413	0.8009	0.9626	0.8757	0.9403	0.8373	0.9474
S3	85	β	1.4277	0.9727	1.0209	0.9939	1.4277	0.9727	1.3268	0.9443

Table 10. Interval Estimates: AL and CP at $\theta = 3, \beta = 1.5, \tau = 0.15, \eta = 0.8$, and $n = 50, 100$.

			AT-I PHCS				Type I PHCS			
			MLE		Bayes		MLE		Bayes	
Scheme	m	Par.	AL	CP	AL	CP	AL	CP	AL	CP
						$n = 50$				
S1	30	θ	2.3326	0.9093	1.7323	0.9750	2.1052	0.8092	2.0036	0.9395
S1	30	β	1.7842	0.8530	1.1176	1.0000	2.2898	0.8644	2.0419	0.9708
S2	30	θ	2.5655	0.9168	1.8469	0.9823	2.3099	0.8085	2.1141	0.9428
S2	30	β	2.0597	0.8439	1.1747	1.0000	2.3421	0.8626	2.0905	0.9688
S3	30	θ	2.4725	0.9113	1.8242	0.9749	2.1941	0.7933	2.0669	0.9384
S3	30	β	2.0086	0.8309	1.1707	0.9990	2.2706	0.8549	2.0437	0.9749
S1	35	θ	2.3375	0.9142	1.7338	0.9738	2.1220	0.8180	1.9823	0.9435
S1	35	β	1.7962	0.8588	1.1206	1.0000	2.0884	0.8724	1.9158	0.9707
S2	35	θ	2.4765	0.9086	1.8046	0.9758	2.2277	0.8025	2.0576	0.9433
S2	35	β	1.9859	0.8445	1.1633	1.0000	2.1600	0.8634	1.9779	0.9674
S3	35	θ	2.4472	0.9134	1.8073	0.9736	2.1981	0.8089	2.0391	0.9535
S3	35	β	1.9938	0.8448	1.1668	0.9989	2.1182	0.8669	1.9508	0.9694
S1	40	θ	2.3397	0.9116	1.7338	0.9740	2.1273	0.8252	1.9736	0.9459
S1	40	β	1.7966	0.8606	1.1198	1.0000	1.9639	0.8762	1.8334	0.9657
S2	40	θ	2.4494	0.9080	1.7840	0.9791	2.2145	0.8161	2.0235	0.9467
S2	40	β	1.9479	0.8631	1.1535	1.0000	2.0160	0.8736	1.8798	0.9645
S3	40	θ	2.4198	0.9135	1.7748	0.9800	2.1881	0.8291	2.0013	0.9441
S3	40	β	1.9481	0.8745	1.1517	0.9989	1.9913	0.8808	1.8566	0.9641
						$n = 100$				
S1	65	θ	1.7214	0.9304	1.3866	0.9761	1.6553	0.9054	1.5398	0.9574
S1	65	β	1.2660	0.9470	0.9479	0.9906	1.4792	0.9428	1.4037	0.9647
S2	65	θ	1.8705	0.9175	1.4757	0.9804	1.8011	0.8928	1.6471	0.9567
S2	65	β	1.3919	0.9320	1.0001	0.9938	1.5084	0.9402	1.4410	0.9660
S3	65	θ	1.8112	0.9198	1.4529	0.9866	1.7312	0.8911	1.6122	0.9527
S3	65	β	1.3767	0.9229	0.9938	0.9949	1.4703	0.9332	1.4110	0.9702
S1	75	θ	1.7170	0.9252	1.3868	0.9761	1.6655	0.9096	1.5423	0.9522
S1	75	β	1.2601	0.9397	0.9474	0.9906	1.3878	0.9501	1.3337	0.9667
S2	75	θ	1.8210	0.9246	1.4508	0.9773	1.7510	0.9019	1.6159	0.9587
S2	75	β	1.3460	0.9339	0.9836	0.9938	1.4081	0.9411	1.3623	0.9638
S3	75	θ	1.7905	0.9311	1.4306	0.9784	1.7332	0.9095	1.5866	0.9496
S3	75	β	1.3523	0.9414	0.9804	0.9949	1.3950	0.9516	1.3379	0.9640
S1	85	θ	1.7235	0.9281	1.3874	0.9753	1.6805	0.9157	1.5410	0.9558
S1	85	β	1.2630	0.9466	0.9455	0.9908	1.3230	0.9579	1.2785	0.9599
S2	85	θ	1.7856	0.9289	1.4239	0.9773	1.7324	0.9135	1.5820	0.9537
S2	85	β	1.3252	0.9526	0.9691	0.9938	1.3431	0.9567	1.2992	0.9619
S3	85	θ	1.7760	0.9254	1.4239	0.9796	1.7276	0.9162	1.5741	0.9458
S3	85	β	1.3527	0.9622	0.9831	0.9959	1.3554	0.9611	1.3085	0.9550

8. Real data

This section utilizes a real dataset to illustrate the applicability and efficacy of the provided approaches. This real dataset example is presented solely as an illustrative application to demonstrate how the proposed inferential framework can be implemented in practice when an SS-PALT structure is imposed artificially for methodological demonstration. The dataset, initially documented by Murphy et al. [51], comprises 50 right-skewed failure times (in weeks). These observations denote the lifetimes of identical units functioning under comparable settings and act as a standard for evaluating the adaptability and accuracy of the proposed model. The documented failure times are as follows: 0.374, 2.405, 2.169, 2.301, 2.506, 5.119, 1.26, 0.495, 1.933, 1.373, 1.092, 0.09, 1.295, 0.008, 0.564, 0.959, 4.446, 0.084, 0.921, 1.76, 1.022, 3.087, 1.926, 3.926, 1.395, 0.134, 1.892, 2.135, 1.414, 0.688, 2.32, 8.596, 1.858, 1.921, 0.686, 0.058, 0.061, 1.284, 2.971, 2.808, 3.669, 3.492, 0.681, 0.48, 0.017, 0.245, 0.353, 0.535, 2.598, and 0.238. This dataset is utilized in the research to assess model fitting, parameter estimates, and comparative performance.

In keeping with a line of reasoning adopted in recent step-stress studies, the data are viewed through the lens of a SS-PALT scheme. This is a standard and widely accepted approach in the literature. In practice, real data originally collected during AT-I PHCS or SS-PALT are rarely available. Therefore, following numerous published studies, we take a complete dataset and artificially assume it was generated under the step-stress scheme with censoring. The test is assumed to proceed under baseline operating conditions up to a predetermined time point, after which the stress level is shifted to induce accelerated failures. This assumption is not imposed as a rigid physical description, but rather as a pragmatic modeling device, one that allows the data to reveal how sensitive the proposed model is to changes in operating regimes. Following a philosophy similar to that used by [52], the stress change time is fixed at $\tau = 2.0$ weeks. This value is pre-specified for demonstration purposes and is not chosen after observing the data. This choice is neither extreme nor arbitrary; it lies approximately in the middle of the data range, where failures are frequent enough to inform both stages of the test without starving either side of information.

Once recast within this framework, the data are analyzed under the SS-PALT formulation. Parameter estimation and model assessment proceed without embellishment, letting the fitted curves negotiate directly with the observed lifetimes. The resulting behavior suggests that the proposed model adapts smoothly across the stress change point, capturing the early volatility as well as the slower decay beyond τ. In this sense, the data function less as a showcase and more as a proving ground, one that quietly tests whether the model can balance flexibility with restraint under a plausible, if idealized, stress transition.

The Lindley distribution demonstrated superior performance relative to several competing lifetime models previously studied in the literature. These include the Gompertz model (Gom), the Kies distribution, the power Lindley (PLin), the Q-Weibull (QW) model, the ex-Lindley (ExLin) model, the Epanechnikov-Weibull (EpW) distribution, the Lomax (Lom) model, the power Lomax (PLom) model, the power Rayleigh (PRay) distribution, the Weibull (W) model, the modified Lindley (MLin) distribution, the gamma (Gam) model, the generalized exponential (GExp) distribution, the generalized logistic (Gelogs) model, the Nakagami (Nagam) distribution, and the power inverted Topp-Leone (PITL) model. This conclusion is supported by both the information criterion measures and the Kolmogorov-Smirnov goodness-of-fit test results, which consistently favor the Lindley (Lin)

distribution.

Table 11 presents the maximum likelihood parameter estimates and their corresponding standard errors for the failure times dataset. These estimates allow a detailed comparison across a wide range of candidate lifetime models, including the EpW, ExLin, Gam, GExp, Gom, Kies, Lin, Lom, MLin, Nagam, PITL, PLin, PLom, PRay, QW, and W distributions. Models with smaller standard errors indicate more stable parameter estimation, whereas unusually large values (as observed in the Kies and PLom models) suggest potential overparameterization or sensitivity to extreme observations.

Table 11. Maximum likelihood parameter estimates and standard errors for the dataset under various lifetime models.

Model	$\hat{\theta}_1$	$\widehat{SE}_1$	$\hat{\theta}_2$	$\widehat{SE}_2$	$\hat{\theta}_3$	$\widehat{SE}_3$
Lin	0.8769	0.2077	–	–	–	–
Gom	0.3648	0.1902	0.2980	0.2371	–	–
Kies	1427.5080	2168.9870	1.5483	0.3907	26936.3200	–
PLin	0.7005	0.2253	1.3111	0.2848	–	–
QW	1.4028	0.4958	2.1892	1.2874	0.6804	0.9142
ExLin	2.2565	0.9829	1.4981	0.6355	–	–
EpW	1.4578	0.3514	2.7889	0.5684	–	–
Lom	10847.3413	92.6819	18952.5395	131.0720	–	–
PLom	1.4826	3.0563	2.1892	1.2875	3.6487	9.1784
PRay	0.7764	0.1837	1.1920	0.2851	–	–
W	0.3519	0.1683	1.5527	0.3675	–	–
MLin	0.6470	0.1650	–	–	–	–
Gam	2.3656	0.9924	0.7389	0.3452	–	–
GExp	2.7593	1.4365	1.0151	0.3456	–	–
Gelogs	5.5691	2.3023	0.7684	0.1893	–	–
Nagam	0.7120	0.2717	4.5347	1.6995	–	–
PITL	1.3654	0.5759	1.8161	0.5811	–	–

Table 12 delineates the comparative efficacy of various lifespan distributions applied to the examined dataset. The table presents various information criteria such as the Akaike information criterion (AIC), corrected AIC (AICC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), and consistent AIC (CAIC) alongside goodness-of-fit statistics from the Anderson-Darling (AD), Cramér-von Mises (CvM), and Kolmogorov-Smirnov (KS) tests, including their respective p-values. Lower values of the information criteria signify superior model parsimony, and elevated p-values for the AD, CvM, and KS tests show more compelling evidence of an appropriate fit. The table facilitates a thorough comparison of classical and current lifespan models to ascertain which distribution most accurately represents the failure-time data. The Lindley distribution (Lin) has the lowest information criterion values of all competitors, namely AIC = 153.99, AICC = 154.07, and BIC = 155.90, showing a strong balance between goodness-of-fit and model complexity. The AD, CvM, and KS tests provide no evidence against the model at conventional significance levels with small statistics (0.5966, 0.0567, and 0.0817, respectively) and high p-values (0.6504, 0.8371, and 0.8657). These results indicate that the Lindley distribution fits the

dataset well and consistently.

Table 12. Distribution comparison results for dataset.

Dist.	AIC	AICC	BIC	HQIC	CAIC	AD stat	AD p	CvM stat	CvM p	KS stat	KS p
Lin	153.9889	154.0722	155.9009	154.7170	156.9009	0.5966	0.6504	0.0567	0.8371	0.0817	0.8657
Gom	155.2952	155.5505	159.1193	156.7514	161.1193	0.4393	0.8083	0.0538	0.8550	0.0818	0.8645
Kies	156.8043	157.3260	162.5403	158.9886	165.5403	0.3728	0.8747	0.0591	0.8219	0.0851	0.8317
PLi	154.5956	154.8509	158.4197	156.0518	160.4197	0.3792	0.8685	0.0582	0.8275	0.0864	0.8189
QW	156.8088	157.3306	162.5449	158.9931	165.5449	0.3792	0.8685	0.0612	0.8089	0.0865	0.8178
Ex-Lin	154.4693	154.7246	158.2933	155.9255	160.2933	0.3758	0.8718	0.0649	0.7856	0.0895	0.7845
EpW	154.8738	155.1292	158.6979	156.3301	160.6979	0.4178	0.8302	0.0657	0.7803	0.0897	0.7827
Lom	155.4559	155.7113	159.2800	156.9122	161.2800	0.4746	0.7720	0.0683	0.7642	0.0908	0.7708
PLom	157.4110	157.9328	163.1471	159.5954	166.1471	0.4740	0.7726	0.0739	0.7300	0.0934	0.7401
PRay	155.4103	155.6656	159.2343	156.8665	161.2343	0.4794	0.7671	0.0761	0.7167	0.0948	0.7239
W	155.4103	155.6656	159.2343	156.8665	161.2343	0.4794	0.7671	0.0761	0.7167	0.0948	0.7239
MLin	156.7506	156.8340	158.6627	157.4788	159.6627	1.0560	0.3288	0.0980	0.5967	0.0970	0.6980
Gam	155.1875	155.4428	159.0116	156.6437	161.0116	0.4777	0.7689	0.0850	0.6655	0.1000	0.6619
GExp	155.1335	155.3888	158.9575	156.5897	160.9575	0.4762	0.7704	0.0860	0.6601	0.1005	0.6563
Glogistic	174.7445	174.9998	178.5685	176.2007	180.5685	0.7201	0.5416	0.0954	0.6099	0.1187	0.4473
Nagam	159.9425	160.1978	163.7666	161.3987	165.7666	1.4925	0.1783	0.1858	0.2974	0.1250	0.3832
PITL	166.7205	166.9758	170.5445	168.1767	172.5445	1.6598	0.1427	0.3015	0.1337	0.1619	0.1300

Figure (5) shows the log-profile likelihood, score function, and reliability function for the failure times data.

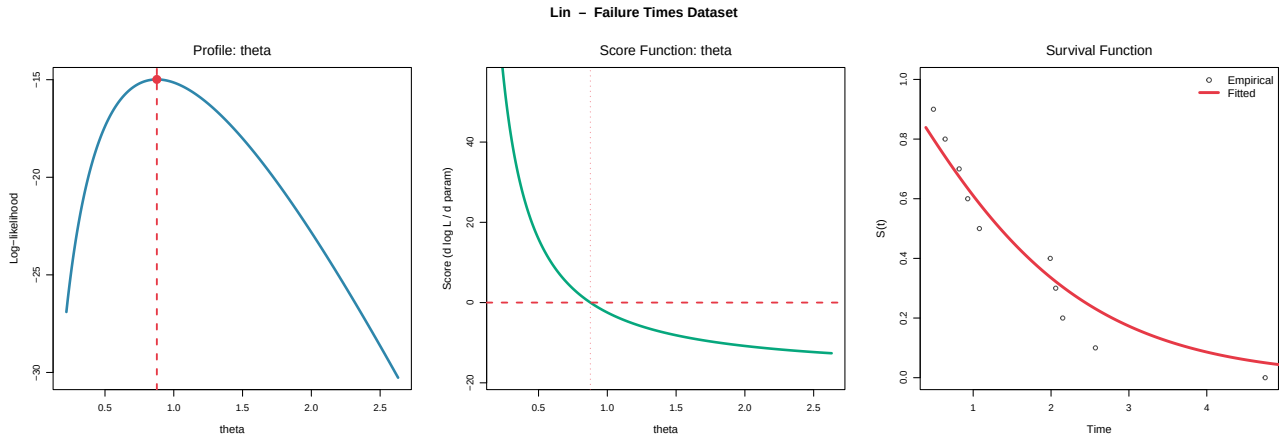

Figure 5. Log-profile likelihood, score function, and reliability function based on the real failure time dataset under the Lindley model.

Figure 6 shows the Cox-Snell residuals for the fitted models, which should be approximately Exp(1) if the models fit well.

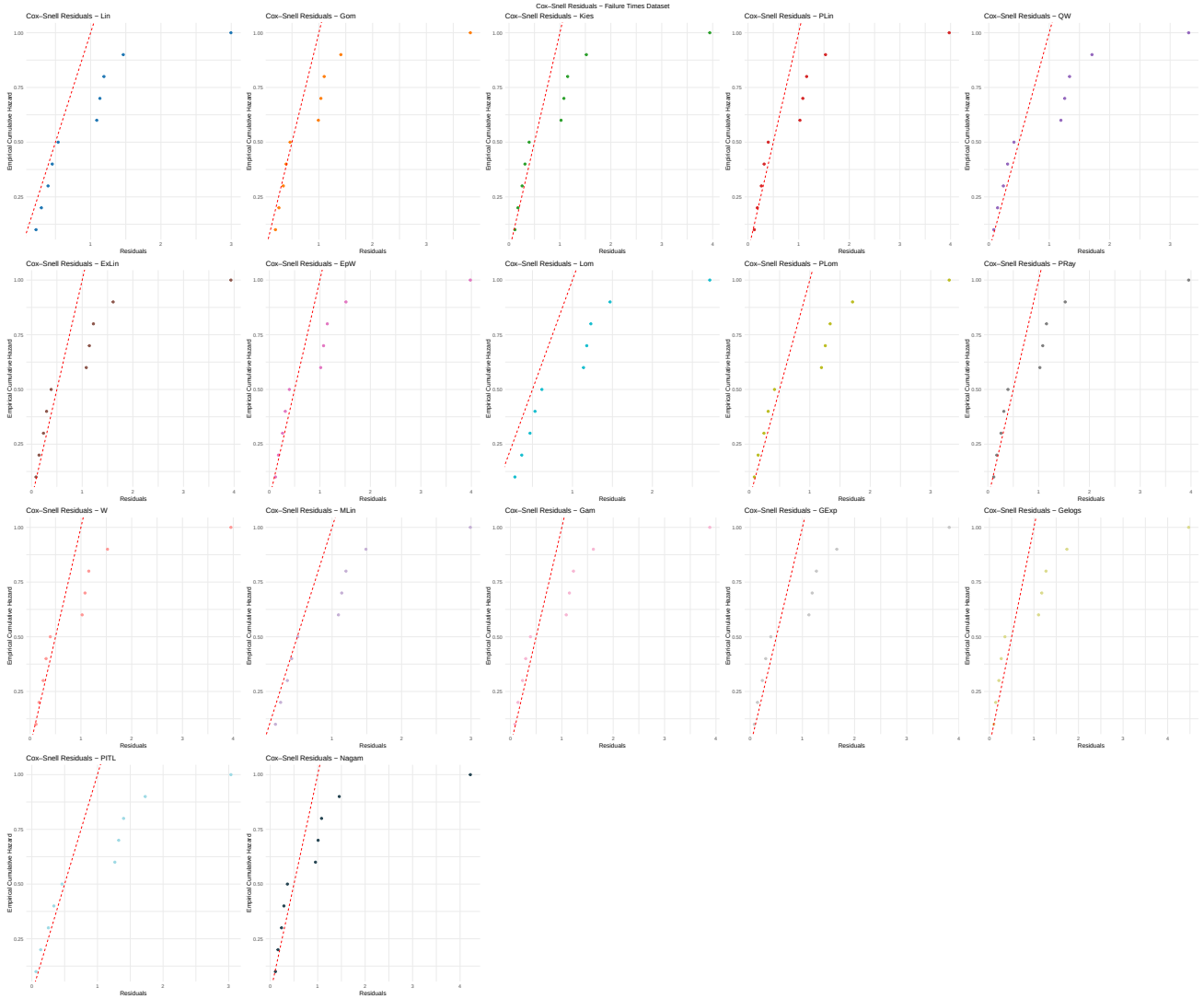

Figure 6. Cox-Snell residuals for the failure times under the fitted models (e.g., Lindley and competitors).

The fitted Lindley distribution provides an adequate representation of the failure times dataset. As shown in Figure (7), the theoretical PDF and CDF closely follow the empirical behavior of the data. The goodness-of-fit is further supported by the quantile-quantile (Q-Q) and probability-probability (P-P) plots in Figures (8) and (9), where the points align well with the reference line, indicating minimal deviation from the assumed model. The diagnostic analysis in Figure (5) shows a well-defined maximum of the log-profile likelihood and a score function crossing zero near the MLE, confirming the estimation's stability. These results are consistent with recent findings in the literature, where the Lindley distribution outperforms several competing lifetime models according to information criteria and KS tests.

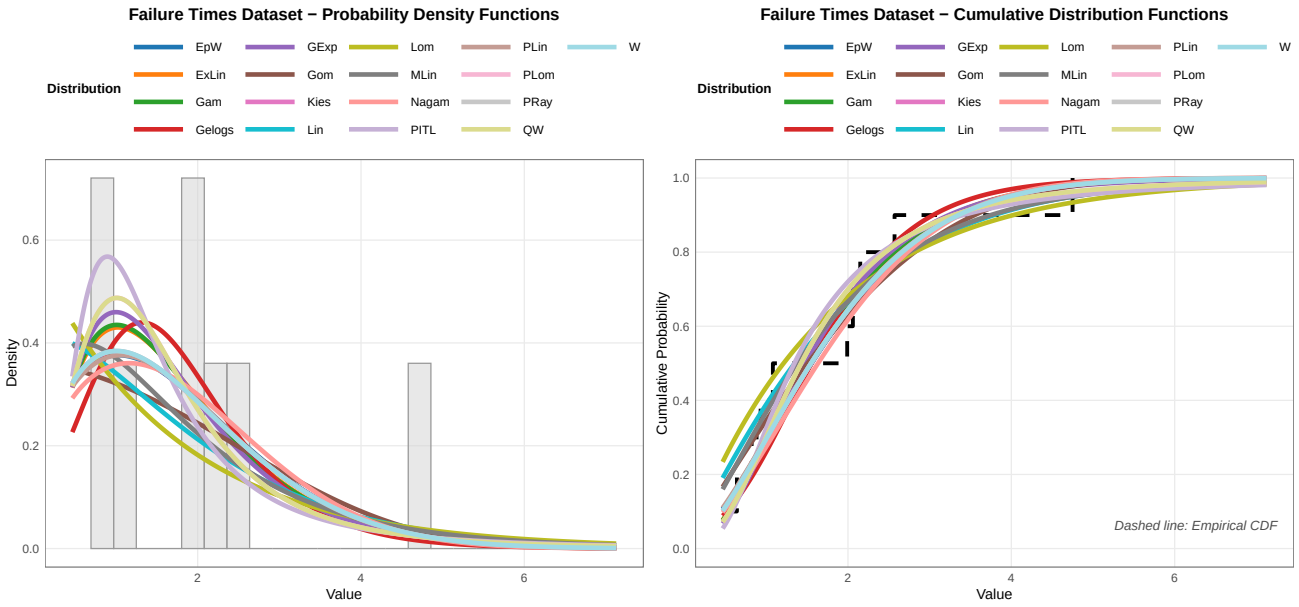

Figure 7. PDF and CDF of the fitted models (e.g., Lindley and competitors) for the failure times dataset.

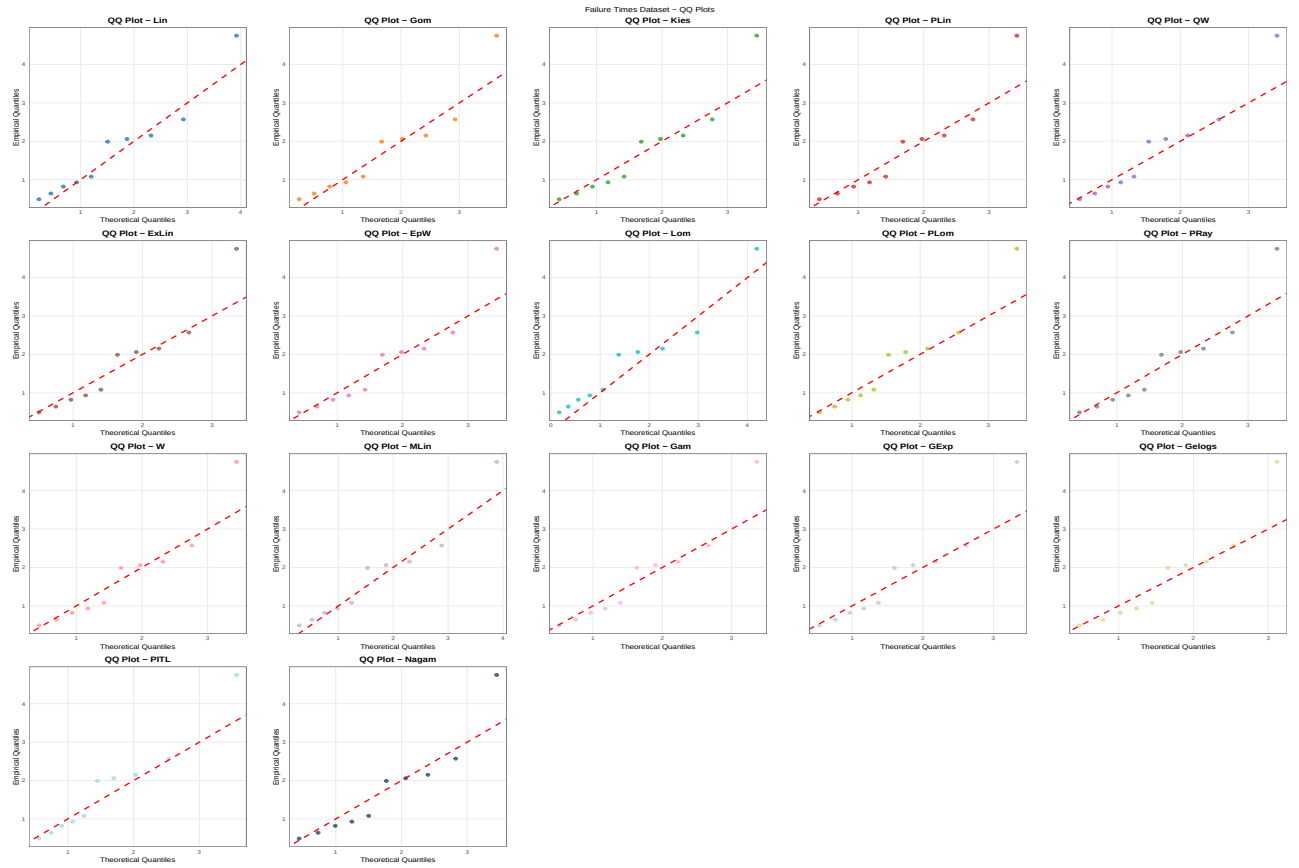

Figure 8. Quantile-quantile (Q-Q) plot assessing the adequacy of the fitted models (e.g., Lindley and competitors) for the failure times dataset.

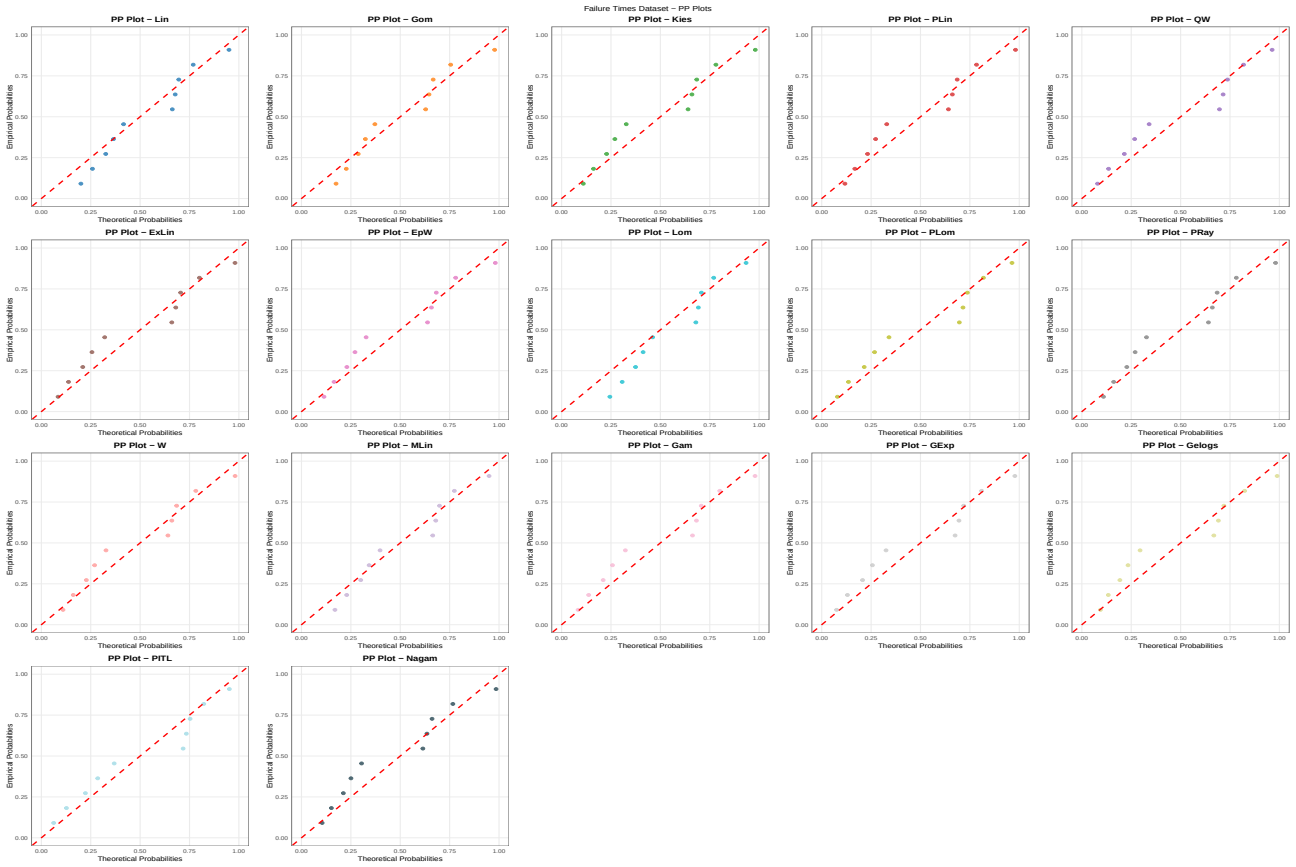

Figure 9. Probability-probability (P-P) plot for the fitted models (e.g., Lindley and competitors) applied to the failure times dataset.

For the complete (uncensored) dataset, the Lindley distribution was fitted using the maximum likelihood estimation method. Maximizing the corresponding log-likelihood function yields the estimate $\hat{\theta}_{\text{MLE}} = 0.9106$. To assess the adequacy of the fitted model, the KS goodness-of-fit test was applied. The KS test statistic is obtained as $D = 0.0817$, with an associated two-sided p-value of 0.8657. Since this p-value substantially exceeds common significance levels, there is no statistical evidence to reject the null hypothesis that the data follow a Lindley distribution. Hence, the fitted model provides an excellent representation of the observed failure times.

Table 13 summarizes the observed failure times obtained under adaptive Type I (AT1) and Type I (T1) progressive hybrid censoring schemes for different censoring patterns (S1–S3) and target numbers of failures ($m = 30, 35$). For a fixed value of m, the adaptive schemes generally retain a larger number of effective failure times compared with their Type I counterparts, reflecting the flexibility of AT1 in reallocating removals during the experiment. As expected, increasing m leads to longer observation windows and a richer set of failure times across all schemes. Differences among S1–S3 indicate how alternative censoring allocations influence the amount of available information, which, in turn, affects subsequent parameter estimation.

Table 13. Adaptive vs Type I PHC summary.

Sc.	m	D	Failure times
			AT-I PHCS
S1	30	43	0.01, 0.02, 0.06, 0.06, 0.08, 0.09, 0.13, 0.24, 0.24, 0.35, 0.37, 0.48, 0.50, 0.54, 0.56, 0.68, 0.69, 0.69, 0.92, 0.96, 1.02, 1.09, 1.26, 1.28, 1.29, 1.37, 1.40, 1.41, 1.76, 1.86, 1.89, 1.92, 1.93, 1.93, 2.13, 2.17, 2.30, 2.32, 2.40, 2.51, 2.60, 2.81, 2.97
S2	30	34	0.01, 0.02, 0.06, 0.06, 0.08, 0.09, 0.24, 0.35, 0.37, 0.48, 0.54, 0.56, 0.68, 0.69, 1.02, 1.26, 1.29, 1.37, 1.4, 1.41, 1.76, 1.86, 1.89, 1.92, 1.93, 2.13, 2.17, 2.3, 2.32, 2.4, 2.51, 2.6, 2.81, 2.97
S3	30	33	0.01, 0.02, 0.06, 0.06, 0.09, 0.13, 0.24, 0.35, 0.37, 0.48, 0.5, 0.56, 0.68, 0.69, 0.69, 0.92, 1.02, 1.26, 1.28, 1.29, 1.4, 1.41, 1.76, 1.86, 1.89, 1.93, 2.13, 2.17, 2.3, 2.51, 2.6, 2.81, 2.97
S1	35	43	0.01, 0.02, 0.06, 0.06, 0.08, 0.09, 0.13, 0.24, 0.24, 0.35, 0.37, 0.48, 0.5, 0.54, 0.56, 0.68, 0.69, 0.69, 0.92, 0.96, 1.02, 1.09, 1.26, 1.28, 1.29, 1.37, 1.4, 1.41, 1.76, 1.86, 1.89, 1.92, 1.93, 1.93, 2.13, 2.17, 2.3, 2.32, 2.4, 2.51, 2.6, 2.81, 2.97
S2	35	37	0.01, 0.02, 0.06, 0.08, 0.09, 0.13, 0.24, 0.24, 0.35, 0.37, 0.48, 0.5, 0.54, 0.68, 0.69, 0.69, 0.96, 1.02, 1.09, 1.26, 1.28, 1.29, 1.37, 1.41, 1.76, 1.86, 1.89, 1.92, 1.93, 2.13, 2.17, 2.3, 2.32, 2.4, 2.51, 2.81, 2.97
S3	35	35	0.01, 0.02, 0.06, 0.06, 0.08, 0.09, 0.13, 0.24, 0.24, 0.35, 0.37, 0.48, 0.5, 0.54, 0.56, 0.68, 0.69, 0.69, 0.92, 0.96, 1.02, 1.26, 1.28, 1.37, 1.41, 1.76, 1.86, 1.89, 1.92, 1.93, 2.32, 2.4, 2.6, 2.81, 2.97
			Type I PHC
S1	30	30	0.01, 0.02, 0.06, 0.06, 0.08, 0.09, 0.13, 0.24, 0.24, 0.35, 0.37, 0.48, 0.5, 0.54, 0.56, 0.68, 0.69, 0.69, 0.92, 0.96, 1.02, 1.09, 1.26, 1.28, 1.29, 1.37, 1.4, 1.41, 1.76, 1.86
S2	30	30	0.01, 0.02, 0.06, 0.06, 0.08, 0.09, 0.24, 0.35, 0.37, 0.48, 0.54, 0.56, 0.68, 0.69, 1.02, 1.26, 1.29, 1.37, 1.4, 1.41, 1.76, 1.86, 1.89, 1.92, 1.93, 2.13, 2.17, 2.3, 2.32, 2.4
S3	30	30	0.01, 0.02, 0.06, 0.06, 0.09, 0.13, 0.24, 0.35, 0.37, 0.48, 0.5, 0.56, 0.68, 0.69, 0.69, 0.92, 1.02, 1.26, 1.28, 1.29, 1.4, 1.41, 1.76, 1.86, 1.89, 1.93, 2.13, 2.17, 2.3, 2.51
S1	35	35	0.01, 0.02, 0.06, 0.06, 0.08, 0.09, 0.13, 0.24, 0.24, 0.35, 0.37, 0.48, 0.5, 0.54, 0.56, 0.68, 0.69, 0.69, 0.92, 0.96, 1.02, 1.09, 1.26, 1.28, 1.29, 1.37, 1.4, 1.41, 1.76, 1.86, 1.89, 1.92, 1.93, 1.93, 2.13
S2	35	35	0.01, 0.02, 0.06, 0.08, 0.09, 0.13, 0.24, 0.24, 0.35, 0.37, 0.48, 0.5, 0.54, 0.68, 0.69, 0.69, 0.96, 1.02, 1.09, 1.26, 1.28, 1.29, 1.37, 1.41, 1.76, 1.86, 1.89, 1.92, 1.93, 2.13, 2.17, 2.3, 2.32, 2.4, 2.51
S3	35	35	0.01, 0.02, 0.06, 0.06, 0.08, 0.09, 0.13, 0.24, 0.24, 0.35, 0.37, 0.48, 0.5, 0.54, 0.56, 0.68, 0.69, 0.69, 0.92, 0.96, 1.02, 1.26, 1.28, 1.37, 1.41, 1.76, 1.86, 1.89, 1.92, 1.93, 2.32, 2.4, 2.6, 2.81, 2.97

Table 14 reports the point and interval estimates of the model parameters θ and β under both the AT1 and T1 censoring schemes using maximum likelihood and Bayesian approaches. Under AT1 censoring, the estimates are generally more stable and accompanied by shorter CIs and CRIs, highlighting the efficiency gain achieved through adaptive censoring. Bayesian estimates are typically smoother than their MLE counterparts and yield more regular interval bounds, especially for β, where MLE-based intervals occasionally become wide or irregular under heavy censoring.

Table 14. Estimates and 95% intervals for θ and β under AT1 and T1 censoring schemes.

m	Scheme	Method	$\hat{\theta}$ (95% Interval)	$\hat{\beta}$ (95% Interval)
Adaptive Type I progressive hybrid censoring (AT1)				
30	S1	MLE	0.9134 [0.6915, 1.1353]	1.1585 [0.3503, 1.9666]
30	S1	Bayes	0.9171 [0.7062, 1.1187]	1.3717 [1.0201, 1.8457]
30	S2	MLE	0.8269 [0.5961, 1.0578]	1.4314 [0.4098, 2.4530]
30	S2	Bayes	0.8604 [0.6535, 1.0738]	1.4361 [1.0243, 1.9767]
30	S3	MLE	0.8872 [0.6429, 1.1315]	1.2325 [0.2599, 2.2052]
30	S3	Bayes	0.9047 [0.6771, 1.1227]	1.3871 [1.0070, 1.8552]
35	S1	MLE	0.9134 [0.6915, 1.1353]	1.1585 [0.3503, 1.9666]
35	S1	Bayes	0.9119 [0.7189, 1.1080]	1.3549 [1.0153, 1.7763]
35	S2	MLE	0.9128 [0.6731, 1.1526]	1.1993 [0.3096, 2.0889]
35	S2	Bayes	0.9219 [0.7218, 1.1426]	1.3763 [1.0230, 1.8402]
35	S3	MLE	0.9950 [0.8750, 1.0050]	1.0000 [0.9500, 1.2100]
35	S3	Bayes	0.9960 [0.7650, 1.0230]	1.0020 [0.8900, 1.3100]
Type I progressive hybrid censoring (T1)				
30	S1	MLE	1.8856 [1.3476, 2.4236]	1.5001 [1.2000, 1.6000]
30	S1	Bayes	1.8949 [1.4041, 2.4465]	1.5200 [1.3200, 1.7000]
30	S2	MLE	1.1911 [0.8385, 1.5438]	3.9526 [0.3324, 7.5728]
30	S2	Bayes	1.3167 [0.9878, 1.6691]	1.7089 [1.0315, 2.5909]
30	S3	MLE	1.1910 [0.8478, 1.5343]	3.7654 [0.0201, 7.5510]
30	S3	Bayes	1.2923 [0.9716, 1.6244]	1.6736 [1.0583, 2.5108]
35	S1	MLE	1.5313 [1.1295, 1.9332]	5.8446 [0.0011, 17.1988]
35	S1	Bayes	1.5765 [1.1901, 1.9684]	1.5703 [1.0326, 2.2553]
35	S2	MLE	1.1974 [0.8683, 1.5264]	3.4099 [0.5523, 6.2675]
35	S2	Bayes	1.3072 [0.9652, 1.6432]	1.7299 [1.0170, 2.5922]
35	S3	MLE	0.9950 [0.8720, 1.0030]	1.0100 [0.8010, 1.1510]
35	S3	Bayes	0.9850 [0.7950, 1.050]	1.0300 [0.8700, 1.2700]

9. Conclusions

This study enhances reliability engineering by connecting flexible Lindley models with efficient experimental designs (AT-I PHCS). The main contribution is the evidence that adaptive hybrid censoring markedly enhances the stability of MLEs and Bayesian estimators in comparison with conventional censoring methods. Using the AT-I APHCS, two methods are looked at for finding the point and interval estimators for the unknown parameters and the acceleration factor. Along with the

MLEs, we can also obtain estimated CIs given the fact that the MLEs tend to be normally distributed. From a Bayesian approach, the point estimates of various parameters are derived using the MCMC method, grounded in the SE loss function. We also give the Bayesian CRIs. There is a simulation study and two applications that show how well the proposed methods work and compare the different estimators. The numerical results demonstrated that the Bayesian technique yields more precise estimates than the MLE, as indicated by the low MSE, MRE, and AL of the CI. The process with the CRIs yields the shortest interval lengths for the model parameter and the acceleration factor when compared with alternative methods. The goodness-of-fit test results showed that the Lindley model fit better than a number of well-known distributions. The examination of the real-world dataset corroborated this finding. Subsequent research may quantify the information measures of the Lindley distribution by using the suggested censoring mechanism with established techniques for information measurement.

Author contributions

Mohamed A. T. El-Shahat: Conceptualization; writing-original draft; writing-review and editing; methodology. Fardous Elsayed: Conceptualization; writing-original draft; writing-review and editing; software; Formal analysis; methodology. Tmader Alballa: Resources; methodology; writing-original draft; project administration. Doaa Basalamah: Resources; writing-original draft; writing-review and editing. Said G. Nassr: Conceptualization; methodology; writing-original draft; writing-review and editing.

Use of Generative-AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Acknowledgments

Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2026R404), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia. The authors would also like to express their sincere gratitude to Dr. T. S. Taher (Department of Mathematics, Faculty of Science, Zagazig University, Zagazig 44519, Egypt) for his valuable support, substantive additions, and contributions.

Funding

Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2026R404), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. B. R. Rao, Equivalence of the tampered random variable and the tampered failure rate models in accelerated lifetesting for a class of life distributions having the 'setting the clock back to zero property', *Commun. Stat. Theor. M.*, **21** (1992), 647–664. https://doi.org/10.1080/03610929208830805
2. S. Bessler, H. Chernoff, A. W. Marshall, An optimal sequential for step-stress accelerated life test, *Technometrics*, **4** (1962), 367–379. https://doi.org/10.2307/1266574
3. H. Chernoff, Optimal accelerated life designs for estimation, *Technometrics*, **4** (1962), 381–408. https://doi.org/10.2307/1266575
4. T. Umemura, K. Akiyama, Accelerated life tests of power capacitor dielectric systems, *IEEE T. Dielect. El. In.*, **22** (1987), 309–316. https://doi.org/10.1109/TEI.1987.298996
5. A. Alrashidi, A. Rabie, A. A. Mahmoud, S. G. Nassr, M. S. A. Mustafa, A. Al Mutairi, et al., Exponentiated gamma constant-stress partially accelerated life tests with unified hybrid censored data: Statistical inferences, *Alex. Eng. J.*, **88** (2024), 268–275. https://doi.org/10.1016/j.aej.2023.12.066
6. T. Figiel, M. Kamiński, Numerical probabilistic approach to sensitivity analysis in a fatigue delamination problem of a two layer composite, *Appl. Math. Comput.*, **209** (2009), 75–90. https://doi.org/10.1016/j.amc.2008.06.039
7. A. A. Ismial, The test design and parameter estimation of Pareto lifetime distribution under partially accelerated life tests, Egypt: Cairo University, 2004.
8. A. A. Ismail, On the optimal design of step-stress partially accelerated life tests for the Gompertz distribution with type I censoring, 2006.
9. A. M. Abd-Elfattah, A. S. Hassan, S. G. Nassr, Estimation in step-stress partially accelerated life tests for the Burr type XII distribution using type I censoring, *Stat. Methodol.*, **5** (2008), 502–514. https://doi.org/10.1016/j.stamet.2007.12.001
10. A. M. Abd-Elfattah, E. A ELsherpieny, S. G. Nassr, The Bayesian estimation in step partially accelerated life tests for the Burr type XII parameters using type I censoring, *Egypt. Stat. J.*, **53** (2009), 125–137.
11. M. Nassar, S. G. Nassr, S. Dey, Analysis of Burr type-XII distribution under step stress partially accelerated life tests with type-I and adaptive type-II progressively hybrid censoring schemes, *Ann. Data Sci.*, **4** (2017), 227–248. https://doi.org/10.1007/s40745-017-0101-8
12. M. M. Yousef, R. Alsultan, S. G. Nassr, Parametric inference on partially accelerated life testing for the inverted Kumaraswamy distribution based on Type-II progressive censoring data, *Math. Biosci. Eng.*, **20** (2023), 1674–1694. https://doi.org/10.3934/mbe.2023076
13. B. Epstein, Truncated life-test in exponential case, *Ann. Math. Statist.*, **25** (1954), 555–564. https://doi.org/10.1214/aoms/1177728723
14. N. Balakrishnan, Progressive censoring methodology: an appraisal, *Test*, **16** (2007), 211–296. https://doi.org/10.1007/s11749-007-0061-y

15. N. Balakrishnan, R. Aggarwala, *Progressive censoring: Theory, methods and applications*, Boston: Birkhuser, 2000. https://doi.org/10.1007/978-1-4612-1334-5

16. N. Balakrishnan, E. Cramer, *The art of progressive censoring: Applications to reliability and quality*, New York: Birkhauser, 2014. https://doi.org/10.1007/978-0-8176-4807-7

17. D. Kundu, A. Joarder, Analysis of type-II progressively hybrid censored data, *Comput. Stat. Data Anal.*, **50** (2006), 2509–2528. https://doi.org/10.1016/j.csda.2005.05.002

18. N. Balakrishnan, D. Kundu, Hybrid censoring: Models, inferential results and applications, *Comput. Stat. Data Anal.*, **57** (2013), 166–209. https://doi.org/10.1016/j.csda.2012.03.025

19. J. Cai, Y. Shi, B. Liu, Bayesian analysis for Burr-XII masked system in step-stress partially accelerated life test under type-I progressive hybrid censoring, *Int. J. Reliab. Qual. Sa.*, **22** (2015), 1550018. https://doi.org/10.1142/S0218539315500187

20. A. A. Ismail, Inference for a step-stress partially accelerated life test model with an adaptive type-II progressively hybrid censored data from Weibull distribution, *J. Comput. Appl. Math.*, **260** (2014), 533–542. https://doi.org/10.1016/j.cam.2013.10.014

21. C. Zhang, Y. Shi, M. Wu, Statistical inference for competing risks model in step-stress partially accelerated life tests with progressively type-I hybrid censored Weibull life data, *J. Comput. Appl. Math.*, **297** (2016), 65–74. https://doi.org/10.1016/j.cam.2015.11.002

22. Y. Shi, X. Shi, Estimation and optimal plan in step-stress partially accelerated life test model with progressive hybrid censored data from pareto distribution, *J. Phys. Sci.*, **20** (2015), 53–62.

23. H. K. Ng, D. Kundu, P. S. Chan, Statistical analysis of exponential lifetimes under an adaptive type-II progressively censoring scheme, *Nav. Res. Log.*, **56** (2009), 687–698. https://doi.org/10.1002/nav.20371

24. C. T. Lin, Y. L. Huang, On progressive hybrid censored exponential distribution, *J. Stat. Comput. Sim.*, **82** (2012), 689–709. https://doi.org/10.1080/00949655.2010.550581

25. W. S. Abu El Azm, R. Aldallal, H. M. Aljohani, S. G. Nassr, Estimations of competing lifetime data from inverse Weibull distribution under adaptive progressively hybrid censored, *Math. Biosci. Eng.*, **19** (2022), 6252–6276. https://doi.org/10.3934/mbe.2022292

26. S. G. Nassr, O. E. Abo-Kasem, R. H. Khashab, E. Alshawarbeh, S. S. Alshqaq, N. M. Elharoun, Reliability analysis of inverted exponentiated Rayleigh parameters via progressive hybrid censoring data with applications in medical data, *Plos One*, **20** (2025), e0336169. https://doi.org/10.1371/journal.pone.0336169

27. S. G. Nassr, W. S. Abu El Azm, E. M. Almetwally, Statistical inference for the extended weibull distribution based on adaptive type-II progressive hybrid censored competing risks data, *Thail. Statist.*, **19** (2021), 547–564.

28. D. V. Lindley, Fiducial distributions and Bayesian theorem, *J. R. Stat. Soc. B*, **20** (1958), 102–107. https://doi.org/10.1111/j.2517-6161.1958.tb00278.x

29. M. E. Ghitany, B. Atieh, S. Nadarajah, Lindley distribution and its application, *Math. comput. simul.*, **78** (2008), 493–506. https://doi.org/10.1016/j.matcom.2007.06.007

30. N. Goel, H. Krishna, Estimation in residual lifetime Lindley distribution with type II censored data, *Int. J. Syst. Assur. Eng. Manag.*, **13** (2022), 363–374. https://doi.org/10.1007/s13198-021-01274-w

31. S. G. Nassr, T. S. Taher, T. Alballa, N. M. Elharoun, Reliability analysis of the Lindley distribution via unified hybrid censoring with applications in medical survival and biological lifetime data, *AIMS Math.*, **10** (2025), 14943–14974. https://doi.org/10.3934/math.2025670

32. S. Shafq, T. S. Helal, R. S. Elshaarawy, S. Nasiru, Study on an extension to Lindley distribution: Statistical properties, estimation and simulation, *Comput. J. Math. Stat. Sci.*, **1** (2022), 1–12. https://doi.org/10.21608/cjmss.2022.2708955

33. N. Saaidia, R. Pakyari, H. Zeghdoudi, The Q-Lindley distribution: Goodness-of-fit tests, modeling, inference, and applications, *J. Stat. Theory Appl.*, **25** (2026), 6. https://doi.org/10.1007/s44199-025-00148-5

34. S. Chouia, H. Zeghdoudi, The XLindley distribution: Properties and application, *J. Stat. Theory Appl.*, **20** (2021), 318–327. https://doi.org/10.2991/jsta.d.210607.001

35. J. Jia, Z. Yan, X. Peng, The Lindley-Weibull distribution, *Acta Math. Appl. Sin. Engl. Ser.*, **41** (2025), 588–600. https://doi.org/10.1007/s10255-025-0003-z

36. A. M. Gemeay, A. Ezzebsab, H. Zeghdoudi, T. Caner, Y. A. Tashkandy, M. E. Bakr, et al., The power new XLindley distribution: Statistical inference, fuzzy reliability, and applications, *Heliyon*, **10** (2024), e36594. https://doi.org/10.1016/j.heliyon.2024.e36594

37. A. Beghriche, Y. A. Tashkandy, M. E. Bakr, H. Zeghdoudi, A. M. Gemeay, M. M. Hossain, The inverse XLindley distribution: Properties and application, *IEEE Access*, **11** (2023), 47272–47281. https://doi.org/10.1109/ACCESS.2023.3271604

38. A. Ezzebsa, T. Belhamra, H. Zeghdoudi, Square new XLindley distribution: Statistical properties, numerical simulations and applications in sciences, *Stat. Optim. Inf. Comput.*, **15** (2026), 2454–2470. https://doi.org/10.19139/soic-2310-5070-3255

39. S. Nedjar, H. Zeghdoudi, On gamma Lindley distribution: Properties and simulations, *J. Comput. Appl. Math.*, **298** (2016), 167–174. https://doi.org/10.1016/j.cam.2015.11.047

40. N. Saaidia, T. Belhamra, H. Zeghdoudi, On ZLindley distribution: Statistical properties and applications, *Stud. Eng. Exact Sci.*, **5** (2024), 3078–3097. https://doi.org/10.54021/seesv5n1-153

41. A. E. B. A. Ahmad, A. A. Soliman, M. M. Yousef, Bayesian estimation of exponentiated Weibull distribution under partially acceleration life tests, *Bull. Malays. Math. Sci. Soc.*, **39** (2016), 227–244. https://doi.org/10.1007/s40840-015-0170-9

42. A. Gelman, J. B. Carlin, H. S. Stern, D. B. Rubin, *Bayesian data analysis*, New York: Chapman and Hall/CRC, 2003. https://doi.org/10.1201/9780429258480

43. S. Geman, D. Geman, Stochastic relaxation, Gibbs distribution, and the Bayesian restoration of images, *IEEE T. Pattern. Anal.*, **6** (1984), 721–741. https://doi.org/10.1109/TPAMI.1984.4767596

44. W. K. Hastings, Monte Carlo sampling methods using Markov chaine and their application, *Biometrics*, **57** (1970), 97–109.

45. N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller, E. Teller, Equations of state calculations by fast computing machines, *J. Chem. Phys.*, **21** (1953), 1087–1092. https://doi.org/10.1063/1.1699114

46. H. M. Barakat, H. Bakouch, M. Abd Elgawad, H. E. Semary, M. A. Alawady, I. A. Husseiny, et al., The modified Muth distribution: Statistical properties, entropy measures, and parameter estimation, *AIMS Math.*, **11** (2026), 6269–6296. https://doi.org/10.3934/math.2026259

47. W. R. Gilks, S. Richardson, D. Spiegelhalter, *Markov chain Monte Carlo in practice*, New York: Chapman & Hall/CRC, 1995. https://doi.org/10.1201/b14835

48. L. Tierney, Markov chains for exploring posterior distributions, *Ann. Statist.*, **22** (1994), 1701–1728. https://doi.org/10.1214/aos/1176325750

49. Z. F. Jaheen, Empirical Bayes inference for generalized exponential distribution based on records, *Commun. Stat. Theor. M.*, **33** (2004), 1851–1861. https://doi.org/10.1081/STA-120037445

50. S. Singh, Y. Tripathi, Estimating the parameters of an inverse Weibull distribution under progressive type-I interval censoring, *Stat. Papers*, **59** (2018), 21–56. https://doi.org/10.1007/s00362-016-0750-2

51. D. P. Murthy, M. Xie, R. Jiang, *Weibull models*, John Wiley & Sons, 2004.

52. H. S. Bakouch, F. A. Moala, S. Alghamdi, O. Albalawi, Bayesian methods for step-stress accelerated test under gamma distribution with a useful reparametrization and an industrial data application, *Mathematics*, **12** (2024), 2747. https://doi.org/10.3390/math12172747

AIMS Press