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Research article

Certain subclasses of analytic functions: Fekete-Szegö inequalities involving q-Bessel functions and Carlson-Shaffer operator

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Abstract: In this paper, we introduce a new linear operator involving the q-analogue of generalized Bessel functions and the q-analogue of the Carlson-Shaffer operator. By using this operator, we define and investigate certain subclasses of analytic functions in the open unit disk. We obtain coefficient estimates and establish Fekete-Szegö-type inequalities for the newly introduced function classes. Moreover, by employing the Hadamard product, we study several geometric properties and derive necessary as well as sufficient conditions for these classes. The results obtained in this work generalize and extend a number of earlier results in the existing literature.

Keywords: analytic functions; coefficient estimates; Fekete-Szegö inequalities; Hadamard product; Poisson distribution; q-Bessel functions; q-Carlson-Shaffer operator
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1. Introduction and preliminaries

In recent years, the study of geometric properties of analytic functions has attracted considerable attention due to its fundamental role in geometric function theory and its applications to various subclasses of univalent functions [1]. Among the important problems in this area, the Fekete-Szegö problem occupies a central position because of its close connection with coefficient estimates and geometric behavior of analytic functions [2, 3]. Motivated by the increasing interest in q-calculus and special functions, the present work investigates subclasses of analytic functions associated with the q-analogue of generalized Bessel functions [4–7] together with the q-analogue of the Carlson-Shaffer operator [8, 9]. These operators have proved to be useful in studying geometric properties, such as starlikeness and convexity of complex order.

The theory of analytic function classes associated with q-operators has developed rapidly in recent years. Several researchers, including Murugusundaramoorthy [10], Orhan and Cotîrlă [11], and Al-Sa'di [12], have introduced and studied various subclasses defined through differential and convolution operators involving q-analogues. In particular, coefficient-related problems for these classes have received significant attention. The estimation of initial Taylor-Maclaurin's coefficients plays an important role in understanding the geometric structure and growth properties of analytic and bi-univalent functions [13, 14]. Furthermore, Al-Sa'di and Srivastava et al. [12, 15] investigated the Fekete-Szegö functional for several subclasses of analytic functions and obtained important coefficient inequalities.

The present investigation contributes to this line of research by introducing new subclasses of analytic functions associated with the above-mentioned operators and deriving coefficient estimates together with Fekete-Szegö type inequalities. In addition, the Hadamard product technique is employed to establish necessary and sufficient conditions for certain geometric properties. The obtained results generalize and unify several earlier results available in the literature and may stimulate further studies related to higher-order coefficient problems and other operator-defined subclasses.

Let $\mathbb{D}$ denote the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\},$$

and let $\mathcal{A}$ be the class of functions analytic in $\mathbb{D}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (z \in \mathbb{D}). \tag{1.1}$$

Further, let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of normalized univalent functions in $\mathbb{D}$. In geometric function theory, the subclasses of starlike and convex functions play a fundamental role. A function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^*$ of starlike functions if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \qquad (z \in \mathbb{D}),$$

and $f \in \mathcal{A}$ belongs to the class C of convex functions if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \qquad (z \in \mathbb{D}).$$

These characterizations were established by Alexander [16].

For $\rho \in \mathbb{C} \setminus \{0\}$, Nasr and Aouf [17] (see also [18, 19]) introduced the class of starlike functions of complex order ρ, denoted by $\mathcal{S}^*(\rho)$, defined by

$$\mathcal{S}^*(\rho) = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{1}{\rho}\left(\frac{zf'(z)}{f(z)} - 1\right)\right) > 0, \quad (z \in \mathbb{D}) \right\}.$$

If $g \in \mathcal{A}$ is defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \qquad (z \in \mathbb{D}), \tag{1.2}$$

then the Hadamard product (or convolution) of f and g is defined by

$$(f \star g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g \star f)(z). \tag{1.3}$$

Now, let $f, F \in \mathcal{A}$ be analytic functions in the open unit disk $\mathbb{D}$. The function f is said to be subordinate to F, written as $f(z) \prec F(z)$, if there exists an analytic Schwarz function w in $\mathbb{D}$ satisfying $w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{D})$, such that $f(z) = F(w(z)) \quad (z \in \mathbb{D})$. In particular, if F is univalent in $\mathbb{D}$, then the subordination relation $f(z) \prec F(z)$ is equivalent to $f(0) = F(0) \quad \text{and} \quad f(\mathbb{D}) \subset F(\mathbb{D})$. For further details on the theory of differential subordination and its applications, we refer to Miller and Mocanu [20].

Next, we recall the generalized Bessel function of the first kind of order p, denoted by $\omega_{\gamma,b,p}(z)$, which is defined as a particular solution of the second-order linear homogeneous differential equation (see [21])

$$z^2\omega''(z) + bz\omega'(z) + [\gamma z^2 - p^2 + (1-b)p]\omega(z) = 0, \quad (\gamma; b; p \in \mathbb{C}).$$

The function $\omega_{\gamma,b,p}(z)$ admits the series representation

$$\omega_{\gamma,b,p}(z) = \sum_{n=0}^{\infty} \frac{(-\gamma)^n}{n!\Gamma(p+n+(b+1)/2)} \left(\frac{z}{2}\right)^{2n+p}, \quad (z \in \mathbb{C}), \tag{1.4}$$

where Γ denotes the Euler Gamma function. This representation provides a unified framework for several classical special functions, including the standard Bessel function, modified Bessel function, and spherical Bessel function corresponding to suitable choices of the parameters b and γ.

Deniz [22] and Deniz et al. [23] (see also [21, 24–26]) introduced the function $\varphi_{\gamma,b,p}(z)$ associated with $\omega_{\gamma,b,p}(z)$ by

$$\varphi_{\gamma,b,p}(z) = 2^p \Gamma\left(p + \frac{b+1}{2}\right) z^{1-p/2} \omega_{\gamma,b,p}(\sqrt{z}), \tag{1.5}$$

where $\omega_{\gamma,b,p}(z)$ is given by (1.4).

The present investigation is motivated by the broad applicability of Bessel functions in various branches of mathematics, physics, and engineering. In particular, Bessel functions naturally arise in the solutions of Laplace and Helmholtz equations in spherical and cylindrical coordinate systems, especially in problems involving wave propagation and potential theory. Recently, increasing attention has been devoted to the study of q-analogues of special functions, leading to substantial developments in the theory of q-Bessel functions. In this direction, functions of two variables and generalized q-Bessel functions were studied in [27, 28]. Furthermore, Srivastava [29] investigated several recent developments concerning q-Bessel polynomials and their generalizations.

For convenience, we write

$$\varphi(z) = \varphi_{\gamma,b,p}(z).$$

Then, from (1.5), we obtain

$$\varphi(z) = z + \sum_{n=2}^{\infty} \frac{(-\gamma)^{n-1}\Gamma(k)}{4^{n-1}\Gamma(k+n-1)} \frac{z^n}{(n-1)!}, \tag{1.6}$$

where

$$k = p + \frac{b+1}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-,$$

and $\mathbb{Z}_0^- = \{0, -1, -2, \ldots\}$.

The symbol $(\delta)_m$ denotes the Pochhammer symbol (or shifted factorial), defined by

$$(\delta)_m = \frac{\Gamma(\delta + m)}{\Gamma(\delta)} = \begin{cases} 1, & \text{if } m = 0, \\ \delta(\delta + 1) \cdots (\delta + m - 1), & \text{if } m \in \mathbb{N}. \end{cases}$$

Quantum calculus, also known as q-calculus, is a generalization of classical calculus in which the notion of limits is replaced by finite difference operators. It encompasses both q-calculus and h-calculus, where h is commonly associated with Planck's constant and q denotes the quantum parameter. In recent years, q-calculus has attracted considerable attention due to its wide range of applications in mathematics, physics, and operator theory. The subject was systematically developed by Jackson [30], who introduced the concepts of the q-derivative and q-integral. Later, the geometric interpretation of q-analysis was connected with the theory of quantum groups and integrable systems. Applications of q-calculus in operator theory and approximation theory have also been investigated extensively; see, for example, [31]. Furthermore, Ismail et al. [32] established important connections between geometric function theory and q-analysis, thereby initiating several developments in the study of analytic and univalent functions associated with q-operators.

Now we will present some basic definitions and concepts from q-calculus that will be required for this investigation.

For $0 < q < 1$, the function $\varphi(z)$ defined by (1.6) admits the Jackson q-derivative operator (or q-difference operator), given by (see [30])

$$\begin{aligned} \varphi_q(z) = \partial_q(\varphi(z)) &= \frac{\varphi(z) - \varphi(qz)}{(1-q)z} \\ &= \partial_q \left[z + \sum_{n=2}^{\infty} \frac{(-\gamma)^{n-1}\Gamma(k)}{4^{n-1}\Gamma(k+n-1)} \frac{z^n}{(n-1)!} \right] \\ &= 1 + \sum_{n=2}^{\infty} \frac{(-\gamma)^{n-1}\Gamma_q(k)}{4^{n-1}\Gamma_q(k+n-1)} \frac{[n,q]}{(n-1)!} z^{n-1}. \end{aligned} \tag{1.7}$$

Consequently,

$$z\varphi_q(z) = z + \sum_{n=2}^{\infty} \frac{(-\gamma)^{n-1}\Gamma_q(k)}{4^{n-1}\Gamma_q(k+n-1)} \frac{[n,q]}{(n-1)!} z^n, \qquad (z \in \mathbb{D}).$$

Here, the q-integer $[n, q]$ is defined by

$$[n,q] := \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}, \qquad [0,q] := 0. \tag{1.8}$$

Using (1.8), we recall the following related concepts.

(i) **q-shifted factorial.** For a nonnegative integer n,

$$[n,q]! = \begin{cases} 1, & \text{if } n = 0, \\ [1,q][2,q]\cdots[n,q], & \text{if } n \in \mathbb{N}. \end{cases}$$

(ii) **q-generalized Pochhammer symbol.** For $k > 0$ and $n \in \mathbb{N}_0$,

$$[k,q]_n = \begin{cases} 1, & \text{if } n = 0, \\ [k,q][k+1,q]\cdots[k+n-1,q], & \text{if } n \in \mathbb{N}. \end{cases}$$

Now, for $k > 0$, $\lambda > -1$, and $0 < q < 1$, we define the function

$$\mathcal{N}_{k,q}^{\lambda} : \mathbb{D} \to \mathbb{C}$$

by

$$\mathcal{N}_{k,q}^{\lambda}(z) = z + \sum_{n=2}^{\infty} \frac{(-\gamma)^{n-1}\Gamma_q(k)}{4^{n-1}(n-1)!\Gamma_q(k+n-1)} \frac{[n,q]!}{[\lambda+1,q]_{n-1}} z^n, \qquad (z \in \mathbb{D}).$$

Remark 1.1. A straightforward computation shows that

$$\mathcal{N}_{k,q}^{\lambda} \star \mathcal{F}_{q,\lambda+1}(z) = z\varphi_q(z) \quad (z \in \mathbb{D}),$$

where

$$\mathcal{F}_{q,\lambda+1}(z) = z + \sum_{n=2}^{\infty} \frac{[\lambda+1,q]_{n-1}}{[n-1,q]!} z^n \quad (z \in \mathbb{D}).$$

Next, we define the function $\phi_q(\rho,\beta;z)$ in terms of the q-generalized Pochhammer symbol by

$$\phi_q(\rho,\beta;z) = z + \sum_{n=2}^{\infty} \frac{([\rho]_q)_{n-1}}{([\beta]_q)_{n-1}} z^n,$$

where $\rho \in \mathbb{R}, \beta \in \mathbb{R} \setminus \mathbb{Z}_0^-, z \in \mathbb{D}$, and $\mathbb{Z}_0^- = \{0,-1,-2,\ldots\}$.

Associated with this function, we define the linear operator $\mathrm{L}_q(\rho,\beta;z) : \mathcal{A} \longrightarrow \mathcal{A}$ by means of the Hadamard product as follows:

$$\mathrm{L}_q(\rho,\beta)(z) = \phi_q(\rho,\beta;z) \star f(z) = z + \sum_{n=2}^{\infty} \frac{([\rho]_q)_{n-1}}{([\beta]_q)_{n-1}} a_n z^n,$$

where $\mathrm{L}_q(\rho,\beta)$ corresponds to the Carlson-Shaffer operator [9].

Generalized Bessel functions and the operators associated with them have attracted considerable attention in geometric function theory. Various differential subordination and superordination results involving generalized Bessel function operators have been investigated by Baricz et al. [26] and several subsequent authors. More recently, higher-order differential subordination results associated with generalized Bessel function operators and Carlson–Shaffer-type operators have also been studied. Motivated by these developments, we introduce a new linear operator that combines the q-derivative operator, the q-analogue of the Carlson–Shaffer operator, and the q-analogue of the generalized Bessel function, defined as follows.

Definition 1.2. For $f \in \mathcal{A}$, we define the operator $\mathcal{T}_q^k(\rho,\beta) : \mathcal{A} \longrightarrow \mathcal{A}$, by the Hadamard product of the q-analogue of the Carlson-Shaffer operator and the q-analogue of generalized Bessel functions as

$$\mathcal{T}_q^k(\rho,\beta)f(z) = \mathcal{N}_{k,q}^{\lambda}(z) \star \mathrm{L}_q(\rho,\beta)f(z) = z + \sum_{n=2}^{\infty} \mathrm{A}_n a_n z^n, \tag{1.9}$$

where $k > 0$, $\lambda > -1$, $0 < q < 1$, and

$$\mathrm{A}_n = \frac{(-\gamma)^{n-1}\Gamma_q(k)([\rho]_q)_{n-1}}{4^{n-1}(n-1)!\Gamma_q(k+n-1)([\beta]_q)_{n-1}} \cdot \frac{[n,q]!}{[\lambda+1,q]_{n-1}}, \quad (z \in \mathbb{D}). \tag{1.10}$$

For additional properties and applications of the corresponding classical operator, which serves as the limiting case of the present quantum-extension operator, we refer the reader to [33].

Remark 1.3. From definition (1.9), we obtain the identity

$$z\partial_q\left(\mathcal{T}_q^{k+1}(\rho,\beta)f(z)\right) = \frac{1}{q^{k-1}}\left([k]_q\mathcal{T}_q^k(\rho,\beta)f(z) - [k-1]_q\mathcal{T}_q^{k+1}(\rho,\beta)f(z)\right). \tag{1.11}$$

Furthermore, by letting $q \to 1^-$, we obtain

$$\begin{aligned}\lim_{q\to 1^-} \mathcal{T}_q^k(\rho,\beta)f(z) &= \mathcal{N}_{k,1}^{\lambda}(z) \star \mathrm{L}_1(\rho,\beta)f(z) \\ &= \mathcal{T}^k(\rho,\beta)f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(-\gamma)^{n-1}\Gamma(k)(\rho)_{n-1}}{4^{n-1}(n-1)!\Gamma(k+n-1)(\beta)_{n-1}} \frac{n!}{(\lambda+1)_{n-1}} a_n z^n, \qquad (z \in \mathbb{D}).\end{aligned}$$

Fekete-Szegö problem

The Fekete-Szegö inequality plays a fundamental role in the theory of univalent and analytic functions. It was first introduced by Fekete and Szegö [2] in 1933 in connection with the celebrated Bieberbach conjecture. The problem of obtaining coefficient estimates analogous to the Fekete-Szegö inequality for various subclasses of analytic and univalent functions is commonly referred to as the Fekete-Szegö problem.

Over the years, considerable attention has been devoted to determining sharp bounds for the initial coefficients a_2, a_3, and, in particular, for the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ where $f \in \mathcal{A}$ is of the form (1.1) and $\mu \in \mathbb{R}$ or $\mu \in \mathbb{C}$. This line of investigation was initiated by Fekete and Szegö themselves [2]. The quantity $|a_3 - \mu a_2^2|$ is now known as the Fekete-Szegö coefficient functional. In particular, the special case $|a_3 - a_2^2|$ has been extensively studied, while the generalized functional $|a_3 - \mu a_2^2|$ provides a broader framework for coefficient estimation problems.

It is well known that for functions $f \in \mathcal{S}$ of the form (1.1), the classical Fekete-Szegö theorem states that

$$|a_3 - \mu a_2^2| \leq 1,$$

and the estimate is sharp [2]. Subsequently, Pfluger [34] extended this result to complex values of μ and proved that

$$|a_3 - \mu a_2^2| \leq 1 + 2\left|\exp\left(\frac{-2\mu}{1-\mu}\right)\right|.$$

For the subclasses of starlike and convex functions, the Fekete-Szegö problem was solved by Keogh and Merkes [3]. Later, Orhan et al. [35] investigated several special cases associated with starlike and convex functions of order μ. Since then, many authors have contributed to this area by establishing upper bounds for the Fekete-Szegö functional in different subclasses of $\mathcal{A}$. For example, Deniz et al. [36], Kavitha et al. [37] and Kanas & Darwish [38] obtained the Fekete-Szegö inequalities for certain families of analytic functions which were defined using some well-known differential operators. Also, refer to Al-Hawary et al. [39] and the references provided therein for the similar results pertaining to various subclasses of bi-univalent functions.

We denote by $\mathcal{P}$ the well-known Carathéodory class consisting of analytic functions $\mathfrak{p}$ in the open unit disk $\mathbb{D}$ satisfying

$$\Re(\mathfrak{p}(z)) > 0, \quad (z \in \mathbb{D}),$$

and having the series representation

$$\mathfrak{p}(z) = 1 + \sum_{j=1}^{\infty} c_j z^j. \tag{1.12}$$

The following lemmas will be required in the proofs of our main results.

Lemma 1.4. *Let $\mathfrak{p} \in \mathcal{P}$ be of the form* (1.12)*. Then, for $\eta \in \mathbb{C}$, the following inequalities hold:*

$$|c_n| \leq 2, \quad \text{for } n \geq 1, \tag{1.13}$$

$$|c_2 - \eta c_1^2| \leq 2\max\{1; |2\eta - 1|\}. \tag{1.14}$$

The inequality (1.13) is the classical result of Carathéodory [40], whereas (1.14) was obtained by Keogh and Merkes [3] (see also [41]).

Lemma 1.5. *[42] Let $\mathfrak{p} \in \mathcal{P}$ be given by* (1.12)*. Then, for any $\eta \in \mathbb{R}$, the following sharp estimate holds:*

$$|c_2 - \eta c_1^2| \leq \begin{cases} -4\eta + 2, & \text{if } \eta \leq 0, \\ 2, & \text{if } 0 \leq \eta \leq 1, \\ 4\eta - 2, & \text{if } \eta \geq 1. \end{cases}$$

Moreover, the equality cases are characterized as follows:

- *If $\eta < 0$ or $\eta > 1$, then equality holds if and only if*

$$\mathfrak{p}(z) = \frac{1+z}{1-z},$$

  or one of its rotations.
- *If $0 < \eta < 1$, then equality holds if and only if*

$$\mathfrak{p}(z) = \frac{1+z^2}{1-z^2}.$$

- *If $\eta = 0$, then equality holds if and only if*

$$\mathfrak{p}(z) = \left(\frac{1}{2} + \frac{\tau}{2}\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{\tau}{2}\right)\frac{1-z}{1+z}, \quad (0 \leq \tau \leq 1),$$

  or one of its rotations.

- *If $\eta = 1$, then equality holds if and only if*

$$\frac{1}{\mathfrak{p}(z)} = \left(\frac{1}{2} + \frac{\tau}{2}\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{\tau}{2}\right)\frac{1-z}{1+z}, \quad (0 \le \tau \le 1).$$

As observed in [42], although the above estimates are sharp, they can be refined in the case $0 < \eta < 1$ as follows:

$$|c_2 - \eta c_1^2| + \eta |c_1|^2 \le 2, \quad 0 < \eta \le \tfrac{1}{2}, \tag{1.15}$$

$$|c_2 - \eta c_1^2| + (1-\eta)|c_1|^2 \le 2, \quad \tfrac{1}{2} \le \eta \le 1. \tag{1.16}$$

Lemma 1.6. *[13] Let $\mathfrak{p} \in \mathcal{P}$ be given by* (1.12). *If*

$$0 \le M \le 1$$

and

$$M(2M-1) \le N \le M,$$

then

$$|c_3 - 2Mc_1c_2 + Nc_1^3| \le 2.$$

Lemma 1.7. *[44] Let $\mathfrak{p} \in \mathcal{P}$ be of the form* (1.12). *Suppose that the real constants u, v, w, and l satisfy*

$$0 < u < 1, \qquad 0 < l < 1,$$

and

$$8l(1-l)\left[(uv - 2w)^2 + (u(l+u) - v)^2\right] + u(1-u)(v - 2lu)^2 \le 4u^2(1-u)^2 l(1-l).$$

Then,

$$|wc_1^4 + lc_2^2 + 2uc_1c_3 - \frac{3}{2}vc_1^2c_2 - c_4| \le 2.$$

The primary objective of this paper is to establish Fekete-Szegö type inequalities for functions belonging to the class $\mathcal{S}_{k,q}^{\lambda,\upsilon}(\xi, \Phi)$. In addition, we investigate several applications of the obtained results to certain subclasses of analytic functions defined via convolution with normalized analytic functions. In particular, we derive the Fekete-Szegö inequalities for subclasses of analytic functions associated with the Poisson distribution.

The paper is organized as follows. In Section 1, we present the necessary preliminaries, definitions, and auxiliary results required throughout the paper. Section 2 is devoted to the derivation of the main results and several related corollaries concerning Fekete-Szegö type inequalities for the newly introduced function class. In Section 3, we discuss several applications involving analytic functions associated with the Poisson distribution. Finally, concluding remarks and possible directions for future research are presented in Section 4.

2. Fekete-Szegö-type inequality results

In this section, we derive coefficient estimates for the initial coefficients a_2, a_3, a_4, and a_5 for functions belonging to the class $\mathcal{S}_{k,q}^{\lambda,\upsilon}(\xi, \Phi)$. We are now in a position to introduce the following subclass of analytic functions.

Definition 2.1. Let

$$\Phi(z) = 1 + B_1 z + B_2 z^2 + \cdots, \quad (z \in \mathbb{D};\ B_1 > 0),$$

where $\Phi(z)$ is a univalent function that is starlike with respect to the point 1, maps the unit disk $\mathbb{D}$ onto a region contained in the right half-plane, and is symmetric with respect to the real axis.

For $\xi \in \mathbb{C}^*$ and $0 \leq \upsilon < 1$, a function $f \in \mathcal{A}$ is said to belong to the class

$$\mathcal{S}_{k,q}^{\lambda,\upsilon}(\xi, \Phi),$$

if the function

$$1 + \frac{1}{\xi}\left[\frac{z(\mathcal{T}_q^k(\rho,\beta)f(z))'}{(1-\upsilon)\mathcal{T}_q^k(\rho,\beta)f(z) + \upsilon z(\mathcal{T}_q^k(\rho,\beta)f(z))'} - 1\right]$$

is analytic in $\mathbb{D}$ and satisfies the subordination condition

$$1 + \frac{1}{\xi}\left[\frac{z(\mathcal{T}_q^k(\rho,\beta)f(z))'}{(1-\upsilon)\mathcal{T}_q^k(\rho,\beta)f(z) + \upsilon z(\mathcal{T}_q^k(\rho,\beta)f(z))'} - 1\right] \prec \Phi(z),$$

where

$$k > 0, \quad \lambda > -1, \quad 0 < q < 1.$$

Moreover, by taking the limit as $q \to 1^-$, we obtain

$$\lim_{q \to 1^-} \mathcal{S}_{k,q}^{\lambda,\upsilon}(\xi, \Phi) = \mathcal{S}_k^{\lambda,\upsilon}(\xi, \Phi),$$

where

$$\mathcal{S}_k^{\lambda,\upsilon}(\xi, \Phi) = \left\{ f \in \mathcal{A} : 1 + \frac{1}{\xi}\left[\frac{z(\mathcal{T}^k(\rho,\beta)f(z))'}{(1-\upsilon)\mathcal{T}^k(\rho,\beta)f(z) + \upsilon z(\mathcal{T}^k(\rho,\beta)f(z))'} - 1\right] \prec \Phi(z) \right\},$$

for

$$k > 0, \quad \xi \in \mathbb{C}^*, \quad 0 \leq \upsilon < 1.$$

Remark 2.2. The class $S_{k,q}^{\lambda,\upsilon}(\xi, \Phi)$ unifies and extends several previously studied subclasses of analytic functions.

1. By taking the limit $q \to 1^-$, the q-analogue Carlson–Shaffer operator $T_q^k(\rho,\beta)$ reduces to the corresponding classical Carlson–Shaffer type operator. Consequently, the class $S_{k,q}^{\lambda,\upsilon}(\xi, \Phi)$ reduces to its classical counterpart.
2. Further, by choosing $\rho = 1$ and $\beta = 1$, the operator simplifies to a special case associated with the q-analogue of Bessel functions, and the corresponding coefficient estimates reduce to those obtained for related subclasses considered in [5].

Theorem 2.3. *Let* $f \in \mathcal{S}_{k,q}^{\lambda,\upsilon}(\xi,\Phi)$*, where* f *is given by* (1.1) *and* $\Phi(z) = 1 + B_1 z + B_2 z^2 + \cdots, z \in \mathbb{D}$ *satisfies Definition 2.1. Then, the following coefficient estimates hold:*

$$|a_2| \leq \frac{|\xi| B_1}{(1-\upsilon)|\mathrm{A}_2|}, \tag{2.1}$$

$$|a_3| \leq \frac{|\xi| B_1}{2(1-\upsilon)|\mathrm{A}_3|} \cdot \max\left\{1; \left|\frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon}\right|\right\}, \tag{2.2}$$

$$|a_4| \leq \frac{|\xi| B_1}{6(1-\upsilon)|A_4|}\left[2 + \frac{|\xi| B_1}{1-\upsilon}(3+4\upsilon) \cdot \max\left\{1, \left|\frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon}\right|\right\} + \frac{2|\xi|^2 B_1^2 (1+\upsilon)^2}{(1-\upsilon)^2}\right], \tag{2.3}$$

and

$$\begin{aligned} |a_5| \leq \frac{|\xi| B_1}{8(1-\upsilon)|A_5|} \Bigg[& 2 + \frac{2(2+3\upsilon)|\xi| B_1}{1-\upsilon} + \frac{(1+2\upsilon)|\xi| B_1}{4(1-\upsilon)}|k_1|^2 + \frac{(1+\upsilon)(1+2\upsilon)|\xi|^2 B_1^2}{(1-\upsilon)^2}|k_1| \\ & + \frac{|\xi|^2 B_1^2 (2+3\upsilon)(7+8\upsilon)}{2(1-\upsilon)^2} \cdot \max\left\{1, \left|\frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon}\right|\right\} + \frac{|\xi|^3 B_1^3 (1+\upsilon)^2 (7+10\upsilon)}{(1-\upsilon)^3}\Bigg], \end{aligned} \tag{2.4}$$

where A_n $(n = 2,3,4,5)$ *is defined by* (1.10).

Proof. Let $f \in \mathcal{S}_{k,q}^{\lambda,\upsilon}(\xi,\Phi)$. Then, by the principle of subordination, there exists a Schwarz function ψ, analytic in $\mathbb{D}$, satisfying $\psi(0) = 0$, and $|\psi(z)| < 1, (z \in \mathbb{D})$ such that

$$1 + \frac{1}{\xi}\left[\frac{z(\mathcal{T}_q^k(\rho,\beta) f(z))'}{(1-\upsilon)\mathcal{T}_q^k(\rho,\beta) f(z) + \upsilon z(\mathcal{T}_q^k(\rho,\beta) f(z))'} - 1\right] = \Phi(\psi(z)) \quad (z \in \mathbb{D}). \tag{2.5}$$

Define the function

$$\mathfrak{k}_1(z) = \frac{1+\psi(z)}{1-\psi(z)} = 1 + c_1 z + c_2 z^2 + \cdots, z \in \mathbb{D}, \tag{2.6}$$

which clearly belongs to the Carathéodory class $\mathcal{P}$.

Further, define

$$\begin{aligned} \mathfrak{H}(z) =& 1 + \frac{1}{\xi}\left[\frac{z(\mathcal{T}_q^k(\rho,\beta) f(z))'}{(1-\upsilon)\mathcal{T}_q^k(\rho,\beta) f(z) + \upsilon z(\mathcal{T}_q^k(\rho,\beta) f(z))'} - 1\right] \\ =& 1 + d_1 z + d_2 z^2 + \cdots, \quad (z \in \mathbb{D}). \end{aligned} \tag{2.7}$$

By combining (2.5) and (2.6), we get

$$\mathfrak{H}(z) = \Phi\left(\frac{\mathfrak{k}_1(z) - 1}{\mathfrak{k}_1(z) + 1}\right) \quad (z \in \mathbb{D}). \tag{2.8}$$

Using the series expansion

$$\frac{\mathfrak{k}_1(z) - 1}{\mathfrak{k}_1(z) + 1} = \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \left(c_3 + \frac{c_1^3}{4} - c_1 c_2\right) z^3 + \cdots\right],$$

we obtain

$$\begin{aligned}
&\Phi\left(\frac{\mathfrak{k}_1(z)-1}{\mathfrak{k}_1(z)+1}\right)\\
&=1+\frac{1}{2}B_1c_1z+\left[\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2\right]z^2+\cdots \quad (z\in\mathbb{D}).
\end{aligned}$$

From (2.8), we have

$$\begin{aligned}
d_1 =&\frac{1}{2}B_1c_1\\
d_2 =&\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2,\\
d_3 =&\frac{1}{2}B_1\left(c_3-c_1c_2+\frac{c_1^3}{4}\right)+\frac{1}{2}B_2\left(c_2-\frac{c_1^2}{2}\right)c_1+\frac{c_1^3}{8}B_3,\\
d_4 =&\frac{1}{2}B_1D+\frac{1}{4}B_2C+\frac{3B_3}{8}\left(c_1^2c_2-\frac{1}{2}c_1^4\right)+\frac{c_1^4}{16}B_4,
\end{aligned}$$

where

$$\begin{aligned}
C &= 2c_1c_3+c_2^2-3c_1^2c_2+\frac{3c_1^4}{4},\\
D &= c_4-c_1c_3-\frac{c_2^2}{2}+\frac{3c_1^2c_2}{4}-\frac{c_1^4}{8}.
\end{aligned}$$

On the other hand, by comparing the corresponding coefficients obtained from the series expansion of the operator $\mathcal{T}_q^k(\rho,\beta)$, we establish relationships between the coefficients d_n and a_n. Consequently, the coefficients a_2, a_3, a_4, and a_5 can be expressed in terms of c_1, c_2, c_3, and c_4 as follows:

$$a_2=\frac{\xi B_1c_1}{2(1-\upsilon)\mathrm{A}_2}, \tag{2.9}$$

$$a_3=\frac{\xi B_1}{4(1-\upsilon)\mathrm{A}_3}\left[c_2-\frac{c_1^2}{2}+\frac{B_2}{2B_1}c_1^2+\frac{\xi B_1(1+\upsilon)c_1^2}{2(1-\upsilon)}\right], \tag{2.10}$$

$$\begin{aligned}
a_4 =&\frac{\xi B_1}{6(1-\upsilon)A_4}\left[c_3-c_1c_2+\frac{c_1^3}{4}+\frac{B_2}{B_1}\left(c_2-\frac{c_1^2}{2}\right)c_1+\frac{B_3}{4B_1}c_1^3\right.\\
&\left.+\frac{\xi B_1c_1(3+4\upsilon)}{4(1-\upsilon)}\left(c_2-\frac{c_1^2}{2}+\frac{B_2}{2B_1}c_1^2+\frac{\xi B_1(1+\upsilon)}{2(1-\upsilon)}c_1^2\right)-\frac{\xi^2B_1^2(1+\upsilon)^2}{4(1-\upsilon)^2}c_1^3\right]
\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
a_5 =& \frac{\xi B_1}{8(1-\upsilon)A_5}\left\{c_4-c_1c_3-\frac{c_2^2}{2}+\frac{3c_1^2c_2}{4}-\frac{c_1^4}{8}+\frac{B_2}{2B_1}\left(2c_1c_3+c_2^2-3c_1^2c_2+\frac{3c_1^4}{4}\right)\right.\\
&+\frac{3B_3}{4B_1}\left(c_1^2c_2-\frac{1}{2}c_1^4\right)+\frac{B_4}{8B_1}c_1^4+\frac{(1+2\upsilon)\xi B_1}{4(1-\upsilon)}\left[k_1^2+\frac{(1+\upsilon)^2\xi^2B_1^2}{4(1-\upsilon)^2}c_1^4+\frac{(1+\upsilon)\xi B_1}{(1-\upsilon)}c_1^2k_1\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(2+3\upsilon)\xi B_1 c_1}{4(1-\upsilon)}\Bigg[c_3 - c_1c_2 + \frac{c_1^3}{4} + \frac{B_2}{B_1}\left(c_2 - \frac{c_1^2}{2}\right)c_1 + \frac{B_3}{4B_1}c_1^3 \\
&+\frac{\xi B_1 c_1(3+4\upsilon)}{4(1-\upsilon)}\left(k_1 + \frac{\xi B_1(1+\upsilon)}{2(1-\upsilon)}c_1^2\right) - \frac{\xi^2 B_1^2(1+\upsilon)^2}{4(1-\upsilon)^2}c_1^3\Bigg] \\
&-\frac{(1+\upsilon)(2+3\upsilon)\xi^2 B_1^2 c_1^2}{4(1-\upsilon)^2}\left(k_1 + \frac{\xi B_1(1+\upsilon)}{2(1-\upsilon)}c_1^2\right) + \frac{\xi^3 B_1^3(1+\upsilon)^3}{8(1-\upsilon)^3}c_1^4\Bigg\},
\end{aligned}
\tag{2.12}
$$

where $k_1 = c_2 - \frac{c_1^2}{2} + \frac{B_2}{2B_1}c_1^2$.

From (2.9) and Lemma 1.4, we have

$$
|a_2| = \frac{|\xi|B_1|c_1|}{|(1-\upsilon)\mathrm{A}_2|} \le \frac{|\xi|B_1}{(1-\upsilon)|\mathrm{A}_2|},
$$

which proves (2.1).

Next, from (2.10), we obtain

$$
\begin{aligned}
|a_3| &= \left|\frac{\xi B_1}{4(1-\upsilon)\mathrm{A}_3}\left[c_2 - \frac{c_1^2}{2} + \frac{B_2}{2B_1}c_1^2 + \frac{\xi B_1(1+\upsilon)c_1^2}{2(1-\upsilon)}\right]\right| \\
&= \left|\frac{\xi B_1}{4(1-\upsilon)\mathrm{A}_3}(c_2 - \eta_1 c_1^2)\right|,
\end{aligned}
$$

where

$$
\eta_1 = \frac{1}{2}\left[1 - \frac{B_2}{B_1} - \frac{\xi B_1(1+\upsilon)}{1-\upsilon}\right].
$$

Applying Lemma 1.4, we get

$$
|a_3| \le \frac{|\xi|B_1}{2(1-\upsilon)|\mathrm{A}_3|}\cdot\max\left\{1;\left|\frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon}\right|\right\},
$$

which establishes (2.2).

Rewriting (2.11), we have

$$
a_4 = \frac{\xi B_1}{6(1-\upsilon)A_4}\left[c_3 - \left(1 - \frac{B_2}{B_1}\right)c_1c_2 + \left(\frac{1}{4} - \frac{B_2}{2B_1} + \frac{B_3}{4B_1}\right)c_1^3 + \frac{\xi B_1(3+4\upsilon)c_1}{4(1-\upsilon)}\left(c_2 - \eta c_1^2\right) - \Psi c_1^3\right],
$$

where

$$
\eta = \frac{1}{2}\left(1 - \frac{B_2}{B_1} - \frac{\xi B_1(1+\upsilon)}{(1-\upsilon)}\right), \quad \Psi = \frac{\xi^2 B_1^2(1+\upsilon)^2}{4(1-\upsilon)^2}.
$$

Let $M = \frac{1}{2}\left(1 - \frac{B_2}{B_1}\right)$ and $N = \frac{1}{4} - \frac{B_2}{2B_1} + \frac{B_3}{4B_1}$.

Taking the modulus and using the triangle inequality on a_4, we obtain:

$$
|a_4| \le \frac{|\xi|B_1}{6(1-\upsilon)|A_4|}\left[|c_3 - Mc_1c_2 + Nc_1^3| + \frac{|\xi|B_1(3+4\upsilon)|c_1|}{4(1-\upsilon)}|c_2 - \eta c_1^2| + |\Psi||c_1|^3\right].
$$

By applying Lemmas 1.4 and 1.6, we obtain

$$|a_4| \leq \frac{|\xi|B_1}{6(1-\upsilon)A_4}\left[2 + \frac{|\xi|B_1}{(1-\upsilon)}(3+4\upsilon)\cdot \max\{1, |2\eta - 1|\} + 8|\Psi|\right].$$

Substituting the values of η and Ψ, we obtain

$$|a_4| \leq \frac{|\xi|B_1}{6(1-\upsilon)A_4}\left[2 + \frac{|\xi|B_1}{(1-\upsilon)}(3+4\upsilon)\cdot \max\left\{1, \left|\frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{(1-\upsilon)}\right|\right\} + \frac{2|\xi|^2 B_1^2(1+\upsilon)^2}{(1-\upsilon)^2}\right].$$

Proceeding similarly, taking the modulus on both sides of (2.12) and applying the triangle inequality, we obtain

$$|a_5| \leq \frac{|\xi|B_1}{8(1-\upsilon)|A_5|}\left(T_1 + T_2 + T_3 + T_4 + T_5\right),$$

where

$$\begin{aligned}
T_1 =& \left|c_4 - c_1c_3 - \frac{c_2^2}{2} + \frac{3}{4}c_1^2c_2 - \frac{1}{8}c_1^4 + \frac{B_2}{2B_1}\left(2c_1c_3 + c_2^2 - 3c_1^2c_2 + \frac{3}{4}c_1^4\right)\right.\\
&\left. + \frac{3B_3}{4B_1}\left(c_1^2c_2 - \frac{1}{2}c_1^4\right) + \frac{B_4}{8B_1}c_1^4\right|,\\
T_2 =& \frac{(1+2\upsilon)|\xi|B_1}{4(1-\upsilon)}\left|k_1^2 + \frac{(1+\upsilon)^2\xi^2B_1^2}{4(1-\upsilon)^2}c_1^4 + \frac{(1+\upsilon)\xi B_1}{1-\upsilon}c_1^2k_1\right|,\\
T_3 =& \frac{(2+3\upsilon)|\xi|B_1|c_1|}{4(1-\upsilon)}\left|c_3 - c_1c_2 + \frac{1}{4}c_1^3 + \frac{B_2}{B_1}\left(c_2 - \frac{1}{2}c_1^2\right)c_1 + \frac{B_3}{4B_1}c_1^3\right.\\
&\left. + \frac{\xi B_1(3+4\upsilon)}{4(1-\upsilon)}c_1\left(k_1 + \frac{\xi B_1(1+\upsilon)}{2(1-\upsilon)}c_1^2\right) - \frac{\xi^2B_1^2(1+\upsilon)^2}{4(1-\upsilon)^2}c_1^3\right|,\\
T_4 =& \frac{(1+\upsilon)(2+3\upsilon)|\xi|^2B_1^2}{4(1-\upsilon)^2}|c_1|^2\left|k_1 + \frac{\xi B_1(1+\upsilon)}{2(1-\upsilon)}c_1^2\right|,\\
T_5 =& \frac{|\xi|^3B_1^3(1+\upsilon)^3}{8(1-\upsilon)^3}|c_1|^4.
\end{aligned}$$

Now, by Lemma 1.7, provided the corresponding parameters satisfy its hypotheses, we have

$$T_1 \leq 2.$$

Moreover, Lemma 1.4 yields

$$|c_1| \leq 2, \qquad |c_1|^2 \leq 4, \qquad |c_1|^4 \leq 16.$$

Hence,

$$T_2 \leq \frac{(1+2\upsilon)|\xi|B_1}{4(1-\upsilon)}|k_1|^2 + \frac{(1+\upsilon)(1+2\upsilon)|\xi|^2B_1^2}{(1-\upsilon)^2}|k_1| + \frac{(1+\upsilon)^2(1+2\upsilon)|\xi|^3B_1^3}{(1-\upsilon)^3}.$$

Next, write

$$k_1 + \frac{\xi B_1(1+\upsilon)}{2(1-\upsilon)}c_1^2 = c_2 - \eta c_1^2,$$

where

$$\eta = \frac{1}{2} - \frac{B_2}{2B_1} - \frac{\xi B_1(1+\upsilon)}{2(1-\upsilon)}.$$

Applying Lemma 1.4, we obtain

$$\left|k_1 + \frac{\xi B_1(1+\upsilon)}{2(1-\upsilon)}c_1^2\right| \leq 2\Gamma,$$

where

$$\Gamma = \max\left\{1, \left|\frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon}\right|\right\}.$$

Furthermore, if the parameters in Lemma 1.6 satisfy

$$0 \leq M \leq 1, \qquad M(2M-1) \leq N \leq M,$$

then

$$|c_3 - 2Mc_1c_2 + Nc_1^3| \leq 2.$$

Consequently,

$$\begin{aligned} T_3 \leq & \frac{2(2+3\upsilon)|\xi|B_1}{1-\upsilon} + \frac{(2+3\upsilon)(3+4\upsilon)}{2}\frac{|\xi|^2B_1^2}{(1-\upsilon)^2}\Gamma \\ & + \frac{2(2+3\upsilon)(1+\upsilon)^2}{(1-\upsilon)^2}|\xi|^3B_1^3. \end{aligned}$$

Similarly,

$$T_4 \leq 2(1+\upsilon)(2+3\upsilon)\frac{|\xi|^2B_1^2}{(1-\upsilon)^2}\Gamma,$$

and

$$T_5 \leq 2\frac{|\xi|^3B_1^3(1+\upsilon)^3}{(1-\upsilon)^3}.$$

Combining the above estimates, we conclude that

$$\begin{aligned} |a_5| \leq & \frac{|\xi|B_1}{8(1-\upsilon)|A_5|}\left[2 + \frac{2(2+3\upsilon)|\xi|B_1}{1-\upsilon} + \frac{(1+2\upsilon)|\xi|B_1}{4(1-\upsilon)}|k_1|^2 + \frac{(1+\upsilon)(1+2\upsilon)|\xi|^2B_1^2}{(1-\upsilon)^2}|k_1|\right. \\ & \left. + \frac{|\xi|^2B_1^2(2+3\upsilon)(7+8\upsilon)}{2(1-\upsilon)^2} \cdot \max\left\{1, \left|\frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon}\right|\right\} + \frac{|\xi|^3B_1^3(1+\upsilon)^2(7+10\upsilon)}{(1-\upsilon)^3}\right]. \end{aligned}$$

This completes the proof. □

We now establish the following Fekete-Szegö-type inequality results.

Theorem 2.4. *Let $f \in \mathcal{S}_{k,q}^{\lambda,\upsilon}(\xi,\Phi)$, where f is given by* (1.1). *Then, for $\mu \in \mathbb{C}$, the Fekete-Szegö inequality holds:*

$$|a_3 - \mu a_2^2| \leq \frac{|\xi|B_1}{2(1-\upsilon)|\mathrm{A}_3|} \max\left\{1; \left|\frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon} - \frac{2\mu\xi B_1 \mathrm{A}_3}{(1-\upsilon)\mathrm{A}_2^2}\right|\right\},$$

where A_2 *and* A_3 *are given by* (1.10).

Proof. From (2.9) and (2.10), we obtain

$$a_3 - \mu a_2^2 = \frac{\xi B_1}{4(1-\upsilon)\mathrm{A}_3}(c_2 - \eta_2 c_1^2), \tag{2.13}$$

where

$$\eta_2 = \frac{1}{2}\left[1 - \frac{B_2}{B_1} - \frac{\xi B_1(1+\upsilon)}{1-\upsilon} + \frac{2\mu\xi B_1 \mathrm{A}_3}{(1-\upsilon)\mathrm{A}_2^2}\right]. \tag{2.14}$$

Applying Lemma 1.4 to estimate

$$|c_2 - \eta_2 c_1^2|,$$

we obtain

$$|a_3 - \mu a_2^2| \leq \frac{|\xi|B_1}{2(1-\upsilon)|\mathrm{A}_3|} \max\left\{1; \left|\frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon} - \frac{2\mu\xi B_1 \mathrm{A}_3}{(1-\upsilon)\mathrm{A}_2^2}\right|\right\}.$$

This completes the proof. □

The following corollary follows immediately from Theorem 2.4 by taking the limit $q \to 1^-$.

Corollary 2.5. *Let $f \in \mathcal{S}_{k,1}^{\lambda,\upsilon}(\xi,\Phi)$, where f is given by* (1.1). *Then, for $\mu \in \mathbb{C}$, the following Fekete-Szegö inequality holds:*

$$|a_3 - \mu a_2^2| \leq \frac{8|\xi|B_1(\lambda+1)_2(\beta)_2(k)_2}{3\gamma^2(1-\upsilon)(\rho)_2} \max\left\{1; \left|\frac{B_2}{B_1} + \frac{\xi B_1}{(1-\upsilon)}\left((1+\upsilon) - \frac{3\mu(\rho+1)(\lambda+1)\beta k}{2\rho(k+1)(\beta+1)(\lambda+2)}\right)\right|\right\}.$$

By employing a method analogous to the proof of Theorem 2.3 in conjunction with Lemma 1.5, we derive the following result.

Theorem 2.6. *Let $f \in \mathcal{S}_{k,q}^{\lambda,\upsilon}(\xi,\Phi)$, where f is given by* (1.1)*, with $\mu \in \mathbb{R}$ and $\xi > 0$. Then, the following Fekete-Szegö inequality holds:*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\xi B_1}{2(1-\upsilon)|\mathrm{A}_3|}\left[\frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon} - \frac{2\mu\xi B_1\mathrm{A}_3}{(1-\upsilon)\mathrm{A}_2^2}\right], & \text{if } \mu \leq x_1, \\ \frac{\xi B_1}{2(1-\upsilon)|\mathrm{A}_3|}, & \text{if } x_1 \leq \mu \leq x_2, \\ \frac{-\xi B_1}{2(1-\upsilon)|\mathrm{A}_3|}\left[\frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon} - \frac{2\mu\xi B_1\mathrm{A}_3}{(1-\upsilon)\mathrm{A}_2^2}\right], & \text{if } \mu \geq x_2, \end{cases}$$

where

$$x_1 = \frac{(1-\upsilon)\mathrm{A}_2^2}{2\xi B_1 \mathrm{A}_3}\left[-1 + \frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon}\right] \tag{2.15}$$

and

$$x_2 = \frac{(1-\upsilon)\mathrm{A}_2^2}{2\xi B_1 \mathrm{A}_3}\left[1 + \frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon}\right], \tag{2.16}$$

with A_2 *and* A_3 *given by* (1.10).

Proof. Proceeding as in the proof of Theorem 2.4, we obtain (2.13) and (2.14).

- From Lemma 1.5, we have
$$|c_2 - \eta c_1^2| \leq -4\eta + 2, \quad \text{if } \eta \leq 0.$$
Using (2.14), the condition $\eta \leq 0$ is equivalent to $\mu \leq x_1$. Substituting this estimate into (2.13), we obtain the first part of the theorem.
- For the case $0 \leq \eta \leq 1$, Lemma 1.5 gives
$$|c_2 - \eta c_1^2| \leq 2.$$
By (2.14), this corresponds to the range $x_1 \leq \mu \leq x_2$. Substituting into (2.13) yields the second estimate.
- Finally, for $\eta \geq 1$, Lemma 1.5 implies
$$|c_2 - \eta c_1^2| \leq 4\eta - 2.$$
Again, from (2.14), the condition $\eta \geq 1$ is equivalent to $\mu \geq x_2$. Substituting into (2.13) gives the third estimate.

Hence, the proof is complete. □

The following corollary follows from Theorem 2.6 by taking the limit $q \to 1^-$.

Corollary 2.7. *Let $f \in \mathcal{S}_{k,1}^{\lambda,\upsilon}(\xi, \Phi)$, where f is given by* (1.1)*, with $\mu, \gamma \in \mathbb{R}$ and $\xi > 0$. Then,*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{8|\xi|B_1(\lambda+1)_2(\beta)_2(k)_2}{3\gamma^2(1-\upsilon)(\rho)_2}\left[\frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon} - \frac{3\mu\xi B_1\beta k(\rho+1)(\lambda+1)}{2\rho(1-\upsilon)(k+1)(\beta+1)(\lambda+2)}\right], & \text{if } \mu \leq y_1, \\ \frac{8|\xi|B_1(\lambda+1)_2(\beta)_2(k)_2}{3\gamma^2(1-\upsilon)(\rho)_2}, & \text{if } y_1 \leq \mu \leq y_2, \\ \frac{-8|\xi|B_1(\lambda+1)_2(\beta)_2(k)_2}{3\gamma^2(1-\upsilon)(\rho)_2}\left[\frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon} - \frac{3\mu\xi B_1\beta k(\rho+1)(\lambda+1)}{2\rho(1-\upsilon)(k+1)(\beta+1)(\lambda+2)}\right], & \text{if } \mu \geq y_2, \end{cases}$$

where
$$y_1 = \frac{2\rho(1-\upsilon)(k+1)(\beta+1)(\lambda+2)}{3\xi k B_1\beta(\rho+1)(\lambda+1)}\left[-1 + \frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon}\right] \tag{2.17}$$
and
$$y_2 = \frac{2\rho(1-\upsilon)(k+1)(\beta+1)(\lambda+2)}{3\xi k B_1\beta(\rho+1)(\lambda+1)}\left[1 + \frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon}\right]. \tag{2.18}$$

The subsequent results are obtained by applying inequalities (1.15) and (1.16) through arguments analogous to those used in the proof of Theorem 2.3.

Theorem 2.8. *Let $f \in \mathcal{S}_{k,q}^{\lambda,\upsilon}(\xi, \Phi)$, where f is given by* (1.1)*, and let $\mu \in \mathbb{R}$ with $\xi > 0$. Then, the following Fekete-Szegö inequalities hold:*

(i) For $x_1 < \mu \leq x_3$,

$$|a_3 - \mu a_2^2| + \frac{(1-\upsilon)\mathrm{A}_2^2}{2\xi B_1\mathrm{A}_3}\left[1 - \frac{B_2}{B_1} - \frac{\xi B_1(1+\upsilon)}{1-\upsilon} + \frac{2\mu\xi B_1\mathrm{A}_3}{(1-\upsilon)\mathrm{A}_2^2}\right]|a_2|^2 \leq \frac{\xi B_1}{2(1-\upsilon)\mathrm{A}_3}. \tag{2.19}$$

(ii) For $x_3 < \mu \leq x_2$,

$$|a_3 - \mu a_2^2| + \frac{(1-\upsilon)\mathrm{A}_2^2}{2\xi B_1 \mathrm{A}_3}\left[1 + \frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon} - \frac{2\mu\xi B_1 \mathrm{A}_3}{(1-\upsilon)\mathrm{A}_2^2}\right]|a_2|^2 \leq \frac{\xi B_1}{2(1-\upsilon)\mathrm{A}_3}. \tag{2.20}$$

Here, x_1 *and* x_2 *are given by* (2.15) *and* (2.16)*, respectively and*

$$x_3 = \frac{(1-\upsilon)\mathrm{A}_2^2}{2\xi B_1 \mathrm{A}_3}\left[\frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon}\right],$$

where A_n $(n = 2, 3)$ *is defined by* (1.10).

Proof. Using the same procedure as in the proof of Theorem 2.3, we obtain Eqs (2.13) and (2.14). Moreover, from (2.9), we have

$$c_1 = \frac{2(1-\upsilon)\mathrm{A}_2}{\xi B_1} a_2. \tag{2.21}$$

(i) To prove (2.19), we apply inequality (1.15). Substituting (2.13), (2.14), and (2.21) into (1.15), we obtain inequality (2.19). Furthermore, the condition

$$0 < \eta_2 \leq \frac{1}{2}$$

is equivalent to

$$x_1 < \mu \leq x_3.$$

(ii) Similarly, applying inequality (1.16) together with (2.13), (2.14), and (2.21), we arrive at inequality (2.20). In this case, the condition

$$\frac{1}{2} \leq \eta_2 < 1$$

corresponds to

$$x_3 < \mu \leq x_2.$$

Hence, the proof is complete. □

The following corollary is obtained from Theorem 2.8 by taking the limit $q \to 1^-$.

Corollary 2.9. *Let* $f \in \mathcal{S}_{k,1}^{\lambda,\upsilon}(\xi, \Phi)$*, where* f *is given by* (1.1)*, and let* $\mu, \gamma \in \mathbb{R}$ *with* $\xi > 0$*. Then, the following Fekete–Szegö inequalities hold:*

(i) For $y_1 < \mu \leq y_3$,

$$\begin{aligned} &|a_3 - \mu a_2^2| \\ &\quad + \frac{2\rho(1-\upsilon)(k+1)(\beta+1)(\lambda+2)}{3\xi B_1 \beta k(\rho+1)(\lambda+1)}\left[1 - \frac{B_2}{B_1} - \frac{\xi B_1(1+\upsilon)}{1-\upsilon} + \frac{3\mu\xi B_1 \beta k(\rho+1)(\lambda+1)}{2\rho(1-\upsilon)(k+1)(\beta+1)(\lambda+2)}\right]|a_2|^2 \\ &\leq \frac{8\xi B_1(\lambda+1)_2(\beta)_2(k)_2}{3\gamma^2(1-\upsilon)(\rho)_2}. \end{aligned}$$

(ii) For $y_3 < \mu \leq y_2$,

$$\begin{aligned}
&|a_3 - \mu a_2^2| \\
&\quad + \frac{2\rho(1-\upsilon)(k+1)(\beta+1)(\lambda+2)}{3\xi B_1 \beta k(\rho+1)(\lambda+1)}\left[1 + \frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon} - \frac{3\mu\xi B_1\beta k(\rho+1)(\lambda+1)}{2\rho(1-\upsilon)(k+1)(\beta+1)(\lambda+2)}\right]|a_2|^2 \\
&\leq \frac{8\xi B_1(\lambda+1)_2(\beta)_2(k)_2}{3\gamma^2(1-\upsilon)(\rho)_2}.
\end{aligned}$$

Here, y_1 *and* y_2 *are given by* (2.17) *and* (2.18)*, respectively, and*

$$y_3 = \frac{2\rho(1-\upsilon)(k+1)(\beta+1)(\lambda+2)}{3\xi k B_1 \beta(\rho+1)(\lambda+1)}\left[\frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon}\right].$$

3. Applications involving functions associated with the Poisson distribution

In [45], Porwal investigated a power series whose coefficients are associated with the probabilities of the Poisson distribution (see also [46]). This consideration leads to the function

$$\mathfrak{L}_s(z) = z + \sum_{n=2}^{\infty} \frac{s^{n-1}}{(n-1)!} e^{-s} z^n \quad z \in \mathbb{D}, \quad (s > 0).$$

Motivated by this work, Srivastava and Porwal [14] introduced the linear operator $\mathfrak{J}^s : \mathcal{A} \longrightarrow \mathcal{A}$ defined by

$$\mathfrak{J}^s f(z) = \mathfrak{L}_s(z) \star f(z) = z + \sum_{n=2}^{\infty} \frac{s^{n-1}}{(n-1)!} e^{-s} a_n z^n \quad z \in \mathbb{D},$$

where $f \in \mathcal{A}$ is given by (1.1).

Definition 3.1. Let Φ satisfy the conditions stated in Definition 2.1. For fixed parameters $\xi \in \mathbb{C}^*$ and $0 \leq \upsilon < 1$, and for a function $g \in \mathcal{A}$, a function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}_{k,q}^{\lambda,\upsilon}(\xi; g; \Phi)$ if the convolution $f \star g$ belongs to the class $\mathcal{S}_{k,q}^{\lambda,\upsilon}(\xi, \Phi)$. Equivalently,

$$1 + \frac{1}{\xi}\left[\frac{z(\mathcal{T}_q^k(\rho,\beta)(f \star g)(z))'}{(1-\upsilon)\mathcal{T}_q^k(\rho,\beta)(f \star g)(z) + \upsilon z(\mathcal{T}_q^k(\rho,\beta)(f \star g)(z))'} - 1\right]$$

is analytic in the open unit disk $\mathbb{D}$ and satisfies the subordination condition

$$1 + \frac{1}{\xi}\left[\frac{z(\mathcal{T}_q^k(\rho,\beta)(f \star g)(z))'}{(1-\upsilon)\mathcal{T}_q^k(\rho,\beta)(f \star g)(z) + \upsilon z(\mathcal{T}_q^k(\rho,\beta)(f \star g)(z))'} - 1\right] \prec \Phi(z),$$

where $k > 0$, $\lambda > -1$ and $0 < q < 1$.

The class $\mathcal{S}_{k,q}^{\lambda,\upsilon}(\xi; \mathfrak{L}_s; \Phi)$ may be regarded as a special case of the more general class $\mathcal{S}_{k,q}^{\lambda,\upsilon}(\xi; g; \Phi)$, obtained by taking $g = \mathfrak{L}_s$. In this case, the defining condition reduces to $\mathfrak{J}^s f \in \mathcal{S}_{k,q}^{\lambda,\upsilon}(\xi; \Phi)$.

The following results are derived by applying Theorems 2.4 and 2.6 to the convolution function $f \star g$, defined in (1.3).

Theorem 3.2. *Let* $f \in \mathcal{S}_{k,q}^{\lambda,\upsilon}(\xi; g; \Phi)$, *where* f *is given by* (1.1), *and let* $g \in \mathcal{A}$ *be given by* (1.2) *with* $b_2, b_3 \neq 0$. *If* $\mu \in \mathbb{C}$, *then the following Fekete-Szegö inequality holds:*

$$|a_3 - \mu a_2^2| \leq \frac{|\xi| B_1}{2(1-\upsilon)|b_3|\mathrm{A}_3} \max\left\{1; \left|\frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon} - \frac{2\mu\xi B_1 b_3 \mathrm{A}_3}{(1-\upsilon)b_2^2\mathrm{A}_2^2}\right|\right\},$$

where A_n $(n = 2, 3)$ *is defined by* (1.10).

Proof. Since

$$f \in \mathcal{S}_{k,q}^{\lambda,\upsilon}(\xi; g; \Phi),$$

it follows from Definition 3.1 that

$$f \star g \in \mathcal{S}_{k,q}^{\lambda,\upsilon}(\xi, \Phi).$$

Now,

$$(f \star g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Applying Theorem 2.4 to the function $f \star g$, we obtain

$$|a_3 b_3 - \mu(a_2 b_2)^2| \leq \frac{|\xi| B_1}{2(1-\upsilon)\mathrm{A}_3} \max\left\{1; \left|\frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon} - \frac{2\mu\xi B_1 \mathrm{A}_3}{(1-\upsilon)\mathrm{A}_2^2}\right|\right\}.$$

Replacing

$$\mu \quad \text{by} \quad \frac{\mu b_3}{b_2^2},$$

and dividing both sides by $|b_3|$, we arrive at the required inequality. □

Theorem 3.3. *Let* $f \in \mathcal{S}_{k,q}^{\lambda,\upsilon}(\xi; g; \Phi)$, *where* f *is given by* (1.1), *and let* $g \in \mathcal{A}$ *be given by* (1.2) *with* $b_2, b_3 \neq 0$. *Furthermore, let* $\mu, B_2 \in \mathbb{R}$ *and* $\xi > 0$. *Then, the following Fekete-Szegö inequality holds:*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\xi B_1}{2(1-\upsilon)|b_3|\mathrm{A}_3}\left[\frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon} - \frac{2\mu\xi B_1 b_3\mathrm{A}_3}{(1-\upsilon)b_2^2\mathrm{A}_2^2}\right], & \text{if } \mu \leq x_1, \\ \frac{\xi B_1}{2(1-\upsilon)|b_3|\mathrm{A}_3}, & \text{if } x_1 \leq \mu \leq x_2, \\ \frac{-\xi B_1}{2(1-\upsilon)|b_3|\mathrm{A}_3}\left[\frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon} - \frac{2\mu\xi B_1 b_3\mathrm{A}_3}{(1-\upsilon)b_2^2\mathrm{A}_2^2}\right], & \text{if } \mu \geq x_2. \end{cases}$$

Here,

$$x_1 = \frac{(1-\upsilon)b_2^2\mathrm{A}_2^2}{2\xi B_1 b_3 \mathrm{A}_3}\left[-1 + \frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon}\right]$$

and

$$x_2 = \frac{(1-\upsilon)b_3^2\mathrm{A}_2^2}{2\xi B_1 b_3 \mathrm{A}_3}\left[1 + \frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon}\right],$$

where A_n $(n = 2, 3)$ *is defined by* (1.10).

Proof. The proof follows directly by applying Theorem 2.6 to the convolution function

$$(f \star g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n,$$

together with the substitutions

$$a_2 \mapsto a_2 b_2, \qquad a_3 \mapsto a_3 b_3$$

and

$$\mu \mapsto \frac{\mu b_2^2}{b_3}.$$

After simplification, the stated inequalities follow immediately. □

For the particular choice $g = \mathfrak{L}_s$, we have

$$b_2 = se^{-s} \quad \text{and} \quad b_3 = \frac{s^2}{2}e^{-s}.$$

Consequently, Theorems 3.2 and 3.3 yield the following results.

Theorem 3.4. *Let $f \in \mathcal{S}_{k,q}^{\lambda,\upsilon}(\xi; \mathfrak{L}_s; \Phi)$, where f is given by* (1.1)*, and let $\mu \in \mathbb{C}$. Then, the following Fekete-Szegö inequality holds:*

$$|a_3 - \mu a_2^2| \leq \frac{|\xi| B_1}{(1-\upsilon)s^2 e^{-s} \mathrm{A}_3} \max\left\{1; \left|\frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon} - \frac{\mu \xi B_1 \mathrm{A}_3}{(1-\upsilon)e^{-s}\mathrm{A}_2^2}\right|\right\},$$

where A_n $(n = 2, 3)$ is given by (1.10).

Proof. Substituting

$$b_2 = se^{-s} \qquad \text{and} \qquad b_3 = \frac{s^2}{2}e^{-s}$$

into the inequality obtained in Theorem 3.2, we immediately obtain the stated result. □

Theorem 3.5. *Let $f \in \mathcal{S}_{k,q}^{\lambda,\upsilon}(\xi; \mathfrak{L}_s; \Phi)$, where f is given by* (1.1)*, and let $\mu, B_2 \in \mathbb{R}$ with $\xi > 0$. Then, the following Fekete-Szegö inequality holds:*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\xi B_1}{2(1-\upsilon)s^2 e^{-s}\mathrm{A}_3}\left[\frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon} - \frac{\mu\xi B_1 \mathrm{A}_3}{(1-\upsilon)e^{-s}\mathrm{A}_2^2}\right], & \text{if } \mu \leq x_1^*, \\ \frac{\xi B_1}{(1-\upsilon)s^2 e^{-s}\mathrm{A}_3}, & \text{if } x_1^* \leq \mu \leq x_2^*, \\ \frac{-\xi B_1}{(1-\upsilon)s^2 e^{-s}\mathrm{A}_3}\left[\frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon} - \frac{\mu\xi B_1 \mathrm{A}_3}{(1-\upsilon)e^{-s}\mathrm{A}_2^2}\right], & \text{if } \mu \geq x_2^*. \end{cases}$$

Here,

$$x_1^* = \frac{(1-\upsilon)e^{-s}\mathrm{A}_2^2}{\xi B_1 \mathrm{A}_3}\left[-1 + \frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon}\right]$$

and

$$x_2^* = \frac{(1-\upsilon)e^{-s}\mathrm{A}_2^2}{\xi B_1 \mathrm{A}_3}\left[1 + \frac{B_2}{B_1} + \frac{\xi B_1(1+\upsilon)}{1-\upsilon}\right],$$

where A_n $(n = 2, 3)$ is given by (1.10).

Proof. The proof follows directly from Theorem 3.3, by substituting

$$b_2 = se^{-s} \qquad \text{and} \qquad b_3 = \frac{s^2}{2}e^{-s}.$$

□

4. Conclusions

In this study, we introduced a new linear operator associated with the q-analogue of the generalized Bessel function and the q-analogue of the Carlson-Shaffer operator. Using this operator, we established several Fekete-Szegö type inequalities for functions belonging to the class $\mathcal{S}_{k,q}^{\lambda,\upsilon}(\xi;\Phi)$. Furthermore, the obtained results were applied to subclasses of analytic functions associated with the Poisson distribution.

The results presented in this paper extend and unify several earlier investigations on coefficient estimates and Fekete-Szegö problems in geometric function theory. In particular, the derived inequalities provide new coefficient bounds for analytic and univalent function classes defined through convolution operators and subordination conditions.

The present work also suggests several possible directions for future research. One may investigate higher-order Hankel determinants, Toeplitz determinants, and coefficient estimates for related subclasses generated by other special functions or integral operators. Moreover, analogous problems involving fractional operators, (p,q)-calculus, or bi-univalent function classes may also be explored. These extensions may further enhance the applicability of the obtained results in geometric function theory and related areas of complex analysis.

Author contributions

Santosh Mandal: Conceptualization, methodology, validation, formal analysis, investigation, writing-original draft, writing-review and editing; Madan Mohan Soren: Conceptualization, methodology, validation, formal analysis, investigation, writing-original draft, writing-review and editing, visualization, supervision, project administration; K. R. Karthikeyan: Methodology, validation, formal analysis, writing-review and editing, project administration, funding acquisition; Dharmaraj Mohankumar: Methodology, validation, formal analysis, writing-original draft, writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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