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Research article

Transcendental entire solutions of some delay differential equations

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Abstract: The primary focus of this study was to analyze the following three delay differential equations:

$$\omega(z+1)-\omega(z-1)+a(z)\frac{\omega^{(k)}(z)}{\omega(z)}=R(z,\omega(z))=\frac{P(z,\omega)}{Q(z,\omega)},$$

$$\omega(z+1)-\omega(z-1)+a(z)(\frac{\omega'(z)}{\omega(z)})^k=R(z,\omega(z))=\frac{P(z,\omega)}{Q(z,\omega)}$$

and

$$\omega(z+1)+a(z)\frac{\omega^{(k)}(z)}{\omega(z)}=R(z,\omega(z))=\frac{P(z,\omega)}{Q(z,\omega)},$$

where $P(z,\omega)$ and $Q(z,\omega)$ were polynomials in ω with rational coefficients in z and no common roots, $a(z)$ was a rational function, and $k\geq 1$ was an integer. This paper investigated the forms of the transcendental entire solutions for the above equations. Some examples were given to support these results.

Keywords: delay differential equation; entire solutions; Painlevé equations; Nevanlinna theory
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1. Introduction and main results

In what follows, we assume the reader is familiar with the basic notations of Nevanlinna theory, such as the characteristic function $T(r,\omega)$, proximity function $m(r,\omega)$, and counting function $N(r,\omega)$, where ω is a meromorphic function (see [1, 2] for more details). As usual, we use $S(r,\omega)$ to denote any quantity satisfying $S(r, \omega) = o(T(r, \omega))$ as r tends to infinity, possibly outside an exceptional set of finite logarithmic measure. For a meromorphic function g, if $T(r, g) = S(r, \omega)$, we say g is small with respect to ω.

Similar to the fact that second order differential equations can be reduced to Painlevé equations, Halburd et al. [3, 4] proved that if the second order difference equation

$$\omega(z + 1) * \omega(z - 1) = R(z, \omega)$$

has finite order meromorphic solutions, then this difference equation reduces to a short list of canonical equations, including the difference Painlevé I-III equations, where the operation $*$ stands for either addition or multiplication, and $R(z, \omega)$ is rational in ω with small functions of ω as coefficients. The discrete (or difference) Painlevé equations have attracted many researchers; for example, see [3, 5, 6] and the references therein.

It is also worth noting that reductions of integrable differential-difference equations may give rise to delay differential equations with formal continuum limits to Painlevé equations. In [7], Quispel, Capel, and Sahadevan showed the equation

$$\omega(z)[\omega(z + 1) - \omega(z - 1)] + a\omega'(z) = b\omega(z), \tag{1.1}$$

where a and b are constants, can be obtained from the symmetry reduction of the Kac-Van Moerbeke equation, and has a formal continuous limit to the first Painlevé equation $\omega'' = 6\omega^2 + t$. In 2017, Halburd and Korhonen [8] considered an extended version of (1.1) and obtained the following result.

Theorem 1.1. [8] Let ω be a transcendental meromorphic solution of

$$\omega(z + 1) - \omega(z - 1) + a(z)\frac{\omega'(z)}{\omega(z)} = R(z, \omega(z)) = \frac{P(z, \omega(z))}{Q(z, \omega(z))}, \tag{1.2}$$

where $a(z)$ is rational in z, $P(z, \omega)$ is a polynomial in ω having rational coefficients in z, and $Q(z, \omega)$ is a polynomial in ω, and the roots of $Q(z, \omega)$ are nonzero rational functions of z and not the roots of $P(z, \omega)$. If $\rho_2(\omega) < 1$, then

$$\deg_\omega(Q) + 1 = \deg_\omega(P) \leq 3 \text{ or } \deg_\omega(R) \leq 1.$$

In particular, [9,10] respectively considered the entire function solutions of Eq (1.2). More recently, on the one hand, Cao, Chen, and Korhonen [11] considered meromorphic solutions of a higher order version of Eq (1.2), i.e.,

$$\omega(z + 1) - \omega(z - 1) + a(z)\frac{\omega^{(k)}(z)}{\omega(z)} = R(z, \omega(z)) = \frac{P(z, \omega(z))}{Q(z, \omega(z))}, \tag{1.3}$$

and obtained the following result. The growth condition for the solution ω is extended to the case of subnormal growth [12], which is also called minimal hyper-type [13].

Theorem 1.2. [11] Let $\omega(z)$ be a subnormal transcendental meromorphic solution of (1.3), where $a(z)$ is rational in z, $P(z, \omega)$ is a polynomial in ω having rational coefficients in z, and $Q(z, \omega)$ is either of degree zero or a polynomial in ω with roots that are nonzero rational functions of z and are not roots of $P(z, \omega)$.

(i) If $\deg_\omega(Q) \leq 1$, then $\deg_\omega(P) \leq \deg_\omega(Q) + k$;

(ii) If $\deg_\omega(Q) \geq 2$, then $\deg_\omega(Q) < \deg_\omega(P) \leq \deg_\omega(Q) + k$ and $\deg_\omega(P) \leq k + 3$.

In addition, if $\deg_\omega(P) = \deg_\omega(Q) + k$, then $\deg_\omega(Q) \leq 2$.

On the other hand, they considered the meromorphic solutions of the following equation

$$\omega(z+1) + a(z)\frac{\omega^{(k)}(z)}{\omega(z)} = R(z, \omega(z)) = \frac{P(z, \omega(z))}{Q(z, \omega(z))}, \tag{1.4}$$

where k is a positive integer.

Recently, related results on Theorems 1.1 and 1.2 have been obtained in [11, 14–18].

In 2022, Chen and Cao [14] considered replacing the logarithmic derivative term in Eq (1.2) with its k-th power, that is,

$$\omega(z+1) - \omega(z-1) + a(z)\left(\frac{\omega'(z)}{\omega(z)}\right)^k = R(z, \omega(z)) = \frac{P(z, \omega(z))}{Q(z, \omega(z))}, \tag{1.5}$$

where $k \geq 1$ is an integer.

In addition to the above studies on meromorphic solutions, there has also been significant interest in the qualitative properties of delay differential equations, including the exponential stability of time-delay systems with highly nonlinear impulses [19] and the existence of nonoscillatory solutions for higher-order nonlinear mixed neutral differential equations [20]. Motivated by these developments, together with the intrinsic connection between delay differential equations and Painlevé equations, the present paper aims to further investigate the function-theoretic properties of their solutions. Although meromorphic solutions have been studied in previous work, the conditions for the existence of transcendental entire solutions, as well as their explicit analytic representations, remain largely unexplored, particularly in the higher-order setting. This constitutes the central motivation and principal innovation of our study.

The main purpose of this paper is to find transcendental entire solutions of Eqs (1.3)–(1.5), and to determine under what conditions such solutions exist.

For Eq (1.3), we obtain the following theorem.

Theorem 1.3. Let $R(z, \omega(z)) \not\equiv 0$ be an irreducible rational function in $\omega(z)$ with rational coefficients in z, and let $a(z)$ be a rational function and $Q(z, 0) \not\equiv 0$. If $\omega(z)$ is a transcendental entire solution of (1.3), then $\deg_\omega(Q) = 0$ and $\deg_\omega(P) \leq 1$. Furthermore,

(i) If $P(z, \omega(z)) = a_0(z)$, where $a_0(z)$ is rational, then $\omega(z) = ue^{n\pi iz}$, $n \in \mathbb{Z} \setminus \{0\}$, where u is a nonzero constant.

(ii) If $P(z, \omega(z)) = a_1(z)\omega(z) + a_0(z)$, where $a_1(z)$ and $a_0(z)$ are rational such that $a_1(z) \not\equiv 0$, then $\omega(z) = H(z)e^{\lambda z}$, where λ is a nonzero constant, $H(z)$ is a polynomial, and

$$a_1(z) = \frac{e^\lambda H(z+1)}{H(z)} - \frac{e^{-\lambda} H(z-1)}{H(z)}.$$

Remark 1.1. Theorem 1.3 includes the main results of [9, Theorems 3.2, 3.6–3.8]. It is worth noting that the assumption $Q(z, 0) \not\equiv 0$ in our theorems is essential. It prevents $\omega = 0$ from being a root of the denominator, thus avoiding degenerate cases where the rational function degree artificially reduces or unwanted poles are introduced when evaluating entire solutions. The necessity of this condition is illustrated by the following example.

Example 1.1. The equation

$$\omega(z+1) - \omega(z-1) - \frac{1}{\pi i}\frac{\omega'(z)}{\omega(z)} = \frac{2z - \frac{1}{\pi i}}{\omega(z)}$$

has an entire solution $\omega(z) = e^{2\pi iz} + z$.

When the term $\frac{\omega^{(k)}(z)}{\omega(z)}$ is replaced by $\left(\frac{\omega'(z)}{\omega(z)}\right)^k$, the same result as in Theorem 1.3 can be obtained by a proof similar to that of Theorem 1.3, with the only difference being the expression for $a_0(z)$. We have the following theorem.

Theorem 1.4. Let $R(z, \omega(z))(\not\equiv 0)$ be an irreducible rational function in $\omega(z)$ with rational coefficients in z, and let $a(z)$ be a rational function and $Q(z, 0) \not\equiv 0$. If $\omega(z)$ is a transcendental entire solution of (1.5), then $\deg_\omega(Q) = 0$ and $\deg_\omega(P) \leq 1$. Furthermore,

(i) If $P(z, \omega(z)) = a_0(z)$, where $a_0(z)$ is rational, then $\omega(z) = ue^{n\pi iz}$, $n \in \mathbb{Z} \setminus \{0\}$, where u is a nonzero constant.

(ii) If $P(z, \omega(z)) = a_1(z)\omega(z) + a_0(z)$, where $a_1(z)$ and $a_0(z)$ are rational functions such that $a_1(z) \not\equiv 0$, then $\omega(z) = H(z)e^{\lambda z}$, where λ is a nonzero constant, $H(z)$ is a polynomial, and

$$a_1(z) = \frac{e^\lambda H(z+1)}{H(z)} - \frac{e^{-\lambda}H(z-1)}{H(z)}, a_0(z) = a(z)\left(\frac{H'(z)}{H(z)} + \lambda\right)^k.$$

For Eq (1.4), we obtain the following theorem.

Theorem 1.5. Let $R(z, \omega(z))(\not\equiv 0)$ be an irreducible rational function of $\omega(z)$ with rational coefficients in z, let $a(z)$ be a rational function, and suppose $Q(z, 0) \not\equiv 0$. If $\omega(z)$ is a transcendental entire solution of (1.4), then $\deg_\omega(Q) = 0$ and $P(z, \omega(z)) = a_1(z)\omega(z) + a_0(z)$, where $a_1(z)(\not\equiv 0)$ and $a_0(z)$ are rational functions. Furthermore, $\omega(z) = G(z)e^{\mu z}$, where μ is a nonzero constant and $G(z)$ is a polynomial, such that $a_1(z) = \frac{e^\mu G(z+1)}{G(z)}$.

Remark 1.2. Similarly, when the term $\frac{\omega^{(k)}(z)}{\omega(z)}$ is replaced by $\left(\frac{\omega'(z)}{\omega(z)}\right)^k$, the same structural result as Theorem 1.5 can be obtained. The key difference lies in the resulting coefficient $a_0(z)$, which becomes $a_0(z) = a(z)\left(\frac{G'(z)}{G(z)} + \mu\right)^k$. We have the following result.

Theorem 1.6. Let $R(z, \omega(z))(\not\equiv 0)$ be an irreducible rational function of $\omega(z)$ with rational coefficients in z, let $a(z)$ be a rational function, and suppose $Q(z, 0) \not\equiv 0$. If $\omega(z)$ is a transcendental entire solution of

$$\omega(z+1) + a(z)\left(\frac{\omega'(z)}{\omega(z)}\right)^k = R(z, \omega(z)) = \frac{P(z, \omega(z))}{Q(z, \omega(z))}, \tag{1.6}$$

then $\deg_\omega(Q) = 0$ and $P(z, \omega(z)) = a_1(z)\omega(z) + a_0(z)$, where $a_1(z)(\not\equiv 0)$ and $a_0(z)$ are rational functions. Furthermore, $\omega(z) = G(z)e^{\mu z}$, where μ is a nonzero constant and $G(z)$ is a polynomial, such that

$$a_1(z) = \frac{e^\mu G(z+1)}{G(z)},\ a_0(z) = a(z)\left(\frac{G'(z)}{G(z)} + \mu\right)^k.$$

Remark 1.3. Unlike Theorems 1.1 and 1.2, which assume either $\rho_2(\omega) < 1$ or ω is subnormal (i.e., $\limsup_{r\to\infty} \frac{\log T(r,\omega)}{r} = 0$), Theorems 1.3–1.6 do not impose such assumptions. Consequently, our results are independent of those theorems and provide explicit forms of entire solutions to Eqs (1.3)–(1.6).

Some examples are given below to illustrate our results.

Example 1.2. $\omega(z) = e^{2\pi iz}$ satisfies the equation $\omega(z+1) - \omega(z-1) + \frac{z+1}{z-1}\frac{\omega''(z)}{\omega(z)} = -4\pi^2\frac{z+1}{z-1}$. This is the situation (i) of Theorem 1.3.
Example 1.3. $\omega(z) = ze^z$ satisfies the equation $\omega(z+1) - \omega(z-1) + \frac{1}{z-i}\frac{\omega'(z)}{\omega(z)} = \frac{e(z+1)-e^{-1}(z-1)}{z}\omega(z) + \frac{z+1}{z(z-i)}$. This is the situation (ii) of Theorem 1.3.
Example 1.4. $\omega(z) = 2e^{4\pi iz}$ satisfies the equation $\omega(z+1) - \omega(z-1) + \frac{z^2}{z-i}(\frac{\omega'(z)}{\omega(z)})^3 = -64\pi^3 i\frac{z^2}{z-i}$. This is the situation (i) of Theorem 1.4.
Example 1.5. The equation $\omega(z+1) - \omega(z-1) + \frac{(z-i)^2}{z}(\frac{\omega'(z)}{\omega(z)})^2 = \frac{e(z+1-i)-e^{-1}(z-1-i)}{z-i}\omega(z) + \frac{(z+1-i)^2}{z}$ has a solution $\omega(z) = (z-i)e^{z+1}$. This is the situation (ii) of Theorem 1.4.
Example 1.6. $\omega(z) = (z+1)e^z$ satisfies the equation $\omega(z+1) + \frac{z+i}{z}\frac{\omega''(z)}{\omega(z)} = \frac{e(z+2)}{z+1}\omega(z) + \frac{(z+i)(z+3)}{z(z+1)}$. This is the conclusion of Theorem 1.5.
Example 1.7. The equation $\omega(z+1) + (\frac{\omega'(z)}{\omega(z)})^2 = \frac{ez}{z-1}\omega(z) + (\frac{z}{z-1})^2$ has a solution $\omega(z) = (z-1)e^z$. This is the conclusion of Theorem 1.6.

2. Auxiliary results

The Valiron-Mohon'ko identity is a useful tool to estimate the characteristic function of rational functions. Its proof can be found in [21, Theorem 2.2.5].
Lemma 2.1. [21, Theorem 2.2.5] Let ω be a meromorphic function. For any irreducible rational function $R(z,\omega)$ in ω,

$$R(z,\omega) = \frac{P(z,\omega)}{Q(z,\omega)} = \frac{\sum_{i=0}^{p} a_i(z)\omega^i}{\sum_{j=0}^{q} b_j(z)\omega^j}$$

with meromorphic coefficients $a_i(z)$, $b_j(z)$ such that $a_i(z)$ and $b_j(z)$ are small with respect to ω. Then, the Nevanlinna characteristic function of $R(z,\omega(z))$ satisfies

$$T(r, R(z,\omega)) = \deg_\omega(R)T(r,\omega) + S(r,\omega),$$

where $\deg_\omega(R) = \max\{p,q\}$ is the degree of $R(z,\omega)$ in ω.
Lemma 2.2. [2, Theorem 1.51] Let $f_j(z)(j = 1,...,n)(n \geqslant 2)$ be meromorphic functions, and let $g_j(z)(j = 1,...,n)$ be entire functions satisfying
(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;
(ii) when $1 \leqslant i < k \leqslant n$, then $g_i(z) - g_k(z)$ is not a constant;
(iii) when $1 \leqslant j \leqslant n, 1 \leqslant h < k \leqslant n$, then

$$T(r, f_j) = o(T(r, e^{g_h - g_k}))(r \to \infty, r \notin E),$$

where $E \subset (1,\infty)$ is of finite linear measure or logarithmic measure;
Then, $f_j(z) \equiv 0(j = 1,...,n)$.
Lemma 2.3. [2, Theorem 1.56] Let $f_j(z)(j = 1,2,3)$ be meromorphic functions, and $f_1(z)$ be not a constant, satisfying $\sum_{j=1}^{3} f_j \equiv 1$ and

$$\sum_{j=1}^{3} N\left(r, \frac{1}{f_j}\right) + 2\sum_{j=1}^{3} \overline{N}(r, f_j) < (\lambda + o(1))T(r),$$

where $\lambda < 1$ and $T(r) = \max_{1\leq j\leq 3}\left\{T(r, f_j)\right\}$. Then, $f_2(z) \equiv 1$ or $f_3(z) \equiv 1$.

Lemma 2.4. [9, Lemma 2.10] Let $\omega(z) = H(z)e^{h(z)}$, where $H(z)$ is a nonzero small function of $e^{h(z)}$, and $P(z, \omega)$ and $Q(z, \omega)$ be polynomials in ω with coefficients being small functions of ω and $Q(z, 0) \not\equiv 0$. If $\deg_\omega(P) = p \leq \deg_\omega(Q) = q$, then

$$m\left(r, \frac{P(z, \omega)}{Q(z, \omega)}\right) = S(r, e^h) = S(r, \omega).$$

Lemma 2.5. Let ω be a transcendental entire solution of the following equation

$$\omega(z + 1) - \omega(z - 1) + a(z)\frac{\omega^{(k)}(z)}{\omega(z)} = a_0(z), \tag{2.1}$$

where $a(z)$ and $a_0(z)$ are rational functions. Then $\omega(z) = ue^{n\pi iz}$, where u is a nonzero constant and $n \in \mathbb{Z} \setminus \{0\}$.

Proof. If $\omega(z)$ has infinitely many zeros, then $a(z)\frac{\omega^{(k)}(z)}{\omega(z)}$ has infinitely many poles, but $\omega(z + 1) - \omega(z - 1) - a_0(z)$ has only finitely many poles. This is a contradiction, so $\omega(z)$ can be written as

$$\omega(z) = H(z)e^{g(z)}, \tag{2.2}$$

where $H(z)$ is a polynomial, and $g(z)$ is a non-constant entire function. Through calculation, we can obtain

$$\frac{d^k}{dz^k}\left[H(z)e^{g(z)}\right] = e^{g(z)} \sum_{j=0}^{k} \binom{k}{j} H^{(k-j)}(z) \sum_{m=1}^{j} \frac{j!}{m!} \sum_{\substack{\ell_1+\ell_2+\cdots+\ell_m=j \\ \ell_i \geq 1}} \prod_{i=1}^{m} \frac{g^{(\ell_i)}(z)}{\ell_i!}.$$

Therefore, $w^{(k)} = L(z, g')e^g$, where $L(z, g')$ is a differential polynomial in g' with polynomial coefficients, and $L(z, g')$ has a unique highest-degree term $(g')^k$.

Equation (2.1) can be written as $\omega(z + 1) = \omega(z - 1) + A(z)$, where $A(z) = a_0(z) - a(z)\frac{\omega^{(k)}(z)}{\omega(z)}$. From the logarithmic derivative lemma, we can obtain $T(r, \frac{\omega^{(k)}}{\omega}) = T(r, \frac{L(z,g')}{H(z)}) \leq O(T(r, g)) + O(\log r)$. By [5, Lemma 1], we deduce that A is a small function of $\omega(z - 1)$. We claim that $A(z) \equiv 0$. If $A(z) \not\equiv 0$, then from the second fundamental theorem we can obtain

$$\begin{aligned} T(r, \omega(z-1)) \;&\leq\; N(r, \omega(z-1)) + N\left(r, \frac{1}{\omega(z-1)}\right) \\ &+\; N\left(r, \frac{1}{\omega(z-1) + A(z)}\right) + S(r, \omega(z-1)) \\ &\leq\; N\left(r, \frac{1}{\omega(z+1)}\right) + S(r, \omega(z-1)). \end{aligned}$$

Since ω has only finitely many zeros, this leads to a contradiction. Hence, $A(z) \equiv 0$.

$A(z) = a_0(z) - a(z)\frac{\omega^{(k)}(z)}{\omega(z)} \equiv 0$ implies $aL(z, g') = a_0H$. Assume that g' is a transcendental entire function. By applying the differential version of the Clunie Lemma [22] to $aL(z, g') = a_0H$, we obtain $m(r, g') = S(r, g)$, which is a contradiction. Thus, g is a polynomial.

Taking (2.2) into (2.1), we get $H(z + 1)e^{g(z+1)} - H(z - 1)e^{g(z-1)} = 0$, which is

$$e^{g(z+1)-g(z-1)} = \frac{H(z - 1)}{H(z + 1)}. \tag{2.3}$$

Since H is a polynomial, from Eq (2.3), we can conclude that $g(z+1)-g(z-1)$ and $\frac{H(z-1)}{H(z+1)}$ are constants. At this point, H is a constant. Suppose $g(z+1)-g(z-1)=c$; then $g(z)=\frac{c}{2}z+d$, where c and d are constants such that $c\neq 0$. Substituting $g(z)=\frac{c}{2}z+d$ into Eq (2.3) yields $e^c=1$. We deduce that $c=2n\pi i$, $n\in\mathbb{Z}\setminus\{0\}$. At this time, there is $\omega(z)=He^{n\pi iz+d}=ue^{n\pi iz}$, where u is a nonzero constant.

Lemma 2.6. Let ω be a transcendental entire solution of the following equation

$$\omega(z+1)-\omega(z-1)+a(z)\frac{\omega^{(k)}(z)}{\omega(z)}=a_1(z)\omega(z)+a_0(z), \tag{2.4}$$

where $a(z)$, $a_0(z)$, and $a_1(z)(\not\equiv 0)$ are rational functions. Then $\omega(z)=H(z)e^{\lambda z}$, where λ is a nonzero constant, $H(z)$ is a polynomial, and

$$a_1(z)=\frac{e^{\lambda}H(z+1)}{H(z)}-\frac{e^{-\lambda}H(z-1)}{H(z)}.$$

Proof. If $\omega(z)$ has infinitely many zeros, then $a(z)\frac{\omega^{(k)}(z)}{\omega(z)}$ has infinitely many poles, but $\omega(z+1)-\omega(z-1)-a_1(z)\omega(z)-a_0(z)$ has only finitely many poles, which is a contradiction, so $\omega(z)$ can be written as

$$\omega(z)=H(z)e^{g(z)}, \tag{2.5}$$

where $H(z)$ is a polynomial, and $g(z)$ is a non-constant entire function. Taking (2.5) into (2.4), we get

$$H(z+1)e^{g(z+1)}-H(z-1)e^{g(z-1)}-a_1(z)H(z)e^{g(z)}=a_0(z)-a(z)\frac{L(z,g')}{H(z)}, \tag{2.6}$$

where $L(z,g')$ is the same as in Lemma 2.5.

If $a_0(z)-a(z)\frac{L(z,g')}{H(z)}\not\equiv 0$, then (2.6) can be written as

$$\frac{1}{F}f_1+\frac{1}{F}f_2+\frac{1}{F}f_3=1,$$

where

$$F(z)=a_0(z)-a(z)\frac{L(z,g')}{H(z)},\quad f_1(z)=H(z+1)e^{g(z+1)},$$
$$f_2(z)=-H(z-1)e^{g(z-1)},\quad \text{and}\quad f_3(z)=-a_1(z)H(z)e^{g(z)}.$$

Applying Lemma 2.3 to the above equation yields $\frac{1}{F}f_1\equiv 1$ or $\frac{1}{F}f_2\equiv 1$.

When $f_1\equiv F$, $H(z+1)e^{g(z+1)}\equiv a_0(z)-a(z)\frac{L(z,g')}{H(z)}$ implies that $T(r,\omega(z+1))=S(r,\omega)$. At the same time, $f_2=-f_3$ yields $H(z-1)e^{g(z-1)}=-a_1(z)H(z)e^{g(z)}$. Therefore, $T(r,\omega)=T(r,\omega(z+1))+O(\log r)=S(r,\omega)$, which is impossible. The case when $f_2\equiv F$ can be similarly handled as the case when $f_1\equiv F$, leading to a contradiction.

In conclusion, we obtain $a_0(z)-a(z)\frac{L(z,g')}{H(z)}\equiv 0$. By applying the same method as in Lemma 2.5, one can obtain that g is a polynomial. From Eq (2.6), we can obtain

$$H(z+1)e^{g(z+1)}-H(z-1)e^{g(z-1)}-a_1(z)H(z)e^{g(z)}=0. \tag{2.7}$$

If $g(z+1)-g(z-1)$, $g(z+1)-g(z)$, and $g(z-1)-g(z)$ are not constants, applying Lemma 2.2 to Eq (2.7) yields $H(z+1)=H(z-1)\equiv 0$, which is a contradiction. Therefore, at least one of

$g(z+1)-g(z-1)$, $g(z+1)-g(z)$, and $g(z-1)-g(z)$ is a constant. Without loss of generality, let it be $g(z+1)-g(z-1)=c$, where c is a constant. The derivation for other cases is similar and will not be elaborated here. Substituting $g(z+1)=g(z-1)+c$ into Eq (2.7) gives

$$(e^{c}H(z+1)-H(z-1))e^{g(z-1)}-a_1(z)H(z)e^{g(z)}=0. \tag{2.8}$$

We assert that $g(z-1)-g(z)$ is a constant. Otherwise, applying Lemma 2.2 to the Eq (2.8) leads to $a_1(z)H(z)\equiv 0$, which is a contradiction. Assumption $g(z-1)-g(z)=c_1$, c_1 is a constant. Substitute $g(z-1)=g(z)+c_1$ into Eq (2.8) to obtain

$$(e^{c+c_1}H(z+1)-e^{c_1}H(z-1)-a_1(z)H(z))e^{g(z)}=0. \tag{2.9}$$

From (2.9), we get $e^{c+c_1}H(z+1)-e^{c_1}H(z-1)=a_1(z)H(z)$. Now, from $g(z+1)-g(z-1)=c$ and $g(z-1)-g(z)=c_1$, we obtain $g(z+1)-g(z)=c+c_1$. Since $g(z)$ is a polynomial, then $g(z)=(c+c_1)z+d$, where d is a constant. Substituting $g(z)=(c+c_1)z+d$ into $g(z-1)-g(z)=c_1$ implies $c_1=-\frac{c}{2}$, and $c+c_1=\frac{c}{2}$. At this point, $g(z)=\frac{c}{2}z+d$. Let $\lambda=\frac{c}{2}$. Then, we obtain $\omega(z)=H(z)e^{\lambda z+d}$, where λ is a nonzero constant. From Eq (2.9) we get that

$$a_1(z)=\frac{e^{\lambda}H(z+1)}{H(z)}-\frac{e^{-\lambda}H(z-1)}{H(z)}.$$

Lemma 2.7. Let $a(z)$ and $a_0(z)$ be rational functions. Then, equation

$$\omega(z+1)+a(z)\frac{\omega^{(k)}(z)}{\omega(z)}=a_0(z) \tag{2.10}$$

has no transcendental entire solution.

Proof. Let $\omega(z)$ be a transcendental entire solution of the Eq (2.10). If $\omega(z)$ has infinitely many zeros, then $a(z)\frac{\omega^{(k)}(z)}{\omega(z)}$ has infinitely many poles, but $\omega(z+1)-a_0(z)$ has only finitely many poles. This is a contradiction, so $\omega(z)$ can be written as $\omega(z)=H(z)e^{g(z)}$, where H is a polynomial, and $g(z)$ is a non-constant entire function. Taking $\omega(z)=H(z)e^{g(z)}$ into (2.10), we get

$$H(z+1)e^{g(z+1)}+a(z)\frac{L(z,g')}{H(z)}=a_0(z), \tag{2.11}$$

where $L(z,g')$ is the same as in Lemma 2.5. Therefore, $T(r,a_0-a\frac{L(z,g')}{H})=O(T(r,g))+O(\log r)=S(r,e^{g(z+1)})$. Applying Lemma 2.2 to (2.11), we get that $H(z+1)\equiv 0$, which is a contradiction.

Lemma 2.8. Let $a(z)$ and $a_0(z)$ be rational functions. Then, equation

$$\omega(z+1)+a(z)\left(\frac{\omega'(z)}{\omega(z)}\right)^{k}=a_0(z) \tag{2.12}$$

has no transcendental entire solution.

Proof. Let $\omega(z)$ be a transcendental entire solution of the Eq (2.12). An argument analogous to that in Lemma 2.7 yields $\omega(z)=H(z)e^{g(z)}$, where H is a polynomial, and $g(z)$ is a non-constant entire function. Taking $\omega(z)=H(z)e^{g(z)}$ into (2.12), we get

$$H(z+1)e^{g(z+1)}+a(z)\left(\frac{H'(z)}{H(z)}+g'(z)\right)^{k}=a_0(z). \tag{2.13}$$

Therefore, $T\left(r,a_0-a\left(\frac{H'(z)}{H(z)}+g'(z)\right)^{k}\right)=O(T(r,g))+O(\log r)=S(r,e^{g(z+1)})$. Applying Lemma 2.2 to (2.13), we get that $H(z+1)\equiv 0$, which is a contradiction.

3. Proofs of Theorems

Proof of Theorem 1.3. The proof is divided into two parts, namely, $\deg_\omega(Q) = 0$ and $\deg_\omega(Q) \geq 1$. When $\deg_\omega(Q) = 0$ and $\deg_\omega(P) \leq 1$, Lemmas 2.5 and 2.6 give the conclusions (i) and (ii) of Theorem 1.3. Now we will consider the remaining scenarios: the case where $\deg_\omega(Q) = 0$ and $\deg_\omega(P) \geq 2$, as well as the case where $\deg_\omega(Q) \geq 1$.

First, we consider the case that $\deg_\omega(Q) = 0$, $\deg_\omega(P) = p \geq 2$. Then, Eq (1.3) becomes

$$\omega(z+1) - \omega(z-1) + a(z)\frac{\omega^{(k)}(z)}{\omega(z)} = \sum_{i=0}^{p} a_i(z)\omega^i(z), \tag{3.1}$$

where $a(z)$, $a_0(z)$, ... , $a_p(z)$ are rational functions with $a_p(z) \not\equiv 0$. Let $\omega(z)$ be a transcendental entire solution of the above equation. If $\omega(z)$ has infinitely many zeros, then $a(z)\frac{\omega^{(k)}(z)}{\omega(z)}$ has infinitely many poles, but $\omega(z+1) - \omega(z-1) - \sum_{i=0}^{p} a_i(z)\omega^i(z)$ has only finitely many poles, which is a contradiction, so $\omega(z)$ can be written as

$$\omega(z) = H(z)e^{g(z)}, \tag{3.2}$$

where $H(z)$ is a polynomial, and $g(z)$ is a non-constant entire function. Taking (3.2) into (3.1), we get

$$H(z+1)e^{g(z+1)} - H(z-1)e^{g(z-1)} - \sum_{i=1}^{p} a_i(z)H^i(z)e^{ig(z)} = a_0(z) - a(z)\frac{L(z,g')}{H(z)}, \tag{3.3}$$

where $L(z, g')$ is the same as in Lemma 2.5.

We will divide the proof into the following cases.

Case 1. None of $g(z+1)-g(z-1)$, $g(z+1)-ig(z)$ and $g(z-1)-ig(z)$ (for $i = 1, \ldots, p$) is a constant.

Since $a(z)$, $a_0(z)$, $a_1(z)$, ... , $a_p(z)$ are rational functions, $H(z)$ is a polynomial, and $g(z)$ is a non-constant entire function, applying Lemma 2.2 to Eq (3.3) leads to $H(z+1) = H(z-1) \equiv 0$, which is a contradiction.

Case 2. $g(z+1)-g(z) = c$, where c is a constant. Substitute $g(z+1) = g(z)+c$ and $g(z-1) = g(z)-c$ into Eq (3.3) to obtain

$$(e^c H(z+1) - e^{-c}H(z-1))e^{g(z)} + a(z)\frac{L(z,g')}{H(z)} = \sum_{i=0}^{p} a_i(z)H^i(z)e^{ig(z)}. \tag{3.4}$$

From Eq (3.4) and Lemma 2.1, we get

$$\begin{aligned}
pT(r,e^g) &= T(r, \sum_{i=0}^{p} a_i H^i e^{ig}) + S(r,e^g)\\
&= T\left(r, (e^c H(z+1) - e^{-c}H(z-1))e^g + a\frac{L(z,g')}{H(z)}\right) + S(r,e^g)\\
&\leq T(r,e^g) + S(r,e^g),
\end{aligned}$$

which implies that $p \leq 1$. This is in contradiction to $p \geq 2$.

Case 3. $g(z+1) - i_0 g(z) = c$, where c is a constant, $2 \leq i_0 \leq p$.

Substitute $g(z+1)=i_0g(z)+c$ and $g(z-1)=\frac{1}{i_0}g(z)-\frac{c}{i_0}$ into Eq (3.3) to obtain

$$H(z+1)e^{i_0g(z)+c}-H(z-1)e^{\frac{1}{i_0}g(z)-\frac{c}{i_0}}-\sum_{i=0}^{p}a_i(z)H^i(z)e^{ig(z)}=-a(z)\frac{L(z,g')}{H(z)}. \tag{3.5}$$

Applying Lemma 2.2 to Eq (3.5) implies $H(z-1)\equiv 0$, which is impossible.

Case 4. $g(z-1)-g(z)$ is a constant. Using the same method as in Case 2, we obtain a contradiction.

Case 5. $g(z-1)-i_0g(z)$ is a constant, $2\le i_0\le p$. Using the same method as in Case 3, we obtain a contradiction.

Case 6. $g(z+1)-g(z-1)=c$, where c is a constant. Substituting $g(z+1)=g(z-1)+c$ into Eq (3.3) gives

$$(e^cH(z+1)-H(z-1))e^{g(z-1)}=\sum_{i=0}^{p}a_i(z)H^i(z)e^{ig(z)}-a(z)\frac{L(z,g')}{H(z)}. \tag{3.6}$$

Applying Lemma 2.2 to Eq (3.6) leads to $g(z-1)=pg(z)+c_1$ and $a(z)\frac{L(z,g')}{H(z)}-a_0(z)\equiv 0$, where c_1 is a constant.

If g is a transcendental entire function, then $a(z)\frac{L(z,g')}{H(z)}-a_0(z)\not\equiv 0$, which is a contradiction. If g is a polynomial, then $p=1$ since $g(z-1)=pg(z)+c_1$, which is a contradiction to $p\ge 2$.

Next, we consider the case that $\deg_\omega(Q)\ge 1$. Then, Eq (1.3) can be written as

$$\omega(z+1)-\omega(z-1)+a(z)\frac{\omega^{(k)}(z)}{\omega(z)}=\frac{P(z,\omega)}{Q(z,\omega)}=\frac{\sum_{i=0}^{p}a_i(z)\omega^i(z)}{\sum_{j=0}^{q}b_j(z)\omega^j(z)}, \tag{3.7}$$

where $a_p(z)b_q(z)b_0(z)\not\equiv 0$, and $a_i(z)$, $b_j(z)$ are rational functions for $i=0,1,\dots,p$, $j=0,1,\dots,q$. Let $\omega(z)$ be a transcendental entire solution of Eq (3.7). If $\omega(z)$ has infinitely many zeros, then $a(z)\frac{\omega^{(k)}(z)}{\omega(z)}$ has infinitely many poles, but $\omega(z+1)-\omega(z-1)-\frac{\sum_{i=0}^{p}a_i(z)\omega^i(z)}{\sum_{j=0}^{q}b_j(z)\omega^j(z)}$ has only finitely many poles. This is a contradiction, so $\omega(z)$ can be written as $\omega(z)=H(z)e^{g(z)}$, where $H(z)$ is a polynomial, and $g(z)$ is a non-constant entire function.

Case 1. $p\le q$. Applying Lemma 2.4, we get

$$m\left(r,\frac{P(z,\omega)}{Q(z,\omega)}\right)=S(r,\omega)$$

and

$$N\left(r,\frac{P(z,\omega)}{Q(z,\omega)}\right)=N\left(r,\omega(z+1)-\omega(z-1)+a(z)\frac{\omega^{(k)}(z)}{\omega(z)}\right)=O(\log r)=S(r,\omega).$$

Then, $T(r,R(z,\omega))=S(r,\omega)$. Let $R(z,\omega)=\omega(z+1)-\omega(z-1)+a(z)\frac{\omega^{(k)}(z)}{\omega(z)}=R(z)$. Applying Lemma 2.1 to the equation $P(z,\omega)=R(z)Q(z,\omega)$, we obtain $p=q$, and $a_i(z)=R(z)b_i(z)$, $i=0,1,...,p$. This contradicts the fact that $P(z,\omega)$ and $Q(z,\omega)$ are coprime polynomials.

Case 2. $p>q$. Applying the division algorithm, we can write (1.3) in the form

$$\omega(z+1)-\omega(z-1)+a(z)\frac{\omega^{(k)}(z)}{\omega(z)}=P_1(z,\omega)+\frac{P_2(z,\omega)}{Q(z,\omega)}, \tag{3.8}$$

where $\deg_\omega(P_2) = p_2 < q$. We can rewrite (3.8) in the following form

$$G(z,\omega) = \frac{P_2(z,\omega)}{Q(z,\omega)}, \tag{3.9}$$

where $G(z,\omega) = \omega(z+1) - \omega(z-1) + a(z)\frac{\omega^{(k)}(z)}{\omega(z)} - P_1(z,\omega)$. Applying Lemma 2.4 to Eq (3.9), we obtain $m(r, G(z,\omega)) = S(r,\omega)$, and

$$N(r, G(z,\omega)) = N\left(r, \omega(z+1) - \omega(z-1) + a(z)\frac{\omega^{(k)}(z)}{\omega(z)} - P_1(z,\omega)\right) = O(\log r) = S(r,\omega).$$

Then, $T(r, G(z,\omega)) = S(r,\omega)$. Applying Lemma 2.1 to $P_2(z,\omega) = G(z,\omega)Q(z,\omega)$, we obtain $p_2 = q$. This contradicts the fact that $p_2 < q$.

To sum up, when $\deg_\omega(Q) = 0$ and $\deg_\omega(P) \geq 2$ or $\deg_\omega(Q) \geq 1$, Equation (1.3) has no transcendental entire solutions. This completes the proof.

Proof of Theorem 1.5. The proof is divided into two parts, namely, $\deg_\omega(Q) = 0$ and $\deg_\omega(Q) \geq 1$. When $\deg_\omega(Q) = 0$ and $\deg_\omega(P) = 0$, the conclusion is obtained by Lemma 2.7. Now we will consider the remaining scenarios: the case where $\deg_\omega(Q) = 0$ and $\deg_\omega(P) > 0$, as well as the case where $\deg_\omega(Q) \geq 1$. By applying the same method as in the proof of Theorem 1.3, one can obtain that $\omega(z)$ can be written as $\omega(z) = G(z)e^{h(z)}$, where $G(z)$ is a polynomial, and $h(z)$ is a non-constant entire function. Therefore, $w^{(k)} = L(z,h')e^h$, where $L(z,h')$ is a differential polynomial in h' with polynomial coefficients, and its unique highest-degree term is $(h')^k$. Thus, $T(r, \frac{\omega^{(k)}}{\omega}) = T(r, \frac{L(z,h')}{G(z)}) \leq O(T(r,h)) + O(\log r)$.

First, we consider the case that $\deg_\omega(Q) = 0$, $\deg_\omega(P) = 1$. Then, Eq (1.4) becomes

$$\omega(z+1) + a(z)\frac{\omega^{(k)}(z)}{\omega(z)} = a_1(z)\omega(z) + a_0(z), \tag{3.10}$$

where $a(z)$, $a_0(z)$, and $a_1(z)(\not\equiv 0)$ are rational functions. Equation (3.10) can be written as $\omega(z+1) = a_1(z)\omega(z) + A(z)$, where $A(z) = a_0(z) - a(z)\frac{\omega^{(k)}(z)}{\omega(z)}$. Assume that $A(z) \not\equiv 0$. Since A is a small function with respect to $\omega(z)$ and $a_1(z)\omega(z) + A(z)$ has only finitely many zeros, applying the second fundamental theorem of Nevanlinna theory to $\omega(z)$ yields $T(r,\omega) \leq S(r,\omega)$, which is impossible. Hence, $A(z) \equiv 0$. By applying the same method as in Lemma 2.5, we can obtain that h is a polynomial.

Taking $\omega(z) = G(z)e^{h(z)}$ into (3.10), we get

$$G(z+1)e^{h(z+1)} - a_1(z)G(z)e^{h(z)} \equiv 0. \tag{3.11}$$

If $h(z+1) - h(z)$ is not a constant, applying Lemma 2.2 to Eq (3.11) yields $G(z+1) \equiv 0$, which is a contradiction. Therefore, assume $h(z+1) - h(z) = \mu$, where μ is a constant. Substituting $h(z+1) = h(z) + \mu$ into Eq (3.11) gives

$$(e^\mu G(z+1) - a_1(z)G(z))e^{h(z)} \equiv 0. \tag{3.12}$$

From (3.12), we get $e^\mu G(z+1) - a_1(z)G(z) \equiv 0$. Now, from $h(z+1) - h(z) = \mu$, we get that $h(z) = \mu z + d$, where d is a constant. Then, $\omega(z) = G(z)e^d e^{\mu z}$, and from Eq (3.12) we get that

$$a_1(z) = \frac{e^\mu G(z+1)}{G(z)}.$$

This is the conclusion of Theorem 1.5.

Second, we consider the case that $\deg_{\omega}(Q) = 0$, $\deg_{\omega}(P) = p \geq 2$. Then, Eq (1.4) becomes

$$\omega(z+1) + a(z)\frac{\omega^{(k)}(z)}{\omega(z)} = \sum_{i=0}^{p} a_i(z)\omega^i(z), \tag{3.13}$$

where $a(z)$, $a_0(z)$, ... , $a_p(z)$ are rational functions with $a_p(z) \not\equiv 0$. Taking $\omega(z) = G(z)e^{h(z)}$ into (3.13), we get

$$G(z+1)e^{h(z+1)} + a(z)\frac{L(z,h')}{G(z)} = \sum_{i=0}^{p} a_i(z)G^i(z)e^{ih(z)}. \tag{3.14}$$

We will divide the proof into the following cases.

Case 1. If none of $h(z+1)-ih(z)$ is a constant (for $i = 1,\dots,p$), then since $a(z), a_0(z), a_1(z),\dots,a_p(z)$ are rational functions, $G(z)$ is a polynomial, and $h(z)$ is a non-constant entire function. Applying Lemma 2.2 to Eq (3.14) leads to $G(z+1) \equiv 0$, which is a contradiction.

Case 2. If $h(z+1) - h(z) = \mu$, where μ is a constant. Substitute $h(z+1) = h(z) + \mu$ into Eq (3.14) to obtain

$$e^{\mu}G(z+1)e^{h(z)} + a(z)\frac{L(z,h')}{G(z)} = \sum_{i=0}^{p} a_i(z)G^i(z)e^{ih(z)}. \tag{3.15}$$

From Eq (3.15) and Lemma 2.1, we get

$$\begin{aligned} pT(r,e^h) &= T\left(r, \sum_{i=0}^{p} a_i G^i e^{ih}\right) + S(r,e^h) \\ &= T\left(r, e^{\mu}G(z+1)e^h + a\frac{L(z,h')}{G(z)}\right) + S(r,e^h) \\ &\leq T(r,e^h) + S(r,e^h), \end{aligned}$$

which implies that $p \leq 1$. This is in contradiction to $p \geq 2$.

Case 3. If $h(z+1) - i_0h(z) = \mu$, where μ is a constant, $2 \leq i_0 \leq p-1$. Substitute $h(z+1) = i_0h(z) + \mu$ into Eq (3.14) to obtain

$$e^{\mu}G(z+1)e^{i_0h(z)} - \sum_{i=1}^{p} a_i(z)G^i(z)e^{ih(z)} = a_0(z) - a(z)\frac{L(z,h')}{G(z)}. \tag{3.16}$$

From Eq (3.16) and Lemma 2.2, we obtain $a_p(z)G^p(z) \equiv 0$, which is a contradiction.

Case 4. If $h(z+1) - ph(z) = \mu$, where μ is a constant. Substitute $h(z+1) = ph(z) + \mu$ into Eq (3.14) to obtain

$$(e^{\mu}G(z+1) - a_p(z)G^p(z))e^{ph(z)} - \sum_{i=1}^{p-1} a_i(z)G^i(z)e^{ih(z)} = a_0(z) - a(z)\frac{L(z,h')}{G(z)}. \tag{3.17}$$

From Eq (3.17) and Lemma 2.2, we obtain $a_i(z)G^i(z) \equiv 0$ ($i = 1, 2, ..., p-1$), and $a_0(z) - a(z)\frac{L(z,h')}{G(z)} \equiv 0$

If h is a transcendental entire function, then $a(z)\frac{L(z,h')}{G(z)} - a_0(z) \not\equiv 0$, which is a contradiction. If h is a polynomial, then $p = 1$ since $h(z+1) = ph(z) + \mu$, which is a contradiction to $p \geq 2$.

Finally, we consider the case that $\deg_\omega(Q) \geq 1$. Then, Eq (1.4) can be written as

$$\omega(z+1) + a(z)\frac{\omega^{(k)}(z)}{\omega(z)} = \frac{P(z,\omega)}{Q(z,\omega)} = \frac{\sum_{i=0}^{p} a_i(z)\omega^i(z)}{\sum_{j=0}^{q} b_j(z)\omega^j(z)},$$

where $a_p(z)b_q(z)b_0(z) \not\equiv 0$, $a_i(z)$, $b_j(z)$ are rational functions, and $i = 0, 1, ..., p$, $j = 0, 1, ..., q$. Using the same method as in Theorem 1.3, a contradiction is obtained.

Proof of Theorem 1.6. The proof is divided into two parts, namely, $\deg_\omega(Q) = 0$ and $\deg_\omega(Q) \geq 1$. When $\deg_\omega(Q) = 0$ and $\deg_\omega(P) = 0$, the conclusion is obtained by Lemma 2.8. It remains to consider the two cases $\deg_\omega(Q) = 0$ with $\deg_\omega(P) > 0$, and $\deg_\omega(Q) \geq 1$. By applying the same method as in the proof of Theorem 1.5, one can obtain $\omega(z) = G(z)e^{h(z)}$, where $G(z)$ is a polynomial, and $h(z)$ is a non-constant entire function.

First, suppose $\deg_\omega(Q) = 0$, $\deg_\omega(P) = 1$. Then, Eq (1.6) becomes

$$\omega(z+1) + a(z)\left(\frac{\omega'(z)}{\omega(z)}\right)^k = a_1(z)\omega(z) + a_0(z), \tag{3.18}$$

where $a(z)$, $a_0(z)$, and $a_1(z)(\not\equiv 0)$ are rational functions. Similar to the proof of Theorem 1.5, we get

$$a_0(z) - a(z)\left(\frac{\omega'(z)}{\omega(z)}\right)^k \equiv 0. \tag{3.19}$$

Applying the same method as in Lemma 2.5, we obtain that h is a polynomial.

Taking $\omega(z) = G(z)e^{h(z)}$ into (3.18), we get

$$G(z+1)e^{h(z+1)} - a_1(z)G(z)e^{h(z)} \equiv 0. \tag{3.20}$$

If $h(z+1) - h(z)$ is not a constant, applying Lemma 2.2 to Eq (3.20) yields $G(z+1) \equiv 0$, which is a contradiction. Therefore, assume $h(z+1)-h(z) = \mu$, where μ is a constant. Substituting $h(z+1) = h(z)+\mu$ into Eq (3.20) gives

$$(e^{\mu}G(z+1) - a_1(z)G(z))e^{h(z)} \equiv 0. \tag{3.21}$$

From (3.21), we get $e^{\mu}G(z+1) - a_1(z)G(z) \equiv 0$. Now, from $h(z+1) - h(z) = \mu$, we get that $h(z) = \mu z + d$, where $\mu \in \mathbb{C} \setminus \{0\}$. Then, $\omega(z) = G(z)e^d e^{\mu z}$. From Eqs (3.19) and (3.21), we have

$$a_1(z) = \frac{e^{\mu}G(z+1)}{G(z)}, \ a_0(z) = a(z)\left(\frac{G'(z)}{G(z)} + \mu\right)^k.$$

This is the conclusion of Theorem 1.6.

Second, suppose that either $\deg_\omega(Q) = 0$ (with $\deg_\omega(P) = p \geq 2$) or $\deg_\omega(Q) \geq 1$. Using the same method as in Theorem 1.5, we obtain a contradiction.

4. Conclusions

In this paper, we focus on the reduced forms of Eqs (1.3)–(1.6) and derive explicit solution representations without imposing any growth restrictions on the solutions. Our main results are twofold. For Eqs (1.3) and (1.5), whenever they admit entire solutions, we determine their degenerate forms and derive the explicit expressions of the solutions; these are stated in Theorems 1.3 and 1.4, which require no constraints on the order of growth. Similarly, for Eqs (1.4) and (1.6), whenever they admit entire solutions, we obtain their degenerate forms and deduce the explicit solutions as $\omega(z) = G(z)e^{\mu z}$ ($\mu \in \mathbb{C} \setminus \{0\}, G(z)$ polynomial); these results are presented in Theorems 1.5 and 1.6, which also impose no growth conditions.

Looking ahead, we will concentrate on the following two aspects.

(i) References [23,24] studied partial differential equations and presented some interesting results. Our future work will focus on extending Eqs (1.3)–(1.6) to the space C^2, aiming to establish more general conclusions.

(ii) Although we are currently unable to provide sufficient conditions for the existence of solutions to these equations, we intend to address these criteria in a separate future study.

Author contributions

Changwen Peng: Writing-original draft, Methodology, Supervision, Writing-review & editing; Huawei Huang: Writing-review & editing, Investigation, Software; Mengting Xia: Methodology, Validation. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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