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Research article

A generalization of affine fractional L^p Sobolev inequalities

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Abstract: General affine fractional L^p Sobolev inequalities were established by Monika Ludwig and Julián Haddad in their foundational work for fractional parameters $s \in (0, 1)$ and $1 < p < n/s$. In this paper, we extend the admissible range of the fractional parameter to $s \in (0, n) \setminus \mathbb{N}$ and $1 < p < n/s$. The affine fractional L^p Sobolev inequalities are obtained for both the classical fractional case $s \in (0, 1)$ and the higher-order fractional case. In the course of the proof, we define the higher-order fractional L^p Sobolev polar projection body and use the method of symmetric decreasing rearrangement.

Keywords: fractional Sobolev inequality; fractional L^p polar projection body; symmetric decreasing rearrangement; Pólya–Szegö inequality
Mathematics Subject Classification: 26D10, 52A20

1. Introduction

The classical fractional L^p Sobolev inequalities have been extensively studied in the literature; see the monograph [26] and the references therein. In particular, affine fractional L^p Sobolev inequalities were recently established in [18, 25], and anisotropic versions were studied in [17, 24]. They serve as fundamental tools in the theory of function spaces and play a crucial role in various branches of applied mathematics. In particular, these inequalities are indispensable in the analysis of partial differential equations, where they are used to establish existence, uniqueness, regularity, and stability of solutions. They also provide essential compactness embeddings in the calculus of variations, underpin geometric inequalities such as isoperimetric and Sobolev-type inequalities, and are central to the study of nonlinear Schrödinger equations, fluid dynamics, and kinetic theory. For applications to obstacle problems and reaction-diffusion equations, see [3, 8, 31]; for connections to convex geometry and rearrangement inequalities, we refer to [15] and [20], respectively. Letting $0 < s < 1$ and $1 < p < n/s$, the fractional L^p Sobolev inequality states that

$$\|f\|^p_{L^{\frac{np}{n-ps}}(\mathbb{R}^n)} \le \sigma_{n,p,s} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+ps}} dxdy, \tag{1.1}$$

where f belongs to $W^{s,p}(\mathbb{R}^n)$, the fractional L^p Sobolev space consisting of functions $f \in L^p(\mathbb{R}^n)$ for which the right-hand side of (1.1) is finite (see, e.g., [26]). In general, the values of the optimal constants $\sigma_{n,p,s}$ and the forms of the corresponding extremal functions are unknown. However, a conjecture was proposed in [6] regarding this problem. For $p = 1$, the extremal functions of (1.1) are scalar multiples of the indicator functions of balls, with the constants being explicitly determined. For $p = 2$, the constants $\sigma_{n,p,s}$ are obtained via duality from Lieb's sharp Hardy–Littlewood–Sobolev inequalities; see [9, 21]. The asymptotic behavior of $\sigma_{n,p,s}$ as $s \to 1^-$ was studied in [5], and the research on extremal function of (1.1) can be found in [2, 14].

In reference [18], a stronger version of this inequality was proved by introducing the fractional L^p Sobolev polar projection body $\Pi^{*,s}_p f$ and its gauge function

$$\|\xi\|^{ps}_{\Pi^{*,s}_p f} = \int_0^\infty t^{-ps-1} \int_{\mathbb{R}^n} |f(x+t\xi)-f(x)|^p \, dxdt.$$

By combining symmetric decreasing rearrangement, the dual Minkowski mixed volume, and related tools, the following inequality was established:

$$\begin{aligned}\|f\|^p_{L^{\frac{np}{n-ps}}(\mathbb{R}^n)} &\le \sigma_{n,p,s} n\omega_n^{\frac{n+ps}{n}} \left(\frac{1}{n}\int_{\mathbb{S}^{n-1}} \left(\int_0^\infty t^{-ps-1} \int_{\mathbb{R}^n} |f(x+t\xi)-f(x)|^p \, dxdt\right)^{-\frac{n}{ps}} d\xi\right)^{-\frac{ps}{n}} \\ &\le \sigma_{n,p,s} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+ps}} dxdy.\end{aligned}$$

Here, $0 < s < 1$ and $1 < p < n/s$ for $f \in W^{s,p}(\mathbb{R}^n)$, and ω_n is the volume of the n-dimensional unit ball.

The novelty and main contributions of this paper

The affine fractional Sobolev inequalities established by Haddad and Ludwig in [18] are restricted to the case $0 < s < 1$, where the fractional semi-norm is applied directly to the function $f \in W^{s,p}(\mathbb{R}^n)$. In this classical setting, the function f itself is required to belong to $L^p(\mathbb{R}^n)$, and the proof relies on the symmetrization technique and Pólya–Szegö-type inequalities for fractional polar projection bodies.

The present work extends these results to the full range $s \in (0, n) \setminus \mathbb{N}$ with $1 < p < n/s$, where the notation $(0, n) \setminus \mathbb{N}$ denotes the set obtained by removing all natural numbers from the open interval $(0, n)$. This extension is far from straightforward and requires two innovations. Next, we denote by $[s]$ the integer part of s and let $\{s\} = s - [s]$, which is the fractional part of s.

First, the classical fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$ is no longer appropriate when $s > 1$, as it would require $f \in W^{[s],p}(\mathbb{R}^n)$, i.e., all derivatives up to order $[s]$ to be p-integrable. Such a requirement is unnecessarily strong for our purposes and would exclude many functions of interest. Instead, we introduce the space

$$V^{s,p}(\mathbb{R}^n) = \left\{ f \in W^{[s],1}_{\text{loc}}(\mathbb{R}^n) : \nabla^{[s]} f \in W^{\{s\},p}(\mathbb{R}^n) \right\},$$

which only requires the *highest-order* derivatives $\nabla^{[s]} f$ to possess fractional Sobolev regularity, while the lower-order derivatives are merely locally integrable. This space is strictly larger than the classical $W^{s,p}(\mathbb{R}^n)$.

Second, and more importantly, the proof techniques of Haddad and Ludwig must be redeveloped for the higher-order setting. Unlike the case $0 < s < 1$, where the fractional semi-norm is applied to the scalar function f, our setting requires us to apply the fractional semi-norm to the Euclidean norm of the tensor-valued gradient $|\nabla^{[s]} f|$. This introduces new geometric complications: The fractional polar projection bodies $\Pi_p^{*,s}(\nabla_s f)$ are defined in terms of the gauge function

$$\|\xi\|^{p\{s\}}_{\Pi_p^{*,s}(\nabla_s f)} = \int_0^\infty t^{-p\{s\}-1} \int_{\mathbb{R}^n} \left| \left|\nabla^{[s]} f(x+t\xi)\right| - \left|\nabla^{[s]} f(x)\right| \right|^p \, dx\, dt,$$

which involves the intricate coupling between the integer order $[s]$ and the fractional order $\{s\}$ (see Section 3 for details). The corresponding dual mixed volume inequalities and Pólya–Szegő-type principles must be established anew in this higher-order fractional context. These tools, which are absent from the earlier work of Haddad and Ludwig, constitute the main technical contributions of our paper.

As a consequence, we establish the following higher-order affine fractional Sobolev inequalities (see Theorems 7.1 and 7.2 for details), which reduce to the classical inequalities of Haddad and Ludwig when $[s] = 0$ (i.e., $0 < s < 1$) and, in the symmetric case, provide a natural higher-order generalization of their results.

Theorem 1.1. *Let $s \in (0, n) \setminus \mathbb{N}$ and $1 < p < n/s$. For $f \in V_d^{s,p}(\mathbb{R}^n)$,*

$$\begin{aligned}
\left\| \left|\nabla^{[s]} f\right| \right\|^p_{L^{\frac{np}{n-p\{s\}}}(\mathbb{R}^n)} &\leq \sigma_{n,p,s} n \omega_n^{\frac{n+p\{s\}}{n}} \left(\frac{1}{n} \int_{\mathbb{S}^{n-1}} \left(\int_0^\infty t^{-p\{s\}-1} \left\| \left|\nabla^{[s]} f(\cdot + t\xi)\right| - \left|\nabla^{[s]} f(\cdot)\right| \right\|_p^p dt \right)^{-\frac{n}{p\{s\}}} d\xi \right)^{-\frac{p\{s\}}{n}} \\
&\leq \sigma_{n,p,s} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\left| \left|\nabla^{[s]} f(x)\right| - \left|\nabla^{[s]} f(y)\right| \right|^p}{|x-y|^{n+p\{s\}}} dx dy.
\end{aligned}$$

Equality holds in the first inequality if and only if $\left|\nabla^{[s]} f\right| = h_{\{s\},p} \circ \phi$ for some $\phi \in \mathrm{GL}(n)$, where $h_{\{s\},p}$ is an extremal function for inequality (1.1). *For the second inequality, equality is achieved if and only if $\left|\nabla^{[s]} f\right|$ is radially symmetric.*

Here, the function space $V_d^{s,p}(\mathbb{R}^n)$ is defined in Subsection 2.1. For a tensor-valued function $\nabla^{[s]} f : \mathbb{R}^n \to \mathbb{R}^{n^{[s]}}$, we denote that $\left|\nabla^{[s]} f(x)\right|$ is the Euclidean norm of $\nabla^{[s]} f(x)$ as $\nabla_s f$, that is

$$\left|\nabla^{[s]} f(x)\right| = \left[\sum_{\alpha_1+\alpha_2+\cdots+\alpha_n=[s]} \left(\frac{\partial^{[s]} f(x)}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2 \cdots \partial^{\alpha_n} x_n} \right)^2 \right]^{1/2}.$$

Here, each index α_i for $i = 1, 2, \ldots, n$ is a nonnegative integer, and all partial derivatives are interpreted as weak derivatives. Then, we obtain the following mapping:

$$\nabla_s f : \mathbb{R}^n \to \mathbb{R}.$$

This definition is consistently adopted throughout the paper.

As mentioned earlier, we define the s-fractional L^p polar projection body $\Pi_p^{*,s}(\nabla_s f)$ as a star-shaped set determined by the gauge function

$$\|\xi\|^{p\{s\}}_{\Pi_p^{*,s}(\nabla_s f)} = \int_0^\infty t^{-p\{s\}-1} \int_{\mathbb{R}^n} \left| \left|\nabla^{[s]} f(x+t\xi)\right| - \left|\nabla^{[s]} f(x)\right| \right|^p dxdt$$

for $\xi \in \mathbb{R}^n$. Here, $\Pi_p^{*,s} f$ is the L^p polar projection body of f, a convex body associated with f, which was defined using different notation in [18, 25]. We denote $\left|\Pi_p^{*,s}(\nabla_s f)\right|$ as the n-dimensional Lebesgue measure of $\Pi_p^{*,s}(\nabla_s f)$, and the affine fractional Sobolev inequality is given by

$$\begin{aligned} \left\|\left|\nabla^{[s]} f\right|\right\|^p_{L^{\frac{np}{n-p\{s\}}}(\mathbb{R}^n)} &\leq \sigma_{n,p,s} n\omega_n^{\frac{n+p\{s\}}{n}} \left|\Pi_p^{*,s}(\nabla_s f)\right|^{-\frac{p\{s\}}{n}} \\ &\leq \sigma_{n,p,s} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left|\left|\nabla^{[s]} f(x)\right| - \left|\nabla^{[s]} f(y)\right|\right|^p}{|x-y|^{n+p\{s\}}} dxdy. \end{aligned} \tag{1.2}$$

Since both sides of (1.2) are invariant under translations of $\nabla_s f$ and volume-preserving linear transformations $\phi : \mathbb{R}^n \to \mathbb{R}^n$,

$$\Pi_p^{*,s}\left((\nabla_s f)\circ\phi^{-1}\right) = \phi\Pi_p^{*,s}(\nabla_s f),$$

implying that inequality (1.2) has a specific affine invariance property.

In Section 4, we introduce the higher-order fractional asymmetric L^p Sobolev polar projection bodies, which generalize the classical versions for $s \in (0, 1)$ studied in [18]. These functionals are defined as the functional counterparts of the asymmetric L^p polar projection bodies, which were originally defined for convex bodies in [22].

The classical Pólya–Szegö inequality was originally established in [29]. Subsequent works [12, 13] investigated convex symmetrization techniques and the corresponding equality cases. Fractional extensions of this inequality have been developed in [11, 19], while smoothing rearrangements and affine variants are addressed in [4, 32], respectively. On this basis, in Section 5 we derive higher-order Pólya–Szegö-type inequalities for functions involving the Euclidean norm of the gradient. In Section 6, we extend the inequalities for asymmetric and symmetric fractional L^p polar projection bodies.

In Section 7, we prove our main results via anisotropic fractional Sobolev norms, which were introduced in [23, 24] and depend on a star-shaped set $K \subset \mathbb{R}^n$. There has been significant recent interest in anisotropic fractional Sobolev norms. For further details, see [10, 33].

2. Preliminaries

In this section, we provide preliminary knowledge on function spaces, symmetrization, and convexity theory for later analysis.

2.1. Fractional L^p Sobolev spaces

In this part, we introduce some theories related to function space. The following definitions are mainly taken from the monographs [20, 30] and the works [1, 24] on fractional and anisotropic spaces. We recall these basic theories as follows.

Let $p \geq 1$ and $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. Then,

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x\right)^{1/p}.$$

We set $\{f \geq t\} = \{x \in \mathbb{R}^n : f(x) \geq t\}$ for $t \in \mathbb{R}$ and use similar notation for level sets. The function f is non-zero, if $\{f \neq 0\}$ has positive measure, and we identify functions that are equal up to a set of measure zero. For $p \geq 1$, let

$$L^p(\mathbb{R}^n) = \left\{f : \mathbb{R}^n \to \mathbb{R} : f \text{ is measurable}, \|f\|_p < \infty\right\}.$$

We denote

$$W^{1,p}(\mathbb{R}^n) = \{f \in L^p(\mathbb{R}^n) : |\nabla f| \in L^p(\mathbb{R}^n)\},$$

where ∇f is the weak gradient of f. Given $m \in \mathbb{N}$, $W^{m,p}(\mathbb{R}^n)$ is adopted for the classical Sobolev space defined by

$$W^{m,p}(\mathbb{R}^n) = \left\{f \in L^p(\mathbb{R}^n) : |\nabla^k f| \in L^p(\mathbb{R}^n),\ k = 1, \ldots, m\right\},$$

where $\nabla^0 f$ has to be interpreted as f. The space $W^{m,p}(\mathbb{R}^n)$ is a Banach space equipped with the norm

$$\|f\|_{W^{m,p}(\mathbb{R}^n)} = \sum_{k=0}^{m} \|\nabla^k f\|_{L^p(\mathbb{R}^n)}.$$

Here, $\nabla^m f$ denotes the tensor of all weak derivatives of f of order m. If $m = 1$, we simply write ∇f instead of $\nabla^1 f$. Next, let $s \in (0,1)$. The seminorm $|f|_{W^{s,p}(\mathbb{R}^n)}$ of a measurable function f is given by

$$|f|^p_{W^{s,p}(\mathbb{R}^n)} := \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} dxdy.$$

The fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$ is defined by

$$W^{s,p}(\mathbb{R}^n) = \{f \text{ is a measurable function} : |f|_{W^{s,p}(\mathbb{R}^n)} < \infty\}. \tag{2.1}$$

The definitions of the seminorm $|f|_{W^{s,p}(\mathbb{R}^n)}$ and the space $W^{s,p}(\mathbb{R}^n)$ extend naturally to tensor-valued function f by replacing the absolute value of $f(x) - f(y)$ with the norm of the same expression on the right-hand side of equation.

The subspace of functions in (2.1) that vanish at infinity is denoted by $W^{s,p}_d(\mathbb{R}^n)$. Thus,

$$W^{s,p}_d(\mathbb{R}^n) = \{f \in W^{s,p}(\mathbb{R}^n) : |\{|f| > t\}| < \infty \text{ for every } t > 0\}. \tag{2.2}$$

To extend the definition of the classical fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$ to all $s \in (0,n) \setminus \mathbb{N}$ with $1 < p < n/s$, we introduce the following variant space:

$$V^{s,p}(\mathbb{R}^n) = \left\{f \in W^{[s],1}_{\text{loc}}(\mathbb{R}^n) : \nabla^{[s]} f \in W^{\{s\},p}(\mathbb{R}^n)\right\}. \tag{2.3}$$

We emphasize that $V^{s,p}(\mathbb{R}^n)$ does *not* coincide with the standard Besov or Triebel-Lizorkin spaces in general. For space $V^{s,p}(\mathbb{R}^n)$, only the *highest-order* derivatives $\nabla^{[s]} f$ are required to possess fractional Sobolev regularity in L^p, while the lower-order derivatives are merely locally integrable.

In analogy with (2.2), we extend it to $s \in (0,n) \setminus \mathbb{N}$ and $1 < p < n/s$ and denote

$$V^{s,p}_d(\mathbb{R}^n) = \left\{f \in V^{s,p}(\mathbb{R}^n) : \left|\left\{|\nabla^k f| > t\right\}\right| < \infty \text{ for } k = 0, 1, \ldots, [s], \text{ and for every } t > 0\right\}.$$

This paper focuses on the study of the space in (2.3).

2.2. Symmetrization

For the symmetrization theories and symmetric decreasing rearrangement techniques, please refer to reference [2, 7, 20]. Letting $E \subseteq \mathbb{R}^n$, the *indicator function* 1_E is defined by $1_E(x) = 1$ if $x \in E$, and $1_E(x) = 0$ otherwise. For a Borel set $E \subseteq \mathbb{R}^n$ of finite measure, its Schwarz symmetrization $E^\star$ is defined as the closed Euclidean ball centered at the origin having the same measure as E. Now, let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative measurable function such that for every $t > 0$, the superlevel set $\{f \geq t\}$ has finite measure. The *layer cake formula* asserts that

$$f(x) = \int_0^\infty 1_{\{f \geq t\}}(x)\, dt,$$

holds for almost every $x \in \mathbb{R}^n$. The *Schwarz symmetrization* $f^\star$ of f is defined as

$$f^\star(x) = \int_0^\infty 1_{\{f \geq t\}^\star}(x) dt,$$

for $x \in \mathbb{R}^n$, where $\{f \geq t\}^\star$ stands for the Schwarz symmetrization of the set $\{f \geq t\}$. Consequently, $f^\star$ is characterized almost everywhere by being radially symmetric and possessing super-level sets whose measures coincide with those of f. Note that $f^\star$ is also called the *symmetric decreasing rearrangement* of f. If for all x, y with $|x| < |y|$, inequality $f^\star(x) > f^\star(y)$ holds, we say that $f^\star$ is *strictly symmetric decreasing*.

The proofs of our results rely on the Riesz rearrangement inequality (see [20]).

Theorem 2.1. *(Riesz's rearrangement inequality). Let $f, g, k : \mathbb{R}^n \to \mathbb{R}$ be nonnegative measurable functions with superlevel sets of finite measure. Then*

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} f(x)k(x-y)g(y)dxdy \leq \iint_{\mathbb{R}^n \times \mathbb{R}^n} f^\star(x)k^\star(x-y)g^\star(y)dxdy.$$

We will utilize the characterization of equality cases in the Riesz rearrangement inequality, which is from A. Burchard [7].

Theorem 2.2. *(Burchard). Let A, B, and C be sets in $\mathbb{R}^n$ with finite positive measure. Suppose α, β, and γ are the radii of their Schwarz symmetrization $A^\star$, $B^\star$, and $C^\star$. For $|\alpha - \beta| < \gamma < \alpha + \beta$, equality holds in*

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} 1_A(y)1_B(x-y)1_C(x)dxdy \leq \iint_{\mathbb{R}^n \times \mathbb{R}^n} 1_{A^\star}(y)1_{B^\star}(x-y)1_{C^\star}(x)dxdy,$$

if and only if, up to sets of measure zero,

$$A = a + \alpha D,\ B = b + \beta D,\ C = c + \gamma D,$$

where D is a centered ellipsoid, and a, b, and $c = a + b$ are vectors in $\mathbb{R}^n$.

2.3. Star-shaped sets

For the relevant theories and concepts in convex geometric analysis, we refer the reader to [30]. Let $C \subseteq \mathbb{R}^n$ be a *convex set* if $(1-\lambda)\,x + \lambda y \in C$ holds for all x, y belonging to C and $\lambda \in [0,1]$. A *convex body* is a compact convex subset of $\mathbb{R}^n$ with a non-empty interior.

A set $K \subseteq \mathbb{R}^n$ is said to be *star-shaped* (with respect to the origin) if the interval $[0, x] \subseteq K$ for every $x \in K$. The *radial function* $\rho_K : \mathbb{R}^n \setminus \{0\} \to [0, \infty]$ associated with a star-shaped set K is defined as

$$\rho_K(x) = \sup\{\lambda \geq 0 : \lambda x \in K\}.$$

The *gauge function* $\|\cdot\|_K : \mathbb{R}^n \to [0, \infty]$ is given by

$$\|x\|_K = \inf\{\lambda > 0 : x \in \lambda K\}.$$

In particular, we set $\|\cdot\|_{B^n} = |\cdot|$, where B^n denotes the n-dimensional unit ball. Throughout this paper, for an arbitrary n-dimensional vector x, $|x|$ stands for its gauge associated with the unit ball, namely the Euclidean norm, whereas $\|x\|_K$ denotes the gauge function of x with respect to K.

Two star-shaped sets K and L are said to be dilates if their radial functions satisfy $\rho_K = c\rho_L$ almost everywhere on $\mathbb{S}^{n-1}$ for some $c \geq 0$.

We call K a *star body* if its radial function is strictly positive and continuous in $\mathbb{R}^n \setminus \{0\}$. Let $|K|$ denote n-dimensional Lebesgue measure of K. For a star-shaped set $K \subseteq \mathbb{R}^n$ with a measurable radial function, the n-dimensional Lebesgue measure of K satisfies

$$|K| = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho_K(\xi)^n d\xi.$$

Let $\alpha \in \mathbb{R}$. For star-shaped sets $K, L \subseteq \mathbb{R}^n$ endowed with measurable radial functions, the *dual mixed volume* is defined by

$$\tilde{V}_\alpha(K, L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho_K(\xi)^{n-\alpha} \rho_L(\xi)^\alpha \, d\xi. \tag{2.4}$$

If $K = L$ in (2.4), we have

$$\tilde{V}_\alpha(K, K) = |K|.$$

The concept of dual mixed volume for star bodies was introduced by Rolf Schneider [30], who also established the dual mixed volume inequalities (2.5) and (2.6) as consequences of Hölder's inequality. For $0 < \alpha < n$, the dual mixed volume inequality associated with star-shaped sets $K, L \subseteq \mathbb{R}^n$ of finite measure shows that

$$\tilde{V}_\alpha(K, L) \leq |K|^{\frac{n-\alpha}{n}} |L|^{\frac{\alpha}{n}}. \tag{2.5}$$

For $\alpha < 0$ or $\alpha > n$, the dual mixed volume inequality associated with star-shaped sets $K, L \subseteq \mathbb{R}^n$ states that

$$\tilde{V}_\alpha(K, L) \geq |K|^{\frac{n-\alpha}{n}} |L|^{\frac{\alpha}{n}}. \tag{2.6}$$

The equality condition in Hölder's inequality implies equalities in (2.5) and (2.6) for finite $\tilde{V}_\alpha(K, L)$ if and only if K and L are dilates. For the star body, the q-radial sum for $q \neq 0$ of $K, L \subset \mathbb{R}^n$ is defined by

$$\rho^q(K \tilde{+}_q L, \xi) = \rho^q(K, \xi) + \rho^q(L, \xi)$$

for $\xi \in \mathbb{S}^{n-1}$ (cf. [30], Section 9.3). The dual Brunn–Minkowski inequality states that for star bodies $K, L \subset \mathbb{R}^n$ and $q > 0$,

$$|K \tilde{+}_{-q} L|^{-q/n} \geq |K|^{-q/n} + |L|^{-q/n}, \tag{2.7}$$

with equality precisely if K and L are dilates, that is, there is $\lambda > 0$ such that $K = \lambda L$.

3. Fractional L^p Sobolev polar projection bodies

Let $s \in (0, n) \setminus \mathbb{N}$ and $1 < p < n/s$. We define the s-fractional L^p polar projection body $\Pi_p^{*,s}(\nabla_s f)$ as the star-shaped set given by the gauge function

$$\|\xi\|_{\Pi_p^{*,s}(\nabla_s f)}^{p\{s\}} = \int_0^\infty t^{-p\{s\}-1} \int_{\mathbb{R}^n} \left| \left|\nabla^{[s]} f(x+t\xi)\right| - \left|\nabla^{[s]} f(x)\right| \right|^p dxdt \tag{3.1}$$

for $\xi \in \mathbb{R}^n$. We note that $\|\cdot\|_{\Pi_p^{*,s}(\nabla_s f)}$ is a one-homogeneous function on $\mathbb{R}^n$. Indeed, taking $\alpha > 0$ and $\gamma = t\alpha$, we have

$$\begin{aligned}
\|\alpha\xi\|_{\Pi_p^{*,s}(\nabla_s f)}^{p\{s\}} &= \int_0^\infty t^{-p\{s\}-1} \int_{\mathbb{R}^n} \left| \left|\nabla^{[s]} f(x+t\alpha\xi)\right| - \left|\nabla^{[s]} f(x)\right| \right|^p dxdt \\
&= \int_0^\infty \alpha^{p\{s\}+1} \gamma^{-p\{s\}-1} \int_{\mathbb{R}^n} \alpha^{-1} \left| \left|\nabla^{[s]} f(x+\gamma\xi)\right| - \left|\nabla^{[s]} f(x)\right| \right|^p dxd\gamma \\
&= \alpha^{p\{s\}} \int_0^\infty \gamma^{-p\{s\}-1} \int_{\mathbb{R}^n} \left| \left|\nabla^{[s]} f(x+\gamma\xi)\right| - \left|\nabla^{[s]} f(x)\right| \right|^p dxd\gamma \\
&= \alpha^{p\{s\}} \|\xi\|_{\Pi_p^{*,s}(\nabla_s f)}^{p\{s\}} \\
&= \left(\alpha\|\xi\|_{\Pi_p^{*,s}(\nabla_s f)}\right)^{p\{s\}}.
\end{aligned}$$

The analysis above verifies that $\|\cdot\|_{\Pi_p^{*,s}(\nabla_s f)}$ is positively homogeneous of degree one.

Let $K \subset \mathbb{R}^n$ be a star body. For $f \in V_d^{s,p}(\mathbb{R}^n)$, we use the method of integration by substitution and Tonelli's theorem, and we obtain

$$\begin{aligned}
&\iint_{\mathbb{R}^n \times \mathbb{R}^n} \left(\frac{\left| \left|\nabla^{[s]} f(x)\right| - \left|\nabla^{[s]} f(y)\right| \right|}{\|x-y\|_K^{\{s\}}} \right)^p \frac{1}{\|x-y\|_K^n} dxdy \\
= &\iint_{\mathbb{R}^n \times \mathbb{R}^n} \left(\frac{\left| \left|\nabla^{[s]} f(y+z)\right| - \left|\nabla^{[s]} f(y)\right| \right|}{\|z\|_K^{\{s\}}} \right)^p \frac{1}{\|z\|_K^n} dzdy \\
= &\int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} \int_0^\infty \left(\frac{\left| \left|\nabla^{[s]} f(y+t\xi)\right| - \left|\nabla^{[s]} f(y)\right| \right|}{\|t\xi\|_K^{\{s\}}} \right)^p \frac{t^{n-1}}{\|t\xi\|_K^n} dtd\xi dy \\
= &\int_{\mathbb{S}^{n-1}} \|\xi\|_K^{-n-p\{s\}} \int_0^\infty t^{-1-p\{s\}} \int_{\mathbb{R}^n} \left| \left|\nabla^{[s]} f(y+t\xi)\right| - \left|\nabla^{[s]} f(y)\right| \right|^p dydtd\xi
\end{aligned}$$

$$= \int_{\mathbb{S}^{n-1}} \rho_K^{n+p\{s\}}(\xi)\rho_{\Pi_p^{*,s}(\nabla_s f)}^{-p\{s\}}(\xi)d\xi.$$

Hence,

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \left(\frac{\left|\left|\nabla^{[s]}f(x)\right|-\left|\nabla^{[s]}f(y)\right|\right|}{\|x-y\|_K^{\{s\}}}\right)^p \frac{1}{\|x-y\|_K^n}dxdy = n\tilde{V}_{-p\{s\}}\left(K, \Pi_p^{*,s}(\nabla_s f)\right). \tag{3.2}$$

Next, we will establish some basic properties of fractional L^p polar projection bodies.

Proposition 3.1. *Let $f \in V_d^{s,p}(\mathbb{R}^n)$. Then, the set $\Pi_p^{*,s}(\nabla_s f)$ is an origin-symmetric star body. Moreover, for each $s \in (0,n)\setminus\mathbb{N}$, there exists $c > 0$ depending only on f and p such that $\Pi_p^{*,s}(\nabla_s f) \subseteq cB^n$.*

Proof. First, we prove that $\Pi_p^{*,s}(\nabla_s f)$ is symmetric about the origin. For $\xi \in \mathbb{R}^n$ and $t > 0$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \left|\left|\nabla^{[s]}f(x-t\xi)\right|-\left|\nabla^{[s]}f(x)\right|\right|^p dx &= \int_{\mathbb{R}^n} \left|\left|\nabla^{[s]}f(y)\right|-\left|\nabla^{[s]}f(y+t\xi)\right|\right|^p dy\\ &= \int_{\mathbb{R}^n} \left|\left|\nabla^{[s]}f(x+t\xi)\right|-\left|\nabla^{[s]}f(x)\right|\right|^p dx.\end{aligned}$$

Next, we prove that $\Pi_p^{*,s}(\nabla_s f)$ is bounded. We take $r > 1$ large enough and $t > 2r$ such that $\left\|\left|\nabla^{[s]}f\right|\right\|_{L^p(rB^n)} \geq \frac{2}{3}\left\|\left|\nabla^{[s]}f\right|\right\|_p$. Using the method of integration by substitution, we have

$$\begin{aligned}\left\|\left|\nabla^{[s]}f(\cdot+t\xi)\right|-\left|\nabla^{[s]}f(\cdot)\right|\right\|_p &\geq \left\|\left|\nabla^{[s]}f(\cdot+t\xi)\right|-\left|\nabla^{[s]}f(\cdot)\right|\right\|_{L^p(rB^n-t\xi)}\\ &= \left\|\left|\nabla^{[s]}f(\cdot)\right|-\left|\nabla^{[s]}f(\cdot-t\xi)\right|\right\|_{L^p(rB^n)}\\ &\geq \left\|\left|\nabla^{[s]}f\right|\right\|_{L^p(rB^n)} - \left\|\left|\nabla^{[s]}f(\cdot-t\xi)\right|\right\|_{L^p(rB^n)}\\ &\geq \frac{2}{3}\left\|\left|\nabla^{[s]}f\right|\right\|_p - \frac{1}{3}\left\|\left|\nabla^{[s]}f\right|\right\|_p\\ &= \frac{1}{3}\left\|\left|\nabla^{[s]}f\right|\right\|_p.\end{aligned}$$

Moreover, we have

$$\begin{aligned}&\int_0^\infty t^{-p\{s\}-1}\int_{\mathbb{R}^n}\left|\left|\nabla^{[s]}f(x+t\xi)\right|-\left|\nabla^{[s]}f(x)\right|\right|^p dxdt\\ \geq\ & \frac{\left\|\left|\nabla^{[s]}f\right|\right\|_p^p}{3^p}\int_r^\infty t^{-p\{s\}-1}dt\\ \geq\ & \frac{\left\|\left|\nabla^{[s]}f\right|\right\|_p^p}{3^p}\frac{r^{-p\{s\}}}{p\{s\}} \geq c.\end{aligned}$$

This implies that $\Pi_p^{*,s}(\nabla_s f) \subseteq cB^n$ for $c > 0$ independent of s.

Next, we show that $\Pi_p^{*,s}(\nabla_s f)$ contains the origin in its interior. For $\xi, \eta \in \mathbb{R}^n$, by the triangle inequality, we obtain

$$\begin{aligned}&\|\xi+\eta\|_{\Pi_p^{*,s}(\nabla_s f)}^{p\{s\}}\\ =\ & \int_0^\infty t^{-p\{s\}-1}\left\|\left|\nabla^{[s]}f(\cdot+t\xi+t\eta)\right|-\left|\nabla^{[s]}f(\cdot)\right|\right\|_p^p dt\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^\infty t^{-p\{s\}-1}\left(\left\|\left|\nabla^{[s]}f(\cdot+t\xi+t\eta)\right|-\left|\nabla^{[s]}f(\cdot+t\xi)\right|\right\|_p+\left\|\left|\nabla^{[s]}f(\cdot+t\xi)\right|-\left|\nabla^{[s]}f(\cdot)\right|\right\|_p\right)^p dt\\
&\leq \int_0^\infty t^{-p\{s\}-1}2^{p-1}\left(\left\|\left|\nabla^{[s]}f(\cdot+t\xi+t\eta)\right|-\left|\nabla^{[s]}f(\cdot+t\xi)\right|\right\|_p^p+\left\|\left|\nabla^{[s]}f(\cdot+t\xi)\right|-\left|\nabla^{[s]}f(\cdot)\right|\right\|_p^p\right)\mathrm{d}t\\
&\leq \int_0^\infty t^{-p\{s\}-1}2^{p-1}\left(\left\|\left|\nabla^{[s]}f(\cdot+t\eta)\right|-\left|\nabla^{[s]}f(\cdot)\right|\right\|_p^p+\left\|\left|\nabla^{[s]}f(\cdot+t\xi)\right|-\left|\nabla^{[s]}f(\cdot)\right|\right\|_p^p\right)\mathrm{d}t\\
&= 2^{p-1}\|\eta\|_{\Pi_p^{*,s}(\nabla_s f)}^{p\{s\}}+2^{p-1}\|\xi\|_{\Pi_p^{*,s}(\nabla_s f)}^{p\{s\}}. \qquad (3.3)
\end{aligned}
$$

Since $f \in V_d^{s,p}(\mathbb{R}^n)$, using (3.2) with $K = B^n$, we have

$$
\begin{aligned}
\int_{\mathbb{S}^{n-1}}\|\xi\|_{\Pi_p^{*,s}(\nabla_s f)}^{p\{s\}}d\xi &= \tilde{V}_{-p\{s\}}\left(B^n,\Pi_p^{*,s}(\nabla_s f)\right)\\
&= \frac{1}{n}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\left(\frac{\left\|\nabla^{[s]}f(x)\right|-\left|\nabla^{[s]}f(y)\right\|}{\|x-y\|_{B^n}^{\{s\}}}\right)^p\frac{1}{\|x-y\|_{B^n}^n}dxdy\\
&= \frac{1}{n}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\left\|\nabla^{[s]}f(x)\right|-\left|\nabla^{[s]}f(y)\right\|^p}{|x-y|^{n+p\{s\}}}dxdy,
\end{aligned}
$$

which is finite.

Now, we choose $r > 0$ large enough such that the set $A = \left\{\xi \in \mathbb{S}^{n-1} : \|\xi\|_{\Pi_p^{*,s}(\nabla_s f)}^{p\{s\}} < r\right\}$ has positive $(n-1)$-dimensional Hausdorff measure and contains a basis $\{\xi_1, \ldots, \xi_n\} \subseteq A$ of $\mathbb{R}^n$. Without loss of generality, we may assume that $\xi_i = e_i (i = 1, 2, \cdots, n)$ are the canonical basis vectors. For every $x \in \mathbb{R}^n$, we write $x = \sum x_i e_i$. Using (3.3), we note that the norms $|\cdot|_K$ and $|\cdot|_{B^n}$ are equivalent; see [30]. Then,

$$
\|x\|_{\Pi_p^{*,s}(\nabla_s f)} \leq \left(2^{n(p-1)}\sum_{i=1}^n |x_i|^{p\{s\}}\,\|e_i\|_{\Pi_p^{*,s}(\nabla_s f)}^{p\{s\}}\right)^{\frac{1}{p\{s\}}} \leq d|x|, \qquad (3.4)
$$

where $d > 0$ is independent of x. This implies that $\Pi_p^{*,s}(\nabla_s f)$ has the origin as an interior point.

Finally, we show that $\|\cdot\|_{\Pi_p^{*,s}(\nabla_s f)}$ is continuous. Using Hölder's inequality, we obtain

$$
\begin{aligned}
a+b &= r(r^{-1}a)+1\cdot b\\
&\leq (r^q+1)^{1/q}\left[\left(r^{-1}a\right)^p+b^p\right]^{1/p}\\
&= \left(1+r^{p/(p-1)}\right)^{(p-1)/p}\left[\left(r^{-1}a\right)^p+b^p\right]^{1/p}. \qquad (3.5)
\end{aligned}
$$

Here, $a, b, r > 0$ and $1/p + 1/q = 1 (p, q > 1)$. For $\xi, \eta \in \mathbb{R}^n$, by the triangle inequality, (3.5), and (3.4), we take $r = |\eta|^{\frac{p\{s\}}{2}}$. Then

$$
\begin{aligned}
&\|\xi+\eta\|_{\Pi_p^{*,s}(\nabla_s f)}^{p\{s\}}\\
&= \int_0^\infty t^{-1-p\{s\}}\left\|\left|\nabla^{[s]}f(\cdot+t\xi+t\eta)\right|-\left|\nabla^{[s]}f(\cdot)\right|\right\|_p^p dt\\
&\leq \int_0^\infty t^{-1-p\{s\}}\left(\left\|\left|\nabla^{[s]}f(\cdot+t\eta)\right|-\left|\nabla^{[s]}f(\cdot)\right|\right\|_p+\left\|\left|\nabla^{[s]}f(\cdot+t\xi)\right|-\left|\nabla^{[s]}f(\cdot)\right|\right\|_p\right)^p dt
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left(1+|\eta|^{\frac{\{s\}}{2}\cdot\frac{p}{p-1}}\right)^{p-1}\int_0^{\infty} t^{-1-p\{s\}}\left(\frac{\left\|\left|\nabla^{[s]}f(\cdot+t\eta)\right|-\left|\nabla^{[s]}f(\cdot)\right|\right\|_p^p}{|\eta|^{\frac{p\{s\}}{2}}}+\left\|\left|\nabla^{[s]}f(\cdot+t\xi)\right|-\left|\nabla^{[s]}f(\cdot)\right|\right\|_p^p\right)\mathrm{d}t\\
&= \left(1+|\eta|^{\frac{\{s\}}{2}\cdot\frac{p}{p-1}}\right)^{p-1}\left(|\eta|^{-\frac{p\{s\}}{2}}\|\eta\|_{\Pi_p^{*,s}(\nabla_s f)}^{p\{s\}}+\|\xi\|_{\Pi_p^{*,s}(\nabla_s f)}^{p\{s\}}\right)\\
&\leq \left(1+|\eta|^{\frac{\{s\}}{2}\cdot\frac{p}{p-1}}\right)^{p-1}\left(d^{p\{s\}}|\eta|^{\frac{p\{s\}}{2}}+\|\xi\|_{\Pi_p^{*,s}(\nabla_s f)}^{p\{s\}}\right).
\end{aligned}
$$

Thus, we obtain

$$
\|\xi+\eta\|_{\Pi_p^{*,s}(\nabla_s f)}^{p\{s\}} \leq \left(1+|\eta|^{\frac{\{s\}}{2}\cdot\frac{p}{p-1}}\right)^{p-1}\left(d|\eta|^{\frac{p\{s\}}{2}}+\|\xi\|_{\Pi_p^{*,s}(\nabla_s f)}^{p\{s\}}\right). \tag{3.6}
$$

Applying inequality (3.6) for the two vectors $\xi+\eta$ and $-\eta$, we obtain

$$
\begin{aligned}
\|\xi\|_{\Pi_p^{*,s}(\nabla_s f)}^{p\{s\}} &= \|\xi+\eta-\eta\|_{\Pi_p^{*,s}(\nabla_s f)}^{p\{s\}}\\
&\leq \left(1+|-\eta|^{\frac{\{s\}}{2}\cdot\frac{p}{p-1}}\right)^{p-1}\left(d|-\eta|^{\frac{p\{s\}}{2}}+\|\xi+\eta\|_{\Pi_p^{*,s}(\nabla_s f)}^{p\{s\}}\right).
\end{aligned}
$$

Moreover, we have

$$
\|\xi+\eta\|_{\Pi_p^{*,s}(\nabla_s f)}^{p\{s\}} \geq \left(1+|\eta|^{\frac{\{s\}}{2}\cdot\frac{p}{p-1}}\right)^{1-p}\|\xi\|_{\Pi_p^{*,s}(\nabla_s f)}^{p\{s\}}-d|\eta|^{\frac{p\{s\}}{2}}. \tag{3.7}
$$

For $\forall\xi\in\mathbb{R}^n$, taking $|\eta|\to 0$, by (3.6) and (3.7), we get

$$
\|\xi+\eta\|_{\Pi_p^{*,s}(\nabla_s f)}^{p\{s\}} \to \|\xi\|_{\Pi_p^{*,s}(\nabla_s f)}^{p\{s\}}.
$$

The continuity of $\|\cdot\|_{\Pi_p^{*,s}(\nabla_s f)}$ now is proved. □

4. Fractional asymmetric L^p Sobolev polar projection bodies

Let the fractional-order parameter $s\in(0,n)\setminus\mathbb{N}$ and $1<p<n/s$. For $f\in V_d^{s,p}(\mathbb{R}^n)$ and a star body $K\subseteq\mathbb{R}^n$, the asymmetric s-Sobolev polar projection body $\Pi_{p,\pm}^{*,s}(\nabla_s f)$ of f is defined by the following gauge function:

$$
\|\xi\|_{\Pi_{p,\pm}^{*,s}(\nabla_s f)}^{p\{s\}} = \int_0^{+\infty} t^{-1-p\{s\}}\int_{\mathbb{R}^n}\left(\left|\nabla^{[s]}f(x+t\xi)\right|-\left|\nabla^{[s]}f(x)\right|\right)_{\pm}^{p}dxdt.
$$

Here, $\xi\in\mathbb{R}^n$, $a_+=\max\{a,0\}$, and $a_-=\max\{-a,0\}$ for $a\in\mathbb{R}$. Then, we prove

$$
\Pi_{p,-}^{*,s}(\nabla_s f)=\Pi_{p,+}^{*,s}(-\nabla_s f)=-\Pi_{p,+}^{*,s}(\nabla_s f). \tag{4.1}
$$

Since the negative part of f is equal to the positive part of $-f$, we obtain $\Pi_{p,-}^{*,s}(\nabla_s f)=\Pi_{p,+}^{*,s}(-\nabla_s f)$. Then, it holds that

$$
\begin{aligned}
\|\xi\|_{\Pi_{p,-}^{*,s}(\nabla_s f)}^{p\{s\}} &= \int_0^{+\infty} t^{-1-p\{s\}}\int_{\mathbb{R}^n}\left(\left|\nabla^{[s]}f(x+t\xi)\right|-\left|\nabla^{[s]}f(x)\right|\right)_{-}^{p}dxdt\\
&= \int_0^{+\infty} t^{-1-p\{s\}}\int_{\mathbb{R}^n}\left(\left|\nabla^{[s]}f(x)\right|-\left|\nabla^{[s]}f(x+t\xi)\right|\right)_{+}^{p}dxdt
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{+\infty} t^{-1-p\{s\}} \int_{\mathbb{R}^n} \left(\left|\nabla^{[s]} f(y-t\xi)\right| - \left|\nabla^{[s]} f(y)\right|\right)_+^p dydt \\
&= \|-\xi\|^{p\{s\}}_{\Pi^{*,s}_{p,+}(\nabla_s f)} = \|\xi\|^{p\{s\}}_{-\Pi^{*,s}_{p,+}(\nabla_s f)}.
\end{aligned}
$$

Thus, we proved formula (4.1). Next, we only need to investigate the properties of $\Pi^{*,s}_{p,+}(\nabla_s f)$. Also, we have

$$
\begin{aligned}
&\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(\left|\nabla^{[s]} f(x)\right| - \left|\nabla^{[s]} f(y)\right|\right)_+^p}{\|x-y\|_K^{n+p\{s\}}} dxdy \\
= &\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(\left|\nabla^{[s]} f(y+z)\right| - \left|\nabla^{[s]} f(y)\right|\right)_+^p}{\|z\|_K^{n+p\{s\}}} dzdy \\
= &\int_{\mathbb{R}^n}\int_0^{+\infty}\int_{\mathbb{S}^{n-1}} \frac{\left(\left|\nabla^{[s]} f(y+t\xi)\right| - \left|\nabla^{[s]} f(y)\right|\right)_+^p}{\|t\xi\|_K^{n+p\{s\}}} t^{n-1} d\xi dtdy \\
= &\int_{\mathbb{R}^n}\int_0^{+\infty}\int_{\mathbb{S}^{n-1}} t^{-1-p\{s\}} \left(\left|\nabla^{[s]} f(y+t\xi)\right| - \left|\nabla^{[s]} f(y)\right|\right)_+^p \|\xi\|_K^{-n-p\{s\}} d\xi dtdy.
\end{aligned}
$$

By Tonelli's theorem,

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\int_0^{+\infty}\int_{\mathbb{S}^{n-1}} t^{-1-p\{s\}} \left(\left|\nabla^{[s]} f(y+t\xi)\right| - \left|\nabla^{[s]} f(y)\right|\right)_+^p \|\xi\|_K^{-n-p\{s\}} d\xi dtdy \\
= &\int_{\mathbb{S}^{n-1}} \|\xi\|_K^{-n-p\{s\}} \left(\int_0^{+\infty} t^{-1-p\{s\}} \int_{\mathbb{R}^n} \left(\left|\nabla^{[s]} f(y+t\xi)\right| - \left|\nabla^{[s]} f(y)\right|\right)_+^p dydt\right) d\xi \\
= &\int_{\mathbb{S}^{n-1}} \rho_K(\xi)^{n+p\{s\}} \cdot \rho_{\Pi^{*,s}_{p,+}(\nabla_s f)}(\xi)^{-p\{s\}} d\xi = n\tilde{V}_{-p\{s\}}\left(K, \Pi^{*,s}_{p,+}(\nabla_s f)\right).
\end{aligned}
$$

Thus, we obtain

$$
\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(\left|\nabla^{[s]} f(x)\right| - \left|\nabla^{[s]} f(y)\right|\right)_+^p}{\|x-y\|_K^{n+p\{s\}}} dxdy = n\tilde{V}_{-p\{s\}}\left(K, \Pi^{*,s}_{p,+}(\nabla_s f)\right). \tag{4.2}
$$

Proposition 4.1. *Let $f \in V_d^{s,p}(\mathbb{R}^n)$. Then, $\Pi^{*,s}_{p,+}(\nabla_s f)$ contains the origin in its interior, and $\|\xi\|^{p\{s\}}_{\Pi^{*,s}_{p,+}(\nabla_s f)}$ is continuous. Moreover, there exists $c > 0$ depending only on f and p such that $\Pi^{*,s}_{p,+}(\nabla_s f) \subseteq cB^n$ for every $s \in (0,n)\setminus\mathbb{N}$ and $1 < p < n/s$, and $\Pi^{*,s}_{p,+}(\nabla_s f)$ is a star body.*

Proof. First, we prove that $\|\xi\|^{p\{s\}}_{\Pi^{*,s}_{p,+}(\nabla_s f)} (\xi \in \mathbb{S}^{n-1})$ has an upper bound. Using the monotonicity of the power function t^p, we have

$$
\begin{aligned}
\|\xi\|^{p\{s\}}_{\Pi^{*,s}_{p,+}(\nabla_s f)} &= \int_0^{+\infty} t^{-1-p\{s\}} \int_{\mathbb{R}^n} \left(\left|\nabla^{[s]} f(x+t\xi)\right| - \left|\nabla^{[s]} f(x)\right|\right)_+^p dxdt \\
&\leq \int_0^{+\infty} t^{-1-p\{s\}} \int_{\mathbb{R}^n} \left|\left|\nabla^{[s]} f(x+t\xi)\right| - \left|\nabla^{[s]} f(x)\right|\right|^p dxdt = \|\xi\|^{p\{s\}}_{\Pi^{*,s}_{p}(\nabla_s f)}.
\end{aligned}
$$

In the previous section, it was proved that for symmetric polar projection bodies, $\|\xi\|^{p\{s\}}_{\Pi^{*,s}_{p}(\nabla_s f)} \leq (d|\xi|)^{p\{s\}}$ holds, where $d > 0$ is a constant. Thus, $\Pi^{*,s}_{p,+}(\nabla_s f)$ contains the origin in its interior.

Next, we prove $\Pi_{p,+}^{*,s}(\nabla_s f)$ is bounded. According to the Jensen inequality

$$\begin{aligned}(a+b)_+^p &\geq (a)_+^p + p(a)_+^{p-1}b\\ (a+b)_-^p &\geq (a)_-^p - p(a)_-^{p-1}b,\end{aligned}$$

where $a, b \in \mathbb{R}$ and $p > 1$. Indeed, consider the convex function $\phi(t) = (t)_+^p$. Since $p > 1$, ϕ is differentiable on $\mathbb{R}$ with derivative $\phi'(t) = p(t)_+^{p-1}$. By the first-order condition for convex functions (the subgradient inequality), we have for any $a, b \in \mathbb{R}$,

$$\phi(a+b) \geq \phi(a) + \phi'(a) \cdot b,$$

which yields exactly

$$(a+b)_+^p \geq (a)_+^p + p(a)_+^{p-1}b, \quad \forall a, b \in \mathbb{R}.$$

The second inequality for the negative part,

$$(a+b)_-^p \geq (a)_-^p - p(a)_-^{p-1}b,$$

follows analogously by replacing a, b with $-a, -b$ in the first inequality.

Since $\int_{\mathbb{R}^n} \left|\nabla^{[s]}f(x)\right|^p dx > 0$, take $\varepsilon > 0$ small such that

$$\varepsilon + p\varepsilon^{1/p} \left\|\left|\nabla^{[s]}f\right|\right\|_p^{p-1} \leq \frac{1}{2}\int_{\mathbb{R}^n} \left|\nabla^{[s]}f(x)\right|^p dx,$$

and take $r > 0$ large such that $\int_{\mathbb{R}^n\setminus rB^n} \left|\nabla^{[s]}f(x)\right|^p dx < \varepsilon$. For $z \in \mathbb{R}^n \setminus 2rB^n$, by Hölder's inequality, we obtain

$$\begin{aligned}
&\int_{rB^n} \left(\left|\nabla^{[s]}f(x)\right| - \left|\nabla^{[s]}f(x+z)\right|\right)_+^p dx\\
\geq\ &\int_{rB^n} \left(\left|\nabla^{[s]}f(x)\right|\right)_+^p - p\left(\left|\nabla^{[s]}f(x)\right|\right)_+^{p-1} \left|\nabla^{[s]}f(x+z)\right| dx\\
\geq\ &\int_{rB^n} \left|\nabla^{[s]}f(x)\right|^p dx - p\left(\int_{rB^n} \left|\nabla^{[s]}f(x)\right|^p dx\right)^{\frac{p-1}{p}} \left(\int_{rB^n} |\nabla^{[s]}f(x+z)|^p dx\right)^{\frac{1}{p}}\\
\geq\ &\int_{rB^n} \left|\nabla^{[s]}f(x)\right|^p dx - p\left(\int_{\mathbb{R}^n} |\nabla^{[s]}f(x)|^p dx\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n\setminus rB^n} |\nabla^{[s]}f(x)|^p dx\right)^{\frac{1}{p}}\\
\geq\ &\int_{\mathbb{R}^n} \left|\nabla^{[s]}f(x)\right|^p dx - \varepsilon - p\|\nabla^{[s]}f\|_p^{p-1}\varepsilon^{\frac{1}{p}}\\
\geq\ &\frac{1}{2}\int_{\mathbb{R}^n} \left|\nabla^{[s]}f(x)\right|^p dx. \qquad (4.3)
\end{aligned}$$

In the case $\int_{\mathbb{R}^n} \left|\nabla^{[s]}f(x)\right|^p dx = 0$, the previous inequality holds trivially for any $r > 0$. By an analogous calculation and eventually increasing the value of r, we obtain

$$\int_{rB^n - z} \left(\left|\nabla^{[s]}f(x)\right| - \left|\nabla^{[s]}f(x+z)\right|\right)_+^p dx \ = \ \int_{rB^n} \left(\left|\nabla^{[s]}f(x)\right| - \left|\nabla^{[s]}f(x-z)\right|\right)_-^p dx$$

$$\geq \frac{1}{2}\int_{\mathbb{R}^n}\left|\nabla^{[s]}f(x)\right|^p dx. \tag{4.4}$$

By (4.3) and (4.4), it follows that

$$\int_{\mathbb{R}^n}\left(\left|\nabla^{[s]}f(x)\right|-\left|\nabla^{[s]}f(x+z)\right|\right)_+^p dx \geq \frac{1}{2}\left|\left|\left|\nabla^{[s]}f\right|\right|\right|_p^p,$$

for every $z\in\mathbb{R}^n\setminus 2rB^n$ with $r>0$ depending only on f. Then,

$$\begin{aligned}
\|\xi\|^{p\{s\}}_{\Pi^{*,s}_{p,+}(\nabla_s f)} &\geq \int_{2r}^{\infty} t^{-1-p\{s\}}\int_{\mathbb{R}^n}\left(\left|\nabla^{[s]}f(x)\right|-\left|\nabla^{[s]}f(x-t\xi)\right|\right)_+^p dxdt\\
&\geq \frac{1}{2}\int_{2r}^{\infty} t^{-1-p\{s\}}dt\int_{\mathbb{R}^n}\left|\nabla^{[s]}f(x)\right|^p dx\\
&\geq \frac{1}{2}\frac{(2r)^{-p\{s\}}}{p\{s\}}\int_{\mathbb{R}^n}\left|\nabla^{[s]}f(x)\right|^p dx\\
&\geq \frac{(2r)^{-p}}{2p}\left|\left|\left|\nabla^{[s]}f\right|\right|\right|_p^p.
\end{aligned}$$

It is indicates that there exists $c>0$ depending only on f and p such that $\Pi^{*,s}_{p,+}(\nabla_s f)\subseteq cB^n$ for every $s\in(0,n)\setminus\mathbb{N}$ and $1<p<n/s$.

Finally, we prove that $\|\xi\|^{p\{s\}}_{\Pi^{*,s}_{p,+}(\nabla_s f)}$ is continuous. According to the conclusion of the previous section, we know

$$\|\xi\|^{p\{s\}}_{\Pi^{*,s}_{p,+}(\nabla_s f)} \leq \|\xi\|^{p\{s\}}_{\Pi^{*,s}_{p}(\nabla_s f)} \leq d|\xi|,$$

where $d>0$ is an appropriate constant. Next, arbitrarily take a sequence $\{\xi_k\}_{k=1}^{+\infty}\subseteq\mathbb{S}^{n-1}$, satisfying that as $k\to+\infty$, $\xi_k\to\xi$. Taking

$$F_k(t)=t^{-1-p\{s\}}\int_{\mathbb{R}^n}\left(\left|\nabla^{[s]}f(x+t\xi_k)\right|-\left|\nabla^{[s]}f(x)\right|\right)_+^p dx,$$

since $f\in V_d^{s,p}(\mathbb{R}^n)$, using the dominated convergence theorem, we obtain

$$\begin{aligned}
\lim_{k\to+\infty}F_k(t) &= \lim_{k\to+\infty}t^{-1-p\{s\}}\int_{\mathbb{R}^n}\left(\left|\nabla^{[s]}f(x+t\xi_k)\right|-\left|\nabla^{[s]}f(x)\right|\right)_+^p dx\\
&= t^{-1-p\{s\}}\int_{\mathbb{R}^n}\left(\left|\nabla^{[s]}f(x+t\xi)\right|-\left|\nabla^{[s]}f(x)\right|\right)_+^p dx.
\end{aligned}$$

Taking

$$F(t)=t^{-1-p\{s\}}\int_{\mathbb{R}^n}\left(\left|\nabla^{[s]}f(x+t\xi)\right|-\left|\nabla^{[s]}f(x)\right|\right)_+^p dx,$$

then

$$\int_0^{+\infty}F(t)dt\leq\|\xi\|^{p\{s\}}_{\Pi^{*,s}_{p}(\nabla_s f)}<+\infty.$$

Using the dominated convergence theorem once again, we obtain

$$\begin{aligned}
\lim_{k\to+\infty}\int_0^{+\infty}F_k(t)dt &= \int_0^{+\infty}t^{-1-p\{s\}}\int_{\mathbb{R}^n}\left(\left|\nabla^{[s]}f(x+t\xi)\right|-\left|\nabla^{[s]}f(x)\right|\right)_+^p dxdt\\
&= \|\xi\|^{p\{s\}}_{\Pi^{*,s}_{p,+}(\nabla_s f)}.
\end{aligned}$$

Due to the arbitrariness of the sequence $\{\xi_k\}_{k=1}^{+\infty}$, we obtain the continuity of $\|\xi\|^{p\{s\}}_{\Pi^{*,s}_{p,+}(\nabla_s f)}$. □

5. Anisotropic Pólya–Szegö inequalities

In this section, we derive higher-order Pólya–Szegö-type inequalities for functions involving the Euclidean norm of the gradient. Similarly, we present both symmetric and asymmetric versions. Specifically, we consider higher-order fractional parameter $s \in (0,n) \setminus \mathbb{N}$ and $1 < p < n/s$.

Proposition 5.1. *Let $f \in V_d^{s,p}(\mathbb{R}^n)$ and $K \subseteq \mathbb{R}^n$ be a star body, $s \in (0,n) \setminus \mathbb{N}$, and $1 < p < n/s$. Then,*

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(\left|\nabla^{[s]} f(x)\right| - \left|\nabla^{[s]} f(y)\right|\right)_+^p}{\|x-y\|_K^{n+p\{s\}}} dxdy \geq \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(\left|\nabla^{[s]} f(x)\right|^\star - \left|\nabla^{[s]} f(y)\right|^\star\right)_+^p}{\|x-y\|_{K^\star}^{n+p\{s\}}} dxdy.$$

Equality holds if and only if K is a central ellipsoid and $\left|\nabla^{[s]} f\right|$ is a translate of $\left|\nabla^{[s]} f\right|^\star \circ \phi$ for some $\phi \in \mathrm{SL}(n)$.

Proof. First, we claim the following layer cake formula: For every $z \in \mathbb{R}^n$,

$$\frac{1}{\|z\|_K^{n+p\{s\}}} = \int_0^{+\infty} \mathbf{1}_{\psi(t)K}(z)\, dt,$$

where $\psi(t) := t^{-1/(n+p\{s\})}$.

Indeed, for $z \neq 0$, since $\|\cdot\|_K$ is the gauge function of the convex body K, we have the equivalence

$$z \in \psi(t)K \quad \Longleftrightarrow \quad \|z\|_K \leq \psi(t) = t^{-1/(n+p\{s\})}.$$

The function $t \mapsto t^{-1/(n+p\{s\})}$ is strictly decreasing on $(0,\infty)$. Hence, the above condition is equivalent to

$$t \leq \|z\|_K^{-(n+p\{s\})} = \frac{1}{\|z\|_K^{n+p\{s\}}}.$$

Consequently,

$$\int_0^{+\infty} \mathbf{1}_{\psi(t)K}(z)\, dt = \int_0^{\|z\|_K^{-(n+p\{s\})}} 1\, dt = \frac{1}{\|z\|_K^{n+p\{s\}}},$$

which proves the desired identity. The case $z = 0$ is trivial since both sides are infinite (or follows by a standard limiting argument). Denoting $k_t(z) = \mathbf{1}_{\psi(t)K}(z)$, we obtain

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(\left|\nabla^{[s]} f(x)\right| - \left|\nabla^{[s]} f(y)\right|\right)_+^p}{\|x-y\|_K^{n+p\{s\}}} dxdy = \iint_{\mathbb{R}^n\times\mathbb{R}^n} \left(\left|\nabla^{[s]} f(x)\right| - \left|\nabla^{[s]} f(y)\right|\right)_+^p \left(\int_0^{+\infty} k_t(x-y)dt\right) dxdy. \tag{5.1}$$

Consider $\left|\nabla^{[s]} f(\cdot)\right|$ as a function from $\mathbb{R}^n$ to $\mathbb{R}$. It is easy to verify that

$$\left(\left|\nabla^{[s]} f(x)\right| - \left|\nabla^{[s]} f(y)\right|\right)_+^p = \int_0^{+\infty} p\left(\left|\nabla^{[s]} f(x)\right| - r\right)_+^{p-1} \cdot \mathbf{1}_{\{|\nabla^{[s]} f(\cdot)| \leq r\}}(y) dr. \tag{5.2}$$

By (5.1), (5.2), and Tonelli's theorem, we have

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(\left|\nabla^{[s]} f(x)\right| - \left|\nabla^{[s]} f(y)\right|\right)_+^p}{\|x-y\|_K^{n+p\{s\}}} dxdy$$

$$= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \left(\int_0^{+\infty} p\left(\left|\nabla^{[s]}f(x)\right| - r\right)_+^{p-1} \cdot \mathbf{1}_{\{|\nabla^{[s]}f(\cdot)|\le r\}}(y)dr\right)\left(\int_0^{+\infty} k_t(x-y)dt\right)dxdy$$
$$= \iint_{\mathbb{R}_+\times\mathbb{R}_+} \left(\iint_{\mathbb{R}^n\times\mathbb{R}^n} p\left(\left|\nabla^{[s]}f(x)\right| - r\right)_+^{p-1} \cdot \mathbf{1}_{\{|\nabla^{[s]}f(\cdot)|\le r\}}(y)\cdot k_t(x-y)dxdy\right)drdt. \tag{5.3}$$

Therefore, we obtain

$$\begin{aligned}
&\iint_{\mathbb{R}^n\times\mathbb{R}^n} p\left(\left|\nabla^{[s]}f(x)\right| - r\right)_+^{p-1} \cdot \mathbf{1}_{\{|\nabla^{[s]}f(\cdot)|\le r\}}(y)\cdot k_t(x-y)dxdy\\
=\ &\iint_{\mathbb{R}^n\times\mathbb{R}^n} p\left(\left|\nabla^{[s]}f(x)\right| - r\right)_+^{p-1} \cdot \left(\mathbf{1} - \mathbf{1}_{\{|\nabla^{[s]}f(\cdot)|> r\}}(y)\right)\cdot k_t(x-y)dxdy\\
=\ & p\psi(t)^n|K|\int_{\mathbb{R}^n}\left(\left|\nabla^{[s]}f(x)\right| - r\right)_+^{p-1}dx - p\iint_{\mathbb{R}^n\times\mathbb{R}^n}\left(\left|\nabla^{[s]}f(x)\right| - r\right)_+^{p-1} \cdot \mathbf{1}_{\{|\nabla^{[s]}f(\cdot)|> r\}}(y)\cdot k_t(x-y)dxdy.
\end{aligned}$$

According to the Riesz rearrangement inequality (Theorem 2.1), the following inequality holds:

$$\begin{aligned}
&\iint_{\mathbb{R}^n\times\mathbb{R}^n} \left(\left|\nabla^{[s]}f(x)\right| - r\right)_+^{p-1} \cdot \mathbf{1}_{\{|\nabla^{[s]}f(\cdot)|> r\}}(y)\cdot k_t(x-y)dxdy\\
\le\ &\iint_{\mathbb{R}^n\times\mathbb{R}^n} \left(\left|\nabla^{[s]}f(x)\right|^\star - r\right)_+^{p-1} \cdot \mathbf{1}_{\{|\nabla^{[s]}f(\cdot)|^\star> r\}}(y)\cdot k_t^\star(x-y)dxdy.
\end{aligned}$$

Since $f \in V_d^{s,p}(\mathbb{R}^n)$, the measures of level sets of $\left(\left|\nabla^{[s]}f(x)\right| - r\right)_+^{p-1}$ do not change under Schwarz symmetrization of $\left|\nabla^{[s]}f\right|$. Therefore, we obtain

$$\begin{aligned}
&\iint_{\mathbb{R}^n\times\mathbb{R}^n} p\left(\left|\nabla^{[s]}f(x)\right| - r\right)_+^{p-1} \cdot \mathbf{1}_{\{|\nabla^{[s]}f(\cdot)|\le r\}}(y)\cdot k_t(x-y)dxdy\\
\ge\ &\iint_{\mathbb{R}^n\times\mathbb{R}^n} p\left(\left|\nabla^{[s]}f(x)\right|^\star - r\right)_+^{p-1} \cdot \mathbf{1}_{\{|\nabla^{[s]}f(\cdot)|^\star\le r\}}(y)\cdot k_t^\star(x-y)dxdy.
\end{aligned}$$

Combining (5.3), we have

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(\left|\nabla^{[s]}f(x)\right| - \left|\nabla^{[s]}f(y)\right|\right)_+^p}{\|x-y\|_K^{n+p\{s\}}}dxdy \ge \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(\left|\nabla^{[s]}f(x)\right|^\star - \left|\nabla^{[s]}f(y)\right|^\star\right)_+^p}{\|x-y\|_{K^\star}^{n+p\{s\}}}dxdy.$$

Next, we investigate the equality condition of this inequality. By the layer cake formula,

$$\left(\left|\nabla^{[s]}f(x)\right| - r\right)_+^{p-1} = \int_0^{+\infty}(p-1)\left(\tilde{r}-r\right)_+^{p-2}\cdot \mathbf{1}_{\{|\nabla^{[s]}f(\cdot)|>\tilde{r}\}}(x)d\tilde{r}.$$

Thus, we have

$$\begin{aligned}
&\iint_{\mathbb{R}^n\times\mathbb{R}^n} \left(\left|\nabla^{[s]}f(x)\right| - r\right)_+^{p-1} \cdot \mathbf{1}_{\{|\nabla^{[s]}f(\cdot)|> r\}}(y)\cdot k_t(x-y)dxdy\\
=\ &\iint_{\mathbb{R}^n\times\mathbb{R}^n}\left(\int_{\mathbb{R}_+}(p-1)\left(\tilde{r}-r\right)_+^{p-2}\cdot \mathbf{1}_{\{|\nabla^{[s]}f(\cdot)|>\tilde{r}\}}(x)d\tilde{r}\right)\mathbf{1}_{\{|\nabla^{[s]}f(\cdot)|> r\}}(y)k_t(x-y)dxdy
\end{aligned}$$

$$= \quad (p-1)\int_{\mathbb{R}_+}\left(\iint_{\mathbb{R}^n\times\mathbb{R}^n}\mathbf{1}_{\{|\nabla^{[s]}f(\cdot)|>\tilde{r}\}}(x)\cdot\mathbf{1}_{\{|\nabla^{[s]}f(\cdot)|>r\}}(y)k_t(x-y)dxdy\right)(\tilde{r}-r)_+^{p-2}\,d\tilde{r}.$$

We also apply Fubini's theorem at the end of the above equality.

According to Burchard's Theorem 2.2, we know that if the following equation holds for almost everywhere $r, \tilde{r}, t \in \mathbb{R}_+^3$,

$$\begin{aligned}&\iint_{\mathbb{R}^n\times\mathbb{R}^n}\mathbf{1}_{\{|\nabla^{[s]}f(\cdot)|>\tilde{r}\}}(x)\cdot\mathbf{1}_{\{|\nabla^{[s]}f(\cdot)|>r\}}(y)\cdot k_t(x-y)dxdy\\ =\ &\iint_{\mathbb{R}^n\times\mathbb{R}^n}\mathbf{1}_{\{|\nabla^{[s]}f(\cdot)|^\star>\tilde{r}\}}(x)\cdot\mathbf{1}_{\{|\nabla^{[s]}f(\cdot)|^\star>r\}}(y)\cdot k_t^\star(x-y)dxdy,\end{aligned}\tag{5.4}$$

then equality holds in Proposition 5.1.

That is, for almost every $(\tilde{r}, r) \in (0,\infty)^2$, (5.4) holds such that $(\tilde{r}, r, t) \in (0,\infty)^3$ for almost every $t > 0$. For such $(\tilde{r}, r)$ with $\tilde{r} \le r$ and sufficiently large $t > 0$, the assumptions of Theorem 2.2 are satisfied. Hence, there exist a centered ellipsoid D and $a, b \in \mathbb{R}^n$ (depending on $(\tilde{r}, r, t)$) such that

$$\left\{\left|\nabla^{[s]}f(\cdot)\right| > \tilde{r}\right\} = a + \alpha D, \quad \psi(t)K = b + \beta D, \quad \left\{\left|\nabla^{[s]}f(\cdot)\right| > r\right\} = c + \gamma D,$$

where $c = a + b$. Since $K = b/\psi(t) + (|K|/|D|)^{1/n}D$, the centered ellipsoid D is independent of $(\tilde{r}, r, t)$, and a, c are independent of t. This implies that $b = 0$ and K is a multiple of ellipsoid D. Therefore, $a = c$ is a constant vector, which implies that $\left|\nabla^{[s]}f\right|$ is a translate of $\left|\nabla^{[s]}f\right|^\star \circ \phi$ for some $\phi \in \mathrm{SL}(n)$. □

Proposition 5.2. *Let $f \in V_d^{s,p}(\mathbb{R}^n)$ and $K \subseteq \mathbb{R}^n$ be a star body, $s \in (0,n)\setminus\mathbb{N}$, and $1 < p < n/s$. Then,*

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\left|\left|\nabla^{[s]}f(x)\right|-\left|\nabla^{[s]}f(y)\right|\right|^p}{\|x-y\|_K^{n+p\{s\}}}dxdy \ge \iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\left|\left|\nabla^{[s]}f(x)\right|^\star-\left|\nabla^{[s]}f(y)\right|^\star\right|^p}{\|x-y\|_{K^\star}^{n+p\{s\}}}dxdy.$$

Equality holds if and only if K is a central ellipsoid and $\left|\nabla^{[s]}f\right|$ is a translate of $\left|\nabla^{[s]}f\right|^\star \circ \phi$ for some $\phi \in \mathrm{SL}(n)$.

Proof. First, it is easy to verify that the following identity holds:

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\left(\left|\nabla^{[s]}f(x)\right|-\left|\nabla^{[s]}f(y)\right|\right)_-^p}{\|x-y\|_K^{n+p\{s\}}}dxdy = \iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\left(\left|\nabla^{[s]}f(x)\right|-\left|\nabla^{[s]}f(y)\right|\right)_+^p}{\|x-y\|_{-K}^{n+p\{s\}}}dxdy.$$

Observe that the Schwarz symmetrization of the star body K is equal to the Schwarz symmetrization of the star body $-K$, i.e., $K^\star = (-K)^\star$. By Proposition 5.1, we have

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\left(\left|\nabla^{[s]}f(x)\right|-\left|\nabla^{[s]}f(y)\right|\right)_-^p}{\|x-y\|_K^{n+p\{s\}}}dxdy \ge \iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\left(\left|\nabla^{[s]}f(x)\right|^\star-\left|\nabla^{[s]}f(y)\right|^\star\right)_+^p}{\|x-y\|_{K^\star}^{n+p\{s\}}}dxdy.$$

We obtain

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\left|\left|\nabla^{[s]}f(x)\right|-\left|\nabla^{[s]}f(y)\right|\right|^p}{\|x-y\|_K^{n+p\{s\}}}dxdy \ge 2\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\left(\left|\nabla^{[s]}f(x)\right|^\star-\left|\nabla^{[s]}f(y)\right|^\star\right)_+^p}{\|x-y\|_{K^\star}^{n+p\{s\}}}dxdy.$$

Noting that $-K^\star = K^\star$, we have

$$\begin{aligned}
\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(\left|\nabla^{[s]}f(x)\right|^\star - \left|\nabla^{[s]}f(y)\right|^\star\right)_+^p}{\|x-y\|_{K^\star}^{n+p\{s\}}} dxdy &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(\left|\nabla^{[s]}f(x)\right|^\star - \left|\nabla^{[s]}f(y)\right|^\star\right)_-^p}{\|x-y\|_{-K^\star}^{n+p\{s\}}} dxdy \\
&= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(\left|\nabla^{[s]}f(x)\right|^\star - \left|\nabla^{[s]}f(y)\right|^\star\right)_-^p}{\|x-y\|_{K^\star}^{n+p\{s\}}} dxdy.
\end{aligned} \tag{5.5}$$

We obtain

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left|\left|\nabla^{[s]}f(x)\right| - \left|\nabla^{[s]}f(y)\right|\right|^p}{\|x-y\|_{K}^{n+p\{s\}}} dxdy \geq \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left|\left|\nabla^{[s]}f(x)\right|^\star - \left|\nabla^{[s]}f(y)\right|^\star\right|^p}{\|x-y\|_{K^\star}^{n+p\{s\}}} dxdy.$$

Thus, we have completed the proof. From the analysis at the end of Proposition 5.1, we know that equality holds for (5.2) if and only if K is a central ellipsoid and $\left|\nabla^{[s]}f\right|$ is a translate of $\left|\nabla^{[s]}f\right|^\star \circ \phi$ for some $\phi \in \mathrm{SL}(n)$. □

For more applications and content about Pólya-Szegő inequalities, please refer to reference [16, 27, 28].

6. Affine fractional Pólya–Szegö inequalities

We establish affine Pólya–Szegö inequalities for fractional asymmetric and symmetric Sobolev polar projection bodies. According to (4.2), we obtained

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(\left|\nabla^{[s]}f(x)\right|^\star - \left|\nabla^{[s]}f(y)\right|^\star\right)_+^p}{\|x-y\|_{K^\star}^{n+p\{s\}}} dxdy = n\tilde{V}_{-p\{s\}}\left(K^\star, \Pi_{p,+}^{*,s}\left(\nabla_s f\right)^\star\right). \tag{6.1}$$

Proposition 6.1. *Let $f \in V_d^{s,p}(\mathbb{R}^n)$. Then,*

$$\left|\Pi_{p,+}^{*,s}\left(\nabla_s f\right)\right|^{-p\{s\}/n} \geq \left|\Pi_{p,+}^{*,s}\left(\nabla_s f\right)^\star\right|^{-p\{s\}/n}.$$

Equality holds if and only if $\left|\nabla^{[s]}f\right|$ is a translate of $\left|\nabla^{[s]}f\right|^\star \circ \phi$ for some $\phi \in \mathrm{SL}(n)$.

Proof. By Proposition 5.1, (4.2), and the dual mixed volume inequality, for star body $K \subset \mathbb{R}^n$, we obtain

$$\begin{aligned}
\tilde{V}_{-p\{s\}}\left(K, \Pi_{p,+}^{*,s}\left(\nabla_s f\right)\right) &\geq \tilde{V}_{-p\{s\}}\left(K^\star, \Pi_{p,+}^{*,s}\left(\nabla_s f\right)^\star\right) \\
&\geq \left|K^\star\right|^{(n+p\{s\})/n}\left|\Pi_{p,+}^{*,s}\left(\nabla_s f\right)^\star\right|^{-p\{s\}/n} \\
&= |K|^{(n+p\{s\})/n}\left|\Pi_{p,+}^{*,s}\left(\nabla_s f\right)^\star\right|^{-p\{s\}/n}.
\end{aligned}$$

Setting $K = \Pi_{p,+}^{*,s}\left(\nabla_s f\right)$, we have

$$\left|\Pi_{p,+}^{*,s}\left(\nabla_s f\right)\right| = \tilde{V}_{-p\{s\}}\left(\Pi_{p,+}^{*,s}\left(\nabla_s f\right), \Pi_{p,+}^{*,s}\left(\nabla_s f\right)\right)$$

$$\geq \left|\Pi_{p,+}^{*,s}(\nabla_s f)\right|^{(n+p\{s\})/n}\left|\Pi_{p,+}^{*,s}(\nabla_s f)^{\star}\right|^{-p\{s\}/n}.$$

By Proposition 5.1, there is equality in Proposition 6.1 if and only if $\left|\nabla^{[s]}f\right|$ is a translate of $\left|\nabla^{[s]}f\right|^{\star}\circ\phi$ for some $\phi\in\mathrm{SL}(n)$. We have established the inequality. □

The following result is obtained in the same way as Proposition 6.1, by replacing Proposition 5.1 with Proposition 5.2.

Proposition 6.2. *Let $f\in V_d^{s,p}(\mathbb{R}^n)$. Then,*

$$\left|\Pi_{p}^{*,s}(\nabla_s f)\right|^{-p\{s\}/n}\geq\left|\Pi_{p}^{*,s}(\nabla_s f)^{\star}\right|^{-p\{s\}/n}.$$

Equality holds if and only if $\left|\nabla^{[s]}f\right|$ is a translate of $\left|\nabla^{[s]}f\right|^{\star}\circ\phi$ for some $\phi\in\mathrm{SL}(n)$.

7. Affine fractional asymmetric L^p Sobolev inequalities

We establish the following affine fractional asymmetric L^p Sobolev inequalities and verify that they are stronger than (1.1). Proposition 6.1 will play a key role in the proof.

Theorem 7.1. *Let $s\in(0,n)\setminus\mathbb{N}$ and $1<p<n/s$. For $f\in V_d^{s,p}(\mathbb{R}^n)$, then*

$$\begin{aligned}\left\|\left|\nabla^{[s]}f\right|\right\|^p_{L^{\frac{np}{n-p\{s\}}}(\mathbb{R}^n)} &\leq 2\sigma_{n,p,s}n\omega_n^{\frac{n+p\{s\}}{n}}\left|\Pi_{p,+}^{*,s}(\nabla_s f)\right|^{-\frac{p\{s\}}{n}}\\ &\leq 2\sigma_{n,p,s}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\left(\left|\nabla^{[s]}f(x)\right|-\left|\nabla^{[s]}f(y)\right|\right)_+^p}{|x-y|^{n+p\{s\}}}dxdy.\end{aligned}$$

Equality holds in the first inequality if and only if $\left|\nabla^{[s]}f\right|=h_{\{s\},p}\circ\phi$ for some $\phi\in\mathrm{GL}(n)$, where $h_{\{s\},p}$ is an extremal function for inequality (1.1)*. For the second inequality, equality is achieved if and only if $\left|\nabla^{[s]}f\right|$ is radially symmetric.*

Proof. By Proposition 6.1, equality holds for

$$\left|\Pi_{p,+}^{*,s}(\nabla_s f)\right|^{-p\{s\}/n}\geq\left|\Pi_{p,+}^{*,s}(\nabla_s f)^{\star}\right|^{-p\{s\}/n}, \tag{7.1}$$

if and only if $\left|\nabla^{[s]}f\right|$ is a translate of $\left|\nabla^{[s]}f\right|^{\star}\circ\phi$ for some $\phi\in\mathrm{SL}(n)$.

Since $\left|\nabla^{[s]}f\right|^{\star}$ is radially symmetric, $\Pi_{p,+}^{*,s}(\nabla_s f)^{\star}=\Pi_{p,-}^{*,s}(\nabla_s f)^{\star}$ is a ball. Let the ball $\Pi_{p,+}^{*,s}(\nabla_s f)^{\star}$ have radius R so that $\left|\Pi_{p,+}^{*,s}(\nabla_s f)^{\star}\right|=\omega_n R^n$. Then,

$$\begin{aligned}2n\omega_n^{\frac{n+p\{s\}}{n}}\left|\Pi_{p,+}^{*,s}(\nabla_s f)^{\star}\right|^{-\frac{p\{s\}}{n}} &= 2n\omega_n^{1+\frac{p\{s\}}{n}}(\omega_n R^n)^{-\frac{p\{s\}}{n}}\\ &= 2n\omega_n^{1+\frac{p\{s\}}{n}}\omega_n^{-\frac{p\{s\}}{n}}R^{-p\{s\}}\\ &= 2n\omega_n R^{-p\{s\}}.\end{aligned}\tag{7.2}$$

Taking $K^{\star}=\Pi_{p,+}^{*,s}(\nabla_s f)^{\star}$ in (6.1), we have

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\left(\left|\nabla^{[s]}f(x)\right|^{\star}-\left|\nabla^{[s]}f(y)\right|^{\star}\right)_+^p}{\|x-y\|_{K^{\star}}^{n+p\{s\}}}dxdy = n\tilde{V}_{-p\{s\}}\left(\Pi_{p,+}^{*,s}(\nabla_s f)^{\star},\Pi_{p,+}^{*,s}(\nabla_s f)^{\star}\right)$$

$$= \quad n\omega_n R^n.$$

Letting $x \in \mathbb{R}^n$, by the definition of the gauge function, the following equation holds:

$$\|x\|_{R\cdot B^n} = \frac{1}{R}\|x\|_{B^n} = \frac{1}{R}|x|.$$

Consequently, we have

$$\begin{aligned}
\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(\left|\nabla^{[s]}f(x)\right|^\star - \left|\nabla^{[s]}f(y)\right|^\star\right)_+^p}{\|x-y\|_{K^\star}^{n+p\{s\}}} dxdy &= R^{n+p\{s\}}\cdot \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(\left|\nabla^{[s]}f(x)\right|^\star - \left|\nabla^{[s]}f(y)\right|^\star\right)_+^p}{\|x-y\|_{B^n}^{n+p\{s\}}} dxdy \\
&= R^{n+p\{s\}}\cdot \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(\left|\nabla^{[s]}f(x)\right|^\star - \left|\nabla^{[s]}f(y)\right|^\star\right)_+^p}{|x-y|^{n+p\{s\}}} dxdy.
\end{aligned}$$

That is,

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(\left|\nabla^{[s]}f(x)\right|^\star - \left|\nabla^{[s]}f(y)\right|^\star\right)_+^p}{|x-y|^{n+p\{s\}}} dxdy = n\omega_n R^{-p\{s\}}, \tag{7.3}$$

By (7.2) and (7.3), we obtain

$$2n\omega_n^{\frac{n+p\{s\}}{n}} \left|\Pi_{p,+}^{*,s}(\nabla_s f)^\star\right|^{-\frac{p\{s\}}{n}} = 2\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(\left|\nabla^{[s]}f(x)\right|^\star - \left|\nabla^{[s]}f(y)\right|^\star\right)_+^p}{|x-y|^{n+p\{s\}}} dxdy.$$

Hence, from (5.5), we have

$$\begin{aligned}
2n\omega_n^{\frac{n+p\{s\}}{n}} \left|\Pi_{p,+}^{*,s}(\nabla_s f)^\star\right|^{-\frac{p\{s\}}{n}} &= 2\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(\left|\nabla^{[s]}f(x)\right|^\star - \left|\nabla^{[s]}f(y)\right|^\star\right)_+^p}{|x-y|^{n+p\{s\}}} dxdy \\
&= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left|\left|\nabla^{[s]}f(x)\right|^\star - \left|\nabla^{[s]}f(y)\right|^\star\right|^p}{|x-y|^{n+p\{s\}}} dxdy.
\end{aligned} \tag{7.4}$$

The fractional Sobolev inequality (1.1), (7.1), and (7.4) show that

$$\begin{aligned}
\left\|\left|\nabla^{[s]}f\right|\right\|^p_{L^{\frac{np}{n-p\{s\}}}(\mathbb{R}^n)} &= \left\|\left|\nabla^{[s]}f\right|^\star\right\|^p_{L^{\frac{np}{n-p\{s\}}}(\mathbb{R}^n)} \\
&\leq \sigma_{n,p,s}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left|\left|\nabla^{[s]}f(x)\right|^\star - \left|\nabla^{[s]}f(y)\right|^\star\right|^p}{|x-y|^{n+p\{s\}}} dxdy \\
&= 2\sigma_{n,p,s} n\omega_n^{\frac{n+p\{s\}}{n}} \left|\Pi_{p,+}^{*,s}(\nabla_s f)^\star\right|^{-\frac{p\{s\}}{n}} \\
&\leq 2\sigma_{n,p,s} n\omega_n^{\frac{n+p\{s\}}{n}} \left|\Pi_{p,+}^{*,s}(\nabla_s f)\right|^{-\frac{p\{s\}}{n}}.
\end{aligned} \tag{7.5}$$

We have completed the proof of the first inequality of the theorem.

For the second inequality, we set $K = B^n$ in (2.4) and apply the dual mixed volume inequality (2.6). We have

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(\left|\nabla^{[s]}f(x)\right| - \left|\nabla^{[s]}f(y)\right|\right)_+^p}{|x-y|^{n+p\{s\}}} dxdy = n\tilde{V}_{-p\{s\}}\left(B^n, \Pi_{p,+}^{*,s}(\nabla_s f)\right)$$

$$\geq \quad n\omega_n^{\frac{n+p\{s\}}{n}} \left|\Pi_{p,+}^{*,s}(\nabla_s f)\right|^{-\frac{p\{s\}}{n}}. \tag{7.6}$$

There is equality precisely if $\Pi_{p,+}^{*,s}(\nabla_s f)$ is a ball, which is the case when $\left|\nabla^{[s]} f\right|$ is radially symmetric. Combining (7.5) and (7.6), we have completed the proof of Theorem 7.1. □

Note that it follows directly from the definitions of the fractional symmetric and asymmetric L^p polar projection bodies that

$$\Pi_p^{*,s}(\nabla_s f) = \Pi_{p,+}^{*,s}(\nabla_s f) \tilde{+}_{-p\{s\}} \Pi_{p,-}^{*,s}(\nabla_s f). \tag{7.7}$$

Theorem 7.2. *Let $s \in (0,n) \setminus \mathbb{N}$ and $1 < p < n/s$. For $f \in V_d^{s,p}(\mathbb{R}^n)$, then*

$$\begin{aligned} \left\|\left|\nabla^{[s]} f\right|\right\|_{L^{\frac{np}{n-p\{s\}}}(\mathbb{R}^n)}^p &\leq \sigma_{n,p,s} n\omega_n^{\frac{n+p\{s\}}{n}} \left|\Pi_p^{*,s}(\nabla_s f)\right|^{-\frac{p\{s\}}{n}} \\ &\leq \sigma_{n,p,s} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left|\left|\nabla^{[s]} f(x)\right| - \left|\nabla^{[s]} f(y)\right|\right|^p}{|x-y|^{n+p\{s\}}} dxdy. \end{aligned}$$

Equality holds in the first inequality if and only if $\left|\nabla^{[s]} f\right| = h_{\{s\},p} \circ \phi$ for some $\phi \in \mathrm{GL}(n)$, where $h_{\{s\},p}$ is an extremal function for inequality (1.1). *For the second inequality, equality is achieved if and only if $\left|\nabla^{[s]} f\right|$ is radially symmetric.*

Proof. Using the dual Brunn-Minkowski inequalities (2.7), and (7.7), we obtain

$$\begin{aligned} \left|\Pi_p^{*,s}(\nabla_s f)\right|^{-\frac{p\{s\}}{n}} &\geq \left|\Pi_{p,+}^{*,s}(\nabla_s f)\right|^{-\frac{p\{s\}}{n}} + \left|\Pi_{p,-}^{*,s}(\nabla_s f)\right|^{-\frac{p\{s\}}{n}} \\ &= 2\left|\Pi_{p,+}^{*,s}(\nabla_s f)\right|^{-\frac{p\{s\}}{n}} = 2\left|\Pi_{p,-}^{*,s}(\nabla_s f)\right|^{-\frac{p\{s\}}{n}}, \end{aligned} \tag{7.8}$$

where equality holds if the star bodies $\Pi_{p,+}^{*,s}(\nabla_s f)$ and $\Pi_{p,-}^{*,s}(\nabla_s f)$ are dilates. Combining Theorem 7.1 and (7.8), we obtain the proof of the first inequality:

$$\left\|\left|\nabla^{[s]} f\right|\right\|_{L^{\frac{np}{n-p\{s\}}}(\mathbb{R}^n)}^p \leq \sigma_{n,p,s} n\omega_n^{\frac{n+p\{s\}}{n}} \left|\Pi_p^{*,s}(\nabla_s f)\right|^{-\frac{p\{s\}}{n}}.$$

For the second inequality, we can obtain the following result by using the same method in Theorem 7.1. Then,

$$\begin{aligned} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left|\left|\nabla^{[s]} f(x)\right| - \left|\nabla^{[s]} f(y)\right|\right|^p}{|x-y|^{n+p\{s\}}} \mathrm{d}x\, \mathrm{d}y &= n\tilde{V}_{-p\{s\}}\left(B^n, \Pi_p^{*,s}(\nabla_s f)\right) \\ &\geq n\omega_n^{\frac{n+p\{s\}}{n}} \left|\Pi_p^{*,s}(\nabla_s f)\right|^{-\frac{p\{s\}}{n}}. \end{aligned}$$

Similarly, equality holds if $\Pi_p^{*,s}(\nabla_s f)$ is a ball, which is the case when $\left|\nabla^{[s]} f\right|$ is radially symmetric. □

8. Conclusions

We extend affine fractional L^p Sobolev inequalities from the classical range $0 < s < 1$ with $1 < p < n/s$ to all higher-order fractional orders $s \in (0,n) \setminus \mathbb{N}$ with $1 < p < n/s$. By introducing the function space $V_d^{s,p}(\mathbb{R}^n)$, in which only the highest-order derivative tensor $\nabla^{[s]}f$ (and hence its Euclidean norm $\nabla_s f := |\nabla^{[s]}f|$) is required to possess fractional Sobolev regularity, we establish the following sharp inequalities in Section 7:

$$\left\|\left|\nabla^{[s]}f\right|\right\|^p_{L^{\frac{np}{n-p\{s\}}}(\mathbb{R}^n)} \leq \sigma_{n,p,s} n\omega_n^{\frac{n+p\{s\}}{n}} \left|\Pi_p^{*,s}(\nabla_s f)\right|^{-\frac{p\{s\}}{n}} \leq \sigma_{n,p,s} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left|\left|\nabla^{[s]}f(x)\right| - \left|\nabla^{[s]}f(y)\right|\right|^p}{|x-y|^{n+p\{s\}}} dxdy.$$

We also characterize the equality cases: The first inequality becomes equality if and only if $\nabla_s f = h_{\{s\},p} \circ \phi$ for some $\phi \in \mathrm{GL}(n)$, where $h_{\{s\},p}$ is an extremal function for inequality (1.1); while the second inequality becomes equality if and only if $\nabla_s f$ is radially symmetric.

Schwarz symmetrization theory plays a key role throughout. We first analyze in detail the fractional L^p polar projection bodies $\Pi_p^{*,s}(\nabla_s f)$ and $\Pi_{p,\pm}^{*,s}(\nabla_s f)$ and show that they are star bodies with continuous gauges (Sections 3 and 4). Next, we establish higher-order anisotropic Pólya–Szegő-type principles for $|\nabla^{[s]}f|$ (Section 5) and the corresponding affine Pólya–Szegő inequalities for the polar projection bodies (Section 6). Finally, using the dual mixed volume representation of the fractional seminorms and the dual mixed volume inequalities, we obtain the sharp affine fractional Sobolev inequalities stated above.

When $0 < s < 1$ and $1 < p < n/s$, our inequalities become precisely the results of Haddad and Ludwig, thereby providing a complete higher-order generalization.

Author contributions

Wentao Guo: Conceptualization, methodology, formal analysis, writing original draft, writing review and editing; Yuke Li: Formal analysis, writing review and editing; Xiaolin Zeng: Supervision, investigation, writing review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the preparation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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