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Research article

Solvability and stability of nonlinear variable-order fractional boundary value problems via new infinite-partition

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Abstract: This paper studies a class of nonlinear variable-order fractional boundary value problems involving the Riemann–Liouville fractional derivative. An infinite-partition approximation is introduced by replacing the variable-order function with a sequence of piecewise constant functions, transforming the original problem into a family of constant-order fractional boundary value problems. For each approximating problem, an equivalent integral equation and the corresponding Green function are derived. Existence, uniqueness, and Ulam–Hyers stability of solutions are established using Schauder's fixed-point theorem and the Banach contraction principle. It is further proved that the approximating solutions converge uniformly to a solution of the original variable-order problem, providing a rigorous connection between constant-order and variable-order fractional models. A numerical example illustrates the theoretical results.

Keywords: variable-order fractional differential equations; infinite partition method; Riemann–Liouville fractional derivative; Green function; fixed-point theorem; Ulam–Hyers stability
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1. Introduction

Fractional calculus has become an important mathematical framework for describing memory and hereditary effects that cannot be captured by classical integer-order differential equations. During the last two decades, fractional differential equations (FDEs) have found numerous applications in viscoelasticity, anomalous diffusion, heat transfer, electrical circuits, signal processing, control theory, biological systems, and electrochemical processes. In many of these applications, the memory characteristics of the system evolve with time or space due to changes in material properties or environmental conditions. Consequently, a constant fractional order is often insufficient to accurately represent the underlying dynamics.

Variable-order (VO) fractional operators provide a natural generalization of classical fractional operators by allowing the differentiation order to vary continuously with the independent variable [1,2]. This additional flexibility enables the modeling of transition phenomena between different diffusion regimes, adaptive viscoelastic responses, heterogeneous media, and time-dependent memory effects. Because of these advantages, VO fractional models have attracted increasing attention in both theoretical analysis and engineering applications [3–6].

The boundary value problem (BVP) considered in this paper is

$$\begin{cases} D_{0^+}^{q(t)}x(t) + \psi(t, x(t), I_{0^+}^{q(t)}x(t)) = 0, \quad t \in J = [0, T], \\ x(0) = 0, \qquad x(T) = 0, \end{cases} \tag{1.1}$$

where $D_{0^+}^{q(t)}$ and $I_{0^+}^{q(t)}$ denote the Riemann–Liouville (R-L) VO fractional derivative (FD) and fractional integral (FI), respectively. The unknown function $x(t)$ represents the state of the system, while $q(t) \in (1, 2]$ describes the local memory intensity that changes throughout the evolution process. The nonlinear term $\psi(t, x, I_{0^+}^{q(t)}x)$ models external interactions depending simultaneously on the current state and on its accumulated memory through the FI. The homogeneous boundary conditions represent systems constrained at two endpoints, which naturally arise in diffusion, beam deflection, and transport models.

Unlike the Caputo derivative, the R-L VO derivative admits a formulation that naturally incorporates singular memory kernels and is particularly appropriate for problems whose dynamics are governed by hereditary effects acting from the initial instant. Moreover, the integral term appearing in (1.1) is naturally expressed through R-L operators, making this framework well suited for the construction of equivalent integral equations and Green representations. Consequently, the R-L formulation provides a unified setting for studying nonlinear VO BVP.

For constant-order fractional equations, existence and uniqueness results are usually obtained by transforming the differential equation into an equivalent integral equation. This procedure relies heavily on the composition identities

$$I^{\alpha}I^{\beta} = I^{\alpha+\beta}, \qquad D^{\alpha}I^{\alpha} = I,$$

which follow directly from the semigroup property. However, VO operators generally satisfy

$$I_{0^+}^{\alpha(t)}I_{0^+}^{\beta(t)}h(t) \neq I_{0^+}^{\alpha(t)+\beta(t)}h(t),$$

and analogous relations for derivatives also fail. Therefore, the classical derivation of Green functions cannot be applied directly, since repeated integration and inversion of the differential operator are no longer available in closed form. This lack of semigroup structure represents one of the principal analytical difficulties in VO fractional calculus.

To overcome this obstacle, we approximate the continuously varying order function $q(t)$ by a piecewise constant sequence. On each subinterval the problem becomes a classical constant-order fractional BVP, for which an explicit Green function can be constructed. The corresponding local solutions are then assembled, and the convergence of the piecewise approximation guarantees that the resulting solution approaches the solution of the original VO problem. From a physical viewpoint, this discretization represents a medium whose memory properties remain approximately constant over sufficiently small time intervals, which is consistent with practical numerical and engineering approximations.

Several recent contributions have investigated existence, uniqueness, and stability for VO fractional equations by means of fixed-point (FP) techniques [7–10]. Other studies have established Green-function representations for constant-order R-L and Caputo BVPs. Nevertheless, explicit Green functions for nonlinear VO R-L BVPs remain largely unexplored because of the absence of the semigroup property and the difficulty of obtaining an equivalent integral formulation.

Main contributions

The principal objective of this work is to establish a rigorous analytical framework for solving VO fractional BVPs through an infinite partition approach. Unlike our previous work, where the VO function was approximated by a finite piecewise constant decomposition, the present study develops a sequence of increasingly refined partitions whose mesh size tends to zero. This strategy allows the continuously VO problem to be recovered as the limit of a family of constant-order fractional BVPs.

The main contributions of this paper are summarized as follows:

1. We introduce an infinite partition methodology in which the VO function is approximated by a sequence of piecewise constant functions defined on increasingly finer subdivisions of the interval. This construction generates a sequence of constant-order fractional BVPs that progressively approximate the original continuously VO model.
2. For each partition, an explicit Green function is derived and an associated approximate solution is constructed. In contrast to previous finite decomposition approaches, the present framework does not stop at the approximation level but investigates the limiting behavior of the entire sequence of approximate problems.
3. A rigorous convergence analysis is established. We prove that when $n \to \infty$ the sequence of piecewise constant approximations converges to the original VO function, $q_k \to q$ and the associated sequence of approximate solutions converges to the exact solution VO BVP $x_k \to x$.
4. The convergence results provide a mathematically rigorous bridge between classical constant-order fractional models and continuously VO fractional models. Consequently, the exact VO problem can be viewed as the limit of a sequence of explicitly solvable constant-order problems.
5. Compared with our previous work, the essential novelty is the passage from a finite approximation procedure to an infinite refinement process together with the proof of convergence toward the exact continuously VO solution. Therefore, the present paper not only constructs approximate

solutions but also justifies their validity by demonstrating their convergence to the true solution of the original problem.

The core contribution of this work is therefore the development of a convergent infinite-partition framework that transforms VO fractional BVPs into a sequence of constant-order problems while guaranteeing recovery of the exact continuously VO solution in the limit.

The remainder of the paper is organized as follows. Section 2 recalls the necessary preliminaries. Section 3 establishes the Green function and proves the existence and uniqueness results. Section 4 investigates Ulam–Hyers stability (HUS). Section 5 presents a numerical example illustrating the convergence of the piecewise approximation and validating the theoretical results.

2. Preliminaries

This section presents essential definitions, notations, and supporting lemmas needed for our analysis.

Consider the Banach space

$$E = C(J, \mathbb{R})$$

of continuous real-valued functions on J, equipped with the maximum norm

$$\|x\| = \max_{t \in J} |x(t)|.$$

Definition 2.1 (R-L VO integral [11–13]). *For $a < b$ and $q : [a, b] \to (0, \infty)$, the left R-L FI of order $q(t)$ applied to h is*

$$I_{a^+}^{q(t)} h(t) = \frac{1}{\Gamma(q(t))} \int_a^t (t - s)^{q(t)-1} h(s)\, ds, \quad t > a, \tag{2.1}$$

where Γ is the Euler gamma function.

Definition 2.2 (R-L VO derivative [11–13]). *For $a < b$, $n \in \mathbb{N}$, and $q : [a, b] \to (n-1, n)$, the left R-L FD of order $q(t)$ is*

$$D_{a^+}^{q(t)} h(t) = \left(\frac{d}{dt}\right)^n I_{a^+}^{n-q(t)} h(t) = \left(\frac{d}{dt}\right)^n \int_a^t \frac{(t - s)^{n-q(t)-1}}{\Gamma(n - q(t))} h(s)\, ds, \quad t > a. \tag{2.2}$$

Remark. When $q(t) \equiv q$ (constant), expressions (2.1)–(2.2) recover the standard R-L operators [11, 12, 14].

The following properties hold for constant-order operators:

Lemma 2.3 (General solution of homogeneous equation [14]). *If $D_{a^+}^{q} h = 0$ with $q > 0$ and $n - 1 < q \leq n$, then*

$$h(t) = \sum_{k=1}^{n} \rho_k (t - a)^{q-k}, \quad \rho_k \in \mathbb{R}.$$

Lemma 2.4 (Composition formula I [14]). *For $h \in L(a, b)$ with $D_{a^+}^{q} h \in L(a, b)$ and $n - 1 < q \leq n$,*

$$I_{a^+}^{q} \left(D_{a^+}^{q} h(t)\right) = h(t) + \sum_{k=1}^{n} \rho_k (t - a)^{q-k}, \quad \rho_k \in \mathbb{R}.$$

Lemma 2.5 (Composition formula II [14]). *If $q > 0$, then $D^q_{a^+}(I^q_{a^+}h(t)) = h(t)$.*

Lemma 2.6 (Semigroup property [14]). *For $q, \beta > 0$,*

$$I^q_{a^+}(I^\beta_{a^+}h) = I^\beta_{a^+}(I^q_{a^+}h) = I^{q+\beta}_{a^+}h.$$

Remark 2.7 (Semigroup failure [15, 16]). *VO $q(t), v(t)$ violate the semigroup property*

$$I^{q(t)}_{a^+}(I^{v(t)}_{a^+}h) \neq I^{q(t)+v(t)}_{a^+}h.$$

Lemma 2.8 (Integral existence [17]). *If $q : J \to (1, 2]$ is continuous and $x \in C_\delta(J, \mathbb{R}) = \{x \in C((0, T], \mathbb{R}) : t^\delta x(t) \in C(J, \mathbb{R}),\ 0 < \delta < 1\}$, then $I^{q(t)}_{0^+}x(t)$ exists for all $t \in J$.*

Lemma 2.9 (Continuity preservation [17]). *If $q : J \to (1, 2]$ is continuous, then $I^{q(t)}_{0^+} : C(J, \mathbb{R}) \to C(J, \mathbb{R})$.*

Theorem 2.10 (Schauder FP theorem [14]). *Let E be Banach and $Q \subset E$ closed, convex, and nonempty. If $F : Q \to Q$ is continuous and compact, then F admits a FP in Q.*

Definition 2.11 (UHS [3]). *Problem* (1.1) *exhibits UHS if there exists $c_\psi > 0$ such that for every $\epsilon > 0$ and $z \in C(J, \mathbb{R})$ satisfying*

$$\left| D^{q(t)}_{0^+}z(t) + \psi\left(t, z(t), I^{q(t)}_{0^+}z(t)\right)\right| \leq \epsilon, \quad t \in J, \tag{2.3}$$

there is a solution $x \in C(J, \mathbb{R})$ to (1.1) *with*

$$\max_{t\in J} |z(t) - x(t)| \leq c_\psi \epsilon.$$

3. Existence and uniqueness of solutions

To proceed, we impose the subsequent hypotheses:

(H1) Fix an integer $n \in \mathbb{N}$. Partition the interval J into n equal subintervals, each of length w. Let $(q_k(t))^n_{k=1}$ represent the initial n components of a piecewise sequence $(q_n)_{n\in\mathbb{N}}$ that approximates $q(t)$, such that

$$q_k(t) = \begin{cases} q_1, & 0 \leq t < w, \\ q_2, & w \leq t < 2w, \\ q_3, & 2w \leq t < 3w, \\ \vdots & \\ q_n, & (n-1)w \leq t \leq nw, \end{cases}$$

where $1 < q_k \leq 2$ for each $k = 1, 2, \ldots, n$, and $nw = T$.

(H2) Consider the function $t^\delta \psi : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, which is continuous for $0 < \delta < 1$. There are positive constants $c_j > 0$ with $j = 1, 2, 3$, along with $0 < \gamma < 1$ and $0 < \eta < 1$, satisfying

$$t^\delta |\psi(t, y, z)| \leq c_1 + c_2|y|^\gamma + c_3|z|^\eta,$$

for all $y, z \in \mathbb{R}$ and $t \in J$.

(H3) Constants $K, L > 0$ and $0 \leq \delta < 1$ exist such that

$$t^{\delta}|\psi(t, y_1, z_1) - \psi(t, y_2, z_2)| \leq K|y_1 - y_2| + L|z_1 - z_2|,$$

whenever $y_1, y_2, z_1, z_2 \in \mathbb{R}$ and $t \in J$.

We denote

$$E_{kw} := C([0, kw], \mathbb{R}), \quad k \in \{1, 2, ..., n\},$$

the Banach spaces of real-valued continuous functions defined on $[0, kw]$, equipped with the norm

$$\|x_k\|_{E_{kw}} := \sup_{t\in[0,kw]} |x_k(t)|, \quad k \in \{1, 2, ..., n\}.$$

Now, we introduce the modified BVP for FDE of constant order

$$\begin{cases} D^{q_k}_{0^+} x_k(t) = \psi(t, x_k(t), I^{q_k}_{0^+} x_k(t)), & t \in J_{kw}, \\ x_k(0) = 0, \quad x_k(kw) = 0, \end{cases} \tag{3.1}$$

where $J_{kw} := [0, kw],\ k = 1, 2, ..., n$.

Following the approach in [18], if $x_k \in E_{kw}$ is a solution of the BVP (3.1), it can be shown that as $n \to \infty$ (i.e., $w \to 0$) with $k = n$, we have $t = nw$ and the piecewise sequence $(q_n)_{n\in\mathbb{N}}$ converges to the original VO $q(t)$, and $x_k(t) \to x(t)$. Moreover, we have

$$x_n(0) = 0 \to x(0) = 0, \quad x_n(T) = x_n(nw) = 0 \to x(T) = 0.$$

Then, x is a solution of the VO BVP (1).
To establish the existence of solutions to the BVP (3.1), the next auxiliary result is required.

Lemma 3.1. *A function $x_k \in E_{kw}$ solves the BVP* (3.1) *if and only if it fulfills the following integral equation:*

$$x_k(t) = \int_0^{kw} G_k(t, s)\psi(s, x_k(s), I^{q_k}_{0^+} x_k(s))\, ds, \quad t \in J_{kw}. \tag{3.2}$$

Here, $G_k(t, s)$ denotes the Green's function given by

$$G_k(t, s) = \begin{cases} \dfrac{1}{\Gamma(q_k)}\Big[(kw)^{1-q_k} t^{q_k-1}(kw - s)^{q_k-1} - (t - s)^{q_k-1}\Big], & 0 \leq s \leq t \leq kw, \\ \dfrac{1}{\Gamma(q_k)}(kw)^{1-q_k} t^{q_k-1}(kw - s)^{q_k-1}, & 0 \leq t \leq s \leq kw. \end{cases} \tag{3.3}$$

where $k = 1, 2, ..., n$.

Proof. Suppose that $x_k \in E_{kw}$ is a solution of (3.1). Applying the FI operator $I^{q_k}_{0^+}$ to both sides of

$$D^{q_k}_{0^+} x_k(t) = -\psi\Big(t, x_k(t), I^{q_k}_{0^+} x_k(t)\Big),$$

and using Lemma 2.4, we obtain

$$x_k(t) = A_k t^{q_k-1} + B_k t^{q_k-2} - \frac{1}{\Gamma(q_k)} \int_0^t (t - s)^{q_k-1}\psi\Big(s, x_k(s), I^{q_k}_{0^+} x_k(s)\Big)\, ds, \tag{3.4}$$

where $A_k, B_k \in \mathbb{R}$.

Since $1 < q_k \leq 2$ and $x_k(0) = 0$, it follows from (3.4) that

$$B_k = 0.$$

Hence,

$$x_k(t) = A_k t^{q_k-1} - \frac{1}{\Gamma(q_k)} \int_0^t (t-s)^{q_k-1} \psi\Big(s, x_k(s), I_{0^+}^{q_k} x_k(s)\Big)\, ds. \tag{3.5}$$

Using the boundary condition $x_k(kw) = 0$ in (3.5), we obtain

$$A_k (kw)^{q_k-1} = \frac{1}{\Gamma(q_k)} \int_0^{kw} (kw-s)^{q_k-1} \psi\Big(s, x_k(s), I_{0^+}^{q_k} x_k(s)\Big)\, ds,$$

and therefore

$$A_k = \frac{(kw)^{1-q_k}}{\Gamma(q_k)} \int_0^{kw} (kw-s)^{q_k-1} \psi\Big(s, x_k(s), I_{0^+}^{q_k} x_k(s)\Big)\, ds. \tag{3.6}$$

Substituting (3.6) into (3.5) yields

$$\begin{aligned} x_k(t) \quad = \quad & \frac{(kw)^{1-q_k} t^{q_k-1}}{\Gamma(q_k)} \int_0^{kw} (kw-s)^{q_k-1} \psi\Big(s, x_k(s), I_{0^+}^{q_k} x_k(s)\Big)\, ds \\ & - \frac{1}{\Gamma(q_k)} \int_0^t (t-s)^{q_k-1} \psi\Big(s, x_k(s), I_{0^+}^{q_k} x_k(s)\Big)\, ds. \end{aligned}$$

Consequently,

$$x_k(t) = \int_0^{kw} G_k(t,s) \psi\Big(s, x_k(s), I_{0^+}^{q_k} x_k(s)\Big)\, ds,$$

where

$$G_k(t,s) = \begin{cases} \dfrac{1}{\Gamma(q_k)} \Big[(kw)^{1-q_k} t^{q_k-1} (kw-s)^{q_k-1} - (t-s)^{q_k-1}\Big], & 0 \leq s \leq t \leq kw, \\ \dfrac{1}{\Gamma(q_k)} (kw)^{1-q_k} t^{q_k-1} (kw-s)^{q_k-1}, & 0 \leq t \leq s \leq kw. \end{cases}$$

Conversely, assume that $x_k \in E_{kw}$ satisfies (3.2). By Lemma 2.5, applying $D_{0^+}^{q_k}$ to both sides of (3.2) gives

$$D_{0^+}^{q_k} x_k(t) + \psi\Big(t, x_k(t), I_{0^+}^{q_k} x_k(t)\Big) = 0, \qquad t \in J_{kw}.$$

Moreover, from the definition of $G_k(t,s)$, one has

$$x_k(0) = 0, \qquad x_k(kw) = 0.$$

Hence x_k satisfies (3.1). The, proof is complete. □

The subsequent proposition plays a key role:

Proposition 3.2. *Let* $0 < \delta < 1$, *suppose* $t^\delta \psi : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous, and assume* $q(t) : J \to (1, 2]$ *obeys (H1). The Green's function for BVP* (3.1) *then possesses these properties:*

(1) $G_k(t,s) \geq 0$ *for all* $0 \leq t, s \leq kw$,

(2) $\max\limits_{t\in J_{kw}} G_k(t,s) = G_k(s,s),\ s \in J_{kw}$,

(3) $G_k(s,s)$ *has unique maximum given by*

$$\max_{s\in J_{kw}} G_k(s,s) = \frac{1}{\Gamma(q_k)}\Big(\frac{kw}{4}\Big)^{q_k-1}, \tag{3.7}$$

where $k = 1, 2, ..., n$.

Proof. Let $\varphi(t,s) = (kw)^{1-q_k} t^{q_k-1}(kw-s)^{q_k-1} - (t-s)^{q_k-1},\ \ 0 \le s \le t \le kw$.
We see that

$$\begin{aligned}\varphi_t(t,s) &= (q_k-1)\Big[(kw)^{1-q_k} t^{q_k-2}(kw-s)^{q_k-1} - (t-s)^{q_k-2}\Big],\\ &\le (q_k-1)\Big[(kw)^{1-q_k}(t-s)^{q_k-2}(kw)^{q_k-1} - (t-s)^{q_k-2}\Big] = 0.\end{aligned}$$

This implies $\varphi(t,s)$ decreases in t, so $\varphi(t,s) \ge \varphi(kw,s) = 0$ whenever $0 \le s \le t \le kw$.

Hence, $G_k(t,s) \ge 0$ for all $0 \le t, s \le kw$ and $k = 1, \ldots, n$.

As $\varphi(t,s)$ is nonincreasing in t, we have $\varphi(t,s) \le \varphi(s,s)$ for $0 \le s \le t \le kw$.

Now consider the case $0 \le t \le s \le kw$, where

$$(kw)^{1-q_k} t^{q_k-1}(kw-s)^{q_k-1} \le (kw)^{1-q_k} s^{q_k-1}(kw-s)^{q_k-1}.$$

These confirm that $\max\limits_{t\in[0,kw]} G_k(t,s) = G_k(s,s),\ \ s \in [0,kw],\ \ k = 1, ..., n$.
Next, we confirm property (3) from Proposition 3.2. Evidently, the maximum values of $G_k(s,s)$ do not occur at the endpoints 0 or kw, for $k = 1, \ldots, n$.
For $s \in [0,kw],\ \ k = 1, ..., n$, we have

$$\begin{aligned}\frac{dG_k(s,s)}{ds} &= \frac{1}{\Gamma(q_k)}(kw)^{1-q_k}(q_k-1)\Big[s^{q_k-2}(kw-s)^{q_k-1} - s^{q_k-1}(kw-s)^{q_k-2}\Big],\\ &= \frac{1}{\Gamma(q_k)}(kw)^{1-q_k}(q_k-1)s^{q_k-2}(kw-s)^{q_k-2}\Big[kw-2s\Big].\end{aligned}$$

This shows that $G_k(s,s)$ attains its maximum at $s = \frac{kw}{2}$, for $k = 1, \ldots, n$.
Then, for $k = 1, ..., n$,

$$\begin{aligned}\max_{s\in[0,kw]} G_k(s,s) &= G_k\Big(\frac{kw}{2}, \frac{kw}{2}\Big),\\ &= \frac{1}{\Gamma(q_k)}\Big(\frac{kw}{4}\Big)^{q_k-1}.\end{aligned}$$

The initial existence theorem relies on Theorem 2.10.

Theorem 3.3. *Suppose assumptions (H1)–(H3) are satisfied. Then BVP* (3.1) *admits at least one solution in* E_{kw}.

Proof. Convert BVP (3.1) to an equivalent FP problem. Define the operator

$$W : E_{kw} \to E_{kw}$$

by

$$Wx_k(t) = \int_0^{kw} G_k(t,s)\psi(s, x_k(s), I_{0^+}^{q_k} x_k(s))ds,\ t \in J_{kw}. \tag{3.8}$$

The properties of FI, combined with the continuity of $t^{\delta}\psi$, ensure that the operator $W : E_{kw} \to E_{kw}$ from (3.8) is well-defined.
Now define the set

$$B_{R_k} = \{x_k \in E_{kw},\ \|x_k\|_{E_{kw}} \leq R_k\}$$

where

$$\begin{aligned} R_k = \max\Big\{&\frac{3c_1(kw)^{q_k-\delta}}{4^{q_k-1}(1-\delta)\Gamma(q_k)}, \Big(\frac{3c_2(kw)^{q_k-\delta}}{4^{q_k-1}(1-\delta)\Gamma(q_k)}\Big)^{\frac{1}{1-\gamma}}, \\ &\Big(\frac{3c_3(kw)^{q_k-\delta}}{4^{q_k-1}(1-\delta)\Gamma(q_k)}\Big(\frac{(kw)^{q_k}}{\Gamma(q_k+1)}\Big)^{\eta}\Big)^{\frac{1}{1-\eta}}\Big\}. \end{aligned} \tag{3.9}$$

Clearly B_{R_k} is nonempty, bounded, convex, and closed.

We now verify that W meets the hypotheses of Theorem 2.10. The argument proceeds in three steps.
Step 1: $W(B_{R_k}) \subseteq (B_{R_k})$.

For $x \in B_{R_k}$, Proposition 3.2 together with (H2) implies

$$\begin{aligned} |Wx_k(t)| &= \Big|\int_0^{kw} G_k(t,s)\psi(s, x_k(s), I_{0^+}^{q_k} x_k(s))ds\Big| \\ &\leq \int_0^{kw} G_k(t,s)\Big|\psi(s, x_k(s), I_{0^+}^{q_k} x_k(s))\Big|ds \\ &\leq \int_0^{kw} G_k(t,s)s^{-\delta}(c_1 + c_2|x_k(s)|^{\gamma} + c_3|I_{0^+}^{q_k} x_k(s)|^{\eta})ds \\ &\leq \frac{1}{\Gamma(q_k)}\Big(\frac{kw}{4}\Big)^{q_k-1}\int_0^{kw} s^{-\delta}\Big(c_1 + c_2|x_k(s)|^{\gamma} + c_3\Big(\frac{(kw)^{q_k}}{\Gamma(q_k+1)}\Big)^{\eta}|x_k(s)|^{\eta}\Big)ds \\ &\leq \frac{1}{\Gamma(q_k)}\Big(\frac{kw}{4}\Big)^{q_k-1}\Big(\frac{(kw)^{1-\delta}}{1-\delta}\Big)\Big(c_1 + c_2R_k^{\gamma} + c_3\Big(\frac{(kw)^{q_k}}{\Gamma(q_k+1)}\Big)^{\eta}R_k^{\eta}\Big) \\ &\leq \frac{R_k}{3} + \frac{R_k}{3} + \frac{R_k}{3} = R_k. \end{aligned}$$

Which means that $W(B_{R_k}) \subseteq B_{R_k}$.

Step 2: W is continuous.
Let $(x_{k,m})_{m\in\mathbb{N}}$ be a sequence in E_{kw} converging to x_k as $m \to \infty$.
For any $t \in J_{kw}$, Proposition 3.2 and (H3) yield

$$|(Wx_{k,m})(t) - (Wx_k)(t)|$$

$$
\begin{aligned}
&\leq \int_0^{kw} G_k(t,s)\Big|\psi(s,x_{k,m}(s),I_{0^+}^{q_k}x_{k,m}(s)) - \psi(s,x_k(s),I_{0^+}^{q_k}x_k(s))\Big|ds \\
&\leq \frac{1}{\Gamma(q_k)}\Big(\frac{kw}{4}\Big)^{q_k-1}\int_0^{kw}\Big|\psi(s,x_{k,m}(s),I_{0^+}^{q_k}x_{k,m}(s)) - \psi(s,x_k(s),I_{0^+}^{q_k}x_k(s))\Big|ds \\
&\leq \frac{1}{\Gamma(q_k)}\Big(\frac{kw}{4}\Big)^{q_k-1}\int_0^{kw} s^{-\delta}\Big(K|x_{k,m}(s)-x_k(s)| + L|I_{0^+}^{q_k}(x_{k,m}(s)-x_k(s))|\Big)ds \\
&\leq \frac{1}{\Gamma(q_k)}\Big(\frac{kw}{4}\Big)^{q_k-1}\Big[K\|x_{k,m}-x_k\|_{E_{kw}}\int_0^{kw} s^{-\delta}ds + L\|I_{0^+}^{q_k}(x_{k,m}-x_k)\|_{E_{kw}}\int_0^{kw} s^{-\delta}ds\Big] \\
&\leq \frac{(kw)^{q_k-\delta}}{4^{q_k-1}(1-\delta)\Gamma(q_k)}\Big(K + \frac{L(kw)^{q_k}}{\Gamma(q_k+1)}\Big)\|x_{k,m}-x_k\|_{E_{kw}},
\end{aligned}
$$

so

$$
\|(Wx_{k,m}) - (Wx)\|_{E_{kw}} \to 0 \quad as \quad m \to \infty.
$$

Thus, W acts continuously on E_{kw}.

Step 3: W is compact.

To prove that W is compact, it is sufficient to show that $W(B_{R_k})$ is relatively compact in E_{kw}.

Uniform boundedness. From Step 1, we have

$$
W(B_{R_k}) \subset B_{R_k},
$$

and therefore

$$
\|Wx_k\|_{E_{kw}} \leq R_k, \qquad \forall x_k \in B_{R_k}.
$$

Hence, the family $W(B_{R_k})$ is uniformly bounded.

Equicontinuity. Let

$$
\psi^* := \sup_{t\in J_{kw}} t^{\delta}|\psi(t,0,0)|.
$$

Since $x_k \in B_{R_k}$, we have

$$
|x_k(t)| \leq R_k, \qquad t \in J_{kw}.
$$

Moreover, by the properties of the R-L FI,

$$
|I_{0^+}^{q_k}x_k(t)| \leq \frac{(kw)^{q_k}}{\Gamma(q_k+1)}\|x_k\|_{E_{kw}} \leq \frac{(kw)^{q_k}}{\Gamma(q_k+1)}R_k.
$$

Using assumption (H_3) with $(y_2, z_2) = (0, 0)$, we obtain

$$
\begin{aligned}
|\psi(t,x_k(t),I_{0^+}^{q_k}x_k(t))| &\leq |\psi(t,x_k(t),I_{0^+}^{q_k}x_k(t)) - \psi(t,0,0)| + |\psi(t,0,0)| \\
&\leq t^{-\delta}\left(K|x_k(t)| + L|I_{0^+}^{q_k}x_k(t)| + \psi^*\right).
\end{aligned}
$$

Consequently,

$$
|\psi(t,x_k(t),I_{0^+}^{q_k}x_k(t))| \leq M_k\, t^{-\delta},
$$

where

$$
M_k = KR_k + L\frac{(kw)^{q_k}}{\Gamma(q_k+1)}R_k + \psi^*.
$$

Now let $t_1, t_2 \in J_{kw}$ with $t_1 < t_2$. Using the definition of W, we have

$$\begin{aligned}|(Wx_k)(t_2) - (Wx_k)(t_1)| &= \left| \int_0^{kw} \Big(G_k(t_2, s) - G_k(t_1, s)\Big)\psi(s, x_k(s), I_{0^+}^{q_k} x_k(s))\, ds \right| \\ &\leq \int_0^{kw} |G_k(t_2, s) - G_k(t_1, s)|\, |\psi(s, x_k(s), I_{0^+}^{q_k} x_k(s))|\, ds \\ &\leq M_k \int_0^{kw} |G_k(t_2, s) - G_k(t_1, s)| s^{-\delta}\, ds.\end{aligned}$$

Since $G_k(t, s)$ is continuous on the compact set

$$[0, kw] \times [0, kw],$$

it is uniformly continuous. Therefore, for every $\varepsilon > 0$, there exists $\eta > 0$ such that

$$|t_2 - t_1| < \eta \quad \Longrightarrow \quad |G_k(t_2, s) - G_k(t_1, s)| < \frac{\varepsilon(1-\delta)}{M_k (kw)^{1-\delta}}$$

for all $s \in [0, kw]$.

Hence, whenever $|t_2 - t_1| < \eta$,

$$\begin{aligned}|(Wx_k)(t_2) - (Wx_k)(t_1)| &\leq \frac{\varepsilon(1-\delta)}{(kw)^{1-\delta}} \int_0^{kw} s^{-\delta}\, ds \\ &= \frac{\varepsilon(1-\delta)}{(kw)^{1-\delta}} \cdot \frac{(kw)^{1-\delta}}{1-\delta} \\ &= \varepsilon.\end{aligned}$$

The number η depends only on ε and not on the particular choice of $x_k \in B_{R_k}$. Therefore, the family $W(B_{R_k})$ is equicontinuous on J_{kw}.

Application of the Arzelà–Ascoli theorem.

We have shown that $W(B_{R_k})$ is uniformly bounded and equicontinuous in the Banach space

$$E_{kw} = C([0, kw], \mathbb{R}).$$

Hence, by the Arzelà–Ascoli theorem, $W(B_{R_k})$ is relatively compact in E_{kw}. Therefore, W is a compact operator.

Combining Steps 1–3, the operator

$$W : B_{R_k} \to B_{R_k}$$

is continuous and compact. Since B_{R_k} is nonempty, bounded, closed, and convex, Schauder's FP theorem guarantees the existence of at least one FP $x_k \in B_{R_k}$ such that

$$Wx_k = x_k.$$

Consequently, x_k is a solution of the BVP (3.1).

Finally, letting $n \to \infty$, the sequence $(q_n)_{n\in\mathbb{N}}$ converges uniformly to the VO function $q(t)$ and the corresponding solutions x_k converge to a limit function x. Therefore, x is a solution of the original VO BVP (1.1).

We next apply the Banach FP theorem.

Theorem 3.4. *Assume (H1) and (H3) hold, and suppose*

$$\frac{(kw)^{q_k-\delta}}{4^{q_k-1}(1-\delta)\Gamma(q_k)}\left(K+L\frac{(kw)^{q_k}}{\Gamma(q_k+1)}\right)<1. \tag{3.10}$$

Then, BVP (1.1) *possesses a unique solution in* $C([0,T],\mathbb{R})$.

Proof. The Banach contraction principle establishes that the operator W from (3.8) admits a unique FP. For any $x_k, y_k \in E_{kw}$, we derive

$$\begin{aligned}
&|(Wy_k)(t)-(Wx_k)(t)| \\
\leq\ & \int_0^{kw} G_k(t,s)|\psi(s,y_k(s),I_{0^+}^{q_k}y_k(s))-\psi(s,x_k(s),I_{0^+}^{q_k}x_k(s))|ds \\
\leq\ & \int_0^{kw} s^{-\delta}G_k(t,s)(K|y_k(s)-x_k(s)|+L|I_{0^+}^{q_k}(y_k(s)-x_k(s)))|ds \\
\leq\ & \frac{1}{\Gamma(q_k)}\Big(\frac{kw}{4}\Big)^{q_k-1}(K+L\frac{(kw)^{q_k}}{\Gamma(q_k+1)})\|y_k-x_k\|_{E_{kw}}\int_0^{kw}s^{-\delta}ds \\
\leq\ & \frac{(kw)^{1-\delta}}{1-\delta}\frac{1}{\Gamma(q_k)}\Big(\frac{kw}{4}\Big)^{q_k-1}(K+L\frac{(kw)^{q_k}}{\Gamma(q_k+1)})\|y_k-x_k\|_{E_{kw}} \\
\leq\ & \frac{(kw)^{q_k-\delta}}{4^{q_k-1}(1-\delta)\Gamma(q_k)}(K+L\frac{(kw)^{q_k}}{\Gamma(q_k+1)})\|y_k-x_k\|_{E_{kw}}.
\end{aligned}$$

Condition (3.10) thus renders W a contraction mapping. The Banach FP theorem then guarantees a unique FP $x_k \in E_{kw}$, which solves BVP (3.1) uniquely.
Taking $n \to \infty$, the sequence $(q_n)_{n\in\mathbb{N}}$ approach to the original VO $q(t)$ and $x_k(t) \to x(t)$ which is a solution of BVP 1.1.

4. Ulam–Hyers stability

Theorem 4.1. *Assume* (H1)*,* (H2)*, and* (3.10) *hold. Then, BVP* (1.1) *is Ulam–Hyers stable.*

Proof. Let $z \in C(J,\mathbb{R})$ be a function satisfying

$$\left|D_{0^+}^{q(t)}z(t)+\psi(t,z(t),I_{0^+}^{q(t)}z(t))\right| \leq \varepsilon, \qquad t \in J. \tag{4.1}$$

Fix $n \in \mathbb{N}$ and consider the corresponding piecewise constant approximation $(q_n)_{n\in\mathbb{N}}$ of the VO $q(t)$. Let $z_k \in E_{kw}$, $k = 1, 2, \ldots, n$, satisfy

$$\left|D_{0^+}^{q_k}z_k(t)+\psi(t,z_k(t),I_{0^+}^{q_k}z_k(t))\right| \leq \varepsilon, \qquad t \in J_{kw}. \tag{4.2}$$

As $n \to \infty$, the sequence $(q_n)_{n\in\mathbb{N}}$ converges uniformly to the VO $q(t)$ and

$$z_k(t) \longrightarrow z(t).$$

By Theorem 3.4, the BVP (1.1) admits a unique solution $x \in C(J,\mathbb{R})$, obtained as the limit of the unique solutions $x_k \in E_{kw}$ of the approximating problems (3.1). Thus,

$$x_k(t) \longrightarrow x(t) \qquad \text{as } n \to \infty.$$

Moreover, by Lemma 3.1,

$$x_k(t) = \int_0^{kw} G_k(t,s)\psi\big(s, x_k(s), I_{0^+}^{q_k} x_k(s)\big)\, ds.$$

Since (4.2) holds, there exists a continuous function $\omega_k \in C(J_{kw}, \mathbb{R})$ such that

$$|\omega_k(t)| \le \varepsilon, \qquad t \in J_{kw},$$

and

$$D_{0^+}^{q_k} z_k(t) + \psi(t, z_k(t), I_{0^+}^{q_k} z_k(t)) = \omega_k(t), \qquad t \in J_{kw}. \tag{4.3}$$

Applying the FI operator $I_{0^+}^{q_k}$ to both sides of (4.3) and using Lemma 2.2, together with the Green function representation from Lemma 3.1, we obtain

$$z_k(t) = \int_0^{kw} G_k(t,s)\psi\big(s, z_k(s), I_{0^+}^{q_k} z_k(s)\big)\, ds + \int_0^t \frac{(t-s)^{q_k-1}}{\Gamma(q_k)}\omega_k(s)\, ds.$$

Consequently,

$$\begin{aligned}
&\left| z_k(t) - \int_0^{kw} G_k(t,s)\psi\big(s, z_k(s), I_{0^+}^{q_k} z_k(s)\big)\, ds \right| \\
&= \left| \int_0^t \frac{(t-s)^{q_k-1}}{\Gamma(q_k)}\omega_k(s)\, ds \right| \\
&\le \int_0^t \frac{(t-s)^{q_k-1}}{\Gamma(q_k)}|\omega_k(s)|\, ds \\
&\le \varepsilon \int_0^t \frac{(t-s)^{q_k-1}}{\Gamma(q_k)}\, ds \\
&= \varepsilon \frac{t^{q_k}}{\Gamma(q_k+1)} \\
&\le \varepsilon \frac{(kw)^{q_k}}{\Gamma(q_k+1)}.
\end{aligned} \tag{4.4}$$

Now, for any $t \in J_{kw}$, we have

$$\begin{aligned}
|z_k(t) - x_k(t)| &= \left| z_k(t) - \int_0^{kw} G_k(t,s)\psi\big(s, x_k(s), I_{0^+}^{q_k} x_k(s)\big)\, ds \right| \\
&\le \left| z_k(t) - \int_0^{kw} G_k(t,s)\psi\big(s, z_k(s), I_{0^+}^{q_k} z_k(s)\big)\, ds \right| \\
&\quad + \int_0^{kw} G_k(t,s)\Big|\psi\big(s, z_k(s), I_{0^+}^{q_k} z_k(s)\big) - \psi\big(s, x_k(s), I_{0^+}^{q_k} x_k(s)\big)\Big|\, ds.
\end{aligned}$$

Using (4.4), assumption (H3), and Proposition 3.1, we obtain

$$\begin{aligned}
|z_k(t) - x_k(t)| &\le \varepsilon \frac{(kw)^{q_k}}{\Gamma(q_k+1)} \\
&\quad + \frac{1}{\Gamma(q_k)}\Big(\frac{kw}{4}\Big)^{q_k-1} \int_0^{kw} s^{-\delta}\Big(K|z_k(s) - x_k(s)| + L|I_{0^+}^{q_k}(z_k(s) - x_k(s))|\Big)\, ds.
\end{aligned}$$

Since

$$|I_{0^+}^{q_k}(z_k - x_k)(t)| \leq \frac{(kw)^{q_k}}{\Gamma(q_k+1)}\|z_k - x_k\|_{E_{kw}},$$

it follows that

$$\begin{aligned}|z_k(t) - x_k(t)| \leq \varepsilon &\frac{(kw)^{q_k}}{\Gamma(q_k+1)} \\ &+ \frac{1}{\Gamma(q_k)}\Big(\frac{kw}{4}\Big)^{q_k-1}\Big(K\|z_k - x_k\|_{E_{kw}} + L\frac{(kw)^{q_k}}{\Gamma(q_k+1)}\|z_k - x_k\|_{E_{kw}}\Big)\int_0^{kw} s^{-\delta}\, ds.\end{aligned}$$

Since

$$\int_0^{kw} s^{-\delta}\, ds = \frac{(kw)^{1-\delta}}{1-\delta},$$

we obtain

$$|z_k(t) - x_k(t)| \leq \varepsilon\frac{(kw)^{q_k}}{\Gamma(q_k+1)} + \frac{(kw)^{q_k-\delta}}{4^{q_k-1}(1-\delta)\Gamma(q_k)}\Big(K + L\frac{(kw)^{q_k}}{\Gamma(q_k+1)}\Big)\|z_k - x_k\|_{E_{kw}}.$$

Taking the supremum over $t \in J_{kw}$ yields

$$\|z_k - x_k\|_{E_{kw}} \leq \frac{(kw)^{q_k}}{\Gamma(q_k+1)}\varepsilon + \frac{(kw)^{q_k-\delta}}{4^{q_k-1}(1-\delta)\Gamma(q_k)}\Big(K + L\frac{(kw)^{q_k}}{\Gamma(q_k+1)}\Big)\|z_k - x_k\|_{E_{kw}}.$$

Therefore,

$$\|z_k - x_k\|_{E_{kw}}\left(1 - \frac{(kw)^{q_k-\delta}}{4^{q_k-1}(1-\delta)\Gamma(q_k)}\Big(K + L\frac{(kw)^{q_k}}{\Gamma(q_k+1)}\Big)\right) \leq \frac{(kw)^{q_k}}{\Gamma(q_k+1)}\varepsilon.$$

Since condition (3.10) guarantees that

$$1 - \frac{(kw)^{q_k-\delta}}{4^{q_k-1}(1-\delta)\Gamma(q_k)}\Big(K + L\frac{(kw)^{q_k}}{\Gamma(q_k+1)}\Big) > 0,$$

we deduce

$$\|z_k - x_k\|_{E_{kw}} \leq c_\psi \varepsilon,$$

where

$$c_\psi = \max_{k\in\{1,2,\dots,n\}} \frac{(kw)^{q_k}}{\left(1 - \frac{(kw)^{q_k-\delta}}{4^{q_k-1}(1-\delta)\Gamma(q_k)}\Big(K + L\frac{(kw)^{q_k}}{\Gamma(q_k+1)}\Big)\right)\Gamma(q_k+1)}.$$

The constant c_ψ is finite, positive, and independent of ε.
Finally, letting $n \to \infty$ and using the convergences $z_k \to z$ and $x_k \to x$, we obtain

$$\|z - x\| \leq c_\psi \varepsilon.$$

Hence, the BVP (1.1) is Ulam–Hyers stable. □

5. Numerical example

Consider the VO fractional BVP

$$\begin{cases} D_{0^+}^{\frac{4}{3}+\frac{1}{3}t}x(t) = \dfrac{t^{-\frac{1}{4}}}{7e^{t+2}(1+|x(t)|+|I_0^{\frac{4}{3}+\frac{1}{3}t}x(t)|)}, & t \in J := [0,2], \\ x(0) = 0, \qquad x(2) = 0. \end{cases} \tag{5.1}$$

Let

$$q(t) = \frac{4}{3} + \frac{1}{3}t, \qquad t \in [0,2].$$

Then, $q \in C([0,2],\mathbb{R})$ and

$$\frac{4}{3} \le q(t) \le 2.$$

The nonlinear term is given by

$$\psi(t,y,z) = \frac{t^{-1/4}}{7e^{t+2}(1+y+z)}, \qquad (t,y,z) \in [0,2] \times [0,+\infty)^2.$$

For $n \in \mathbb{N}$, divide the interval $[0,2]$ into n equal subintervals and define the piecewise constant approximation

$$q_k = \begin{cases} \frac{4}{3}, & 0 \le t < \frac{2}{n}, \\ \frac{4}{3} + \frac{1}{3}w, & \frac{2}{n} \le t < \frac{4}{n}, \\ \frac{4}{3} + \frac{1}{3}(2w), & \frac{4}{n} \le t < \frac{6}{n}, \\ \vdots & \vdots \\ \frac{4}{3} + \frac{1}{3}(n-1)w, & \frac{2(n-1)}{n} \le t < 2, \end{cases} \tag{5.2}$$

where $nw = 2$.

Equivalently,

$$q_k = \frac{4}{3} + \frac{k-1}{3}w, \qquad k = 1, 2, \ldots, n.$$

Let

$$t \in \left[\frac{2(k-1)}{n}, \frac{2k}{n}\right).$$

Then,

$$q_k = \frac{4}{3} + \frac{1}{3}\frac{2(k-1)}{n},$$

and therefore

$$\begin{aligned} |q(t) - q_k| &= \left|\frac{1}{3}\left(t - \frac{2(k-1)}{n}\right)\right| \\ &\le \frac{2}{3n}. \end{aligned}$$

Hence,

$$\|q - q_k\|_\infty = \sup_{t\in[0,2]} |q(t) - q_k| \le \frac{2}{3n}, \tag{5.3}$$

which implies that $q_k \to q$ uniformly on $[0,2]$ as $n \to \infty$. Consequently, assumption $(H1)$ is satisfied.

The corresponding constant-order BVP is

$$\begin{cases} D_{0^+}^{q_k} x_k(t) = \dfrac{t^{-1/4}}{7e^{t+2}(1 + |x_k(t)| + |I_0^{q_k} x_k(t)|)}, & t \in J_{kw}, \\ x_k(0) = 0, \qquad x_k(kw) = 0. \end{cases} \tag{5.4}$$

Next, for any $(t, y_i, z_i) \in [0,2] \times [0,+\infty)^2$, $i = 1,2$, we obtain

$$\begin{aligned} t^{1/4}|\psi(t,y_1,z_1) - \psi(t,y_2,z_2)| &= \left| \frac{1}{7e^{t+2}} \left(\frac{1}{1+y_1+z_1} - \frac{1}{1+y_2+z_2} \right) \right| \\ &\le \frac{|y_1 - y_2| + |z_1 - z_2|}{(7e^{t+2})(1+y_1+z_1)(1+y_2+z_2)} \\ &\le \frac{1}{7e^2}\Big(|y_1 - y_2| + |z_1 - z_2|\Big). \end{aligned}$$

Thus, assumption $(H3)$ holds with

$$\delta = \frac{1}{4}, \qquad K = L = \frac{1}{7e^2}.$$

Moreover,

$$\begin{aligned} \frac{(kw)^{q_k - \delta}}{4^{q_k - 1}(1-\delta)\Gamma(q_k)} \left(K + L \frac{(kw)^{q_k}}{\Gamma(q_k + 1)} \right) =& \frac{(\frac{2k}{n})^{\frac{4}{3} + \frac{1}{3}\cdot\frac{2(k-1)}{n} - \frac{1}{4}}}{4^{\frac{4}{3}+\frac{1}{3}\cdot\frac{2(k-1)}{n} - 1}(1 - \frac{1}{4})\Gamma(\frac{4}{3} + \frac{1}{3}\cdot\frac{2(k-1)}{n})} (\frac{1}{7e^2} + \frac{1}{7e^2}\frac{(\frac{2k}{n})^{\frac{4}{3}+\frac{1}{3}\cdot\frac{2(k-1)}{n}}}{\Gamma(\frac{4}{3} + \frac{1}{3}\cdot\frac{2(k-1)}{n} + 1)}) \\ \le& 1.023 \times 10^{-5} < 1, \end{aligned}$$

for every $k = 1, \ldots, n$. Hence, condition (3.10) is satisfied.

To quantify the convergence of the approximation $q_k \to q$, Table 1 reports the error

$$e_n = \|q - q_k\|_\infty,$$

together with the experimental convergence rate

$$r_n = \frac{\log(e_n/e_{10n})}{\log(10)}.$$

Table 1. Approximation error and convergence rate for the order function $q(t)$.

n	e_n	r_n
10	6.667×10^{-2}	–
10^2	6.667×10^{-3}	1.000
10^3	6.667×10^{-4}	1.000
10^4	6.667×10^{-5}	1.000
10^5	6.667×10^{-6}	1.000

The results confirm the estimate (5.3) and demonstrate a first-order convergence rate.

Table 2 shows that $M(n)$ rapidly approaches the limiting value

$$M_\infty \approx 0.065032.$$

Indeed,

$$|M(10) - M_\infty| = 5.595 \times 10^{-3}, \qquad |M(10^4) - M_\infty| = 5.0 \times 10^{-6},$$

which quantitatively confirms the stability of condition (3.10) with respect to the discretization parameter n.

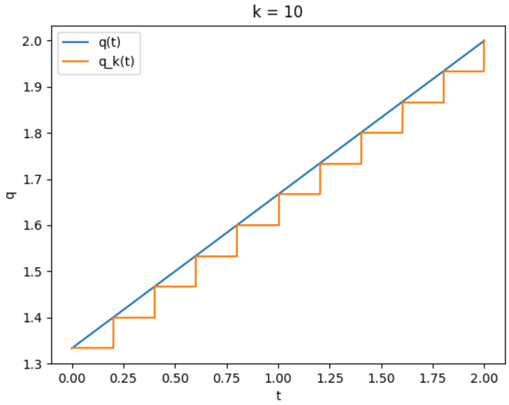
Table 2. Maximum value $M(n) = \max_k e_k(n)$ for $n = 10^i$, $i = 1, \ldots, 10$.

n	$M(n)$
10	0.070 627
10^2	0.065 582
10^3	0.065 089
10^4	0.065 037
10^5	0.065 032
10^6	0.065 032
10^7	0.065 032
10^8	0.065 032
10^9	0.065 032
10^{10}	0.065 032

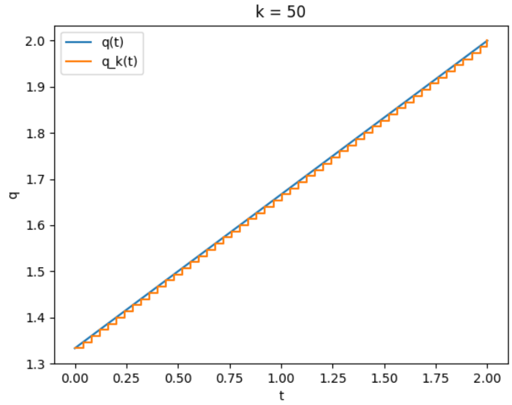
The numerical results are consistent with the theoretical analysis developed in the previous sections. Table 1 confirms that the approximation q_k converges uniformly to the VO function q with first-order rate. Moreover, Table 2 shows that the quantity appearing in condition (3.10) remains uniformly below one and converges to a limiting value as n increases.

Figure 1 illustrates the convergence of the piecewise constant approximations q_k toward q, while Figure 2 displays the stabilization of the corresponding approximate solutions x_k as the discretization is refined. Since no closed-form analytical solution of problem (5.1) is available, Figure 2 should be interpreted as evidence of convergence of the sequence of approximate solutions generated by the constant-order approximating problems rather than convergence to a known exact solution.

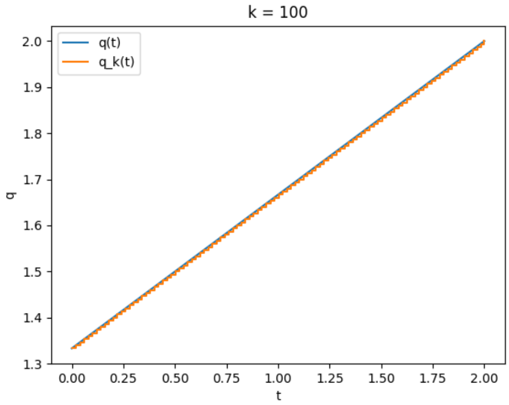
Therefore, the numerical experiments provide quantitative support for the approximation framework developed in this paper and illustrate the effectiveness of the proposed approach for VO fractional BVPs.



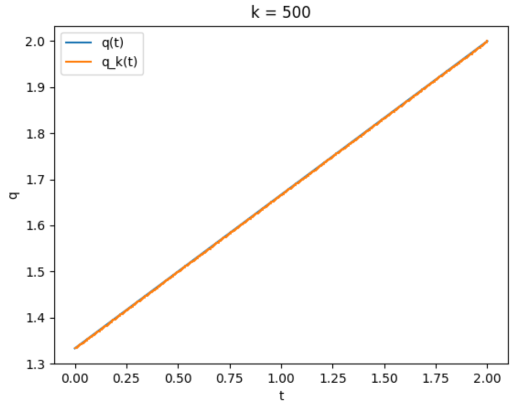
(a) $n = 10$



(b) $n = 50$

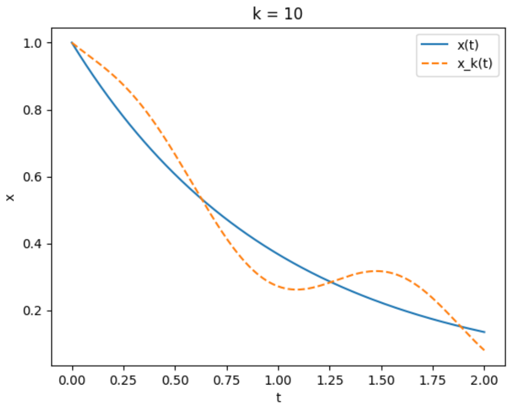


(c) $n = 100$

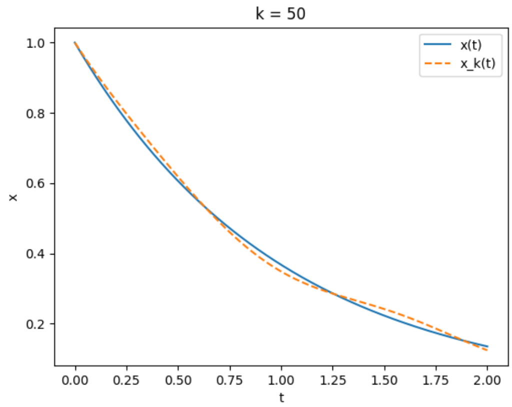


(d) $n = 500$

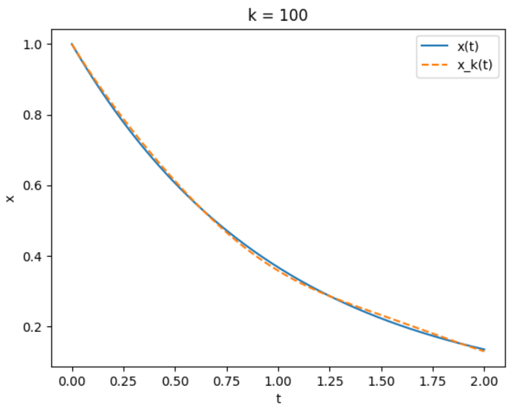
Figure 1. Piecewise constant approximations $q_k(t)$ of the VO function $q(t)$ for different values of n.



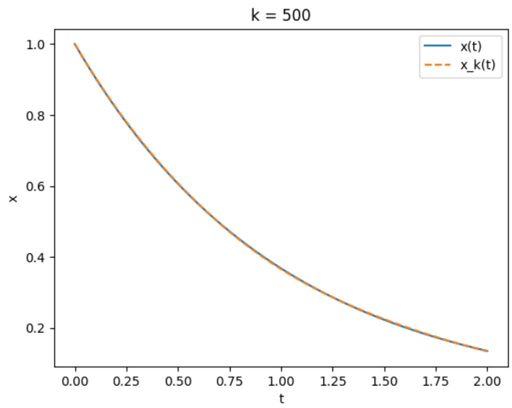
(a) $k = 10$



(b) $k = 50$



(c) $k = 100$



(d) $k = 500$

Figure 2. Approximate solutions $x_k(t)$ corresponding to the constant-order problems (5.4) for different values of k.

6. Conclusions

In this paper, we investigated a class of nonlinear VO fractional BVPs involving the R-L FD. By approximating the VO function with a sequence of piecewise constant orders, the original problem was transformed into a family of constant-order fractional BVPs. For each approximating problem, an equivalent integral formulation was established through the construction of an associated Green function. FP techniques were then employed to derive existence, uniqueness, and Ulam–Hyers stability results. The theoretical findings were further illustrated by a numerical example, which demonstrated the applicability of the proposed framework.

The main advantage of the proposed approach lies in reducing a VO problem to a sequence of constant-order problems, thereby allowing the use of classical tools from FDE. However, several limitations remain. First, although the convergence of the piecewise-constant approximation of the order function was established, a rigorous error analysis between the solutions of the approximating problems and the solution of the original VO problem was not derived. In particular, explicit convergence rates and sharp error bounds for the approximation procedure remain open questions. Second, the analysis was restricted to scalar BVPs on a finite interval and relied on specific Lipschitz-type assumptions on the nonlinear term. Third, the computational study focused on illustrating convergence behavior and did not investigate the numerical accuracy of dedicated discretization schemes.

Several directions for future research naturally arise from this work. One important extension is the development of quantitative error estimates for the piecewise-constant approximation method and the investigation of its convergence order under additional regularity assumptions on the VO function. Another promising direction is the extension of the present framework to systems of VO FDEs, coupled fractional models, and higher-dimensional fractional dynamical systems. It would also be of interest to consider more general boundary conditions, including multi-point, integral, nonlocal, and impulsive boundary conditions.

From the computational viewpoint, future studies may focus on the construction and analysis of efficient numerical algorithms for VO fractional problems. Possible approaches include finite difference methods, finite element methods, spectral methods, collocation techniques, and predictor–corrector schemes adapted to VO operators. The combination of these numerical methods with the approximation framework developed in this paper may provide effective tools for solving complex VO models arising in physics, engineering, biology, and control theory.

Author contributions

I. N. Telli: Writing-original draft, writing-review & editing, conceptualization; B. Telli: Data curation, methodology, software; M. S. Souid: Project administration, supervision, investigation, resources; S. Pinelas: Funding acquisition, visualization, validation, formal analysis. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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