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Research article

Weak traveling wave solutions to a two-species competitive chemotaxis system incorporating Holling-II functional responses

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Abstract: This work focuses on analyzing a two-species competitive chemotaxis model with Holling-II functional responses. First, the stability of the positive equilibrium point E^* is analyzed by applying the continuous Fourier transform and the Routh-Hurwitz criterion. Second, proof by contradiction and the generalized principal eigenvalue argument are employed to prove the non-existence of weak traveling wave solutions for sufficiently small wave speeds c. Finally, the existence of weak traveling wave solutions is proved via the upper and lower solutions method and Schauder's fixed-point theorem for all sufficiently large c.

Keywords: two-species competition model; chemotaxis; traveling wave solution; upper and lower solutions; Schauder's fixed-point theorem
Mathematics Subject Classification: 34C37, 35C07

1. Introduction

Wang and Guo [1] investigated the two-species competitive Keller-Segel chemotaxis system:

$$\begin{cases} u_t = d_1\Delta u - \chi_1\nabla\left(u\nabla w\right) + \delta_1 u\left(1 - u - a_1 v\right), \\ v_t = d_2\Delta v - \chi_2\nabla\left(v\nabla w\right) + \delta_2 v\left(1 - v - a_2 u\right), \\ 0 = \Delta w + \alpha u + \beta v - \gamma w, \end{cases} \tag{1.1}$$

where u and v represent the densities of the two competing species and w represents the substrate concentration. d_1 and d_2 denote the diffusion coefficients of the two species. χ_1 and χ_2 denote the chemotactic sensitivities of the two species. δ_1 and δ_2 denote the growth rates of the two species. a_1 and a_2 denote interspecific linear competitive inhibition coefficients of the two species. α and β denote the rates at which the two species secrete the substrate. γ denotes the degradation rate of the substrate. All of these parameters are positive constants.

Their main results are as follows:

(1) Under the appropriate assumptions, by constructing generalized upper and lower solutions and combining the fixed-point theorem, they proved that system (1.1) has traveling wave solutions connecting E_0 with E^* when wave speed c is sufficiently large, and these traveling wave solutions decay exponentially as $\xi \to \infty$.

(2) Through eigenvalue analysis and proof by contradiction, they further demonstrated that system (1.1) does not have traveling wave solutions when c is sufficiently small.

Interspecific competitive inhibition often exhibits a saturating characteristic in real ecological scenarios, which cannot be fully captured by linear competitive terms. To more accurately describe this biological phenomenon and enrich the theoretical studies of chemotaxis systems with nonlinear competitive interactions, we investigate weak traveling wave solutions to a two-species competitive chemotaxis model incorporating Holling-II functional responses:

$$\begin{cases} u_t = d_1\Delta u - \chi_1\nabla(u\nabla w) + \delta_1 u\left(1 - u - \frac{h_1 v}{1+u}\right), & x \in \mathbb{R}^N,\ t > 0, \\ v_t = d_2\Delta v - \chi_2\nabla(v\nabla w) + \delta_2 v\left(1 - v - \frac{h_2 u}{1+v}\right), & x \in \mathbb{R}^N,\ t > 0, \\ 0 = \Delta w + \alpha u + \beta v - \gamma w, & x \in \mathbb{R}^N,\ t > 0. \end{cases} \tag{1.2}$$

Here, h_1 and h_2 denote the interspecific saturated competitive inhibition coefficients of the two species, respectively. The denominators $1 + u$ and $1 + v$ are introduced to capture the nonlinear effect of competitive inhibition gradually saturating with the increase of the species' own density.

Straightforward verification confirms that system (1.2) has equilibrium points $E_0 = (0, 0, 0)$, $E_1 = \left(1, 0, \frac{\alpha}{\gamma}\right)$, and $E_2 = \left(0, 1, \frac{\beta}{\gamma}\right)$.

Assume $h_1 > 1$ and $h_2 > 1$, then system (1.2) also has a positive equilibrium point $E^* = (u^*, v^*, w^*)$ $(0 < u^*, v^* < 1)$, where the components satisfy

$$\begin{cases} 1 - u^* - \frac{h_1 v^*}{1+u^*} = 0, \\ 1 - v^* - \frac{h_2 u^*}{1+v^*} = 0, \\ \gamma w^* = \alpha u^* + \beta v^*. \end{cases}$$

From these equations, we derive the relations $u^* = \frac{1-v^{*2}}{h_2}$ and $v^* = \frac{1-u^{*2}}{h_1}$, leading to

$$u^* = \frac{1}{h_2}\left[1 - \left(\frac{1-u^{*2}}{h_1}\right)^2\right].$$

Define a continuous function

$$f(u) = \frac{1}{h_2}\left[1 - \left(\frac{1-u^2}{h_1}\right)^2\right] - u,$$

and analyze its zero. By $h_1 > 1$, we have $f(0) = \frac{1}{h_2}\left(1 - \frac{1}{h_1^2}\right) > 0$. By $h_2 > 1$, we have $f(1) = \frac{1}{h_2} - 1 < 0$. Since $f(u)$ is continuous on the interval $[0, 1]$ and $f(0) \cdot f(1) < 0$, differential intermediate value theorem guarantees the existence of $u^* \in (0, 1)$ such that $f(u^*) = 0$. Thus, we have $v^* = \frac{1-u^{*2}}{h_1} > 0$ and $w^* = \frac{\alpha u^* + \beta v^*}{\gamma} > 0$.

Chemotaxis is the directional movement of an individual organism induced by chemical signals, and the Keller-Segel model provides its classic mathematical description [2, 3]. Research over the past two decades has extensively explored the behavior of solutions to chemotaxis-type models. For instance, Feng, Jia, and Zhou [4] analyzed a class of such systems under zero-flux and Dirichlet boundary conditions, establishing non-trivial steady-state solutions using the upper and lower solutions method alongside a monotone iteration scheme. Their work further confirmed the stability of these equilibria via the inverse derivative technique and weighted energy estimates. Li and Wang [5] examined a conserved system derived from the Keller-Segel equations, proving the existence of traveling fronts through phase plane analysis. What's more, Pan, Mu, and Tao [6] investigated a two-species competitive chemotaxis system with a chemical signaling loop, proved the global existence and boundedness of its solutions, and revealed that chemical signaling loops can alter population dynamics.

Within population ecology, the analysis of traveling wave solutions and their non-existence is a key focus on chemotaxis research. Issa, Salako, and Shen [7] studied a two-species chemotaxis system, proving the existence of traveling wave solutions via the upper and lower solutions method and fixed-point theorem, while also showing non-existence of such traveling waves by using proof of contradiction. Li, Tan, and Qiu [8] studied the existence of traveling wavefronts in a modified Leslie-Gower model by applying the perturbation method. Due to the variety and inhomogeneity of ecosystems, the study of different chemotaxis models has more important significance. Along these lines, Li and Guo [9] used perturbation techniques to establish large-speed traveling wave solutions for a reaction-diffusion system incorporating chemotaxis and nonlocal delays. Li and Wang [10] considered a singular Keller-Segel system. First, they applied the geometric singular perturbation theory to prove the existence of traveling wave solutions with small chemical diffusion. Then, they applied spectral analysis to show their linear instability. Finally, they validated wave profile stabilization via numerical simulations. Song and Wang [11] established the existence of bistable traveling wave solutions of a density-suppressed motility model with strong Allee effect using a perturbation argument. For more results, we can see [12–14] and the references therein.

The traveling wave problem for system (1.2) shares identical key properties across both $\mathbb{R}$ and $\mathbb{R}^N$ spatial domains. For analytical simplicity, we focus our subsequent analysis on system (1.2) in $x \in \mathbb{R}$, corresponding to the following system:

$$\begin{cases} u_t = d_1 u_{xx} - \chi_1 (u w_x)_x + \delta_1 u \left(1 - u - \frac{h_1 v}{1+u}\right), \\ v_t = d_2 v_{xx} - \chi_2 (v w_x)_x + \delta_2 v \left(1 - v - \frac{h_2 u}{1+v}\right), \\ 0 = w_{xx} + \alpha u + \beta v - \gamma w. \end{cases} \tag{1.3}$$

It is pointed out that system (1.3) can be written as follows:

$$\begin{cases} u_t = d_1 u_{xx} - \chi_1 u_x w_x + u[\delta_1 - (\frac{\delta_1 h_1}{1+u} - \beta\chi_1)v - \gamma\chi_1 w - (\delta_1 - \alpha\chi_1)u], \\ v_t = d_2 v_{xx} - \chi_2 v_x w_x + v[\delta_2 - (\frac{\delta_2 h_2}{1+v} - \alpha\chi_2)u - \gamma\chi_2 w - (\delta_2 - \beta\chi_2)v], \\ 0 = w_{xx} + \alpha u + \beta v - \gamma w. \end{cases} \tag{1.4}$$

A weak traveling wave solution is a special solution $(u(x,t), v(x,t), w(x,t))$ of the form

$$u(x,t) = u(x-ct), \quad v(x,t) = v(x-ct), \quad w(x,t) = w(x-ct),$$

where $\xi = x - ct$. $u(\xi), v(\xi)$, and $w(\xi)$ are governed by the following ordinary differential equation system:

$$\begin{cases}0 = d_1u'' + (c - \chi_1 w')u' + u[\delta_1 - (\frac{\delta_1 h_1}{1+u} - \beta\chi_1)v - \gamma\chi_1 w - (\delta_1 - \alpha\chi_1)u],\\ 0 = d_2v'' + (c - \chi_2 w')v' + v[\delta_2 - (\frac{\delta_2 h_2}{1+v} - \alpha\chi_2)u - \gamma\chi_2 w - (\delta_2 - \beta\chi_2)v],\\ 0 = w'' + \alpha u + \beta v - \gamma w,\end{cases} \tag{1.5}$$

and

$$0 < u(\xi) \le U_0,\ 0 < v(\xi) \le V_0,\ 0 < w(\xi) \le W_0,\ \forall\, \xi \in \mathbb{R}, \tag{1.6}$$

$$(u(\infty), v(\infty), w(\infty)) = E_0, \tag{1.7}$$

where U_0, V_0, and W_0 are appropriate positive constants. If the solution of system (1.5) satisfies (1.6) and (1.7), it is called the weak traveling wave solution of system (1.3).

Throughout this article, some assumptions are presented as follows:

$(\mathcal{A})$ $h_1, h_2 > 1$;

$(\mathcal{B})$ $\chi_1 < \min\left\{\frac{\delta_1}{\alpha},\ \frac{h_1\delta_1}{\beta(1+U_0)}\right\}, \quad \chi_2 < \min\left\{\frac{\delta_2}{\beta},\ \frac{h_2\delta_2}{\alpha(1+V_0)}\right\}$;

$(\mathcal{C})$ $0 < u^*, v^* < 1$.

This paper is structured as follows: In Section 2, employing small perturbation linearization and continuous Fourier transform, we prove locally asymptotic stability of E^*. In Section 3, we prove the non-existence of weak traveling wave solutions. In Section 4, we verify the existence of weak traveling wave solutions by means of the upper and lower solutions method combined with Schauder's fixed-point theorem.

2. Stability analysis of the positive equilibrium point E^*

Throughout this section, some assumptions are presented as follows:

$(\mathcal{D})$ $4u^*v^* > h_1h_2$;

$(\mathcal{E})$ $d_1 + d_2 > \frac{\alpha\chi_1 u^* + \beta\chi_2 v^*}{\gamma}$;

$(\mathcal{F})$ $d_1d_2 > \frac{d_1\beta\chi_2 v^* + d_2\alpha\chi_1 u^*}{\gamma}$;

$(\mathcal{G})$ $\alpha h_1 > 2\beta u^*$, $\beta h_2 > 2\alpha v^*$.

We introduce a small perturbation around the positive equilibrium point E^* as defined below:

$$u = u^* + \tilde{u}, v = v^* + \tilde{v}, w = w^* + \tilde{w}.$$

Here, $\|\tilde{u}\|, \|\tilde{v}\|, \|\tilde{w}\| \ll 1$. By substituting the above equations into system (1.2) and neglecting the higher-order terms of the perturbation, we obtain the linearized system:

$$\begin{cases}\tilde{u}_t = d_1\Delta\tilde{u} - \chi_1 u^*\Delta\tilde{w} - \frac{2\delta_1 u^{*2}}{1+u^*}\tilde{u} - \frac{\delta_1 h_1 u^*}{1+u^*}\tilde{v},\\ \tilde{v}_t = d_2\Delta\tilde{v} - \chi_2 v^*\Delta\tilde{w} - \frac{2\delta_2 v^{*2}}{1+v^*}\tilde{v} - \frac{\delta_2 h_2 v^*}{1+v^*}\tilde{u},\\ 0 = \Delta\tilde{w} + \alpha\tilde{u} + \beta\tilde{v} - \gamma\tilde{w}.\end{cases} \tag{2.1}$$

We apply a continuous Fourier transform on the spatial variables and define the Fourier modes of the perturbation as follows:

$$\tilde{u}(\boldsymbol{x},t) = \hat{u}(\boldsymbol{k},t)\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tilde{v}(\boldsymbol{x},t) = \hat{v}(\boldsymbol{k},t)\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tilde{w}(\boldsymbol{x},t) = \hat{w}(\boldsymbol{k},t)\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}.$$

Here, $\boldsymbol{k}$ represents the wave vector ($k = |\boldsymbol{k}|$ is the wave number). The Laplacian operator Δ corresponds to $-k^2$ in the frequency domain. We have

$$k^2\hat{w} = \alpha\hat{u} + \beta\hat{v} - \gamma\hat{w},$$

and its solution is given by

$$\hat{w} = \frac{\alpha\hat{u} + \beta\hat{v}}{\gamma + k^2}.$$

By applying the same method used to solve for $\hat{w}$ to the first two equations of system (2.1), we obtain its frequency-domain system:

$$\begin{cases} \frac{d\hat{u}}{dt} = \left(-d_1k^2 - \frac{2\delta_1 u^{*2}}{1+u^*} + \frac{\alpha k^2\chi_1 u^*}{\gamma+k^2}\right)\hat{u} + \left(\frac{\beta k^2\chi_1 u^*}{\gamma+k^2} - \frac{\delta_1 h_1 u^*}{1+u^*}\right)\hat{v}, \\ \frac{d\hat{v}}{dt} = \left(\frac{\alpha k^2\chi_2 v^*}{\gamma+k^2} - \frac{\delta_2 h_2 v^*}{1+v^*}\right)\hat{u} + \left(-d_2k^2 - \frac{2\delta_2 v^{*2}}{1+v^*} + \frac{\beta k^2\chi_2 v^*}{\gamma+k^2}\right)\hat{v}. \end{cases}$$

So, its coefficient matrix is

$$J = \begin{pmatrix} -d_1k^2 - \dfrac{2\delta_1 u^{*2}}{1+u^*} + \dfrac{\alpha k^2\chi_1 u^*}{\gamma+k^2} & \dfrac{\beta k^2\chi_1 u^*}{\gamma+k^2} - \dfrac{\delta_1 h_1 u^*}{1+u^*} \\ \dfrac{\alpha k^2\chi_2 v^*}{\gamma+k^2} - \dfrac{\delta_2 h_2 v^*}{1+v^*} & -d_2k^2 - \dfrac{2\delta_2 v^{*2}}{1+v^*} + \dfrac{\beta k^2\chi_2 v^*}{\gamma+k^2} \end{pmatrix}.$$

We then investigate the local asymptotic stability of E^* for wave number $k = 0$ and $k > 0$, respectively.

(i) For $k = 0$,

$$\begin{aligned} \det|J - \lambda I| &= \left(-\frac{2\delta_1 u^{*2}}{1+u^*} - \lambda\right)\left(-\frac{2\delta_2 v^{*2}}{1+v^*} - \lambda\right) - \left(-\frac{\delta_1 h_1 u^*}{1+u^*}\right)\left(-\frac{\delta_2 h_2 v^*}{1+v^*}\right) \\ &= \lambda^2 - \left(-\frac{2\delta_1 u^{*2}}{1+u^*} - \frac{2\delta_2 v^{*2}}{1+v^*}\right)\lambda + \frac{\delta_1\delta_2 u^* v^*(4u^*v^* - h_1h_2)}{(1+u^*)(1+v^*)}. \end{aligned}$$

By the assumption ($\mathcal{D}$), we obtain that

$$\mathrm{tr}(J) = -\left(\frac{2\delta_1 u^{*2}}{1+u^*} + \frac{2\delta_2 v^{*2}}{1+v^*}\right) < 0,$$

and

$$\det(J) = \frac{\delta_1\delta_2 u^* v^*\,(4u^*v^* - h_1h_2)}{(1+u^*)\,(1+v^*)} > 0.$$

(ii) For $k > 0$,

$$\det|J - \lambda I| = \left(-d_1k^2 - \frac{2\delta_1 u^{*2}}{1+u^*} + \frac{\alpha k^2\chi_1 u^*}{\gamma+k^2} - \lambda\right)\left(-d_2k^2 - \frac{2\delta_2 v^{*2}}{1+v^*} + \frac{\beta k^2\chi_2 v^*}{\gamma+k^2} - \lambda\right)$$

$$-\left(\frac{\beta k^2\chi_1 u^*}{\gamma+k^2}-\frac{\delta_1 h_1 u^*}{1+u^*}\right)\left(\frac{\alpha k^2\chi_2 v^*}{\gamma+k^2}-\frac{\delta_2 h_2 v^*}{1+v^*}\right).$$

By the assumption ($\mathcal{E}$), we have

$$\begin{aligned}
\operatorname{tr}(J) &= \left(-d_1k^2-\frac{2\delta_1 u^{*2}}{1+u^*}-d_2k^2-\frac{2\delta_2 v^{*2}}{1+v^*}+\frac{\alpha k^2\chi_1 u^*}{\gamma+k^2}+\frac{\beta k^2\chi_2 v^*}{\gamma+k^2}\right)\\
&= -k^2\left(d_1+d_2-\frac{\alpha\chi_1 u^*}{\gamma+k^2}-\frac{\beta\chi_2 v^*}{\gamma+k^2}\right)-\frac{2\delta_1 u^{*2}}{1+u^*}-\frac{2\delta_2 v^{*2}}{1+v^*}\\
&< -k^2\left(d_1+d_2-\frac{\alpha\chi_1 u^*+\beta\chi_2 v^*}{\gamma}\right)-\frac{2\delta_1 u^{*2}}{1+u^*}-\frac{2\delta_2 v^{*2}}{1+v^*}\\
&< 0.
\end{aligned}$$

By assumptions ($\mathcal{D}$), ($\mathcal{F}$), and ($\mathcal{G}$), we have

$$\begin{aligned}
\det(J) &= \left(-d_1k^2-\frac{2\delta_1 u^{*2}}{1+u^*}+\frac{\alpha k^2\chi_1 u^*}{\gamma+k^2}\right)\left(-d_2k^2-\frac{2\delta_2 v^{*2}}{1+v^*}+\frac{\beta k^2\chi_2 v^*}{\gamma+k^2}\right)\\
&\quad -\left(\frac{\beta k^2\chi_1 u^*}{\gamma+k^2}-\frac{\delta_1 h_1 u^*}{1+u^*}\right)\left(\frac{\alpha k^2\chi_2 v^*}{\gamma+k^2}-\frac{\delta_2 h_2 v^*}{1+v^*}\right)\\
&= k^4\left(d_1d_2-\frac{d_1\beta\chi_2 v^*}{\gamma+k^2}-\frac{d_2\alpha\chi_1 u^*}{\gamma+k^2}\right)+\frac{\delta_1 u^* k^2\chi_2 v^*\left(\alpha h_1-2\beta u^*\right)}{(1+u^*)(\gamma+k^2)}\\
&\quad +\frac{k^2\chi_1 u^*\delta_2 v^*\left(\beta h_2-2\alpha v^*\right)}{(\gamma+k^2)(1+v^*)}+\frac{\delta_1\delta_2 u^* v^*\left(4u^*v^*-h_1h_2\right)}{(1+u^*)(1+v^*)}+\frac{2d_2k^2\delta_1 u^{*2}}{1+u^*}+\frac{2d_1k^2\delta_2 v^{*2}}{1+v^*}\\
&> k^4\left(d_1d_2-\frac{d_1\beta\chi_2 v^*+d_2\alpha\chi_1 u^*}{\gamma}\right)+\frac{\delta_1 u^* k^2\chi_2 v^*\left(\alpha h_1-2\beta u^*\right)}{(1+u^*)(\gamma+k^2)}\\
&\quad +\frac{k^2\chi_1 u^*\delta_2 v^*\left(\beta h_2-2\alpha v^*\right)}{(\gamma+k^2)(1+v^*)}+\frac{\delta_1\delta_2 u^* v^*\left(4u^*v^*-h_1h_2\right)}{(1+u^*)(1+v^*)}+\frac{2d_2k^2\delta_1 u^{*2}}{1+u^*}\\
&> 0.
\end{aligned}$$

Based on the preceding analysis, we get the following theorem.

Theorem 2.1. *Assume that ($\mathcal{A}$)–($\mathcal{G}$) hold, then tr $(J) < 0$ and $\det(J) > 0$ are satisfied. Therefore, E^* is locally asymptotically stable.*

3. The non-existence of weak traveling wave solutions

This section is devoted to establishing the non-existence of weak traveling wave solutions for system (1.3).

Theorem 3.1. *When $c < \max\left\{2\sqrt{d_1\delta_1},\, 2\sqrt{d_2\delta_2}\right\}$, system (1.3) admits no weak traveling wave solution.*

According to Lemmas 4.8 and 4.9 in [15], similarly, we have the following Lemmas 3.1 and 3.2.

Lemma 3.1. *The following is divided into two parts:*

(i) Let $0 \le c < 2\sqrt{d_1\delta_1}$ *be fixed and let* $\sigma_0 > 0$ *satisfy* $c^2 - 4d_1\delta_1 + 4d_1\sigma_0 < 0$. *Let* $\sigma_p(L)$ *denote the principal eigenvalue of*

$$\begin{cases} d_1\phi_{xx} + c\phi_x + \delta_1\phi = \sigma\phi, \quad 0 < x < L, \\ \phi(0) = \phi(L) = 0, \end{cases} \tag{3.1}$$

then there exists $L > 0$ *such that* $\sigma_p(L) = \sigma_0$.

(ii) Keep c, L *as in (i). Let* $\sigma_p(L; b_1, b_2)$ *stand for the principal eigenvalue of*

$$\begin{cases} d_1\phi_{xx} + (c + b_1(x))\phi_x + (\delta_1 + b_2(x))\phi = \sigma\phi, \quad 0 < x < L, \\ \phi(0) = \phi(L) = 0, \end{cases} \tag{3.2}$$

where $b_1(x), b_2(x)$ *are continuous functions. If some* C^2 *function* $\phi(x) > 0$ *for* $0 < x < L$ *obeys*

$$\begin{cases} d_1\phi_{xx} + (c + b_1(x))\phi_x + (\delta_1 + b_2(x))\phi \le 0, \quad 0 < x < L, \\ \phi(0) \ge 0, \quad \phi(L) \ge 0, \end{cases} \tag{3.3}$$

then $\sigma_p(L; b_1, b_2) \le 0$.

Lemma 3.2. *The following is also divided into two parts:*

(i) Assume that $d > 0$. *Fix* $c < 0$ *and take* $\sigma_0 > 0$ *satisfying* $0 < \sigma_0 < \delta_1$. *Let* $\sigma'_p(L)$ *denote the principal eigenvalue of*

$$\begin{cases} d_1\phi_{xx} + c\phi_x + \delta_1\phi = \sigma\phi, \quad 0 < x < L, \\ \phi_x(0) = \phi(L) = 0, \end{cases} \tag{3.4}$$

then there exists $L > 0$ *such that* $\sigma'_p(L) = \sigma_0$.

(ii) Keep c, L *as in (i). Denote by* $\sigma'_p(L; b_1, b_2)$ *the principal eigenvalue of*

$$\begin{cases} d_1\phi_{xx} + (c + b_1(x))\phi_x + (\delta_1 + b_2(x))\phi = \sigma\phi, \quad 0 < x < L, \\ \phi_x(0) = \phi(L) = 0, \end{cases} \tag{3.5}$$

where $b_1(x), b_2(x)$ *are continuous functions. If some* C^2 *function* $\phi(x) > 0$ *for* $0 < x < L$ *obeys*

$$\begin{cases} d_1\phi_{xx} + (c + b_1(x))\phi_x + (\delta_1 + b_2(x))\phi \le 0, \quad 0 < x < L, \\ \phi_x(0) \le 0, \quad \phi(L) \ge 0, \end{cases} \tag{3.6}$$

then $\sigma'_p(L; b_1, b_2) \le 0$.

Lemma 3.3. *A weak traveling wave solution* $(u(x,t), v(x,t), w(x,t)) = (U(x-ct), V(x-ct), W(x-ct))$ *satisfies condition (1.7) with* $c \in \left(-\infty, 2\sqrt{d_1\delta_1}\right)$, *then* $(U(x), V(x), W(x))$ *is a stationary solution of system (1.5) that satisfies*

$$\lim_{x\to\infty} U(x) = \lim_{x\to\infty} V(x) = \lim_{x\to\infty} W(x) = 0.$$

For arbitrary small $\varepsilon > 0$, *there exists* x_ε *such that*

$$\|U\|_\infty \le \varepsilon, \quad \|V\|_\infty \le \varepsilon, \quad \|W\|_\infty \le \varepsilon \quad \text{for } x \in (x_\varepsilon, \infty). \tag{3.7}$$

For every fixed $L > 0$*, on the interval* $(x_\varepsilon, x_\varepsilon + L)$*,* $W(x)$ *solves the elliptic boundary value problem*

$$\begin{cases} 0 = w_{xx} - w + U + V, \quad x_\varepsilon < x < x_\varepsilon + L, \\ w(x_\varepsilon) = W(x_\varepsilon), \quad w(x_\varepsilon + L) = W(x_\varepsilon + L). \end{cases} \tag{3.8}$$

Define the linear interpolation function

$$\widetilde{W}(x) = W(x_\varepsilon) + \frac{x - x_\varepsilon}{L}(W(x_\varepsilon + L) - W(x_\varepsilon)), \quad x_\varepsilon < x < x_\varepsilon + L,$$

then there exists a constant $A^* > 0$ *depending only on* L *such that*

$$\|W_x\|_{L^1(x_\varepsilon, x_\varepsilon + L)} \le A^* \varepsilon. \tag{3.9}$$

Proof. By the L^p estimates for elliptic equations, we have

$$\|W\|_{W^{2,1}(x_\varepsilon, x_\varepsilon + L)} \le A\left(\|U\|_{L^1(x_\varepsilon, x_\varepsilon + L)} + \|V\|_{L^1(x_\varepsilon, x_\varepsilon + L)} + \|\widetilde{W}\|_{W^{2,1}(x_\varepsilon, x_\varepsilon + L)}\right), \tag{3.10}$$

where $A > 0$ is a constant determined solely by the length L.

By the continuous embedding $W^{2,1}(\Omega) \hookrightarrow W^{1,1}(\Omega)$, where $\Omega = (x_\varepsilon, x_\varepsilon + L)$, we have

$$\|W_x\|_{L^1(x_\epsilon, x_\epsilon + L)} \le \|W\|_{W^{2,1}(x_\epsilon, x_\epsilon + L)}.$$

Combining with (3.10), it follows that

$$\|W_x\|_{L^1(x_\epsilon, x_\epsilon + L)} \le A\left(\|U\|_{L^1(x_\varepsilon, x_\varepsilon + L)} + \|V\|_{L^1(x_\varepsilon, x_\varepsilon + L)} + \|\widetilde{W}\|_{W^{2,1}(x_\varepsilon, x_\varepsilon + L)}\right).$$

We estimate each term on the right-hand side separately:

(1) from (3.7), for U, V:

$$\|U\|_{L^1(x_\varepsilon, x_\varepsilon + L)} \le L\|U\|_\infty \le L\varepsilon, \quad \|V\|_{L^1(x_\varepsilon, x_\varepsilon + L)} \le L\|V\|_\infty \le L\varepsilon;$$

(2) for the linear interpolation function $\widetilde{W}$, we have $\widetilde{W}_{xx} \equiv 0$, hence

$$\|\widetilde{W}\|_{W^{2,1}(x_\varepsilon, x_\varepsilon + L)} = \|\widetilde{W}\|_{L^1(x_\varepsilon, x_\varepsilon + L)} + \|\widetilde{W}_x\|_{L^1(x_\varepsilon, x_\varepsilon + L)} \le L\varepsilon + \varepsilon.$$

Substitute all bounds into the inequality:

$$\|W_x\|_{L^1(x_\epsilon, x_\epsilon + L)} \le A(L\varepsilon + L\varepsilon + L\varepsilon + \varepsilon) = A^* \varepsilon,$$

where $A^* = A(3L + 1) > 0$ depends only on the fixed constant L. □

Remark 3.1. *From Lemma 3.3 and the regularity theory of elliptic equations, the traveling wave profiles satisfy* $U(x), V(x), W(x) \in C^2(\mathbb{R})$*. In addition, the definition of weak traveling waves yields* $0 < U(x) \le U_0$ *and* $0 < V(x) \le V_0$*, which implies* $1 + U(x) \ge 1$ *and* $1 + V(x) \ge 1$*. Hence, Holling-II functional responses*

$$\frac{\delta_1 h_1}{1 + U(x)}, \quad \frac{\delta_2 h_2}{1 + V(x)}$$

are infinitely differentiable and thus continuous on $\mathbb{R}$. Owing to the C^2-regularity of the elliptic equation $W'' = \gamma W - \alpha U - \beta V$, both $W(x)$ and its first derivative $W'(x)$ are continuous functions on $\mathbb{R}$.

In the proof of Theorem 3.1, we will construct a perturbed eigenvalue problem with coefficients of the form

$$b_1(x) = -\chi_1 W'(x), \quad b_2(x) = -\left(\frac{\delta_1 h_1}{1+U} - \beta\chi_1\right)V - \gamma\chi_1 W - (\delta_1 - \alpha\chi_1)U.$$

Based on the foregoing regularity arguments, $b_1(x)$ and $b_2(x)$ are continuous on $\mathbb{R}$, which fully satisfy the continuity assumptions imposed on $b_1(x), b_2(x)$ in Lemmas 3.1 and 3.2.

Proof of Theorem 3.1. Assume on the contrary that system (1.3) has a weak traveling wave solution $(u(x,t), v(x,t), w(x,t)) = (U(x-ct), V(x-ct), W(x-ct))$ satisfying condition (1.7) with $c \in \left(-\infty, 2\sqrt{d_1\delta_1}\right)$, then $(U(x), V(x), W(x))$ is a stationary solution of system (1.5), which satisfies Lemma 3.3.

If $c \in \left[0, 2\sqrt{d_1\delta_1}\right)$, we consider the following eigenvalue problem:

$$\begin{cases} d_1\phi_{xx} + c\phi_x + \delta_1\phi = \sigma\phi, & x_\varepsilon < x < x_\varepsilon + L, \\ \phi(x_\varepsilon) = \phi(x_\varepsilon + L) = 0. \end{cases} \tag{3.11}$$

From Lemma 3.1, there is $L > 0$ such that $\sigma_p(L; c, \delta_1) = p_0 > 0$, where $\sigma_p(L; c, \delta_1)$ is the principal eigenvalue of (3.11). $p_0 > 0$ satisfies $c^2 - 4\delta_1 d_1 + 4d_1 p_0 < 0$, and this L is independent of x_ε. The following eigenvalue problem can be regarded as the disturbed problem of (3.11) when $\|U\|_{L^1}$, $\|V\|_{L^1}$, $\|W\|_{L^1}$, and $\|W_x\|_{L^1}$ are sufficiently small:

$$\begin{cases} \sigma\phi = d_1\phi_{xx} + (c - \chi_1 W_x)\phi_x + \phi\Big[\delta_1 - (\frac{\delta_1 h_1}{1+U} - \beta\chi_1)V - \gamma\chi_1 W - (\delta_1 - \alpha\chi_1)U\Big], & x_\varepsilon < x < x_\varepsilon + L, \\ \ 0 = \phi(x_\varepsilon) = \phi(x_\varepsilon + L). \end{cases} \tag{3.12}$$

Taking x_ε sufficiently large, from (3.7) and (3.9) of Lemma 3.3 and the continuity theory for the principal eigenvalue, we have $\sigma_p^\varepsilon(L; c, \delta_1) > 0$, where $\sigma_p^\varepsilon(L; c, \delta_1)$ denotes the principal eigenvalues of problem (3.12). What's more, note that $U(x) > 0$ satisfies

$$\begin{cases} \ 0 = d_1 U_{xx} + (c - \chi_1 W_x)U_x + U\Big[\delta_1 - (\frac{\delta_1 h_1}{1+U} - \beta\chi_1)V - \gamma\chi_1 W - (\delta_1 - \alpha\chi_1)U\Big], & x_\varepsilon < x < x_\varepsilon + L, \\ U(x_\varepsilon) > 0, \quad U(x_\varepsilon + L) > 0. \end{cases}$$

Then by Lemma 3.1, we have $\sigma_p^\varepsilon(L; c, \delta_1) \leq 0$, a contradiction.

If $c < 0$, since $U(\infty) = 0$, then for every large x_ε, there exists a $\widetilde{x}_\varepsilon > x_\varepsilon$ such that $U_x(\widetilde{x}_\varepsilon) < 0$. Consider the following eigenvalue problem:

$$\begin{cases} \sigma\phi = d_1\phi_{xx} + (c - \chi_1 W_x)\phi_x + \phi\Big[\delta_1 - (\frac{\delta_1 h_1}{1+U} - \beta\chi_1)V - \gamma\chi_1 W - (\delta_1 - \alpha\chi_1)U\Big], & \widetilde{x}_\varepsilon < x < \widetilde{x}_\varepsilon + L, \\ \ 0 = \phi(\widetilde{x}_\varepsilon) = \phi(\widetilde{x}_\varepsilon + L). \end{cases} \tag{3.13}$$

From Lemma 3.2 and the above arguments, there exist $L > 0$ and $0 < \epsilon \ll 1$ such that $\widetilde{\sigma}_p^\varepsilon(L; c, \delta_1) > 0$, where $\widetilde{\sigma}_p^\varepsilon(L; c, \delta_1)$ is the principal eigenvalue of problem (3.13). Note that

$$d_1 U_{xx} + (c - \chi_1 W_x)U_x + U\Big[\delta_1 - (\frac{\delta_1 h_1}{1+U} - \beta\chi_1)V - \gamma\chi_1 W - (\delta_1 - \alpha\chi_1)U\Big] = 0, \widetilde{x}_\varepsilon < x < \widetilde{x}_\varepsilon + L,$$

and $U_x(\widetilde{x_\varepsilon}) < 0$, $U(\widetilde{x_\varepsilon} + L) > 0$. By Lemma 3.2, we have $\widetilde{\sigma}_p^\varepsilon(L; c, \delta_1) \leq 0$, a contradiction. Hence, no traveling wave solution exists if $c < 2\sqrt{d_1\delta_1}$. The same proof shows the non-existence of such waves for $c < 2\sqrt{d_2\delta_2}$. □

4. The existence of weak traveling wave solutions

Introduce

$$C^b_{unif}\left(\mathbb{R}^N\right) = \left\{u \in C\left(\mathbb{R}^N\right) \middle| u(x) \text{ is uniformly continuous in } x \in \mathbb{R}^N \text{and} \sup_{x\in\mathbb{R}^N} |u(x)| < \infty\right\},$$

which is a Banach space under the norm $\|u\|_\infty = \sup_{x\in\mathbb{R}^N} |u(x)|$. Furthermore, for parameters $0 < \mu < 1$ and $0 < \theta < 1$, let

$$C^\mu_{unif}\left(\mathbb{R}^N\right) = \left\{u \in C\left(\mathbb{R}^N\right) \middle| \sup_{x,y\in\mathbb{R}^N, x\neq y} \frac{|u(x) - u(y)|}{|x-y|^\mu} < \infty\right\}, \tag{4.1}$$

endowed with the norm $\|u\|_{\infty,\mu} = \sup_{x\in\mathbb{R}^N} |u(x)| + \sup_{x,y\in\mathbb{R}^N, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\mu}$, and

$$C^\theta\Big((t_1,t_2), C^\mu_{unif}\left(\mathbb{R}^N\right)\Big) = \Big\{u(\cdot) \in C\Big((t_1,t_2), C^\mu_{unif}\left(\mathbb{R}^N\right)\Big) \Big| u(t) \text{ is locally Hölder continuous with exponent } \theta\Big\}.$$

Define variables $\kappa_i\ (i = 1, 2)$ such that $0 < \kappa_i < \sqrt{\frac{\delta_i}{d_i}}$ and $d_i\kappa_i^2 - c\kappa_i + \delta_i = 0 (i = 1, 2)$ when $c > c^*$. Note that these definitions of κ_1, κ_2, and c are well posed, since $\kappa_i \to d_i\kappa_i + \frac{\delta_i}{\kappa_i}$ is a strictly decreasing mapping from $\left(0, \sqrt{\frac{\delta_i}{d_i}}\right)$ to $\left(2\sqrt{d_i\delta_i}, \infty\right)$, $i = 1, 2$. Thus, for each $i = 1, 2$, $\kappa_i \to c = d_i\kappa_i + \frac{\delta_i}{\kappa_i}$ are both bijections from $\left(0, \sqrt{\frac{\delta_i}{d_i}}\right)$ to $\left(2\sqrt{d_i\delta_i}, \infty\right)$. In particular, $\kappa_1 \to \kappa_2$ is a bijection from $\left(0, \sqrt{\frac{\delta_1}{d_1}}\right)$ (or a slightly small interval $\left(0, \sqrt{\frac{\delta_1}{d_1}} - \xi\right)$ if $\sqrt{d_1\delta_1} < \sqrt{d_2\delta_2}$) into $\left(0, \sqrt{\frac{\delta_2}{d_2}}\right)$.

Lemma 4.1. *As $\kappa_1 \to 0$, the limit of the ratio $\frac{\kappa_1}{\kappa_2}$ exists. Moreover, there exist positive constants C and D satisfying*

$$0 < C \leq \lim_{\kappa_1\to 0} \frac{\kappa_1}{\kappa_2} \leq D. \tag{4.2}$$

Proof. From the expression of $\kappa_i (i = 1, 2)$, we have

$$\kappa_i = \frac{c - \sqrt{c^2 - 4d_i\delta_i}}{2d_i},$$

so

$$\frac{\kappa_1}{\kappa_2} = \frac{d_2}{d_1} \cdot \frac{c - \sqrt{c^2 - 4d_1\delta_1}}{c - \sqrt{c^2 - 4d_2\delta_2}} := \frac{d_2}{d_1} F(c),$$

where $F(c) = \frac{c - \sqrt{c^2-4d_1\delta_1}}{c - \sqrt{c^2-4d_2\delta_2}} = \frac{d_1\delta_1}{d_2\delta_2} \cdot \frac{c + \sqrt{c^2-4d_2\delta_2}}{c + \sqrt{c^2-4d_1\delta_1}}$.

We analyze the monotonicity and boundedness of $F(c)$ in three cases:

(i) For $d_1\delta_1 > d_2\delta_2$, we have $F(c) > 1$ and $F'(c) < 0$. So, $F(c)$ is strictly decreasing and bounded below by 1. By the monotone bounded convergence theorem, $\lim_{c\to+\infty} F(c)$ exists.

(ii) For $d_1\delta_1 < d_2\delta_2$, we have $F(c) < 1$ and $F'(c) > 0$. So, $F(c)$ is strictly increasing and bounded below by 1. By the monotone bounded convergence theorem, $\lim_{c\to+\infty} F(c)$ exists.

(iii) For $d_1\delta_1 = d_2\delta_2$, we have $F(c) \equiv 1$. Thus, the limit is clearly 1.

Therefore,

$$\lim_{c\to+\infty} F(c) = \frac{d_1\delta_1}{d_2\delta_2} \cdot \lim_{c\to+\infty} \frac{c + \sqrt{c^2 - 4d_2\delta_2}}{c + \sqrt{c^2 - 4d_1\delta_1}} = \frac{d_1\delta_1}{d_2\delta_2} \cdot 1 = \frac{d_1\delta_1}{d_2\delta_2},$$

and

$$\lim_{\kappa_1\to 0} \frac{\kappa_1}{\kappa_2} = \frac{d_2}{d_1} \cdot \frac{d_1\delta_1}{d_2\delta_2} = \frac{\delta_1}{\delta_2}.$$

Assume $C = \min\left\{\frac{\delta_1}{\delta_2}, \frac{\delta_2}{\delta_1}\right\}, D = \max\left\{\frac{\delta_1}{\delta_2}, \frac{\delta_2}{\delta_1}\right\}$, so we have

$$0 < C \le \lim_{\kappa_1\to 0} \frac{\kappa_1}{\kappa_2} \le D.$$

□

Let $q_1 > 1, q_2 > 1, \eta_1 > 1$, and $\eta_2 > 1$ be the subsequent constants to be determined. The continuous functions $\bar{u}(x), \underline{u}(x), \bar{v}(x)$, and $\underline{v}(x)$ are defined as follows:

$$\bar{u}(x) = \begin{cases} \frac{\delta_1}{\delta_1-\alpha\chi_1}, & x \le b_{\kappa_1}, \\ e^{-\kappa_1 x}, & x > b_{\kappa_1}, \end{cases} \qquad \bar{v}(x) = \begin{cases} \frac{\delta_2}{\delta_2-\beta\chi_2}, & x \le b_{\kappa_2}, \\ e^{-\kappa_2 x}, & x > b_{\kappa_2}, \end{cases}$$
$$\underline{u}(x) = \begin{cases} 0, & x \le a_{\kappa_1}, \\ e^{-\kappa_1 x} - q_1 e^{-\eta_1\kappa_1 x}, & x > a_{\kappa_1}, \end{cases} \qquad \underline{v}(x) = \begin{cases} 0, & x \le a_{\kappa_2}, \\ e^{-\kappa_2 x} - q_2 e^{-\eta_2\kappa_2 x}, & x > a_{\kappa_2}. \end{cases} \tag{4.3}$$

Here,

$$a_{\kappa_1} = \frac{\ln q_1}{(\eta_1 - 1)\kappa_1} > 0, \quad a_{\kappa_2} = \frac{\ln q_2}{(\eta_2 - 1)\kappa_2} > 0,$$

and

$$b_{\kappa_1} = -\frac{\ln\delta_1 - \ln(\delta_1 - \alpha\chi_1)}{\kappa_1} < 0, \quad b_{\kappa_2} = -\frac{\ln\delta_2 - \ln(\delta_2 - \beta\chi_2)}{\kappa_2} < 0.$$

We proceed to introduce a closed subset of $Y^* = \left(C^b_{unif}(\mathbb{R})\right)^2$, which is constructed using the bounds $(\bar{u}, \bar{v})$ and $\left(\underline{u}, \underline{v}\right)$.

Define

$$\Sigma_{k_1,k_2} = \left\{(\phi, \psi) \in \left(C^b_{unif}(\mathbb{R})\right)^2 \middle| \left(\underline{u}, \underline{v}\right) \le (\phi, \psi) \le (\bar{u}, \bar{v})\right\}.$$

Then, for each fixed $(\phi, \psi) \in \Sigma_{k_1,k_2}$, we have

$$0 = w_{xx} - \gamma w + \alpha\phi + \beta\psi. \tag{4.4}$$

So, we consider the following parabolic system:

$$\begin{cases} U_t = d_1 U_{xx} + (c - \chi_1 w_x(x;\phi,\psi))U_x + U\left[\delta_1 - \left(\frac{\delta_1 h_1}{1+U} - \beta\chi_1\right)V - \gamma\chi_1 w(x;\phi,\psi) - (\delta_1 - \alpha\chi_1)U\right], \\ V_t = d_2 V_{xx} + (c - \chi_2 w_x(x;\phi,\psi))V_x + V\left[\delta_2 - \left(\frac{\delta_2 h_2}{1+V} - \alpha\chi_2\right)U - \gamma\chi_2 w(x;\phi,\psi) - (\delta_2 - \beta\chi_2)V\right]. \end{cases} \tag{4.5}$$

Definition 4.1. *A pair of continuous functions $(\bar{u}, \bar{v})$ and $\left(\underline{u}, \underline{v}\right)$ on $\mathbb{R}$ are called upper and lower solutions of system (1.5) if they satisfy the following conditions:*

(i) For for all $x \in \mathbb{R}$,

$$0 \leq \underline{u}(x) \leq \bar{u}(x) \leq U_0, 0 \leq \underline{v}(x) \leq \bar{v}(x) \leq V_0,$$

and U_0 and V_0 are constants, which are defined in (1.6).

(ii) There exists a finite set $\mathbb{D} \subset \mathbb{R}$ satisfying

(a) space $\bar{u}, \underline{u}, \bar{v}, \underline{v} \in C^2(\mathbb{R} \backslash \mathbb{D})$;

(b)regarding $x \in \mathbb{D}$, the left and right limits of $\bar{u}', \underline{u}', \bar{v}'$, and $\underline{v}'$ exist and satisfy

$$\underline{u}'(x^-) \leq \underline{u}'(x^+),\ \bar{u}'(x^-) \geq \bar{u}'(x^+),\ \underline{v}'(x^-) \leq \underline{v}'(x^+),\ \bar{v}'(x^-) \geq \bar{v}'(x^+).$$

(iii) In $\pm\infty$, the first and second derivatives of $\bar{u}, \underline{u}, \bar{v}$, and $\underline{v}$ grow at most exponentially.

(iv) For any continuous functions pair (u, v) with $\underline{u} \leq u \leq \bar{u}$ and $\underline{v} \leq v \leq \bar{v}$, the following inequalities hold:

$$\begin{cases} d_1\bar{u}'' + (c - \chi_1 w'(x;\phi,\psi))\,\bar{u}' + \bar{u}\left[\delta_1 - \left(\frac{\delta_1 h_1}{1+\bar{u}} - \beta\chi_1\right)v - \gamma\chi_1 w(x;\phi,\psi) - (\delta_1 - \alpha\chi_1)\,\bar{u}\right] \leq 0, \\ d_2\bar{v}'' + (c - \chi_2 w'(x;\phi,\psi))\,\bar{v}' + \bar{v}\left[\delta_2 - \left(\frac{\delta_2 h_2}{1+\bar{v}} - \alpha\chi_2\right)u - \gamma\chi_2 w(x;\phi,\psi) - (\delta_2 - \beta\chi_2)\,\bar{v}\right] \leq 0, \\ d_1\underline{u}'' + (c - \chi_1 w'(x;\phi,\psi))\,\underline{u}' + \underline{u}\left[\delta_1 - \left(\frac{\delta_1 h_1}{1+\underline{u}} - \beta\chi_1\right)v - \gamma\chi_1 w(x;\phi,\psi) - (\delta_1 - \alpha\chi_1)\,\underline{u}\right] \geq 0, \\ d_2\underline{v}'' + (c - \chi_2 w'(x;\phi,\psi))\,\underline{v}' + \underline{v}\left[\delta_2 - \left(\frac{\delta_2 h_2}{1+\underline{v}} - \alpha\chi_2\right)u - \gamma\chi_2 w(x;\phi,\psi) - (\delta_2 - \beta\chi_2)\,\underline{v}\right] \geq 0, \end{cases}$$

where $w(x;\phi,\psi)$ is determined by $(\phi,\psi) \in \Sigma_{\kappa_1,\kappa_2}$.

Lemma 4.2. *[1] For each $(\phi,\psi) \in \Sigma_{\kappa_1,\kappa_2}$, $\kappa_1, \kappa_2 \in (0, \sqrt{\gamma})$, and $x \in \mathbb{R}$, we have*

$$0 \leq w(x;\phi,\psi) \leq \frac{\alpha}{\gamma - \kappa_1^2}e^{-\kappa_1 x} + \frac{\beta}{\gamma - \kappa_2^2}e^{-\kappa_2 x}, \tag{4.6}$$

and

$$|w_x(x;\phi,\psi)| \leq \min\left\{\alpha\frac{\kappa_1 + \sqrt{\gamma - \kappa_1^2}}{\gamma - \kappa_1^2}e^{-\kappa_1 x}, \frac{\alpha\delta_1}{\sqrt{\gamma}(\delta_1 - \alpha\chi_1)}\right\} + \min\left\{\beta\frac{\kappa_2 + \sqrt{\gamma - \kappa_2^2}}{\gamma - \kappa_2^2}e^{-\kappa_2 x}, \frac{\beta\delta_2}{\sqrt{\gamma}(\delta_2 - \beta\chi_2)}\right\}. \tag{4.7}$$

Lemma 4.3. *Assume that $(\mathcal{B})$ holds, then there exists a sufficiently small $\epsilon_0 > 0$ such that for each $\kappa_1, \kappa_2 \in (0, \epsilon_0)$, there exists a constant $x_0 > a_{\kappa_2}$ satisfying*

$$\kappa_1\chi_1\left(\alpha\frac{\kappa_1 + \sqrt{\gamma - \kappa_1^2}}{\gamma - \kappa_1^2} + \frac{\beta\delta_2}{\sqrt{\gamma}(\delta_2 - \beta\chi_2)}e^{\kappa_1 x_0}\right) - (\delta_1 - \alpha\chi_1) < 0,$$

and

$$\frac{\beta\chi_1\kappa_1\left(\kappa_2 + \sqrt{\gamma - \kappa_2^2}\right)}{\gamma - \kappa_2^2} - \left(\frac{\delta_1 h_1}{1 + e^{-\kappa_2 x_0}} - \beta\chi_1\right)\left(1 - q_2 e^{(1-\eta_2)\kappa_2 x_0}\right) < 0.$$

Proof. Let $x_0 > 0$ satisfy $q_2 e^{(1-\eta_2)\kappa_2 x_0} = \frac{1}{2}$, namely, $x_0 = \frac{\ln(2q_2)}{\kappa_2(\eta_2-1)} > a_{\kappa_2}$. Since $\lim_{\kappa_2\to 0} \frac{\kappa_2+\sqrt{\gamma-\kappa_2^2}}{\gamma-\kappa_2^2} = \gamma^{-1/2}$, by Lemma 4.1, we have

$$\lim_{\kappa_1\to 0} \frac{\beta\chi_1\kappa_1\left(\kappa_2+\sqrt{\gamma-\kappa_2^2}\right)}{\gamma-\kappa_2^2} = \lim_{\kappa_1,\kappa_2\to 0} \frac{\beta\chi_1\kappa_1\left(\kappa_2+\sqrt{\gamma-\kappa_2^2}\right)}{\gamma-\kappa_2^2} = 0.$$

Through $(\mathscr{B})$, we obtain $\chi_1\beta < \frac{h_1\delta_1}{(1+U_0)}$. Since U_0 is an appropriate positive constant, $U_0 > e^{-\kappa_2 x_0}$. It follows that there exists $\epsilon_1 > 0$ and for any $\kappa_1 \in (0, \epsilon_1)$,

$$\frac{\beta\chi_1\kappa_1\left(\kappa_2+\sqrt{\gamma-\kappa_2^2}\right)}{\gamma-\kappa_2^2} < \left(\frac{\delta_1 h_1}{1+e^{-\kappa_2 x_0}} - \beta\chi_1\right)\left(1-\frac{1}{2}\right) = \left(\frac{\delta_1 h_1}{1+e^{-\kappa_2 x_0}} - \beta\chi_1\right)\left(1 - q_2 e^{(1-\eta_2)\kappa_2 x_0}\right).$$

Let $D > 0$ be the constant introduced in Lemma 4.1. Then, there exists $0 < \epsilon_2 \le \epsilon_1$ such that for any $\kappa_1 \in (0, \epsilon_2)$, $\frac{\kappa_1}{\kappa_2} < D + 1$. Hence, for $\kappa_1 \in (0, \epsilon_2)$,

$$\frac{\beta\delta_2}{\sqrt{\gamma}\,(\delta_2-\beta\chi_2)} e^{k_1 x_0} = \frac{\beta\delta_2}{\sqrt{\gamma}\,(\delta_2-\beta\chi_2)} e^{\frac{1}{\eta_2-1}\ln(2q_2)\frac{k_1}{k_2}} < \frac{\beta\delta_2}{\sqrt{\gamma}\,(\delta_2-\beta\chi_2)} e^{\frac{1}{\eta_2-1}\ln(2q_2)(D+1)} \triangleq S.$$

Hence, there exists $0 < \epsilon_0 \le \epsilon_2$ such that for $\kappa_1 \in (0, \epsilon_0)$,

$$\kappa_1\chi_1\left(\alpha\frac{\kappa_1+\sqrt{\gamma-\kappa_1^2}}{\gamma-\kappa_1^2} + \frac{\beta\delta_2}{\sqrt{\gamma}(\delta_2-\beta\chi_2)} e^{\kappa_1 x_0}\right) < \kappa_1\chi_1\left(\alpha\frac{\kappa_1+\sqrt{\gamma-\kappa_1^2}}{\gamma-\kappa_1^2} + S\right) < (\delta_1 - \alpha\chi_1).$$

□

Lemma 4.4. *There exists a sufficiently small positive constant $\epsilon_0(q_1, q_2, \eta_1, \eta_2)$ such that constants $q_1 > 1$, $q_2 > 1$,*

$$\eta_1 \in \left(1, \min\left\{\frac{\delta_1}{d_1\kappa_1^2}, \frac{\kappa_1+\kappa_2}{\kappa_1}, 1 + \frac{\gamma}{\kappa_1\left(\kappa_1+\sqrt{\gamma-\kappa_1^2}\right)}, 2\right\}\right), \tag{4.8}$$

and

$$\eta_2 \in \left(1, \min\left\{\frac{\delta_2}{d_2\kappa_2^2}, \frac{\kappa_1+\kappa_2}{\kappa_2}, 1 + \frac{\gamma}{\kappa_2\left(\kappa_2+\sqrt{\gamma-\kappa_2^2}\right)}, 2\right\}\right) \tag{4.9}$$

can be chosen to satisfy

$$q_1 I_1 + I_2 + I_3 + q_1 I_4 \le 0, \quad q_2 I_5 + I_6 + I_7 + q_2 I_8 \le 0 \tag{4.10}$$

for all $\kappa_1, \kappa_2 \in (0, \epsilon_0(q_1, q_2, \eta_1, \eta_2))$, where

$$I_1 = d_1(\eta_1\kappa_1)^2 - c\eta_1\kappa_1 + \delta_1,$$

$$I_2 = \delta_1 h_1 - \beta\chi_1 + \frac{\gamma\chi_1\beta}{\gamma - \kappa_2^2} + \frac{\beta\chi_1\kappa_1\left(\kappa_2 + \sqrt{\gamma - \kappa_2^2}\right)}{\gamma - \kappa_2^2},$$

$$I_3 = \frac{\alpha\chi_1\kappa_2\left(\kappa_2 + \sqrt{\gamma - \kappa_2^2}\right)}{\gamma - \kappa_2^2} + \frac{\gamma\chi_1\alpha}{\gamma - \kappa_2^2} + \delta_1 - \alpha\chi_1,$$

$$I_4 = \frac{\beta\chi_1\eta_1\kappa_1\left(\kappa_2 + \sqrt{\gamma - \kappa_2^2}\right)}{\gamma - \kappa_2^2},$$

$$I_5 = d_2(\eta_2\kappa_2)^2 - c\eta_2\kappa_2 + \delta_2,$$

$$I_6 = \delta_2 h_2 - \alpha\chi_2 + \frac{\gamma\chi_2\alpha}{\gamma - \kappa_1^2} + \frac{\alpha\chi_2\kappa_2\left(\kappa_1 + \sqrt{\gamma - \kappa_1^2}\right)}{\gamma - \kappa_1^2},$$

$$I_7 = \frac{\beta\chi_2\kappa_2\left(\kappa_2 + \sqrt{\gamma - \kappa_2^2}\right)}{\gamma - \kappa_2^2} + \frac{\gamma\chi_2\beta}{\gamma - \kappa_2^2} + \delta_2 - \beta\chi_2,$$

and

$$I_8 = \frac{\alpha\chi_2\eta_2\kappa_2\left(\kappa_1 + \sqrt{\gamma - \kappa_1^2}\right)}{\gamma - \kappa_1^2}.$$

Proof. For all sufficiently small κ_1, we have

$$\frac{\delta_1}{d_1\kappa_1^2} > 1, \quad \frac{\kappa_2}{\kappa_1} > 0, \quad \frac{\gamma}{\kappa_1\left(\kappa_1 + \sqrt{\gamma - \kappa_1^2}\right)} > 0,$$

and hence η_1 can be chosen to satisfy (4.8). Similarly, we obtain that η_2 can be chosen to satisfy (4.9).

Since $d_1\kappa_1^2 - c\kappa_1 + \delta_1 = 0$ and $\eta_1 \leq \frac{\delta_1}{d_1\kappa_1^2}$, then $I_1 = d_1\eta_1^2\kappa_1^2 - \eta_1(d_1\kappa_1^2 + \delta_1) + \delta_1 = (\eta_1 - 1)(d_1\eta_1\kappa_1^2 - \delta_1) < 0$. Obviously, I_1 is increasing in k_2 for sufficiently small k_1, so if $(\kappa_1, \kappa_2) \to (0, 0)$, we have $I_1 \to (1 - \eta_1)\delta_1$, $I_2 \to \delta_1 h_1$, $I_3 \to \delta_1$, and $I_4 \to 0$. Then, one can find a positive constant q_1 satisfying $q_1(\eta_1 - 1) > 1 + h_1$, as well as a positive constant ϵ_0 satisfying $-q_1 I_1 - I_2 - I_3 - q_1 I_4 \geq 0$ for all $\kappa_1, \kappa_2 \in (0, \epsilon_0)$. Similarly, a proof can be applied to q_2. □

Theorem 4.1. *Assume that condition ($\mathcal{B}$) holds. If there exists a sufficiently small $\epsilon > 0$ such that for each $\kappa_1, \kappa_2 \in (0, \epsilon)$, $q_i > 1$, and $\eta_i > 1$ $(i = 1, 2)$ satisfying (4.8)–(4.10), then $(\overline{u}, \overline{v})$ and $(\underline{u}, \underline{v})$ are a pair of upper and lower solutions of (4.5), where $w(x; \phi, \psi)$ is obtained by (4.4) with the fixed $(\phi, \psi) \in \Sigma_{\kappa_1,\kappa_2}$.*

Proof. We show

$$\bar{u}'\left(b_{\kappa_1}^-\right) = 0 \geq -\frac{\kappa_1\delta_1}{\delta_1 - \alpha\chi_1} = \bar{u}'\left(b_{\kappa_1}^+\right),$$

$$\bar{v}'\left(b_{k_2}^-\right) = 0 \geq -\frac{k_2\delta_2}{\delta_2 - \beta\chi_2} = \bar{v}'\left(b_{k_2}^+\right),$$

$$\underline{u}'\left(a_{\kappa_1}^-\right) = 0 \leq \kappa_1(\eta_1 - 1)e^{-\frac{\ln q_1}{(\eta_1 - 1)}} = \underline{u}'\left(a_{\kappa_1}^+\right),$$

$$\underline{v}'\left(a_{\kappa_2}^-\right) = 0 \leq \kappa_2(\eta_2 - 1)e^{-\frac{\ln q_2}{(\eta_2-1)}} = \underline{v}'\left(a_{\kappa_2}^+\right).$$

According to (4.7) in Lemma 4.2, when $x \in (b_{k_2}, \infty)$, we have

$$\begin{aligned}
& d_2\bar{v}'' + (c - \chi_2 w')\,\bar{v}' + \bar{v}\left[\delta_2 - \left(\frac{\delta_2 h_2}{1+\bar{v}} - \alpha\chi_2\right)u - \gamma\chi_2 w\,(\cdot;\phi,\psi) - (\delta_2 - \beta\chi_2)\,\bar{v}\right] \\
=& d_2\kappa_2^2 e^{-\kappa_2 x} - \kappa_2\,(c - \chi_2 w_x)\,e^{-\kappa_2 x} + e^{-\kappa_2 x}\left[\delta_2 - \left(\frac{\delta_2 h_2}{1+e^{-\kappa_2 x}} - \alpha\chi_2\right)u - \gamma\chi_2 w\,(\cdot;\phi,\psi) - (\delta_2 - \beta\chi_2)\,e^{-\kappa_2 x}\right] \\
\leq& d_2\kappa_2^2 e^{-\kappa_2 x} - \kappa_2\,(c - \chi_2 w_x)\,e^{-\kappa_2 x} + e^{-\kappa_2 x}\left[\delta_2 - \left(\frac{\delta_2 h_2}{1+e^{-\kappa_2 x}} - \alpha\chi_2\right)\underline{u} - \gamma\chi_2 w\,(\cdot;\phi,\psi) - (\delta_2 - \beta\chi_2)\,e^{-\kappa_2 x}\right] \\
=& \left(d_2\kappa_2^2 - c\kappa_2 + \mu_2\right)e^{-\kappa_2 x} + k_2\chi_2 w_x e^{-\kappa_2 x} - \left(\frac{\delta_2 h_2}{1+e^{-\kappa_2 x}} - \alpha\chi_2\right)e^{-\kappa_2 x}\underline{u} - \gamma\chi_2 w\,(\cdot;\phi,\psi)\,e^{-k_2 x} - (\delta_2 - \beta\chi_2)\,e^{-2\kappa_2 x} \\
\leq& \kappa_2\chi_2\,|w_x|\,e^{-\kappa_2 x} - (\delta_2 - \beta\chi_2)\,e^{-2\kappa_2 x} \\
\leq& \kappa_2\chi_2\left(\alpha\frac{\kappa_1 + \sqrt{\gamma - \kappa_1^2}}{\gamma - \kappa_1^2}e^{-\kappa_1 x} + \beta\frac{\kappa_2 + \sqrt{\gamma - \kappa_2^2}}{\gamma - \kappa_2^2}e^{-\kappa_2 x}\right)e^{-\kappa_2 x} - (\delta_2 - \beta\chi_2)\,e^{-2\kappa_2 x} \\
=& \left[\frac{\kappa_2\chi_2\alpha\left(\kappa_1 + \sqrt{\gamma - \kappa_1^2}\right)}{\gamma - \kappa_1^2}e^{(\kappa_2 - \kappa_1)x} + \frac{\kappa_2\chi_2\beta\left(\kappa_2 + \sqrt{\gamma - \kappa_2^2}\right)}{\gamma - \kappa_2^2} - (\delta_2 - \beta\chi_2)\right]e^{-2\kappa_2 x} \\
\leq& \left[\frac{\kappa_2\chi_2\alpha\left(\kappa_1 + \sqrt{\gamma - \kappa_1^2}\right)}{\gamma - \kappa_1^2}e^{(\kappa_2 - \kappa_1)b_{\kappa_2}} + \frac{\kappa_2\chi_2\beta\left(\kappa_2 + \sqrt{\gamma - \kappa_2^2}\right)}{\gamma - \kappa_2^2} - (\delta_2 - \beta\chi_2)\right]e^{-2\kappa_2 x} \\
\leq& 0.
\end{aligned}$$

Because $\kappa_2 \to 0$, we have $\kappa_1 \to 0$ and

$$(\kappa_2 - \kappa_1)b_{\kappa_2} = -\frac{\ln\delta_2 - \ln(\delta_2 - \beta\chi_2)}{\kappa_2}(\kappa_2 - \kappa_1) \to H,$$

and when $\kappa_1 \to 0$, we obtain

$$\frac{\alpha\chi_2\kappa_2\left(\kappa_1 + \sqrt{\gamma - \kappa_1^2}\right)}{\gamma - \kappa_1^2}e^{(\kappa_2 - \kappa_1)b_{\kappa_2}} + \frac{\beta\chi_2\kappa_2\left(\kappa_2 + \sqrt{\gamma - \kappa_2^2}\right)}{\gamma - \kappa_2^2} \to 0.$$

According to Lemma 4.1, H is a fixed constant independent of κ_1 and κ_2. Hence, there exists $0 < \epsilon_2 < \epsilon_1$ such that for all $\kappa_1, \kappa_2 \in (0, \epsilon_2)$, we have

$$\frac{\alpha\chi_2\kappa_2\left(\kappa_1 + \sqrt{\gamma - \kappa_1^2}\right)}{\gamma - \kappa_1^2}e^{(\kappa_2 - \kappa_1)b_{\kappa_2}} + \frac{\beta\chi_2\kappa_2\left(\kappa_2 + \sqrt{\gamma - \kappa_2^2}\right)}{\gamma - \kappa_2^2} < \delta_2 - \beta\chi_2.$$

Next, for each $x \in (-\infty, b_{k_2})$, we have

$$d_2\bar{v}'' + (c - \chi_2 w')\,\bar{v}' + \bar{v}\left[\delta_2 - \left(\frac{\delta_2 h_2}{1+\bar{v}} - \alpha\chi_2\right)u - \gamma\chi_2 w\,(\cdot;\phi,\psi) - (\delta_2 - \beta\chi_2)\,\bar{v}\right]$$

$$
\begin{aligned}
&=\frac{\delta_2}{\delta_2-\beta\chi_2}\left[\delta_2-\left(\frac{\delta_2 h_2}{1+\frac{\delta_2}{\delta_2-\beta\chi_2}}-\alpha\chi_2\right)u-\gamma\chi_2 w(\cdot;\phi,\psi)-(\delta_2-\beta\chi_2)\frac{\delta_2}{\delta_2-\beta\chi_2}\right]\\
&\le\frac{\delta_2}{\delta_2-\beta\chi_2}\left[\delta_2-\left(\frac{\delta_2 h_2}{1+\frac{\delta_2}{\delta_2-\beta\chi_2}}-\alpha\chi_2\right)\underline{u}-\gamma\chi_2 w(\cdot;\phi,\psi)-(\delta_2-\beta\chi_2)\frac{\delta_2}{\delta_2-\beta\chi_2}\right]\\
&\le-\frac{\delta_2\gamma\chi_2 w}{\delta_2-\beta\chi_2}\le 0.
\end{aligned}
$$

For each $\kappa_1, \kappa_2 \in (0, \epsilon_2)$, one can find a corresponding positive constant x_0 such that the inequalities in Lemma 4.3 hold. Meanwhile, we assume ϵ_2 is small enough to ensure that for $\kappa_1, \kappa_2 \in (0, \epsilon_2)$, the inequality

$$
\frac{\alpha\chi_1\kappa_1\left(\kappa_1+\sqrt{\gamma-\kappa_1^2}\right)}{\gamma-\kappa_1^2}\le\delta_1-\alpha\chi_1
$$

holds.

By Lemma 4.3 and (4.7) in Lemma 4.2, when $x \in (x_0, \infty)$, we have

$$
\begin{aligned}
&d_1\bar{u}''+(c-\chi_1 w')\bar{u}'+\bar{u}\left[\delta_1-\left(\frac{\delta_1 h_1}{1+\bar{u}}-\beta\chi_1\right)v-\gamma\chi_1 w(\cdot;\phi,\psi)-(\delta_1-\alpha\chi_1)\bar{u}\right]\\
=&d_1k_1^2e^{-k_1x}-k_1(c-\chi_1 w_x)e^{-k_1x}+e^{-k_1x}\left[\delta_1-\left(\frac{\delta_1 h_1}{1+e^{-k_1x}}-\beta\chi_1\right)v-\gamma\chi_1 w(\cdot;\phi,\psi)-(\delta_1-\alpha\chi_1)e^{-k_1x}\right]\\
\le&d_1k_1^2e^{-k_1x}-k_1(c-\chi_1 w_x)e^{-k_1x}+e^{-k_1x}\left[\delta_1-\left(\frac{\delta_1 h_1}{1+e^{-k_1x}}-\beta\chi_1\right)\underline{v}-\gamma\chi_1 w(\cdot;\phi,\psi)-(\delta_1-\alpha\chi_1)e^{-k_1x}\right]\\
=&\left(d_1k_1^2-ck_1+\delta_1\right)e^{-k_1x}+k_1\chi_1 w_xe^{-k_1x}-\left(\frac{\delta_1 h_1}{1+e^{-k_1x}}-\beta\chi_1\right)e^{-k_1x}\underline{v}-\gamma\chi_1 w(\cdot;\phi,\psi)e^{-k_1x}-(\delta_1-\alpha\chi_1)e^{-2k_1x}\\
\le&k_1\chi_1|w_x|e^{-k_1x}-\left(\frac{\delta_1 h_1}{1+e^{-k_1x}}-\beta\chi_1\right)e^{-k_1x}\left(e^{-k_2x}-q_2e^{-\eta_2k_2x}\right)-(\delta_1-\alpha\chi_1)e^{-2k_1x}\\
\le&k_1\chi_1\left(\alpha\frac{k_1+\sqrt{\gamma-k_1^2}}{\gamma-k_1^2}e^{-k_1x}+\beta\frac{k_2+\sqrt{\gamma-k_2^2}}{\gamma-k_2^2}e^{-k_2x}\right)e^{-k_1x}-\left(\frac{\delta_1 h_1}{1+e^{-k_1x}}-\beta\chi_1\right)e^{-k_1x}\left(e^{-k_2x}-q_2e^{-\eta_2k_2x}\right)-(\delta_1-\alpha\chi_1)e^{-2k_1x}\\
\le&\left[k_1\chi_1\beta\frac{k_2+\sqrt{\gamma-k_2^2}}{\gamma-k_2^2}-\left(\frac{\delta_1 h_1}{1+e^{-k_1x_0}}-\beta\chi_1\right)\left(1-q_2e^{(1-\eta_2)k_2x_0}\right)\right]e^{-(k_1+k_2)x}+\left[k_1\chi_1\alpha\frac{k_1+\sqrt{\gamma-k_1^2}}{\gamma-k_1^2}-(\delta_1-\alpha\chi_1)\right]e^{-2k_1x}\\
\le&0.
\end{aligned}
$$

By Lemma 4.3 and (4.7) in Lemma 4.2, when $x \in (b_{k_1}, x_0)$, we have

$$
\begin{aligned}
&d_1\bar{u}''+(c-\chi_1 w')\bar{u}'+\bar{u}\left[\delta_1-\left(\frac{\delta_1 h_1}{1+\bar{u}}-\beta\chi_1\right)v-\gamma\chi_1 w(\cdot;\phi,\psi)-(\delta_1-\alpha\chi_1)\bar{u}\right]\\
=&d_1k_1^2e^{-k_1x}-k_1(c-\chi_1 w_x)e^{-k_1x}+e^{-k_1x}\left[\delta_1-\left(\frac{\delta_1 h_1}{1+e^{-k_1x}}-\beta\chi_1\right)v-\gamma\chi_1 w(\cdot;\phi,\psi)-(\delta_1-\alpha\chi_1)e^{-k_1x}\right]\\
\le&d_1k_1^2e^{-k_1x}-k_1(c-\chi_1 w_x)e^{-k_1x}+e^{-k_1x}\left[\delta_1-\left(\frac{\delta_1 h_1}{1+e^{-k_1x}}-\beta\chi_1\right)\underline{v}-\gamma\chi_1 w(\cdot;\phi,\psi)-(\delta_1-\alpha\chi_1)e^{-k_1x}\right]
\end{aligned}
$$

$$
\begin{aligned}
&=\left(d_1k_1^2-ck_1+\delta_1\right)e^{-k_1x}+k_1\chi_1w_xe^{-k_1x}-\left(\frac{\delta_1h_1}{1+e^{-k_1x}}-\beta\chi_1\right)e^{-k_1x}\underline{v}-\gamma\chi_1w\left(\cdot;\phi,\psi\right)e^{-k_1x}-\left(\delta_1-\alpha\chi_1\right)e^{-2k_1x}\\
&\le k_1\chi_1\left|w_x\right|e^{-k_1x}-\left(\delta_1-\alpha\chi_1\right)e^{-2k_1x}\\
&\le k_1\chi_1\left(\alpha\frac{k_1+\sqrt{\gamma-k_1^2}}{\gamma-k_1^2}e^{-k_1x}+\frac{\beta\delta_2}{\sqrt{\gamma}\left(\delta_2-\beta\chi_2\right)}\right)e^{-k_1x}-\left(\delta_1-\alpha\chi_1\right)e^{-2k_1x}\\
&\le\left[k_1\chi_1\left(\alpha\frac{k_1+\sqrt{\gamma-k_1^2}}{\gamma-k_1^2}+\frac{\beta\delta_2}{\sqrt{\gamma}\left(\delta_2-\beta\chi_2\right)}e^{-k_1x_0}\right)-\left(\delta_1-\alpha\chi_1\right)\right]e^{-2k_1x}\\
&\le 0.
\end{aligned}
$$

At last, for each $x\in(-\infty,b_{k_1})$, we have

$$
\begin{aligned}
&d_1\bar{u}''+\left(c-\chi_1w'\right)\bar{u}'+\bar{u}\left[\delta_1-\left(\frac{\delta_1h_1}{1+\bar{u}}-\beta\chi_1\right)v-\gamma\chi_1w\left(\cdot;\phi,\psi\right)-\left(\delta_1-\alpha\chi_1\right)\bar{u}\right]\\
=&\frac{\delta_1}{\delta_1-\alpha\chi_1}\left[\delta_1-\left(\frac{\delta_1h_1}{1+\frac{\delta_1}{\delta_1-\alpha\chi_1}}-\beta\chi_1\right)v-\gamma\chi_1w\left(\cdot;\phi,\psi\right)-\left(\delta_1-\alpha\chi_1\right)\frac{\delta_1}{\delta_1-\alpha\chi_1}\right]\\
\le&\frac{\delta_1}{\delta_1-\alpha\chi_1}\left[\delta_1-\left(\frac{\delta_1h_1}{1+\frac{\delta_1}{\delta_1-\alpha\chi_1}}-\beta\chi_1\right)\underline{v}-\gamma\chi_1w\left(\cdot;\phi,\psi\right)-\left(\delta_1-\alpha\chi_1\right)\frac{\delta_1}{\delta_1-\alpha\chi_1}\right]\\
=&-\frac{\delta_1\gamma\chi_1w}{\delta_1-\alpha\chi_1}\le 0.
\end{aligned}
$$

By Lemma 4.2, when $x\in(a_{k_1},\infty)$, we have

$$
\begin{aligned}
&d_1\underline{u}''+\left(c-\chi_1w'\right)\underline{u}'+\underline{u}\left[\delta_1-\left(\frac{\delta_1h_1}{1+\underline{u}}-\beta\chi_1\right)v-\gamma\chi_1w\left(\cdot;\phi,\psi\right)-\left(\delta_1-\alpha\chi_1\right)\underline{u}\right]\\
\ge& d_1k_1^2e^{-k_1x}-d_1q_1(\eta_1k_1)^2e^{-\eta_1k_1x}+\left(c-\chi_1w_x\right)\left(-k_1e^{-k_1x}+q_1\eta_1k_1e^{-\eta_1k_1x}\right)\\
&+\left(e^{-k_1x}-q_1e^{-\eta_1k_1x}\right)\left[\delta_1-\left(\frac{\delta_1h_1}{1+\underline{u}}-\beta\chi_1\right)\bar{v}-\gamma\chi_1w\left(\cdot;\phi,\psi\right)-\left(\delta_1-\alpha\chi_1\right)e^{-k_1x}\right]\\
\ge&\left(d_1k_1^2-ck_1+\delta_1\right)e^{-k_1x}-q_1\left[d_1(\eta_1k_1)^2-c\eta_1k_1+\delta_1\right]e^{-\eta_1k_1x}\\
&-\chi_1\left|w_x\right|\left(k_1e^{-k_1x}+q_1\eta_1k_1e^{-\eta_1k_1x}\right)-\left(\frac{\delta_1h_1}{1+e^{-k_1x}-q_1e^{-\eta_1k_1x}}-\beta\chi_1\right)\left(e^{-k_1x}-q_1e^{-\eta_1k_1x}\right)e^{-k_2x}\\
&-\gamma\chi_1w\left(e^{-k_1x}-q_1e^{-\eta_1k_1x}\right)-\left(\delta_1-\alpha\chi_1\right)\left(e^{-k_1x}-q_1e^{-\eta_1k_1x}\right)^2\\
\ge&-q_1\left[d_1(\eta_1k_1)^2-c\eta_1k_1+\delta_1\right]e^{-\eta_1k_1x}-\chi_1\left(\alpha\frac{k_1+\sqrt{\gamma-k_1^2}}{\gamma-k_1^2}e^{-k_1x}+\beta\frac{k_2+\sqrt{\gamma-k_2^2}}{\gamma-k_2^2}e^{-k_2x}\right)\left(k_1e^{-k_1x}+q_1\eta_1k_1e^{-\eta_1k_1x}\right)\\
&-\left(\delta_1h_1-\beta\chi_1\right)\left(e^{-k_1x}-q_1e^{-\eta_1k_1x}\right)e^{-k_2x}\;-\gamma\chi_1\left(\frac{\alpha}{\gamma-k_1^2}e^{-k_1x}+\frac{\beta}{\gamma-k_2^2}e^{-k_2x}\right)\left(e^{-k_1x}-q_1e^{-\eta_1k_1x}\right)\\
&-\left(\delta_1-\alpha\chi_1\right)\left(e^{-k_1x}-q_1e^{-\eta_1k_1x}\right)^2.
\end{aligned}
$$

Note that

$$-\left(\delta_1 h_1-\beta\chi_1\right)\left(e^{-k_1x}-q_1e^{-\eta_1k_1x}\right)e^{-k_2x}\geq-\left(\delta_1 h_1-\beta\chi_1\right)e^{-(k_1+k_2)x},$$

and

$$\begin{aligned}&-\gamma\chi_1\left(\frac{\alpha}{\gamma-k_1^2}e^{-k_1x}+\frac{\beta}{\gamma-k_2^2}e^{-k_2x}\right)\left(e^{-k_1x}-q_1e^{-\eta_1k_1x}\right)\\ \geq&-\gamma\chi_1\left(\frac{\alpha}{\gamma-k_1^2}e^{-k_1x}+\frac{\beta}{\gamma-k_2^2}e^{-k_2x}\right)e^{-k_1x}+q_1\frac{\gamma\chi_1\alpha}{\gamma-k_1^2}e^{-(\eta_1+1)k_1x}\end{aligned}$$

hold.

We finally have

$$\begin{aligned}&d_1\underline{u}''+\left(c-\chi_1w'\right)\underline{u}'+\underline{u}\left[\delta_1-\left(\frac{\delta_1h_1}{1+\underline{u}}-\beta\chi_1\right)v-\gamma\chi_1w\left(\cdot;\phi,\psi\right)-\left(\delta_1-\alpha\chi_1\right)\underline{u}\right]\\ \geq&-q_1\left[d_1(\eta_1k_1)^2-c\eta_1k_1+\delta_1\right]e^{-\eta_1k_1x}-\left[\left(\delta_1h_1-\beta\chi_1\right)+\frac{\gamma\chi_1\beta}{\gamma-k_2^2}+\frac{\beta\chi_1k_1\left(k_2+\sqrt{\gamma-k_2^2}\right)}{\gamma-k_2^2}\right]e^{-(k_1+k_2)x}\\ &-\left[\frac{\alpha\chi_1k_1\left(k_1+\sqrt{\gamma-k_1^2}\right)}{\gamma-k_1^2}+\frac{\gamma\chi_1\alpha}{\gamma-k_1^2}+\left(\delta_1-\alpha\chi_1\right)\right]e^{-2k_1x}-q_1\frac{\chi_1\eta_1k_1\beta\left(k_2+\sqrt{\gamma-k_2^2}\right)}{\gamma-k_2^2}e^{-(k_2+\eta_1k_1)x}\\ &+q_1\left[\frac{\gamma\chi_1\alpha}{\gamma-k_1^2}-\frac{\alpha\chi_1\eta_1k_1\left(k_1+\sqrt{\gamma-k_1^2}\right)}{\gamma-k_1^2}+2\left(\delta_1-\alpha\chi_1\right)\right]e^{-(\eta_1+1)k_1x}-q_1^2\left(\delta_1-\alpha\chi_1\right)e^{-2\eta_1k_1x}.\end{aligned}$$

Note that $e^{-k_1x}>q_1e^{-\eta_1k_1x}$ for $x\in(a_{k_1},\infty)$, so

$$q_1^2\left(\delta_1-\alpha\chi_1\right)e^{-2\eta_1k_1x}<q_1\left(\delta_1-\alpha\chi_1\right)e^{-(\eta_1+1)k_1x}.$$

Then we have

$$\begin{aligned}&d_1\underline{u}''+\left(c-\chi_1w'\right)\underline{u}'+\underline{u}\left[\delta_1-\left(\frac{\delta_1h_1}{1+\underline{u}}-\beta\chi_1\right)v-\gamma\chi_1w\left(\cdot;\phi,\psi\right)-\left(\delta_1-\alpha\chi_1\right)\underline{u}\right]\\ \geq&-q_1Q_1e^{-\eta_1k_1x}-Q_2e^{-(k_1+k_2)x}-Q_3e^{-2k_1x}-q_1Q_4e^{-(k_2+\eta_1k_1)x}\\ &+q_1\left[\frac{\gamma\chi_1\alpha}{\gamma-k_1^2}-\frac{\alpha\chi_1\eta_1k_1\left(k_1+\sqrt{\gamma-k_1^2}\right)}{\gamma-k_1^2}+\left(\delta_1-\alpha\chi_1\right)\right]e^{-(\eta_1+1)k_1x}.\end{aligned}$$

It follows from (4.8) that

$$e^{-\eta_1k_1x}>e^{-(k_1+k_2)x},e^{-\eta_1k_1x}>e^{-2k_1x}.$$

Hence, thanks to Lemma 4.4, we have

$$-q_1Q_1e^{-\eta_1\kappa_1x}-Q_2e^{-(\kappa_1+\kappa_2)x}-Q_3e^{-2\kappa_1x}-q_1Q_4e^{-(\kappa_2+\eta_1\kappa_1)x}\geq0.$$

On the other hand,

$$
\begin{aligned}
&\frac{\gamma\chi_1\alpha}{\gamma-k_1^2}-\frac{\alpha\chi_1\eta_1k_1\left(k_1+\sqrt{\gamma-k_1^2}\right)}{\gamma-k_1^2}+(\delta_1-\alpha\chi_1)\\
=&(\delta_1-\alpha\chi_1)-\frac{\alpha\chi_1k_1\left(k_1+\sqrt{\gamma-k_1^2}\right)}{\gamma-k_1^2}+\frac{\gamma\chi_1\alpha}{\gamma-k_1^2}-\frac{(\eta_1-1)\,\alpha\chi_1k_1\left(k_1+\sqrt{\gamma-k_1^2}\right)}{\gamma-k_1^2}\\
\geq&\frac{\gamma\chi_1\alpha}{\gamma-k_1^2}-\frac{(\eta_1-1)\,\alpha\chi_1k_1\left(k_1+\sqrt{\gamma-k_1^2}\right)}{\gamma-k_1^2}\\
\geq&0
\end{aligned}
$$

provided that

$$\eta_1-1\leq\frac{\gamma}{k_1\left(k_1+\sqrt{\gamma-k_1^2}\right)}$$

holds. And for $x\in(-\infty,a_{k_1})$, we have

$$d_1\underline{u}''+(c-\chi_1w')\,\underline{u}'+\underline{u}\left[\delta_1-\left(\frac{\delta_1h_1}{1+\underline{u}}-\beta\chi_1\right)v-\gamma\chi_1w\,(\cdot;\phi,\psi)-(\delta_1-\alpha\chi_1)\,\underline{u}\right]=0.$$

At last, the same arguments as above show that for $x\in(a_{k_2},\infty)$,

$$d_2\underline{v}''+(c-\chi_2w')\,\underline{v}'+\underline{v}\left[\delta_2-\left(\frac{\delta_2h_2}{1+\underline{v}}-\alpha\chi_2\right)u-\gamma\chi_2w\,(\cdot;\phi,\psi)-(\delta_2-\beta\chi_2)\,\underline{v}\right]\geq0$$

under the assumptions on η_2 and q_2 given in Lemma 4.3. And for $x\in(-\infty,a_{k_2})$,

$$d_2\underline{v}''+(c-\chi_2w')\,\underline{v}'+\underline{v}\left[\delta_2-\left(\frac{\delta_2h_2}{1+\underline{v}}-\alpha\chi_2\right)u-\gamma\chi_2w\,(\cdot;\phi,\psi)-(\delta_2-\beta\chi_2)\,\underline{v}\right]=0.$$

□

Lemma 4.5. *Assume that condition ($\mathcal{B}$) is satisfied. Then, for every constant $z>0$ and each pair $(\phi,\psi)\in\Sigma_{k_1,k_2}$, the following system:*

$$
\begin{cases}
U_t=d_1U_{xx}+(c-\chi_1w_x(x;\phi,\psi))\,U_x & \\
\qquad+U[\delta_1-(\dfrac{\delta_1h_1}{1+U}-\beta\chi_1)V-\gamma\chi_1w(x;\phi,\psi)-(\delta_1-\alpha\chi_1)U], & |x|<z,\ t\geq0,\\
V_t=d_2V_{xx}+(c-\chi_2w_x(x;\phi,\psi))\,V_x & \\
\qquad+V[\delta_2-(\dfrac{\delta_2h_2}{1+V}-\alpha\chi_2)U-\gamma\chi_2w(x;\phi,\psi)-(\delta_2-\beta\chi_2)V], & |x|<z,\ t\geq0,\\
U(x,t)=\bar{u}(x),\ V(x,t)=\underline{v}(x), & x=\pm z,\ t\geq0,\\
U(x,0)=\bar{u}(x),\ V(x,0)=\underline{v}(x), & |x|\leq z
\end{cases}
\tag{4.11}
$$

has a unique positive solution $(U(x,t), V(x,t))$ for $t > 0$, where $\bar{u}(x), \underline{v}(x)$ are defined in (4.2). This solution satisfies

(i) $(U(t), V(t)) \in \Sigma_{k_1,k_2}$ in the sense $|x| \leq z$ for $t > 0$;

(ii) U is decreasing, and V is increasing for $t > 0$;

(iii) As $t \to \infty$, the functions U and V converge uniformly on $(-z, z)$ to a unique pair of limits $u^*(x)$ and $v^*(x)$. $(u^*(x), v^*(x))$ satisfies

$$\begin{cases} 0 = d_1 U_{xx} + (c - \chi_1 w_x(x;\phi,\psi))\, U_x & \\ \quad + U[\delta_1 - (\dfrac{\delta_1 h_1}{1+U} - \beta\chi_1)V - \gamma\chi_1 w(x;\phi,\psi) - (\delta_1 - \alpha\chi_1)U], & |x| < z, \\ 0 = d_2 V_{xx} + (c - \chi_2 w_x(x;\phi,\psi))\, V_x & \\ \quad + V[\delta_2 - (\dfrac{\delta_2 h_2}{1+V} - \alpha\chi_2)U - \gamma\chi_2 w(x;\phi,\psi) - (\delta_2 - \beta\chi_2)V], & |x| < z, \\ U(x) = \bar{u}(x),\ V(x) = \underline{v}(x), & x = \pm z. \end{cases} \tag{4.12}$$

Proof. The proof is similar to [1], and the addition of the Holling-II functional responses requires only minor adjustments while remaining fully compatible with the original proof process. □

Corollary 4.1. *Suppose the assumptions of Lemma 4.5 hold. If $(\phi,\psi) \in \Sigma_{\kappa_1,\kappa_2}$, then the stationary solution (u^*, v^*) constructed in Lemma 4.5 lies in $\Sigma_{\kappa_1,\kappa_2}$ over the domain $|x| \leq z$.*

Lemma 4.6. *Let the hypotheses of Theorem 4.1 be satisfied. For any $(\phi,\psi) \in \Sigma_{\kappa_1,\kappa_2}$ and any $z > 0$, the elliptic system (4.12) has exactly one positive solution $(u^*, v^*) \in \Sigma_{\kappa_1,\kappa_2}$ on $|x| \leq z$ that is maximal with respect to the U-component. This solution satisfies*

$$u^*(x) \geq \widetilde{u}(x), \quad v^*(x) \leq \widetilde{v}(x) \quad \text{for } x \in (-z, z) \tag{4.13}$$

for every other positive solution $(\widetilde{u}, \widetilde{v}) \in \Sigma_{\kappa_1,\kappa_2}$ to (4.12), where (u^*, v^*) refers to the solution derived in Lemma 4.5.

Proof. Introduce the following linear operators and nonlinear reaction terms:

$$\mathcal{L}_i(\varphi) = \varphi_t - d_i\varphi_{xx} - (c - \chi_i w_x(x;\phi,\psi))\varphi_x, \quad i = 1, 2,$$

$$f_1(M,N) = M[\delta_1 - (\frac{\delta_1 h_1}{1+M} - \beta\chi_1)N - \gamma\chi_1 w(x;\phi,\psi) - (\delta_1 - \alpha\chi_1)M],$$

and

$$f_2(M,N) = N[\delta_2 - (\frac{\delta_2 h_2}{1+N} - \alpha\chi_2)M - \gamma\chi_2 w(x;\phi,\psi) - (\delta_2 - \beta\chi_2)N].$$

Take any $(\widetilde{u}, \widetilde{v}) \in \Sigma_{\kappa_1,\kappa_2}$ defined on $|x| \leq z$ that solves (4.12). It then satisfies

$$\begin{cases} \mathcal{L}_1(\widetilde{u}) = f_1(\widetilde{u}, \widetilde{v}), \\ \mathcal{L}_2(\widetilde{v}) = f_2(\widetilde{u}, \widetilde{v}). \end{cases}$$

The solution $(U(x,t), V(x,t))$ established in Lemma 4.5 obeys the system

$$\begin{cases} \mathcal{L}_1(U) = f_1(U, V), \\ \mathcal{L}_2(V) = f_2(U, V). \end{cases}$$

Note that $\widetilde{u}(x) \le \overline{u}(x) = U(x, 0)$ and $V(x, 0) = \underline{v}(x) \le \widetilde{v}(x)$. By the comparison principle for parabolic systems, we deduce that $\widetilde{u} \le U(x, t)$ and $\widetilde{v} \ge V(x, t)$ hold for all $t \ge 0$. Passing to the limit $t \to +\infty$, we obtain $\widetilde{u} \le u^*$ and $\widetilde{v} \ge v^*$. □

By Lemma 4.6, we conclude that for any given $z > 0$ and any pair $(\phi, \psi) \in \Sigma_{\kappa_1,\kappa_2}$, the elliptic system (4.12) possesses a pair of solutions (u^*, v^*). So, we define a mapping Φ on the subset $\Sigma_{\kappa_1,\kappa_2} \subseteq Y^*$ to Y^* by

$$\Phi(\phi, \psi) = (u^*, v^*).$$

Given any fixed $z > 0$, u^* and v^* are initially defined on $(-z, z)$, where they satisfy $u^*(x) = \overline{u}(x)$ and $v^*(x) = \underline{v}(x)$ at the endpoints $x = \pm z$. We then extend both functions to the whole real line $\mathbb{R}$ by setting $u^*(x) = \overline{u}(x)$ and $v^*(x) = \underline{v}(x)$ for all $x \notin (-z, z)$. Unless otherwise specified, in all subsequent discussion, u^* and v^* will be understood to refer to these extended functions on $\mathbb{R}$. We now introduce the following norm on $C^b_{\mathrm{unif}}(\mathbb{R})$:

$$\|u\|_* = \sum_{n=1}^{\infty} \frac{1}{2^n} \|u\|_{L^\infty([-n,n])}.$$

It is straightforward to verify that, in the space $C^b_{\mathrm{unif}}(\mathbb{R})$, convergence of a sequence $\{u_n\}_{n\ge1}$ to u under the norm $\|\cdot\|_*$ is equivalent to the locally uniform convergence of $\{u_n\}_{n\ge1}$ to u.

Theorem 4.2. *Assume that the condition ($\mathcal{B}$) holds. Then, for every constant $z > 0$, the elliptic system*

$$\begin{cases} 0 = d_1 u_{xx} + (c - \chi_1 w_x(x; \phi, \psi))\, u_x & \\ \qquad + u[\delta_1 - (\dfrac{\delta_1 h_1}{1+u} - \beta\chi_1)v - \gamma\chi_1 w(x; \phi, \psi) - (\delta_1 - \alpha\chi_1)u], & |x| < z, \\ 0 = d_2 v_{xx} + (c - \chi_2 w_x(x; \phi, \psi))\, v_x & \\ \qquad + v[\delta_2 - (\dfrac{\delta_2 h_2}{1+v} - \alpha\chi_2)u - \gamma\chi_2 w(x; \phi, \psi) - (\delta_2 - \beta\chi_2)v], & |x| < z, \\ u(x) = \bar{u}(x),\ v(x) = \underline{v}(x), & x = \pm z \end{cases} \tag{4.14}$$

admits at least one solution $(u, v) \in \Sigma_{\kappa_1,\kappa_2}$.

Proof. The existence of solutions to the elliptic system (4.14) is established via an application of Schauder's fixed-point theorem. The overall proof framework follows that of Theorem 3.1 in [1]. Here, only the key differences in proving the compactness and continuity of the operator Φ are explained, corresponding to the Holling-II nonlinearity introduced in this paper.

First, we ensure Holling-II functional responses satisfy uniform boundedness. For any solution (u, v) in a bounded subset of $\Sigma_{\kappa_1,\kappa_2}$, the bounds given by the upper and lower solutions guarantee the existence of constants $u_{\min}, u_{\max}, v_{\min}, v_{\max} > 0$ satisfying

$$0 < u_{\min} \le u(x) \le u_{\max}, \quad 0 < v_{\min} \le v(x) \le v_{\max}, \quad \forall x \in [-z, z].$$

Under this boundedness, Holling-II functional responses satisfy uniform boundedness:

$$\left|\frac{\delta_1 h_1}{1+u(x)}\right| \le \frac{\delta_1 h_1}{1+u_{\min}} < +\infty, \quad \left|\frac{\delta_2 h_2}{1+v(x)}\right| \le \frac{\delta_2 h_2}{1+v_{\min}} < +\infty.$$

Thus, the entire right-hand side of (4.14) belongs to L^∞, and the interior L^p estimate of elliptic operators and the Sobolev embedding theorem remain valid. The solution sequence is uniformly bounded in $C^{1,\rho}((-y,y))$ for any $\rho \in (0,1)$, and a convergent subsequence can be extracted by the Arzel-Ascoli theorem. From these, we can conclude that Φ is compact.

Second, we require Holling-II functional responses to satisfy monotonicity and Lipschitz continuity. $f(u) = \frac{\delta_1 h_1}{1+u}$ is strictly decreasing in u, and $g(v) = \frac{\delta_2 h_2}{1+v}$ is strictly decreasing in v, since

$$f'(u) = -\frac{\delta_1 h_1}{(1+u)^2} < 0, \quad g'(v) = -\frac{\delta_2 h_2}{(1+v)^2} < 0, \quad \forall u, v > 0.$$

So, the validity of the comparison principle for the associated parabolic system preserves the ordering of solutions. What's more, in the bounded set $[u_{\min}, u_{\max}]$ and $[v_{\min}, v_{\max}]$, for any $u_1, u_2 \in [u_{\min}, u_{\max}]$, we have

$$\left|\frac{\delta_1 h_1}{1+u_1} - \frac{\delta_1 h_1}{1+u_2}\right| = \delta_1 h_1 \frac{|u_1 - u_2|}{(1+u_1)(1+u_2)} \le L_1 |u_1 - u_2|,$$

where $L_1 = \delta_1 h_1/(1+u_{\min})^2$ is a constant. The same holds for $\frac{\delta_2 h_2}{1+v}$. This guarantees the continuity of the coefficients of the operator, which is used together with Schauder's estimate to carry out the proof by contradiction. From these, we can conclude that Φ is continuous.

Finally, through Schauder's fixed-point theorem, we know that (4.14) has a solution $(u,v) \in \Sigma_{\kappa_1,\kappa_2}$ in the sense $|x| \le z$. □

Theorem 4.3. *Suppose that condition ($\mathscr{B}$) holds, then system (1.3) possesses a weak traveling wave solution $(u(x,t), v(x,t), w(x,t)) = (u(\xi), v(\xi), w(\xi))$. Additionally, the solution satisfies $\lim_{\xi\to\infty}(u(\xi), v(\xi), w(\xi)) = E_0$.*

Proof. Let an increasing sequence $\{z_n\}_{n\ge 1}$ converge to ∞. For each fixed n, we consider the boundary value problem on the interval $(-z_n, z_n)$:

$$\begin{cases} d_1 u''_{z_n} + c u'_{z_n} - \chi_1 (u_{z_n} w'_{z_n})' + \delta_1 u_{z_n} \left(1 - u_{z_n} - \frac{h_1 v_{z_n}}{1+u_{z_n}}\right) = 0, \\ d_2 v''_{z_n} + c v'_{z_n} - \chi_2 (v_{z_n} w'_{z_n})' + \delta_2 v_{z_n} \left(1 - v_{z_n} - \frac{h_2 u_{z_n}}{1+v_{z_n}}\right) = 0, \end{cases}$$

where $u_{z_n}(\pm z_n) = v_{z_n}(\pm z_n) = 0$.

Based on the standard existence theory of elliptic equations, this boundary value problem possesses a solution (u_{z_n}, v_{z_n}) for every n. Owing to the saturation property of the Holling-II functional responses, the nonlinear terms are uniformly bounded. This, together with Schauder's estimate for elliptic operators, implies that $\|u_{z_n}\|_{C^{1+\rho}(-z_n,z_n)}$ and $\|v_{z_n}\|_{C^{1+\rho}(-z_n,z_n)}$ are bounded uniformly for any $z_n > 0$ with $\rho \in (0,1)$. By a standard diagonal process, we extract a subsequence, denoted by $\{(u_{z_n}, v_{z_n})\}$, satisfying $(u_{z_n}, v_{z_n}) \to (u,v)$ in $C^1(K)$ and weakly in $W^{2,p}(K)$ for any compact set $K \subset \mathbb{R}$ and $p > 1$. (u,v) is a weak solution of (1.5), and hence it is a classical solution by elliptic regularity theory. The component w is uniquely determined by (1.3), yielding a weak traveling wave solution.

Next, we show $\lim_{\xi\to\infty}(u(\xi), v(\xi), w(\xi)) = E_0$. Given $\xi \to \infty$, we construct the upper solutions $\bar{u}(\xi) = e^{-k_1\xi}$ and $\bar{v}(\xi) = e^{-k_2\xi}$, which satisfy $\lim_{\xi\to\infty} e^{-k_1\xi} = 0$ and $\lim_{\xi\to\infty} e^{-k_2\xi} = 0$, where $k_1, k_2 > 0$. Since $0 \le u(\xi) \le \bar{u}(\xi)$ and $0 \le v(\xi) \le \bar{v}(\xi)$ for all ξ, $\lim_{\xi\to\infty} u(\xi) = 0$ and $\lim_{\xi\to\infty} v(\xi) = 0$. Through a priori estimates in (4.6), we know $w(\xi)$ is bounded, and it satisfies the equation $0 = w'' - \gamma w + \alpha u + \beta v$, so $\lim_{\xi\to\infty} w(\xi) = 0$. In summary, we have $\lim_{\xi\to\infty}(u(\xi), v(\xi), w(\xi)) = E_0$. □

5. Conclusions and further discussion

This work is concerned with the traveling wave problems of the competitive chemotaxis system with Holling-II functional responses. First, the continuous Fourier transform method and the Routh-Hurwitz criterion are employed to examine the locally asymptotic stability of positive equilibrium point E^*. Then, we use the generalized principal eigenvalue argument and proof by contradiction to prove that system (1.3) admits no traveling wave solutions when wave speed $c < c^*$, where $c^* = \max\left\{2\sqrt{d_1\delta_1}, 2\sqrt{d_2\delta_2}\right\}$ is the critical threshold derived from the linearized analysis around the trivial equilibrium E_0. Finally, the upper and lower solutions method together with Schauder's fixed-point theorem are used to prove the existence of weak traveling wave solutions for system (1.3) when $c > c^*$.

Theoretically, this paper breaks the limitation of unbounded competition in linear models and improves the theory of weak traveling waves under saturable competition. Ecologically, the saturable terms more realistically describe the natural law that competitive inhibition tends to saturate as population density increases. The obtained conclusions on the stability of coexistence states and critical wave speeds provide more reliable theoretical support for predicting spatio-temporal dynamics such as microbial diffusion and biological invasion.

However, there are still several key open problems that remain unsolved in the current work. First, the exact minimal wave speed for the fully nonlinear system (1.3) cannot be determined.

Second, due to the strong coupling effect of the chemotaxis terms and the nonlinear saturation of the Holling-II competitive interactions, the global asymptotic stability of E^* for system (1.2) has not been established. These unresolved issues will be the focus of our subsequent research work.

Author contributions

Shuyang Yu wrote the first version of the paper, Yujuan Jiao proposed the problem and proved the main results, and Ziji Dang checked the calculations. All authors read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

No potential conflict of interest was reported by the authors.

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