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Research article

Blowup dynamics for the Euler-Smoluchowski-Poisson equation: Quantized collapse and subvortex interactions

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Abstract: We investigate the relaxation dynamics of point vortices described by the Euler-Smoluchowski-Poisson equation. Our study reveals a quantization mechanism for collapse masses and demonstrates that the Hamiltonian of the point vortex system governs the dynamics of subcollapses, both in finite-time and infinite-time blowup scenarios. By analyzing the interplay between the Euler term and the Smoluchowski-Poisson structure, we establish that the quantized blowup mechanism and recursive hierarchy observed in the absence of the Euler term persist when it is introduced. Furthermore, we clarify the role of the Euler term in shaping the microscopic dynamics of subcollapses during blowup. Our results extend the understanding of blowup phenomena in this system and highlight the robustness of its underlying geometric and analytical structures.

Keywords: Euler-Smoluchowski-Poisson equation; blowup of the solution; system of point vortices; Onsager's theory; recursive hierarchy
Mathematics Subject Classification: 35Q82, 82B21

1. Introduction

The study of vortex dynamics is fundamental to our understanding of fluid behavior across various scales, from weather patterns to turbulence. A particularly elegant approach to this problem involves studying systems of point vortices and their mean-field descriptions. In this paper, we examine the blowup mechanism of the Euler-Smoluchowski-Poisson equation

$$u_t + \beta\nabla\cdot(u\nabla^{\perp}v) = \nabla\cdot(\nabla u - u\nabla v), \ -\Delta v = u \quad \text{in } \Omega\times(0,T) \tag{1.1}$$

with

$$\frac{\partial u}{\partial \nu} - u\left(\frac{\partial v}{\partial \nu} + \beta\frac{\partial v}{\partial \tau}\right)\bigg|_{\partial\Omega} = 0, \quad v|_{\partial\Omega} = 0 \tag{1.2}$$

and

$$u|_{t=0} = u_0(x) > 0, \tag{1.3}$$

where $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, $|\beta| < 1$,

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix}, \quad \nabla^\perp = \begin{pmatrix} -\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} \end{pmatrix},$$

for $x = (x_1, x_2)$,

$$\nu = \begin{pmatrix} \nu^1 \\ \nu^2 \end{pmatrix}$$

is the outer unit vector, and

$$\tau = \begin{pmatrix} \nu^2 \\ -\nu^1 \end{pmatrix}.$$

Henceforth, we write $x^\perp = (x_2, -x_1)$ for $x = (x_1, x_2)$.

This system is formulated by [6] to describe the relaxation dynamics of the mean field of many point vortices from quasistationary to stationary, using Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy and factorization. The constant β is associated with the vortex intensity, and henceforth, $\beta\nabla \cdot (u\nabla^\perp v)$ is called the Euler term. Equation (1.1) is also introduced in theoretical biology as a system of chemotaxis [28]. Its general form was studied by [7] on the whole space $\Omega = \mathbf{R}^2$, where the optimal mass $\lambda = \|u_0\|_1$ is derived for the blowup of the solution in finite time, by the method of second moment.

If $\beta = 0$, it is the Smoluchowski-Poisson equation describing the material transport of the closed system in thermodynamics [25, 27]. There is total mass conservation with free energy decreasing; this induces the Boltzmann-Poisson equation as a stationary state; scaling invariance results in the critical dimension and mass; and a weak form ensures the quantized blowup mechanism with recursive hierarchy to both blowup in finite and infinite time. Our purpose is to confirm that these structures still hold in the case of $\beta \neq 0$ and to clarify the effect of the Euler term in the subcollapse dynamics.

First, if $u_0 = u_0(x)$ is smooth there is a classical solution local-in-time, of which maximal existence time is denoted by $T = T_{\max} \in (0, +\infty]$. It holds that $u = u(\cdot, t) > 0$ on $\overline{\Omega}$ and

$$\begin{aligned} \frac{d}{dt}\int_\Omega u &= \int_\Omega \nabla \cdot (\nabla u - u\nabla v - \beta u \nabla^\perp v) = \int_{\partial\Omega} \nu \cdot (\nabla u - u\nabla v - \beta u\nabla^\perp v) \\ &= \int_{\partial\Omega} \frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} - \beta u \frac{\partial v}{\partial \tau}\, ds = 0, \end{aligned}$$

and therefore,

$$\lambda = \|u(\cdot, t)\|_1 \tag{1.4}$$

is independent of t. Here and henceforth, we write

$$\int_{\partial\Omega} \ldots\, ds$$

for

$$\int_{\partial\Omega} \{\ldots\}\, ds$$

without confusion.

Second, writing the first equation of (1.1) as in

$$\begin{aligned} u_t &= \nabla\cdot(\nabla u - u\nabla v - \beta u\nabla^\perp v) \\ &= \nabla\cdot(u\nabla\log u - u\nabla v - u\beta\nabla^\perp v) \\ &= \nabla\cdot u(\nabla(\log u - v) - \beta\nabla^\perp v) \end{aligned}$$

we obtain

$$\begin{aligned} \int_\Omega u_t(\log u - v) &= \int_\Omega [\nabla\cdot(u\nabla(\log u - v) - \beta\nabla^\perp v)](\log u - v) \\ &= \int_\Omega -u|\nabla(\log u - v)|^2 + \beta\nabla^\perp v\cdot\nabla(\log u - v)\,dx \end{aligned} \tag{1.5}$$

by the first equality of (1.2). Here, we have

$$\int_\Omega u_t\log u = \frac{d}{dt}\int_\Omega u(\log u - 1).$$

It also holds that

$$\begin{aligned} \int_\Omega u_t v &= \iint_{\Omega\times\Omega} G(x,x')u_t(x,t)u(x',t)\,dxdx' \\ &= \frac{1}{2}\frac{d}{dt}\iint_{\Omega\times\Omega} G(x,x')u\otimes u, \end{aligned} \tag{1.6}$$

where

$$u\otimes u = u(x,t)u(x',t)$$

and $G = G(x,x')$ is the Green function to the Poisson equation

$$-\Delta v = u \text{ in } \Omega, \quad v|_{\partial\Omega} = 0.$$

In fact the Green function satisfies

$$G(x,x') = G(x',x), \tag{1.7}$$

which implies (1.6).

We have, on the other hand,

$$\nabla^\perp v\cdot\nabla v = 0$$

and also

$$\int_\Omega u\nabla^\perp v\cdot\nabla\log u = \int_\Omega \nabla^\perp v\cdot\nabla u = \int_{\partial\Omega}\frac{\partial v}{\partial\tau}u - \int_\Omega[\nabla\cdot(\nabla^\perp v)]u = 0$$

by the second equality of (1.2). Equality (1.5) is now reduced to

$$\frac{d}{dt}\mathcal{F}(u) = -\int_\Omega u|\nabla(\log u - v)|^2 \le 0 \tag{1.8}$$

where

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\iint_\Omega G(x,x')u\otimes u$$

stands for the free energy of Helmholtz [25]. The stationary state is, therefore, formulated by

$$u > 0, \ \log u - v = \text{constant}, \qquad \int_\Omega u = \lambda,$$

that is, the Boltzmann-Poisson equation

$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v} \text{ in } \Omega, \quad v|_{\partial\Omega} = 0. \tag{1.9}$$

This equation arises in Onsager's theory of point vortices [17], derived from the point vortex Hamiltonian [2, 18],

$$H(x_1, \cdots, x_\ell) = \frac{1}{2}\sum_{j=1}^{\ell} R(x_j) + \sum_{1\leq i<j\leq \ell} G(x_i, x_j), \tag{1.10}$$

where

$$R(x) = \left[G(x, x') + \frac{1}{2\pi}\log|x - x'| \right]_{x'=x}$$

stands for the Robin function. There is a quantized blowup mechanism for the family of solutions (λ_k, v_k), $k = 1, 2, \cdots$, where the location of blowup points is prescribed by (1.10) [16].

Systems (1.1)–(1.3) describe the relaxation dynamics from the quasistationary to the stationary, which exhibits the profile of *recursive hierarchy*, that is, control of the Hamiltonian to the blowup mechanism both in finite time and infinite time. Here we recall that (1.9) is the mean field equation of the stationary state of many point vorticies derived from the theory of statistical mechanics. Actually, the kinetics of point vortices is subject to the Hamilton system associated with the Hamiltonian H_ℓ [18].

Recent advances in blowup analysis for the Keller-Segel system have been developed by Chen-Li-Wang [3], who established a comprehensive blowup theory for the parabolic-elliptic Keller-Segel system as a parabolic counterpart to the Liouville equation. Their approach, which uses convergence analysis and quantized mass concentration at blowup points, shares similar structural features with our analysis of the Euler-Smoluchowski-Poisson system, particularly regarding the 8π quantization phenomenon and the role of scaling invariance in determining critical masses.

Equation (1.1) on the whole space $\Omega = \mathbf{R}^n$,

$$u_t + \beta\nabla\cdot(u\nabla^\perp v) = \nabla\cdot(\nabla u - u\nabla v), \ -\Delta u = v \quad \text{in } \mathbf{R}^n \times (0, T), \tag{1.11}$$

is scaling invariant under the transformation

$$u_\sigma(x, t) = \sigma^2 u(\sigma x, \sigma^2 t), \quad v_\sigma(x, t) = v(\sigma x, \sigma^2 t),$$

where $\sigma > 0$ is a constant. This scaling invariance implies, first, the critical dimension $n = 2$ by

$$\|u_\sigma\|_1 = \sigma^{2-n}\|u\|_1$$

for $u_\sigma(x) = \sigma^2 u(\sigma x)$.

Second, the free energy $\mathcal{F}$ to (1.11) with $n = 2$ becomes

$$\mathcal{F}_*(u) = \int_{\mathbf{R}^2} u(\log u - 1) - \frac{1}{2}\iint_{\mathbf{R}^2\times\mathbf{R}^2} \Gamma(x - x')u \otimes u,$$

where

$$\Gamma(x) = \frac{1}{2\pi} \log \frac{1}{|x|} \tag{1.12}$$

stands for the fundamental solution to the two-dimensional Laplacian satisfying

$$-\Delta \Gamma = \delta_0(dx)$$

for

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \quad x = (x_1, x_2).$$

Therefore, it holds that

$$\begin{aligned}
\mathcal{F}_*(u_\sigma) &= \int_{\mathbf{R}^2} u_\sigma(\log u_\sigma - 1) - \frac{1}{4\pi} \iint_{\mathbf{R}^2 \times \mathbf{R}^2} \log \frac{1}{|x - x'|} u_\sigma \otimes u_\sigma \\
&= \int_{\mathbf{R}^2} u(\log u + 2\log\sigma - 1) - \frac{1}{4\pi} \iint_{\mathbf{R}^2 \times \mathbf{R}^2} \left(\log \frac{1}{|x - x'|} + \log \sigma \right) u \otimes u \\
&= (2\lambda - \frac{\lambda^2}{4\pi}) \log \sigma + \mathcal{F}_*(u)
\end{aligned}$$

and the critical mass $\lambda \equiv \|u\|_1 = 8\pi$ arises from the requirement

$$\mathcal{F}_*(u_\sigma) = \mathcal{F}_*(u).$$

The weak form of (1.1) and (1.2) arises for

$$\varphi \in C^2(\overline{\Omega}), \quad \left. \frac{\partial \varphi}{\partial \nu} + \beta \frac{\partial \varphi}{\partial \tau} \right|_{\partial\Omega} = 0, \tag{1.13}$$

by (1.7) as in

$$\frac{d}{dt} \int_\Omega \varphi u = \int_\Omega \Delta\varphi \cdot u + \frac{1}{2} \iint_{\Omega \times \Omega} (\rho_\varphi(x, x') + \beta \rho_\varphi^\perp(x, x')) u \otimes u, \tag{1.14}$$

where

$$\begin{aligned}
\rho_\varphi(x, x') &= \nabla\varphi(x) \cdot \nabla_x G(x, x') + \nabla\varphi(x') \cdot \nabla_{x'} G(x, x') \\
\rho_\varphi^\perp(x, x') &= \nabla\varphi(x) \cdot \nabla_x^\perp G(x, x') + \nabla\varphi(x') \cdot \nabla_{x'}^\perp G(x, x').
\end{aligned} \tag{1.15}$$

Interior and boundary behaviors of $G = G(x, x')$ are derived from the elliptic regularity and conformal transformation ([25] pp. 84–90). We thus obtain

$$\begin{aligned}
& G(x, x') = \Gamma(x - x') + K(x, x') \\
& K \in C^{2+\theta, 1+\theta}(\Omega \times \overline{\Omega}) \cap C^{1+\theta, 2+\theta}(\Omega \times \overline{\Omega})
\end{aligned} \tag{1.16}$$

for $0 < \theta < 1$. It also holds that

$$\begin{aligned}
& G(x, x') = E(X, X') + L(x, x') \\
& L \in C^{2+\theta}((\overline{\Omega} \cap B(x_0, R)) \times (\overline{\Omega} \cap B(x_0, R)))
\end{aligned} \tag{1.17}$$

for $x_0 \in \partial\Omega$ and $0 < R \ll 1$, where

$$X : \overline{\Omega} \cap B(x_0, R) \to \overline{\mathbf{R}}^2_+ = \{(X_1, X_2) \mid X_2 \geq 0\}$$

is a conformal mapping satisfying $X(x_0) = 0$ and

$$E(X, X') = \Gamma(X - X') - \Gamma(X - X'_*)$$

with $X_* = (X_1, -X_2)$ for $X = (X_1, X_2)$ [26]. These properties ensure $\rho_\varphi + \beta\rho_\varphi^\perp \in L^\infty(\Omega \times \Omega)$ ([25] pp. 84–90).

With these mathematical structures in place, we can now state our main results. The following two theorems establish the quantized blowup mechanism for systems (1.1)–(1.3), extending the known results for the case of $\beta = 0$ to the general case with the Euler term. These results demonstrate the deep connections between the partial differential equation (PDE) system and the underlying point vortex dynamics.

The first theorem asserts that blowup in finite time arises with the formation of delta functions $\delta_{x_0}(dx)$, $x_0 \in \mathcal{S}$, of which coefficients $m(x_0)$ are so quantized as 8π times an integer. This theorem also shows that this event occurs only if the total initial mass is so disquantized as $\lambda = \|u_0\|_1 \notin 8\pi\mathbf{N}$, where $\mathbf{N} = \{1, 2, \cdots\}$. This result is consistent with a criterion for the blowup in finite time, known for $\beta = 0$, that is, concentration of the initial value $u_0 = u_0(x)$ at a point with a mass greater than 8π ([14, 21]).

Theorem 1.1. *(blowup in finite time) If $T < +\infty$ it holds that*

$$u(x,t)dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx \quad \text{in } \mathcal{M}(\overline{\Omega}) = C(\overline{\Omega})' \tag{1.18}$$

as $t \uparrow T$, where

$$\mathcal{S} = \{x_0 \in \overline{\Omega} \mid \exists x_k \to x_0,\ t_k \uparrow T,\ u(x_k, t_k) \to +\infty\} \tag{1.19}$$

denotes the blowup set. It holds that

$$\mathcal{S} \subset \Omega, \quad \sharp\mathcal{S} < +\infty, \quad m(x_0) \in 8\pi\mathbf{N},\ \mathbf{N} = \{1, 2, \cdots\} \tag{1.20}$$

and

$$0 < f = f(x) \in C(\overline{\Omega} \setminus \mathcal{S}) \cap L^1(\Omega). \tag{1.21}$$

The second theorem asserts that the blowup in infinite time occurs only if the total initial mass is so quantized as $\lambda = \|u_0\|_1 \in 8\pi\mathbf{N}$, and there is a critical point of the point vortex Hamiltonian H_ℓ. This feature is also observed in the blowup family of stationary solutions [16].

Theorem 1.2. *(blowup in infinite time) If*

$$T = +\infty,\ \limsup_{t\uparrow+\infty} \|u(\cdot, t)\|_\infty = +\infty,$$

it holds that

$$\lambda \equiv \|u_0\|_1 = 8\pi\ell$$

for some $\ell \in \mathbf{N}$. There is $x_* \in \Omega^\ell \setminus D_\ell$ such that

$$\nabla H_\ell(x_*) = 0,$$

where $H_\ell = H$ stands for the point vortex Hamiltonian defined by (1.10) and

$$D_\ell = \{(x, \cdots, x) \in \Omega^\ell \mid x \in \Omega\} \tag{1.22}$$

denotes the diagonal set.

There are several applications of these theorems as in the case of $\beta = 0$. First, if $\lambda = 8\pi$, blowup in finite time does not arise by Theorem 1.1. From (1.4) and (1.8), on the other hand, if the solution exists global-in-time and is uniformly bounded, its ω-limit set is contained in the set of stationary solutions, and in particular, there must exist a solution to (1.9) [9]. If $\lambda = 8\pi$, therefore, it holds that $T = +\infty$, and blowup in infinite time occurs if Ω is close to a disc ([27] p.124).

Second, if $\lambda \equiv \|u_0\|_1 \notin 8\pi\mathbf{N}$ and equation (1.9) does not admit a solution, the solution to (1.1) makes blowup in finite time. If Ω is convex, on the other hand, the Hamiltonian H_ℓ for $\ell \geq 2$ does not take any critical point by [8], and therefore, if $\lambda > 8\pi$ and Ω is close to a disc, the solution to (1.1) makes blowup in finite time. In spite of these similarities with the case of $\beta = 0$, the Euler term $\beta\nabla \cdot (u\nabla^\perp v)$ affects the microscopic blowup mechanism, that is, the subcollapse dynamics both in blowup in finite time and infinite time (§3.11).

The system of Eq (1.1) induces

$$u_t + \beta\nabla^\perp v \cdot \nabla u = \Delta u - \nabla v \cdot \nabla u + u^2,$$

and therefore, its spatially homogeneous part takes the form

$$u_t = u^2.$$

If a blowup arises at $t = T$ of this ordinary differential equation (ODE), it follows that

$$u(t) = (T - t)^{-1},$$

and for this reason, the blowup rate $O((T - t)^{-1})$ is referred to as Type I. We have, however, always a Type II blowup rate indicated by

$$\lim_{t\to T}(T - t)\|u(\cdot, t)\|_\infty = +\infty.$$

This profile is more refined according to the boundedness of the free energy $\mathcal{F}(u)$ (§3.11).

Under these conditions, detecting the actual blowup rate is the first step to construct blowup solution by the method of matched asymptotic expansion, and this process has been done for $\beta = 0$ by [10, 13, 5] with and without radial symmetry, respectively. To our knowledge, however, it still remains open to detect the actual blowup rate and to construct blowup solution for the case of $\beta \neq 0$.

Exclusion of the boundary blowup is a consequence of the form of the Poisson part associated with the Dirichlet boundary condition,

$$-\Delta v = u, \;\; v|_{\partial\Omega} = 0$$

in (1.1) and (1.2). If this part is involved by a Neumann boundary condition such as

$$-\Delta v + v = u, \;\; \left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0$$

or

$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u, \ \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0, \ \int_\Omega v = 0,$$

actual boundary blowup in finite time is expected. Then the collapsed mass $m(x_0)$, $x_0 \in \mathcal{S}$ may be so quantized as $m(x_0) \in m_*(x_0)\mathbf{N}$, where

$$m_*(x_0) = \begin{cases} 8\pi, & x_0 \in \Omega, \\ 4\pi, & x_0 \in \partial\Omega. \end{cases}$$

This feature is confirmed for $\beta = 0$, see [27] for example, and we expect it even for $\beta \neq 0$. Showing an actual collision of subcollapses on the boundary is more challenging.

Relatives of the Euler-Smoluchowski-Poisson Eq (1.1) are observed widely in the theory of statistical mechanics, particularly in the field of astrophysics. We refer to [4] for readers who are interested in this topic.

2. Organization of the paper

This paper is composed of three sections. Theorems 1.1 and 1.2 are proven in Sections 3 and 4, respectively. Although these results are not involved by β explicitly, the Euler term $\beta\nabla\cdot(u\nabla^\perp v)$ causes technical difficulties. We show that this term is controllable as in the argument [27] developed for $\beta = 0$ except for the last part, the dynamics of subcollapses involved by the Euler term, (3.27) and (4.10).

For the blowup in finite time, we localize the ε-regularity for the existence of the solution, uniformly bounded and global-in-time, to ensure the formation of finite collapses by the weak form (1.14) (§3.1–§3.3). Furthermore, this weak form induces the notion of the weak solution, and the principle of its generation from the bounded sequence (§3.4). Then we introduce the backward self-similar transformation to take its weak limit $\zeta(dy, s)$ with the key properties of the parabolic envelope (§3.5 and §3.6). As a consequence is the exclusion of the boundary blowup (§3.7). Taking the scaling back $A(dy', s')$ of $\zeta(dy, s)$ and its scaling limit, we observe that the singular part of $A(dy', s')$ is composed of a sum of delta functions, called subcollapses, with the quantized mass 8π (§3.8). Then we prepare a scaling invariant ε-regularity to deduce the vanishing of the absolutely continuous part of $\zeta(dy, s)$, which results in the collapse mass quantization $m(x_0) \in 8\pi\mathbf{N}$, $x_0 \in \mathcal{S}$ (§3.9 and §3.10). We finally detect the subcollapse dynamics using local second moment (§3.11).

For the blowup in infinite time, we take the translation limit $\mu(dx, t)$ of the solution, instead of the scaling limit with the formation of collapses (§4.1). Then, we exclude the boundary blowup and the collapse mass quantization using scaling limit (§4.2 and §4.3). A rough description of the dynamics of the singular part of $\mu(dx, t)$ is derived from the local second moment, and then the *defect second moment* ensures the vanishing of its absolutely continuous part (§4.4 and §4.5). Here, the control of the Euler term is crucial. The collapse dynamics are finally confirmed with the aid of the theory of dynamical systems (§4.6).

3. Blowup in finite time

3.1. ε-regularity

Here we show the following proposition obtained by [11] for $\beta = 0$ for the case of Neumann boundary condition in the Poisson part. See also [1, 15, 29] for a refined estimate of ε_0 in this case, and [14] for its optimality.

We follow the estimates of L^p-norms executed for $\beta = 0$ ([25] Chapter 11).

Proposition 3.1. *(ε-regularity) There is $\varepsilon_0 > 0$ such that*

$$\lambda \equiv \|u_0\|_1 < \varepsilon_0 \quad \Rightarrow \quad T = +\infty, \ \|u(\cdot, t)\|_\infty \leq C.$$

Proof. Letting $p > 0$, we obtain

$$\begin{aligned} \frac{1}{p+1}\frac{d}{dt}\|u\|_{p+1}^{p+1} &= \int_\Omega u_t u^p = \int_\Omega [\nabla \cdot (\nabla u - u\nabla v - \beta u \nabla^\perp v)]u^p \\ &= -\int_\Omega (\nabla u - u\nabla v - \beta u\nabla^\perp v) \cdot \nabla u^p, \end{aligned}$$

with

$$\int_\Omega \nabla u \cdot \nabla u^p = \frac{4p}{(p+1)^2}\int_\Omega |\nabla u^{\frac{p+1}{2}}|^2,$$

$$\begin{aligned} \int_\Omega u\nabla v \cdot \nabla u^p &= \frac{p}{p+1}\int_\Omega \nabla v \cdot \nabla u^{p+1} \\ &= \frac{p}{p+1}\int_{\partial\Omega} \frac{\partial v}{\partial \nu} \cdot u^{p+1} - \frac{p}{p+1}\int_\Omega (\Delta v) u^{p+1} \\ &\leq \frac{p}{p+1}\int_\Omega u^{p+2} \end{aligned}$$

and

$$\begin{aligned} \int_\Omega u\nabla^\perp v \cdot \nabla u^p &= \frac{p}{p+1}\int_\Omega \nabla u^{p+1} \cdot \nabla^\perp v \\ &= \frac{p}{p+1}\int_{\partial\Omega} u^{p+1}\frac{\partial v}{\partial \tau} - \frac{1}{p+1}\int_\Omega u^{p+1}\nabla \cdot (\nabla^\perp v) \\ &= 0, \end{aligned}$$

which implies

$$\frac{1}{p+1}\frac{d}{dt}\|u\|_{p+1}^{p+1} \leq -\frac{p}{(p+1)^2}\|\nabla u^{\frac{p+1}{2}}\|_2^2 + \frac{p}{p+1}\|u\|_{p+2}^{p+2}. \tag{3.1}$$

Inequality (3.1) is the same as the one for $\beta = 0$, and then Moser's iteration scheme ensures the result. □

3.2. Localization

We provide the following proposition to localize Proposition 3.1.

Proposition 3.2. *There is $\varepsilon_0 > 0$ such that*

$$\sup_{0 \leq t < T} \|u(\cdot, t)\|_{L^1(\Omega \cap B(x_0, R))} < \varepsilon_0 \ \Rightarrow \ \sup_{0 \leq t < T} \|u(\cdot, t)\|_{L^2(\Omega \cap B(x_0, R/2))} \leq C$$

for any $x_0 \in \overline{\Omega}$ and $0 < R \ll 1$.

Proof. Given $\varphi = \varphi(x)$ in (1.13), we obtain

$$\begin{aligned}\frac{1}{p+1}\frac{d}{dt}\int_\Omega u^{p+1}\varphi &= \int_\Omega [\nabla\cdot(\nabla u - u\nabla v - \beta u\nabla^\perp v)]u^p\varphi \\ &= -\int_\Omega (\nabla u - u\nabla v - \beta u\nabla^\perp v)\cdot\nabla(u^p\varphi). \end{aligned} \tag{3.2}$$

It holds that

$$\begin{aligned}&\int_\Omega u\nabla^\perp v\cdot\nabla(u^p\varphi) = \int_\Omega u\nabla^\perp v\cdot(\varphi\nabla u^p + u^p\nabla\varphi)\\ &= \int_\Omega\left(\frac{p}{p+1}\nabla u^{p+1}\cdot\nabla^\perp v\right)\varphi + u^{p+1}\nabla^\perp v\cdot\nabla\varphi\, dx\\ &= \int_{\partial\Omega}\frac{p}{p+1}u^{p+1}\frac{\partial v}{\partial\tau}\varphi + \int_\Omega -\frac{p}{p+1}u^{p+1}\nabla\cdot(\varphi\nabla^\perp v) + u^{p+1}\nabla^\perp v\cdot\nabla\varphi\, dx\\ &= \frac{p}{p+1}\int_\Omega u^{p+1}\nabla^\perp v\cdot\nabla\varphi, \end{aligned} \tag{3.3}$$

and the term

$$\int_\Omega (\nabla u - u\nabla v)\cdot\nabla(u^p\varphi)$$

is treated as in the case of $\beta = 0$.

Here we use the smooth cutoff function $\varphi = \varphi_{x_0,R}$ for $x_0 \in \overline{\Omega}$ and $0 < R \ll 1$, satisfying (1.13), $0 \le \varphi \le 1$

$$\varphi = \begin{cases} 1 & \text{in } \Omega\cap B(x_0,R) \\ 0 & \text{in } \Omega\setminus B(x_0,R), \end{cases}$$

and

$$|\nabla\varphi| \le A\varphi^{5/6}, \quad |\nabla^2\varphi| \le B\varphi^{2/3},$$

where $A = CR^{-1}$ and $B = CR^{-2}$ ([25] p.222).

Thus, for $p = 1$ we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\varphi + \int_\Omega |\nabla u|^2\varphi + u\nabla u\cdot\nabla\varphi\, dx\\ &\quad= \int_\Omega u(\nabla v\cdot\nabla u)\varphi + u^2\nabla v\cdot\nabla\varphi + \frac{\beta}{2}u^2\nabla^\perp v\cdot\nabla\varphi\, dx\end{aligned}$$

([25] p.225) with

$$\int_\Omega u^2\nabla^\perp v\cdot\nabla\varphi = \int_\partial u^2 v\frac{\partial\varphi}{\partial\nu} - \int_\Omega v\nabla^\perp\cdot(u^2\nabla\varphi) = -\int_\Omega v\nabla^\perp u^2\cdot\nabla\varphi.$$

This term is treated similarly to

$$\int_\Omega v\nabla u^2\cdot\nabla\varphi$$

([25] p.227), and the other terms are the same as in the case of $\beta = 0$ ([25] pp.226 and 227). Then we obtain the result. □

Applying (3.2) and (3.3) to $p = 2$, we now obtain the following fact as in Proposition 3.2 ([25] p.229).

Proposition 3.3. *Under the assumption of Proposition 3.1, it holds that*

$$\sup_{0\le t<T} \|u(\cdot,t)\|_{L^3(\Omega\cap B(x_0,R/4))} \le C,$$

which implies

$$\sup_{0\le t<T} \|\nabla v(\cdot,t)\|_{L^\infty(\Omega\cap B(x_0,R/8))} \le C. \tag{3.4}$$

Once (3.4) arises, Moser's iteration scheme works ([25] pp.229–235), which results in the following lemma.

Lemma 3.1. *(localized ε-regularity) There is $\varepsilon_0 > 0$ such that*

$$\lim_{R\downarrow 0}\limsup_{t\uparrow T} \|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} < \varepsilon_0 \ \Rightarrow\ x_0 \notin \mathcal{S}.$$

Lemma 3.1 implies the following fact as a corollary.

Lemma 3.2. *It holds that*

$$\limsup_{t\uparrow T}\int_\Omega u(\log u-1)\cdot\varphi_{x_0,R} < +\infty,\ 0<R\ll 1 \quad\Rightarrow\quad x_0\notin\mathcal{S}.$$

Proof. Given $\varphi = \varphi_{x_0,R}$, we have

$$\begin{aligned} \frac{d}{dt}\int_\Omega u(\log u-1)\varphi &= \int_\Omega (u_t\log u)\varphi \\ &= -\int_\Omega (\nabla u - u\nabla v - \beta u\nabla^\perp v)\cdot\nabla(\varphi\log u) \end{aligned} \tag{3.5}$$

with

$$\begin{aligned} &\int_\Omega u\nabla^\perp v\cdot\nabla(\varphi\log u) = \int_\Omega (\nabla^\perp v\cdot\nabla u)\varphi + (u\log u)\nabla^\perp v\cdot\nabla\varphi\, dx \\ &= \int_{\partial\Omega}\frac{\partial v}{\partial\tau}\cdot u\varphi + \int_\Omega -u\nabla\cdot(\varphi\nabla^\perp v) + (u\log u)\nabla^\perp v\cdot\nabla\varphi\, dx \\ &= \int_\Omega u(\log u-1)\nabla^\perp v\cdot\nabla\varphi. \end{aligned}$$

The other terms of the right-hand side of (3.5) are treated similarly to the case of $\beta = 0$ and the Neumann boundary condition to the Poisson part ([25] pp.235–237). In fact we have

$$\begin{aligned} &\int_\Omega u\nabla v\cdot\nabla(\varphi\log u) \\ &\quad= \int_{\partial\Omega}\frac{\partial v}{\partial\nu}u\varphi + \int_\Omega -u\nabla\cdot(\varphi\nabla v) + (u\log u)\nabla v\cdot\nabla\varphi\, dx \\ &\quad\le \int_\Omega u(\log u-1)\nabla v\cdot\nabla\varphi + u^2\varphi\, dx, \end{aligned}$$

which implies the result. □

3.3. Monotonicity formula

Equality (1.14) with $\rho_\varphi, \rho_\varphi^\perp \in L^\infty(\Omega\times\Omega)$ implies

$$\left|\frac{d}{dt}\int_\Omega \varphi u\right| \leq C(\lambda+\lambda^2)\|\nabla\varphi\|_{C^1} \tag{3.6}$$

for $\varphi = \varphi(x)$ in (1.13), which takes a similar role of the monotonicity formula in the study of harmonic maps ([24] p.259). Then we obtain the following lemma as in the case of $\beta = 0$ ([25] pp.237 and 238, [27] p.66) together with the finiteness of the blowup points [20].

Lemma 3.3. *(formation of collapses) It holds that*

$$\sharp\mathcal{S} < +\infty$$

for the blowup set $\mathcal{S}$ defined by (1.19) and (1.18) as $t \uparrow T$ with

$$m(x_0) \geq \varepsilon_0, \quad 0 \leq f = f(x) \in L^1(\Omega)\cap C(\overline{\Omega}\setminus\mathcal{S}).$$

Then there arises the positivity of $f = f(x)$, (1.21), because $u = u(x,t)$ satisfies a parabolic equation on $\overline{\Omega}\times[0,T]\setminus(\overline{\Omega}\setminus\mathcal{S})\times\{T\}$ ([27] p.83).

3.4. Weak solutions

Now we define the weak solution to (1.1) and (1.2) as in the case of $\beta = 0$ ([25] Chapter 13, [27] p.67).

Definition 3.1. *(weak solution) We assert that*

$$0 \leq \mu = \mu(dx,t) \in C_*([0,T],\mathcal{M}(\overline{\Omega}))$$

is a weak solution to (1.1) and (1.2) if there is $0 \leq N = N(\cdot,t) \in L^\infty_*([0,T],\mathcal{X}')$, called the multiplicated operator, such that

$$t \in [0,T] \mapsto \langle\varphi,\mu(dx,t)\rangle, \quad \varphi \in \mathcal{Y}$$

is absolutely continuous, and

$$\begin{aligned}&\frac{d}{dt}\langle\varphi,\mu\rangle = \langle\Delta\varphi,\mu\rangle + \frac{1}{2}\langle\rho_\varphi+\beta\rho_\varphi^\perp, \mathcal{N}(\cdot,t)\rangle\\ &N(\cdot,t)|_{C(\overline{\Omega}\times\overline{\Omega})} = \mu(dx,t)\otimes\mu(dx,t), \quad a.e.\ t\in[0,T]\end{aligned} \tag{3.7}$$

for $\rho_\varphi = \rho_\varphi(x,x')$ and $\rho_\varphi^\perp = \rho_\varphi^\perp(x,x')$ defined by (1.15), where

$$\mathcal{Y} = \{\varphi \in C^2(\overline{\Omega}) \mid \frac{\partial\varphi}{\partial\nu} + \beta\frac{\partial\varphi}{\partial\tau}\Big|_{\partial\Omega} = 0\}, \quad \mathcal{X} = [\mathcal{X}_0]^{L^\infty(\Omega\times\Omega)},$$

and

$$\mathcal{X}_0 = \{\rho_\varphi + \beta\rho_\varphi^\perp + \psi \mid \varphi\in\mathcal{Y},\ \psi\in C(\overline{\Omega}\times\overline{\Omega}).$$

Then it holds that

$$\mu(\overline{\Omega}, t) = \mu(\overline{\Omega}, 0) \equiv \lambda, \quad 0 \le t \le T$$

and

$$\left| \frac{d}{dt} \langle \varphi, \mu(dx, t) \rangle \right| \le C(\lambda + \lambda^2) \|\nabla\varphi\|_{C^1}.$$

If $u = u(x, t)$ is a classical solution, then $\mu(dx, t) = u(x, t)dx$ is a weak solution with the associated multiplicated operator

$$N(\cdot, t) = u(x, t)u(x', t)dxdx'.$$

It holds that

$$\|N(\cdot, t)\|_{\mathcal{X}'} = \lambda^2, \quad \lambda = \|u_0\|_1.$$

Because $\mathcal{X}$ is separable, we obtain the following lemma as in the case of $\beta = 0$ ([22], [25] pp.283–286).

Lemma 3.4. *(generation of the weak solution) If*

$$\mu_k(dx, t) \in C_*([0, T], \mathcal{M}(\overline{\Omega})), \quad k = 1, 2, \cdots,$$

is a sequence of weak solutions to (1.1) and (1.2) with the associated multiplicated operators $N_k(\cdot, t) \in L^\infty_*(\mathcal{X}')$, $k = 1, 2, \cdots$, *satisfying*

$$0 \le \lambda_k = \mu_k(\overline{\Omega}, 0) \le C, \quad \|N_k(\cdot, t)\|_{\mathcal{X}'} \le C,$$

there is a subsequence denoted by the same symbol, such that

$$\begin{aligned} &\mu_k(dx, t) \rightharpoonup \exists \mu(dx, t) \quad \text{in } C_*([0, T], \mathcal{M}(\overline{\Omega})) \\ &N_k(\cdot, t) \rightharpoonup \exists N(\cdot, t) \qquad \text{in } L^\infty_*([0, T], \mathcal{X}'), \end{aligned}$$

where $\mu(dx, t)$ *is a weak solution to (1.1) and (1.2) with the associated multiplicated operator* $\mathcal{N}(\cdot, t)$.

3.5. Scaling limit

Given $x_0 \in \mathcal{S}$, we introduce the backward self-similar transformation

$$y = (x - x_0)/(T - t)^{1/2}, \; s = -\log(T - t), \; z(y, s) = (T - t)u(x, t) \tag{3.8}$$

in (1.1) and (1.2), to reach

$$\begin{aligned} &z_s + \beta\nabla \cdot (z\nabla^\perp w) = \nabla \cdot \left(\nabla z - z\nabla(w + \frac{|y|^2}{4}) \right) \quad \text{in} \bigcup_{s > -\log T} \Omega_s \times \{s\} \\ &\frac{\partial z}{\partial \nu} - z \left\{ \frac{\partial}{\partial \nu}(w + \frac{|y|^2}{4}) + \beta \frac{\partial w}{\partial \tau} \right\} = 0 \qquad \text{on} \bigcup_{s > -\log T} \partial\Omega_s \times \{s\} \end{aligned}$$

for

$$w(y, s) = \int_{\Omega_s} G_s(y, y')z(y', s)dy', \quad G_s(y, y') = G(x, x'),$$

where $\Omega_s = (T - t)^{-1/2}(\Omega - \{x_0\})$. We take zero extension of z where it is not defined. Then we apply Lemma 3.4 to $z = z(y, s)$ to obtain the following fact:

Lemma 3.5. *(scaling limit) Any $s_k \uparrow +\infty$ admits a subsequence, denoted by the same symbol, such that*

$$z_k(y,s)dy \rightharpoonup \exists\zeta(dy,s) \quad \text{in } C_*(-\infty,+\infty;\mathcal{M}(\mathbf{R}^2)) \tag{3.9}$$

for $z_k(y,s) = z(y,s+s_k)$, where

$$\mathcal{M}(\mathbf{R}^2) = C_\infty(\mathbf{R}^2)', \quad C_\infty(\mathbf{R}^2) = \{\varphi \in C(\mathbf{R}^2 \cup \{\infty\} \mid \varphi(\infty) = 0\},$$

and $\mathbf{R}^2 \cup \{\infty\}$ is the one-point compactification of $\mathbf{R}^2$.

Then we use the boundary behavior of $G(x,x')$, (1.16) and (1.17), to obtain the following lemmas as in the case of $\beta = 0$ ([26]).

Lemma 3.6. *(interior blowup point) If $x_0 \in \Omega$, the above $\zeta(dy,s)$ in (3.9) is a weak solution to*

$$z_s + \beta\nabla\cdot(z\nabla^\perp\Gamma * z) = \nabla\cdot(\nabla z - z\nabla(\Gamma * z + \frac{|y|^2}{4})) \quad \text{in } \mathbf{R}^2\times(-\infty,+\infty) \tag{3.10}$$

defined similarly, using $\rho^0_\varphi(y,y')$ and $\rho^{0\perp}_\varphi(y,y')$ for $\rho_\varphi(x,x')$ and $\rho^\perp_\varphi(x,x')$ in (3.7), where

$$\begin{aligned}\rho^0_\varphi(y,y') &= \nabla\varphi(y)\cdot\nabla_y\Gamma(y-y') + \nabla\varphi(y')\cdot\nabla_{y'}\Gamma(y-y') \\ &= \frac{1}{2\pi}\frac{(y-y')}{|y-y'|^2}\cdot(\nabla\varphi(y)-\nabla\varphi(y'))\end{aligned}$$

and

$$\begin{aligned}\rho^{0\perp}_\varphi(y,y') &= \nabla\varphi(y)\cdot\nabla^\perp_y\Gamma(y-y') + \nabla\varphi(y')\cdot\nabla^\perp_{y'}\Gamma(y-y') \\ &= -\frac{1}{2\pi}\frac{(y-y')^\perp}{|y-y'|^2}\cdot(\nabla\varphi(y)-\nabla\varphi(y')).\end{aligned}$$

Lemma 3.7. *(boundary blowup point) If $x_0 \in \partial\Omega$, the above $\zeta(dy,s)$ in (3.9) satisfies supp $\zeta(dy,s) \subset \overline{L}$, where L is a half-space such that $0 \in \overline{L}$, and is a weak solution to*

$$z_s + \beta\nabla\cdot(z\nabla^\perp E * z) = \nabla\cdot(\nabla z - z\nabla(E * z + \frac{|y|^2}{4})) \quad \text{in } L\times(-\infty,+\infty)$$

for

$$E(y,y') = \Gamma(y-y') - \Gamma(y-y'_*),$$

where y'_* is the reflection of y' with respect to L. This weak solution is thus associated with $\rho^{1,\beta}_\varphi = \rho^1_\varphi(y,y') + \beta\rho^{1\perp}_\varphi(y,y')$, where

$$\begin{aligned}\rho^1_\varphi(y,y') &= \nabla\varphi(y)\cdot\nabla_y E(y,y') + \nabla\varphi(y')\cdot\nabla_{y'}E(y,y') \\ \rho^{1\perp}_\varphi(y,y') &= \nabla\varphi(y)\cdot\nabla^\perp_y E(y,y') + \nabla\varphi(y')\cdot\nabla^\perp_{y'}E(y,y').\end{aligned}$$

3.6. Parabolic envelope

The following fact plays a fundamental role in the proof of collapse mass quantization, $m(x_0) \in 8\pi\mathbf{N}$, in (1.20). The proof is similar to the case of $\beta = 0$ ([19], [25] pp.309, [27] pp.89 and 90).

Lemma 3.8. *It holds that*

$$\zeta(\mathbf{R}^2, s) = m(x_0), \ \langle |y|^2, \zeta(dy, s)\rangle \leq C, \quad -\infty < s < +\infty. \tag{3.11}$$

Proof. Letting $\varphi = \varphi_{x_0,R} \in \mathcal{Y}$ be the smooth cutoff function, we have

$$\|\nabla\varphi\|_{C^1} \leq CR^{-2}$$

and hence,

$$\left|\frac{d}{dt}\int_\Omega u\varphi_{x_0,R}\right| \leq C_\lambda R^{-2}, \quad 0 < R \leq 1$$

by (3.6), which implies

$$\left|\langle \varphi_{x_0,R}, u(\cdot,t)dx\rangle - \langle \varphi_{x_0,R}, \mu(dx,T)\rangle\right| \leq C_\lambda (T-t)/R^2 \tag{3.12}$$

Let $s_k \uparrow +\infty$ be such that (3.9). Fix $s \in \mathbf{R}$ and define $t_k \in (0,T)$ by

$$s_k + s = -\log(T - t_k).$$

Fix $b > 0$. It holds that $t_k \uparrow T$, and therefore, $R = b(T-t_k)^{1/2} \leq 1$ for $k \gg 1$. Putting this $R = R_k$ to (3.12) for $t = t_k$, we obtain

$$\left|\langle \varphi_{x_0,b}, z(\cdot, s+s_k)dy\rangle - \langle \varphi_{x_0,be^{-(s+s_k)/2}}, \mu(dx,T)\rangle\right| \leq C_\lambda/b^2,$$

and therefore,

$$\left|\langle \varphi_{x_0,b}, \zeta(dy,s)\rangle - \mu(\{x_0\},T)\right| \leq C_\lambda/b^2.$$

It thus holds that

$$m(x_0) = \zeta(\mathbf{R}^2, s)$$

with $b \uparrow +\infty$.

We have

$$\left|\frac{d}{dt}\int_\Omega u|x-x_0|^2\varphi_{x_0,R}\right| \leq C_\lambda$$

if $x_0 \in \Omega$, which implies

$$\langle |y|^2, \zeta(dy,s)\rangle \leq C \tag{3.13}$$

similarly. If $x_0 \in \partial\Omega$, we use the conformal mapping $X : \overline{\Omega} \cap B(x_0,R) \to \mathbf{R}^2_+$ with $X(x_0) = 0$ for $0 < R \ll 1$, and a smooth cutoff function $\eta = \eta(X_1,X_2)$, satisfying $\frac{\partial\eta}{\partial X_2} + \beta\frac{\partial\eta}{\partial X_1}\Big|_{X_2=0} = 0$ in $\mathbf{R}^2$ ([25] p.91). Letting $\varphi_R = X^*\eta$, we similarly obtain

$$\left|\frac{d}{dt}\int_\Omega u|A(X)|^2\varphi_R\right| \leq C_\lambda$$

and (3.13) similarly, where $A(X) = X_1^2 + X_2^2 - 2\beta X_1X_2$. □

3.7. Exclusion of boundary blowup

The exclusion of boundary blowup is proven as in the case of $\beta = 0$ ([26]).

Lemma 3.9. *It holds that $\partial\Omega \cap \mathcal{S} = \emptyset$.*

Proof. If $x_0 \in \partial\Omega \cap \mathcal{S}$, Lemma 3.7 ensures the following property: Let

$$\mathcal{Y}_1 = \{\varphi \in C_0^2(\overline{L}) \mid \left. \frac{\partial\varphi}{\partial\nu} + \beta\frac{\partial\varphi}{\partial\tau}\right|_{\partial L} = 0\}$$

and

$$\mathcal{E} = [\mathcal{E}_1]^{L^\infty(\Omega\times\Omega)}, \quad \mathcal{E}_1 = \{\psi + \rho_\varphi^{1,\beta} \mid \psi \in C_0(\overline{L}\times\overline{L}),\ \varphi \in \mathcal{Y}_1\},$$

where $\rho_\varphi^{1,\beta} = \rho_\varphi^1 + \beta\rho_\varphi^{1\perp}$. Then the mapping

$$s \in (-\infty, +\infty) \ \mapsto \ \langle \varphi, \zeta(dy, s)\rangle, \quad \varphi \in \mathcal{Y}_1$$

is absolutely continuous, and it holds that

$$\frac{d}{ds}\langle \varphi, \zeta(dy, s)\rangle = \langle \Delta\varphi + \frac{y}{2}\cdot\nabla\varphi, \zeta(dy, s)\rangle + \frac{1}{2}\langle \rho_\varphi^{1,\beta}, \mathcal{K}(\cdot, s)\rangle \quad \text{a.e. } s$$

with $0 \le \mathcal{K} = \mathcal{K}(\cdot, s) \in L_*^\infty(-\infty, +\infty); \mathcal{E}')$ satisfying

$$\mathcal{K}(\cdot, s)|_{C_0(\overline{L}\times\overline{L})} = \zeta(dy, s) \otimes \zeta(dy, s), \quad \text{a.e. } s.$$

We also have

$$\zeta \in C_*(-\infty, +\infty; \mathcal{M}(\mathbf{R}^2)), \quad \operatorname{supp} \zeta(\cdot, s) \subset \overline{L}$$

and

$$\zeta(\mathbf{R}^2, s) = m(x_0) > 0,\ \langle |y|^2, \zeta(dy, s)\rangle \le C, \quad -\infty < s < +\infty.$$

Then, we obtain

$$\int_{s_1}^{s_2} \langle \Delta\varphi + \frac{y}{2}\cdot\nabla\varphi, \zeta(dy, s)\rangle + \frac{1}{2}\langle \rho_\varphi^{1,\beta}, \mathcal{K}(\cdot, s)\rangle\, ds = [\langle \varphi, \zeta(dy, s)\rangle]_{s_1}^{s_2}$$

for $-\infty < s_1 < s_2 < +\infty$.

Here, we can take

$$\varphi = A(y)\xi_R(y), \quad \xi_R(y) = \xi(y/R),$$

where $\xi = \xi(y_1, y_2) \in C_0^\infty(\mathbf{R}^2)$ satisfies $\left. \frac{\partial\xi}{\partial y_2} + \beta\frac{\partial\xi}{\partial y_1}\right|_{y_2=0} = 0$, $0 \le \xi \le 1$ and

$$\xi = \begin{cases} 1 & \text{on } B(0,1), \\ 0 & \text{in } \mathbf{R}^2 \setminus B(0,2). \end{cases}$$

We make $R \uparrow +\infty$ and use the dominated convergence theorem, noting

$$\Delta A = 4, \quad y \cdot A = 2A, \quad \langle 1 + |y|^2, \zeta(dy, s)\rangle \le C$$

([26]). First, the equality

$$\lim_{R\uparrow+\infty}\int_{s_1}^{s_2}\langle\rho_\varphi^1, \mathcal{K}(\cdot,s)\rangle\,ds = 0 \tag{3.14}$$

follows from the result on $\beta = 0$ because $y_*'^\perp\cdot y = -y_*^\perp\cdot y'$ ([27] pp.103–105). Second, we obtain

$$\lim_{R\uparrow+\infty}\int_{s_1}^{s_2}\langle\rho_\varphi^{1\perp}, \mathcal{K}(\cdot,s)\rangle\,ds = 0. \tag{3.15}$$

To see this property, we use

$$\rho_{|y|^2}^{1\perp} = I - II$$

for

$$\begin{aligned} I &= \nabla^\perp\Gamma(y-y')\cdot(\nabla|y|^2-\nabla|y'|^2) \\ II &= \nabla^\perp\Gamma(y-y_*')\cdot\nabla|y|^2+\nabla^\perp\Gamma(y'-y_*)\cdot\nabla|y'|^2. \end{aligned}$$

It holds that

$$I = -\frac{1}{2\pi}\frac{(y-y')^\perp}{|y-y'|^2}\cdot 2(y-y') = 0 \tag{3.16}$$

and

$$\begin{aligned} II &= -\frac{1}{2\pi}\frac{(y-y_*)^\perp}{|y-y_*'|^2}\cdot 2y - \frac{1}{2\pi}\frac{(y'-y_*)^\perp}{|y'-y_*|^2}\cdot 2y' \\ &= \frac{1}{\pi}\left(\frac{y_*'^\perp\cdot y}{|y-y_*'|^2}+\frac{y_*^\perp\cdot y'}{|y'-y_*|^2}\right) = 0 \end{aligned}$$

because of

$$|y-y_*'| = |y'-y_*|, \quad y_*'^\perp\cdot y = -y_*^\perp\cdot y'.$$

Then we obtain

$$\frac{dI}{ds} = 4m(x_0)+I, \quad I(s) = \langle A(y), \zeta(dy,s)\rangle \le C$$

as in the case of $\beta = 0$ ([27] p.106) by $m(x_0) = \zeta(\mathbf{R}^2, s) \ge \varepsilon_0$, which implies

$$\lim_{s\uparrow+\infty} I(s) = +\infty,$$

a contradiction. □

3.8. Interior blowup point

Having (3.11) with $\mathcal{S}\subset\Omega$, we take $x_0\in\mathcal{S}$. There is a weak solution

$$\zeta = \zeta(dy,s)\in C_*(-\infty,+\infty;\mathcal{M}(\mathbf{R}^2))$$

to (3.10) generated by the classical solution $u = u(x,t)$ as in (3.8) and (3.9). By the ε-regularity (Lemma 3.1) and the monotonicity formula to u ((3.6)), if

$$\lim_{R\downarrow 0}\langle B_R(x_0), \zeta(dy,s)\rangle < \varepsilon_0,$$

this $\zeta(dy, s)$ is regular near x_0, where $s \in \mathbf{R}$ is arbitrary. The regular part of $\zeta(dy, s)$, therefore, takes the form

$$\zeta^s(dy, s) = \sum_{j=1}^{\ell(s)} \tilde{m}_j(s)\delta_{y_j(s)}(dy), \quad \tilde{m}(s) \geq \varepsilon_0,$$

with

$$\ell(s) \leq m(x_0)/\varepsilon_0, \quad |y_j(s)| \leq C$$

by the second inequality of (3.11) and Chebyshev's inequality.

Here we define $A(dy', s') \in C_*(-\infty, 0; \mathcal{M}(\mathbf{R}^2))$ by the scaling back of $\zeta(dy, s)$ ([25] p.317),

$$\zeta(dy, s) = e^{-s}A(dy', s'), \quad y' = e^{-s/2}y, \quad s' = -e^{-s}, \tag{3.17}$$

to reach the weak solution to

$$A_{s'} + \beta\nabla' \cdot (A\nabla'^{\perp}\Gamma * A) = \nabla' \cdot (\nabla' A - A\nabla'\Gamma * A) \quad \text{in } \mathbf{R}^2 \times (-\infty, 0), \tag{3.18}$$

satisfying $A(dy', s') \geq 0$, $A(\mathbf{R}^2, s') = m(x_0)$, and

$$A^s(dy', s') = \sum_{j=1}^{\ell(s')} \tilde{m}_j(s')\delta_{y'_j(s')}(dy'), \tag{3.19}$$

where $A^s(dy', s')$ denotes the singular part of $A(dy', s)$. Then we take the dilation (forward scaling) of $A(dy', s')$, taking $s'_0 < 0$ and $1 \leq j \leq \ell(s'_0)$ ([27] p.73), by

$$\tilde{A}_\mu(dy, s) = \mu^2 A(dy', s'), \quad y' = \mu y + y'_j(s'_0), \ s' = \mu^2 s + s'_0,$$

with $\mu > 0$. This $\tilde{A}_\mu(dy, s)$ is a weak solution to

$$\tilde{A}_{\mu s} + \beta\nabla \cdot (\tilde{A}_\mu \nabla^{\perp}\Gamma * \tilde{A}_\mu) = \nabla \cdot (\nabla \tilde{A}_\mu - \tilde{A}_\mu \nabla\Gamma * \tilde{A}_\mu) \text{ in } \mathbf{R}^2 \times (-\infty, -\mu^{-2}s'_0).$$

Let $\mu = \mu_k \downarrow 0$. Passing to a subsequence, we obtain

$$\tilde{A}_{\mu_k}(dy, s) \rightharpoonup \tilde{A}(dy, s) \quad \text{in } C_*(-\infty, +\infty; \mathcal{M}(\mathbf{R}^2))$$

with

$$\tilde{A}(dy, s) = \tilde{m}_j(s_0)\delta_0(dy), \tag{3.20}$$

where $\tilde{A}(dy, s)$ is a weak solution to

$$\tilde{A}_s + \beta\nabla \cdot (\tilde{A}\nabla^{\perp}\Gamma * \tilde{A}) = \nabla \cdot (\nabla \tilde{A} - \tilde{A}\nabla\Gamma * \tilde{A}) \quad \text{in } \mathbf{R}^2 \times (-\infty, +\infty). \tag{3.21}$$

Then we use the following property.

Lemma 3.10. *(Liouville property) If*

$$\tilde{A} = \tilde{A}(dy, s) \in C_*(-\infty, +\infty; \mathcal{M}(\mathbf{R}^2))$$

is a weak solution to (3.21) it holds that $\tilde{A}(\mathbf{R}^2, s) = m$ *with* $m = 8\pi$ *or* $m = 0$.

Proof. The proof is similar to the case of $\beta = 0$ ([27] p.90). In fact, the argument [12] for the classical solution is applicable to the above weak solution. First we show the result using the local second moment, assuming that $\tilde{A}(dy, 0)$ is concentrated on $y = 0$ as in

$$\langle |y|^2, \tilde{A}(dy, 0)\rangle \ll 1.$$

Then we use the scaling invariance of (3.21),

$$\tilde{A}_\mu(dy', s') = \mu^2 A(dy, s), \quad y = \mu y', \quad s = \mu^2 s'$$

to remove this assumption. As for the Euler term to execute the first step, we apply the argument used for the proof of Lemma 3.9, noting (3.16). □

By (3.20) and (3.21), we obtain $\tilde{m}_j(s_0) = 8\pi$, and therefore,

$$A^s(dy', s') = 8\pi \sum_{j=1}^{\ell(s')} \delta_{y'_j(s')}(dy')$$

in (3.19). Thus we obtain

$$\zeta^s(dy, s) = 8\pi \sum_{j=1}^{\ell(s)} \delta_{y_j(s)}(dy), \ |y_j(s)| \leq C, \quad -\infty < s < +\infty. \tag{3.22}$$

3.9. Improved ε-regularity

The following proposition is proven similarly to the case of $\beta = 0$. Indeed, the key inequality

$$\frac{dJ}{dt} + 3J^{3/2} \leq C_R, \quad J = \int_{\Omega\cap B(x_0,R)} u(\log u - 1) + 1 \, dx$$

is derived similarly by the proof of Lemma 3.2 for the Euler term, which implies

$$J(t) \leq t^{-2}, \quad 0 < t \leq t_1, \ t_1^{-3} = C_R.$$

In fact, the constant $C = C_\tau$ there does not depend on $\|u_0\|_{L^\infty(\Omega\cap B(x_0,R))}$ by this inequality. This technique is used in [23]. See [25] pp.250–255 for the Neumann boundary condition to the Poisson part and [27] pp.85–87 for $x_0 \in \Omega$, which is sufficient for our later argument.

Proposition 3.4. *In (1.1)–(1.3), there are* $0 < R_0 \ll 1$, $0 < \varepsilon_0 \ll 1$, *and* $0 < t_0 \ll 1$, *such that if*

$$\|u_0\|_{L^1(\Omega\cap B(x_0,R))} < \varepsilon_0, \quad 0 < R \leq R_0, \ x_0 \in \overline{\Omega}$$

any $\tau \in (0, \min\{t_0, T\})$ *admits* $C = C_\tau > 0$ *such that*

$$\sup_{\tau\leq t<t_0} \|u(\cdot, t)\|_{L^\infty(\Omega\cap B(x_0,R/2))} \leq C.$$

Applying scaling invariance and the monotonicity formula, Proposition 3.4 takes the form of the following lemma, see [27] p.87 for $\beta = 0$.

Lemma 3.11. *(Improved ε-regularity) There are ε_0, σ_0, and C such that if $u = u(dx,t)$ is a weak solution to*

$$u_t + \beta\nabla\cdot u\nabla^{\perp}\Gamma * u = \nabla\cdot(\nabla u - u\nabla\Gamma * u) \quad \text{in } \mathbf{R}^2\times(-T,T)$$

generated by the classical solution and if

$$\langle B(x_0,2R), u_0(dx)\rangle < \varepsilon_0, \quad u_0 = u(dx,0),$$

it holds that

$$\sup_{t\in[-\sigma_0R^2,\sigma_0R^2]\cap(-T,T)} \|u(\cdot,t)\|_{L^\infty(B(x_0,R))} \le CR^{-2}.$$

3.10. Residual vanishing

Lemma 3.11 implies the following fact for the weak solution $\zeta = \zeta(dy,s)$ to (3.10) formed by (3.9) ([27] p.91).

Lemma 3.12. *It holds that*

$$\zeta(B(y_0,2r),s) < \varepsilon_0 \quad\Rightarrow\quad \|\zeta(\cdot,s)\|_{L^\infty(B(x_0,r))} \le Cr^{-2}.$$

By (3.11) and Chebyshev's inequality, we obtain

$$\|\zeta(\cdot,s)\|_{L^\infty(\mathbf{R}^2\setminus B_R)} \le C, \quad -\infty < s < +\infty \tag{3.23}$$

for $R \gg 1$. This inequality actually follows from the uniform estimate in k,

$$\|z_k(\cdot,s)\|_{L^\infty(\mathbf{R}^2\setminus B_R)} \le C, \quad -\infty < s < +\infty \tag{3.24}$$

ensured by the assumption in Lemma 3.12.

Now we complete the following.

Proof of Theorem 1.1. We have only to show $m(x_0) \in 8\pi\mathbf{N}$, which is derived from $\zeta^{ac}(dy,s) = 0$ by (3.22), where $\zeta^{ac}(dy,s)$ denotes the absolutely continuous part of $\zeta(dy,s)$. In fact, the term $\frac{|y|^2}{4}$ in (3.10) carries $\zeta^{ac}(dy,s)$ to $y=\infty$, while $\zeta^{ac}(dy,s)$ is thin near $y=\infty$ by the second equality of (3.11). The result then follows as $s\uparrow+\infty$, where $\mathbf{R}^2\cup\{\infty\}$ denotes the one point compactification of $\mathbf{R}^2$.

To realize this idea, we use the outer second moment, which satisfies

$$\frac{d}{ds}\left\langle \left(\frac{|y|^2}{R^2}-1\right)_+, \zeta(dy,s)\right\rangle \ge 0, \quad -\infty < s < +\infty \tag{3.25}$$

for $R \gg 1$. In fact, we have

$$\int_{\Omega_s} |\nabla_y^{\perp} G_s(y,y')z_k(y',s)|dy' \le C, \quad |y| \ge 2R,$$

similarly to

$$\int_{\Omega_s} |\nabla_y G_s(y,y')z_k(y',s)|dy' \le C, \quad |y| \ge 2R$$

by (3.24). Then (3.25) follows as in the case of $\beta = 0$ ([27] pp.92–94).

This inequality implies

$$\lim_{s\uparrow+\infty}\left\langle (\frac{|y|^2}{R^2}-1)_+, \zeta(dy,s)\right\rangle = +\infty$$

if

$$\left\langle (\frac{|y|^2}{R^2}-1)_+, \zeta(dy,s_0)\right\rangle > 0, \quad -\infty < \exists s_0 < +\infty,$$

which contradicts (3.11). We thus obtain

$$\zeta(dy,s) = 0 \quad \text{in } (\mathbf{R}^2 \setminus B_R)\times(-\infty,+\infty),$$

and hence $\zeta^{ac}(dy,s) \equiv 0$ by (3.23) and the unique continuation theorem for parabolic equations. □

Remark 3.1. *The residual vanishing property established here is analogous to similar results in the blowup analysis of Keller-Segel systems [3]. The key difference lies in the treatment of the Euler term, which requires careful analysis of the rotational components in the subcollapse dynamics, as demonstrated below in our collision analysis in Proposition 3.5.*

3.11. Subcollapse dynamics

Because the bound C in (3.22) is independent of $s_k \uparrow +\infty$ or the selection of its subsequence, we obtain a Type II blowup rate for any $x_0 \in \mathcal{S}$:

$$\lim_{t\uparrow T}(T-t)\|u(\cdot,t)\|_{L^\infty(B(x_0,b(T-t)^{1/2})} = +\infty, \quad \exists b > 0. \tag{3.26}$$

If

$$\lim_{t\uparrow T}\mathcal{F}(u(\cdot,t)) > -\infty$$

Inequality (3.26) is improved as

$$\lim_{t\uparrow T}(T-t)\|u(\cdot,t)\|_{L^\infty(B(x_0,b(T-t)^{1/2})} = +\infty, \forall b > 0$$

for any $x_0 \in \mathcal{S}$. The proof is the same as in the case $\beta = 0$ ([27] pp.121–123) and is omitted.

For $\beta = 0$, actual rate of convergence is shown by [10, 13] for radially symmetric case. Nonradial stable and unstable blowup patterns with various blowup rates are also observed by [5] for this case in the context of the collision of subcollapses.

We have shown

$$\zeta(dy,s) = 8\pi\sum_{j=1}^{\ell}\delta_{y_j(s)}(dy), \quad |y_j(s)| \le C, \ -\infty < s < +\infty$$

in (3.9). Then its scaling back $A(dy',s')$, defined by (3.17), takes the form

$$A(dy',s') = 8\pi\sum_{j=1}^{\ell}\delta_{y_j'(s')}(dy'), \quad |y_j'(s')| \le C(-s')^{1/2}, \quad s' < 0.$$

Henceforth, we call these

$$8\pi\delta_{y_j'(s')}(dy'), \quad 1 \le j \le \ell,$$

the subcollapses.

This $A(dy',s')$ is a weak solution to (3.18), and we can apply the method of local second moment to follow the movement of $y'_j = y'_j(s')$, $1 \leq j \leq \ell$, as in the case of $\beta = 0$ ([27] pp.120 and 121). Then we obtain

$$(y'_j - a) \cdot \frac{dy'_j}{ds'} + 8\pi\beta(y'_j - a) \cdot \nabla'^{\perp}_j H^0_\ell(y'_1, \cdots, y'_\ell) = 8\pi(y'_j - a) \cdot \nabla'_j H^0_\ell(y'_1, \cdots, y'_\ell)$$

for any $a \in \mathbf{R}^2$, and hence

$$\frac{dy'_j}{ds'} + 8\pi\beta\nabla'^{\perp}_j H^0_\ell(y'_1, \cdots, y'_\ell) = 8\pi\nabla'_j H^0_\ell(y'_1, \cdots, y'_\ell), \quad 1 \leq \ell \leq j, \tag{3.27}$$

similarly, where

$$H^0_\ell(y'_1, \cdots, y'_\ell) = \begin{cases} \sum_{1\leq i<j\leq \ell} \Gamma(y'_i - y'_j), & \ell \geq 2 \\ 0, & \ell = 1. \end{cases}$$

It thus holds that

$$A(dy', s') = 8\pi\delta_0(dy'), \quad -\infty < s' < 0$$

if $\ell = 1$. There then arises the following fact.

Proposition 3.5. *(Collision of subcollapses) If $\ell = 2$, it holds that*

$$A(dy', s') = 8\pi\left(\delta_{2(-s')^{1/2}e^{\imath(-\frac{\beta}{2}\log(-s')+c)}}(dy') + \delta_{-2(-s')^{1/2}e^{\imath(-\frac{\beta}{2}\log(-s')+c)}}(dy')\right)$$

and hence

$$\zeta(dy, s) = 8\pi\left(\delta_{2e^{\imath\frac{\beta}{2}(s+c)}}(dy) + \delta_{-2e^{\imath\frac{\beta}{2}(s+c)}}(dy)\right), \quad -\infty < s < +\infty,$$

where $c \in \mathbf{R}$ is a constant.

Proof. We have

$$\begin{aligned} \frac{dy_1}{ds'} + 8\pi\beta\nabla'^{\perp}\Gamma(y'_1 - y'_2) &= 8\pi\nabla'\Gamma(y'_1 - y'_2) \\ \frac{dy_2}{ds'} + 8\pi\beta\nabla'^{\perp}\Gamma(y'_2 - y'_1) &= 8\pi\nabla'\Gamma(y'_2 - y'_1), \end{aligned}$$

which means

$$\begin{aligned} \frac{dy'_1}{ds'} + 4\pi\beta\frac{(y'_1 - y'_2)^{\perp}}{|y'_1 - y'_2|^2} &= -4\frac{y'_1 - y'_2}{|y'_1 - y'_2|^2} \\ \frac{dy'_2}{ds'} + 4\pi\beta\frac{(y'_2 - y'_1)^{\perp}}{|y'_2 - y'_1|^2} &= -4\frac{y'_2 - y'_1}{|y'_2 - y'_1|^2} \end{aligned}$$

with

$$\lim_{s'\uparrow 0} y'_1(s') = \lim_{s'\uparrow 0} y'_2(s') = 0.$$

Because

$$\frac{d}{ds'}(y'_1 + y'_2) = 0,$$

we can put

$$y' = y_1' = -y_2'.$$

It then follows that

$$\frac{dy'}{ds'} + 2\beta\frac{y'^{\perp}}{|y'|^2} = -2\frac{y'}{|y'|^2}, \quad \lim_{s'\uparrow 0} y'(s') = 0. \tag{3.28}$$

Writing

$$y' = r\omega, \quad r = |y'|,$$

we obtain

$$\dot{r}\omega + r\dot{\omega} + 2\beta\frac{\omega^{\perp}}{r} = -2\frac{\omega}{r}. \tag{3.29}$$

by (3.28). Because $|\omega|^2 = 1$, it holds that $\dot{\omega}\cdot\omega = 0$, and therefore,

$$\dot{\omega} \parallel \omega^{\perp}.$$

Equality (3.29) thus splits into

$$\dot{\omega} + 2\beta\frac{\omega^{\perp}}{r^2} = 0, \quad \dot{r} = -\frac{2}{r}. \tag{3.30}$$

By the second equalities of (3.29) and (3.30), we obtain

$$r = 2(-s')^{1/2}.$$

Then we put $\omega = e^{\imath\theta}$. It holds that $\omega^{\perp} = \imath\omega$, and therefore,

$$\dot{\theta} = -\frac{2\beta}{r^2} = \frac{\beta}{2s'}$$

from the first equality of (3.30). It thus holds that

$$\theta = -\frac{\beta}{2}\log(-s') + c, \quad c \in \mathbf{R},$$

and hence

$$y'(s') = 2(-s')^{1/2}e^{\imath(-\frac{\beta}{2}\log(-s')+c)}.$$

Then the result follows. □

4. Blowup in infinite time

4.1. *Generation of the weak solution*

Here we assume

$$T = +\infty, \ \exists t_k \uparrow +\infty, \ \lim_{k\to\infty} \|u(\cdot, t_k)\|_{\infty} = +\infty \tag{4.1}$$

in (1.1)–(1.3). Then, passing to a subsequence, we obtain

$$u(x, t + t_k)dx \rightharpoonup \exists\mu(dx, t) \in C_*(-\infty, +\infty; \mathcal{M}(\overline{\Omega})), \tag{4.2}$$

and this $\mu(dx, t)$ is a weak solution. By Proposition 3.4 it holds that

$$\mu(dx, t) = \sum_{x \in \mathcal{S}_t} m_t(x_0)\delta_{x_0}(dx) + f(x, t)dx, \tag{4.3}$$

where

$$m_t(x_0) \geq \varepsilon_0 > 0, \quad 0 \leq f = f(\cdot, t) \in L^1(\Omega)$$

and

$$\mathcal{S}_t = \{x_0 \in \overline{\Omega} \mid \exists x_k \to x_0, \ \lim_{k\to\infty} u(x_k, t + t_k) = +\infty\}$$

for each t [23]. By the parabolic regularity and unique continuation theorem, it also holds that

$$0 < f = f(\cdot, t) \in C(\overline{\Omega} \setminus \mathcal{S}_t).$$

4.2. Exclusion of boundary blowup

The following property is proven similarly to the case of blowup in finite time (Lemma 3.9).

Lemma 4.1. *It holds that $\mathcal{S}_t \subset \Omega$ in (4.3).*

Proof. We assume $x_0 \in \partial\Omega \cap \mathcal{S}_T$ and put

$$\zeta(dy, s) = (T - t)\mu(dx, t), \ y = (x - x_0)/(T - t)^{1/2}, \ s = -\log(T - t)$$

for $t < T$. We take zero extension of $\zeta(dy, s)$ where it is not defined. Then, any $s_k \uparrow +\infty$ admits a subsequence denoted by the same symbol such that

$$\zeta(dy, s + s_k) \rightharpoonup \exists\tilde{\zeta}(dy, s) \quad \text{in } C_*(-\infty, +\infty; \mathcal{M}(\mathbf{R}^2)).$$

It holds that

$$\tilde{\zeta}(\mathbf{R}^2, s) = m_T(x_0) \geq \varepsilon_0$$

by (4.3) and also

$$\langle |y|^2, \tilde{\zeta}(dy, s)\rangle \leq C, \ -\infty < s < +\infty$$

similarly to Lemmas 3.7 and 3.8. We have $\tilde{\zeta}(\cdot, s) \subset \overline{L}$ and this $\tilde{\zeta}(dy, s)$ is a weak solution to

$$z_s + \beta\nabla \cdot (z\nabla^\perp E * z) = \nabla \cdot (\nabla z - z\nabla(E * z + \frac{|y|^2}{4})) \text{ in } L \times (-\infty, +\infty)$$
$$\left.\frac{\partial z}{\partial \nu} - z(\frac{\partial}{\partial \nu}(E * z + \frac{|y|^2}{4}) + \beta\frac{\partial}{\partial \tau}(E * z))\right|_{\partial L} = 0,$$

where $L \subset \mathbf{R}^2$, where L is a half-space satisfying $0 \in \overline{L}$. Then we obtain a contradiction by Lemma 3.9. □

4.3. Collapse mass quantization

The following property is also shown similarly to the case of blowup in infinite time (Lemma 3.10).

Lemma 4.2. *It holds that* $m_t(x_0) = 8\pi$ *in (4.3).*

Proof. We assume $x_0 = 0 \in \mathcal{S}_0$ without loss of generality. Then we take the dilation of $\mu(dx, t)$,

$$\mu_\sigma(dx, t) = \mu^2 \mu(dx'.t'), \quad x' = \sigma x, \ t' = \sigma^2 t,$$

where $\sigma > 0$ is a constant. Any $\sigma_k \downarrow 0$ admits a subsequence, denoted by the same symbol, such that

$$\mu_{\sigma_k}(dx, t) \rightharpoonup \tilde{\mu}(dx, t) \quad \text{in } C_*(-\infty, +\infty; \mathcal{M}(\mathbf{R}^2)),$$

and this $\tilde{\mu}(dx, t)$ is a weak solution to (3.21). It holds that $\tilde{\mu}(\mathbf{R}^2, 0) = m_0(0)$ by (4.3), and therefore, $m_0(0) = 8\pi$ by Lemma 3.10. We thus obtain $m_t(x_0) = 8\pi$ for any $x_0 \in \mathcal{S}_t$ and $-\infty < t < +\infty$. Similarly to the case of $\beta = 0$ ([27] pp.111–114), $\ell(t) \in \mathbf{N}$ is locally constant, and hence is independent of t: $\ell(t) = \ell$.

4.4. Collapse dynamics—rough estimate

The singular part of $\mu(dx, t)$ in (4.2) thus takes the form of

$$\mu^s(dx, t) = 8\pi \sum_{i=1}^{\ell(t)} \delta_{x_i(t)}(dx). \tag{4.4}$$

Then we obtain the following fact.

Lemma 4.3. *The number* $\ell(t) = \sharp \mathcal{S}_t$ *is independent of* t*, denoted by* ℓ*. It holds that*

$$\overline{\mathcal{O}} \subset \Omega^\ell \setminus D_\ell, \tag{4.5}$$

where D_ℓ *is the diagonal set defined by (1.22) and*

$$\mathcal{O} = \{x(t) \mid -\infty < t < +\infty\}, \quad x(t) = (x_1(t), \dots, x_\ell(t)) \tag{4.6}$$

for $x_i(t)$*,* $1 \le i \le \ell$ *defined by (4.4). It holds, furthermore, that*

$$|\dot{x}(t)| \le C, \quad -\infty < t < +\infty. \tag{4.7}$$

Proof. Given $x_0 \in \mathcal{S}_{t_0} \subset \Omega$, we have $0 < r, \delta \ll 1$ such that

$$\sharp(\mathcal{S}_t \cap B(x_0, 4r)) \le 1, \quad |t - t_0| < \delta$$

by $\mu(dx, t) \in C_*(-\infty, +\infty; \mathcal{M}(\overline{\Omega}))$ and (4.4). Then we take

$$\xi = |x - x_0|^2 \varphi_{x_0, 2r},$$

where $\varphi_{x_0,2r}$ is the smooth cutoff function around $x = x_0$. It follows that

$$|\langle \Delta\xi, \mu(dx, t)\rangle| \le C, \ |\langle \rho_\xi, \nu(t)\rangle| \le C, \ |\langle \rho_\xi^\perp, \nu(t)\rangle| \le C, \quad |t - t_0| < \delta$$

for ρ_ξ and $\rho_\xi^\perp$ defined by (1.15), where $v(\cdot,t)$ denotes the multiplicated operator associated with $\mu(dx,t)$. Then, similarly to the case of $\beta = 0$ ([27] pp.111–114), $\ell(t) \in \mathbf{N}$ is locally constant, and hence is independent of t, $\ell(t) = \ell$, and it holds that (4.7).

To show (4.5), we assume the contrary,

$$\lim_{k\to\infty} \mathrm{dist}(x_j(t_k), \partial\Omega) = 0 \tag{4.8}$$

for some $1 \leq j \leq \ell$, or

$$\lim_{k\to\infty} |x_i(t_k) - x_j(t_k)| = 0 \tag{4.9}$$

for some $1 \leq i < j \leq \ell$, with $t_k \to \pm\infty$. Then, passing to a subsequence, we obtain

$$\mu(dx, t+t_k) \rightharpoonup \exists\tilde{\mu}(dx,t) \quad \text{in } C_*(-\infty,+\infty; \mathcal{M}(\overline{\Omega})).$$

This $\tilde{\mu}(dx,t)$ is a weak solution to (1.1) and (1.2), and therefore, its singular part takes the form of

$$\tilde{\mu}^s(dx,t) = 8\pi \sum_{j=1}^{\tilde{\ell}} \delta_{\tilde{x}_j(t)}(dx).$$

We have $\tilde{x}_j(t) \in \Omega$, $1 \leq j \leq \tilde{\ell}$, by Lemma 4.1, which contradicts (4.8). It also holds that $\tilde{x}_i(t) \neq \tilde{x}_j(t)$ for $i \neq j$ by Lemma 4.2, which contradicts (4.9). We thus obtain (4.5). □

4.5. Residual vanishing

Here we show the following fact.

Lemma 4.4. *It holds that $\mu^{ac}(dx,t) = 0$, where $\mu^{ac}(dx,t) = f(x,t)dx$ denotes the absolutely continuous part of $\mu(dx,t)$.*

Proof. We fix $1 \leq i \leq \ell$ and write $x_i = x_i(t)$ in (4.4). We also have

$$u_k(\cdot,t) = u(\cdot, t+t_k), \quad v_k(\cdot,t) = v(\cdot,t+t_k)$$

in (1.1) and (1.2). By Lemma 4.3, there is $0 < r \ll 1$ such that

$$B(x_i,r) \subset \Omega, \ B(x_i,r) \cap \mathcal{S}_t = \{x_i\}, \quad -\infty < t < +\infty.$$

Because $B(x_i,r)$ moves uniformly with the velocity $\dot{x}_i$, we obtain

$$\frac{d}{dt}\int_{B(x_i,r)} |x-x_i|^2 u_k = \int_{B(x_i,r)} \frac{\partial}{\partial t}(|x-x_i|^2 u_k) + \dot{x}_i \cdot \nabla(|x-x_i|^2 u_k)\, dx$$

by Liouville's theorem, which implies

$$\frac{d}{dt}\int_{B(x_i,r)} |x-x_i|^2 u_k = \int_{B(x_i,r)} |x-x_i|^2 u_{kt} + \dot{x}_i \cdot |x-x_i|^2 \nabla u_k \, dx.$$

Here we obtain

$$\int_{B(x_i,r)} |x-x_i|^2 u_{kt} = \int_{B(x_i,r)} |x-x_i|^2 \nabla\cdot(\nabla u_k - u_k(\nabla v_k + \beta\nabla^\perp v_k))$$

$$
\begin{aligned}
&= \int_{\partial B(x_i,r)} |x-x_i|^2 (\frac{\partial u_k}{\partial \nu} - u_k(\frac{\partial v_k}{\partial \nu} + \beta \nabla^{\perp} v_k)) \\
&\quad - \int_{B(x_i,r)} 2(x-x_i)\cdot(\nabla u_k - u_k(\nabla v_k + \beta\nabla^{\perp} v_k)) \\
&= r^2 \int_{\partial B(x_i,r)} \frac{\partial u_k}{\partial \nu} - u_k(\frac{\partial v_k}{\partial \nu} + \beta \frac{\partial v_k}{\partial \tau})\, ds - 2\int_{\partial B(x_i,r)} (x-x_i)\cdot \nu u_k \\
&\quad + \int_{B(x_i,r)} 4u_k + 2(x-x_i)\cdot u_k(\nabla v_k + \beta \nabla^{\perp} v_k) \\
&\le r^2 \int_{B(x_i,r)} \nabla\cdot(\nabla u_k - u_k(\nabla v_k + \beta\nabla^{\perp} v_k)) \\
&\quad + \int_{B(x_i,r)} 4u_k + 2(x-x_i)\cdot u_k(\nabla v_k + \beta \nabla^{\perp} v_k) \\
&= r^2 \int_{B(x_i,r)} u_{kt} + 4u_k + 2(x-x_k)\cdot u_k(\nabla v_k + \beta\nabla^{\perp} v_k)\, dx
\end{aligned}
$$

and also

$$
\begin{aligned}
&\int_{B(x_i,r)} \dot{x}_i \cdot |x-x_i|^2 \nabla u_k \\
&\quad = \int_{\partial B(x_i,r)} (\cdot x_i\cdot \nu)|x-x_i|^2 u_k - \int_{B(x_i,r)} 2(x-x_i)\cdot \dot{x}_i u_k \\
&\quad = r^2 \int_{\partial B(x_i,r)} (\dot{x}_i\cdot\nu) u_k - \int_{B(x_i,r)} 2(x-x_i)\dot{x}_i \cdot u_k \\
&\quad = \int_{B(x_i,r)} r^2 \dot{x}_i \cdot \nabla u_k - 2(x-x_i)\cdot \dot{x}_i v_k\, dx.
\end{aligned}
$$

It also holds that

$$
\frac{d}{dt}\int_{B(x_i,r)} u_k = \int_{B(x_i,r)} u_{kt} + \dot{x}_i\cdot\nabla u_k\, dx
$$

and therefore,

$$
\begin{aligned}
&\frac{d}{dt}\int_{B(x_i,r)} (|x-x_i|^2 - r^2) u_k \\
&\le \int_{B(x_i,r)} 4u_k + 2(x-x_k)\cdot u_k(\nabla v_k + \beta\nabla^{\perp} v_k) - 2(x-x_i)\cdot \dot{x}_i u_k\, dx.
\end{aligned}
$$

Here we obtain

$$
v_k = v_k^0 + v_k^1 + v_k^2
$$

for

$$
\begin{aligned}
v_k^0(x,t) &= \int_{B(x_i,r)} \Gamma(x-x')u_k(x',t)dx' \\
v_k^1(x,t) &= \int_{B(x_i,r)} K(x,x')u_k(x',t)dx'
\end{aligned}
$$

$$v_k^2(x,t) = \int_{\Omega\setminus B(x_i,r)} G(x,x')u_k(x',t)dx'$$

by (1.16), and it holds that

$$|\nabla v_k^1(x,t)| \leq C, \ |\nabla v_k^2(x,t)| \leq C$$

([27] pp.118 and 119). It also holds that

$$\begin{aligned}
2\int_{B(x_i,r)} (x-x_i)\cdot u_k(\nabla v_k^0 + \beta\nabla^\perp v_l^0) &= 2\cdot \iint_{B(x_i,r)\times B(x_i,r)} (x-x_i) \\
&\quad \cdot\left(-\frac{1}{2\pi}\frac{x-x'}{|x-x'|^2} - \frac{\beta}{2\pi}\frac{(x-x')^\perp}{|x-x'|^2}\right)u_k(x,t)u_k(x',t)dxdx' \\
&= 2\iint_{B_r\times B_r} x\cdot\left(-\frac{1}{2\pi}\frac{x-x'}{|x-x'|^2} - \frac{\beta}{2\pi}\frac{(x-x')^\perp}{|x-x'|^2}\right)\tilde{u}_k(x,t)\tilde{u}_k(x',t)dxdx'
\end{aligned}$$

for

$$\tilde{u}_k(x,t) = u_k(x_i+x,t),$$

and therefore,

$$\begin{aligned}
2\int_{B(x_i,r)} (x-x_i)\cdot u_k(\nabla v_k^0 + \beta\nabla^\perp v_k^0) &= \iint_{B_r\times B_r} (x-x') \\
&\quad \cdot\left(-\frac{1}{2\pi}\frac{x-x'}{|x-x'|^2} - \frac{\beta}{2\pi}\frac{(x-x')^\perp}{|x-x'|^2}\right)\tilde{u}_k(x,t)\tilde{u}_k(x',t)dxdx' \\
&= -\frac{1}{2\pi}\left(\int_{B_r}\tilde{u}(x,t)dx\right)^2 = -\frac{1}{2\pi}\left(\int_{B(x_i,r)} u_k dx\right)^2.
\end{aligned}$$

Using (4.7), we thus obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{B(x_i,r)} (|x-x_i|^2 - r^2)u_k \\
&\quad \leq 4\int_{B(x_i,r)} u_k - \frac{1}{2\pi}\left(\int_{B(x_i,r)} u_k\right)^2 + C\int_{B(x_i,r)} |x-x_i|u_k.
\end{aligned}$$

As $k\to\infty$, it follows that

$$\begin{aligned}
&\frac{d}{dt}\int_{B(x_i,r)} (|x-x_i|^2 - r^2)f \\
&\leq 4(8\pi + \int_{B(x_i,r)} f) - \frac{1}{2\pi}(8\pi + \int_{B(x_i,r)} f)^2 + C\int_{B(x_i,r)} |x-x_i|f \\
&\leq -4\int_{B(x_i,r)} f + C\int_{B(x_i,r)} |x-x_i|f
\end{aligned}$$

in the sense of distributions in t, and therefore,

$$\frac{d}{dt}\int_{B(x_i,r)} (|x-x_i|^2 - r^2))$$

$$\leq -2\int_{B(x_i,r)} f + C'\int_{B(x_i,r)} |x-x_i|^2 f$$
$$= C'\int_{B(x_i,r)} (|x-x_i|^2 - r^2)f + (C'r^2-2)\int_{B(x_i,r)} f.$$

We end up with

$$\frac{dI}{dt} \leq 0, \ -\infty < t < +\infty, \quad I = \int_{B(x_i,r)} (|x-x_i|^2 - r^2)f \leq 0$$

for $0 < r \ll 1$, while $I \geq -C$ is obvious. Then it follows that $I \equiv 0$, and therefore, $f \equiv 0$ by the unique continuation theorem for parabolic equations. □

4.6. Collapse dynamics—recursive hierarchy

We can show the following fact similarly to the case of $\beta = 0$ ([27] pp.120 and 121) as well as that of the blowup in finite time (3.27):

$$\frac{dx_i}{dt} + 8\pi\beta\nabla_{x_i}^{\perp} H_\ell(x_1,\cdots,x_\ell) = 8\pi\nabla_{x_i} H_\ell(x_1,\cdots,x_\ell), \quad 1 \leq i \leq \ell. \tag{4.10}$$

Now we complete the following proof.
Proof of Theorem 1.2. We have readily shown that

$$\mu(dx,t) = 8\pi\sum_{i=1}^{\ell}\delta_{x_i(t)}(dx), \quad 1 \leq i \leq \ell$$

for $\mu(dx,t) \in C_*(-\infty,+\infty;\mathcal{M}(\overline{\Omega})$ generated by (4.2). Hence it follows that

$$\lambda = \mu(\overline{\Omega},t) = 8\pi\ell.$$

The orbit $\mathcal{O}$ defined by (4.6) satisfies (4.5) and (4.7), and it holds that (4.10) in the sense of distributions. Thus $x_i = x_i(t)$, $1 \leq i \leq \ell$, is continuously differentiable, and the ω-limit and α-limit sets, denoted by $\omega(u_0)$ and $\alpha(u_0)$, respectively, are nonempty, compact, connected, and invariant under the flow (4.10) ([9]). It also holds that

$$\begin{aligned}\frac{d}{dt}H_\ell(x_1,\cdots,x_\ell) &= \sum_{i=1}^{\ell}\nabla_{x_i}H_\ell(x_1,\cdots,x_\ell)\frac{dx_i}{dt} \\ &= 8\pi\sum_{i=1}^{\ell}|\nabla_{x_i}H_\ell(x_1,\cdots,x_\ell)|^2 \geq 0, \quad -\infty < t < +\infty,\end{aligned}$$

and hence $-H_\ell = H_\ell(x_1,\cdots,x_\ell)$ acts as a Lyapunov function to (4.10). By the LaSalle principle, therefore, this H_ℓ is constant on $\omega(u_0)$ and $\alpha(u_0)$, which are contained in the set of critical points of H_ℓ, denoted by K. The orbit $\mathcal{O}$ is thus a connecting orbit, and in particular, $K \neq \emptyset$. The proof is complete. □

5. Conclusions

In this paper, we have investigated the blowup dynamics of the Euler-Smoluchowski-Poisson equation, a system that models the relaxation of point vortices. Our primary goal was to determine whether the quantized blowup mechanism and recursive hierarchy, well-established for the Smoluchowski-Poisson equation ($\beta = 0$), persist in the presence of the Euler term ($\beta \neq 0$), and to clarify the role of this term in the dynamics.

Our main results, Theorems 1.1 and 1.2, confirm the robustness of these fundamental structures. We have proven that when blowup occurs in finite time, the collapsing masses are quantized as integer multiples of 8π. Conversely, blowup in infinite time is shown to occur only when the total mass is an integer multiple of 8π and is contingent upon the existence of a critical point of the point vortex Hamiltonian. These findings demonstrate that the underlying geometric and analytical framework governing blowup is preserved despite the additional complexity introduced by the rotational flux.

Crucially, we have also elucidated the specific role of the Euler term. While it does not alter the quantized masses or the overall hierarchical structure, it fundamentally shapes the microscopic dynamics of the subcollapses. As demonstrated in Proposition 3.5, the Euler term introduces a rotational component to the motion of these subcollapses during both finite-time and infinite-time blowup. In the case of a two-point collapse, this manifests as a specific spiraling trajectory, a distinct departure from the dynamics observed when $\beta = 0$.

This work opens several avenues for future research. The construction of blowup solutions with the precise rates and patterns described by our subcollapse dynamics remains a significant open problem, particularly for non-radial configurations. Extending the analysis to systems with Neumann-type boundary conditions for the Poisson part, where boundary blowup and a quantization of 4π are anticipated, presents another challenging and important direction. Finally, exploring the implications of these results for the statistical mechanics of vortices in other physical contexts, such as astrophysics, could provide further insights into the behavior of these complex systems.

Author contributions

Both authors contributed equally to this work. Conceptualization, E. E. and T. S.; methodology, E. E. and T. S.; formal analysis, E. E. and T. S.; investigation, E. E. and T. S.; writing—original draft preparation, E. E. and T. S.; writing—review and editing, E. E. and T. S. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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