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Research article

Spectral radius of biased random walks on regular trees

He Song and Meng Liu*

School of Mathematics and Statistics, Huaiyin Normal University, Huai'an 223300, China

* **Correspondence:** Email: liumeng@hytc.edu.cn.

Abstract: For the biased random walk on d-regular trees, we obtain the spectral radius, first return probability and n-step transition probability, speed. We prove that the spectral radius depends continuously on the bias parameter and is strictly increasing with respect to it, while the speed remains strictly positive and decreases monotonically as the bias increases.

Keywords: biased random walk; spectral radius; continuous
Mathematics Subject Classification: Primary 05C81, 60G50, 60J10; Secondary 05C63, 05C80, 60C05

1. Introduction

Biased random walks, defined by directionally weighted transition probabilities that induce a systematic drift on an underlying structure, constitute a fundamental model in probability theory, statistical physics, computer science, and ecology. They effectively capture asymmetric transport phenomena in physical systems [4, 14, 15] and disordered media [17, 20, 34], directional movement and foraging behavior in ecology [2, 3], and guided exploration and sampling mechanisms in Monte Carlo algorithms [11] and refined in a series of subsequent studies [22, 31, 37], thereby sustaining broad interdisciplinary interest. The specific model, biased random walk RW_λ, where a parameter $\lambda > 0$ quantifies the strength of a bias away from a fixed origin o, serves as a canonical framework for studying how drift influences long-term asymptotic behavior, including escape speed, return probabilities, and spectral characteristics. Furthermore, this model has been extensively employed in the analysis of transport phenomena on percolation clusters [4, 10, 14] and random graph structures [23, 24, 33].

The rigorous study of RW_λ on graphs was profoundly advanced in the 1990s by Lyons [23–25], Pemantle, Peres, and collaborators [26, 27], who established fundamental connections between random walk transience and recurrence, graph growth rate, and geometric properties such as Hausdorff dimension. A key geometric determinant is the growth rate $\mathrm{gr}(G)$ of the graph G. In particular, for

Cayley graphs and regular trees, a sharp transition occurs at $\lambda = \mathrm{gr}(G)$: the walk is transient for $\lambda < \mathrm{gr}(G)$ and recurrent for $\lambda > \mathrm{gr}(G)$ [24, 25]. More recently, RW_λ has continued to attract significant attention, as seen in works such as [1, 5–7, 21]. The transience and recurrence of Galton–Watson trees conditioned on non-extinction were studied by Lyons [23, 24]. Two quantitative measures of central interest are the spectral radius $\rho_G(\lambda)$, defined as the exponential decay rate $\limsup_{n\to\infty} p_\lambda^{(n)}(o, o)^{1/n}$ of the return probability, and the speed $\mathcal{S}(\lambda) = \lim_{n\to\infty} |X_n|/n$, which measures the linear rate of escape. Understanding the exact dependence of $\rho_G(\lambda)$ and $\mathcal{S}(\lambda)$ on λ reveals a deep interplay between graph geometry and stochastic dynamics.

For the lamplighter group, Lyons et al., [26] proved that the speed of RW_λ is either zero or strictly positive depending on the underlying structural and bias conditions. Bergera et al., [9] investigated the monotonicity of the speed for biased random walks on multi-dimensional integer lattices and derived explicit criteria for determining when the speed is strictly positive. Peter explored the influence of the bias parameter λ on the properties of random walks, specifically their spectral radius and speed (see [30], Chapter 9). Song et al., [35] established the continuity of the spectral radius of RW_λ for free product graphs. Liu et al., [29] studied the large deviation principles and invariance principles for biased random walks on multi-dimensional integer lattices. Gantert et al., [19] gave an explicit formula for the speed of the biased random walk on spanning trees of the ladder graph. These developments are also closely connected to advances in spectral methods, including spectral clustering, as explored in works such as [12, 13, 39].

A long-standing question, posed by Lyons et al., [26, Questions], asks whether the speed $\mathcal{S}(\lambda)$ of RW_λ is well-defined and strictly positive for all parameters λ strictly between 1 and the growth rate $\mathrm{gr}(G)$ of a Cayley graph G. This inquiry, often called the Lyons–Pemantle–Peres monotonicity problem, highlights the need for a precise understanding of how the bias parameter influences the asymptotic dynamics. While explicit formulas for $\rho_G(\lambda)$ and $\mathcal{S}(\lambda)$ have been obtained in several special settings, such as integer lattices [34] and Galton–Watson trees [1, 34, 36], a complete picture for the highly symmetric yet non-amenable family of d-regular trees $\mathbb{T}_d$ has remained incomplete. A clear, closed-form description of these quantities, together with a rigorous analysis of their monotonicity and continuity properties, is essential both for intrinsic mathematical interest and for providing benchmarks in more complex disordered environments.

In this paper, we provide a comprehensive analysis of RW_λ on d-regular trees $\mathbb{T}_d$ ($d \geq 2$). We derive explicit expressions for the spectral radius, first-return probabilities, n -step transition probabilities, and the speed. Furthermore, we establish their key analytic properties, including continuity, monotonicity, and precise asymptotic decay rates. Our results not only fill a significant gap in the understanding of biased random walks on homogeneous trees but also offer an affirmative resolution of the Lyons–Pemantle–Peres monotonicity problem in this setting.

Our main results include the following. For $\lambda \in (0, d-1]$, the exact spectral radius of $\mathbb{T}_d$ is continuous and strictly increasing. Asymptotics for the even-step return probabilities revealing a universal $n^{-3/2}$ decay modulated by $\rho_{\mathbb{T}_d}(\lambda)^{2n}$. The speed formula $S(\lambda)$ for $\lambda \in (0, d-1)$, which is positive and strictly decreasing, thereby affirmatively resolving the Lyons–Pemantle–Peres monotonicity problem for regular trees.

These findings not only extend known results for symmetric random walks but also provide a benchmark for analogous studies of biased random walks on less regular structures, such as Galton-Watson trees or percolation clusters. The explicit dependence of $\rho_{\mathbb{T}_d}(\lambda)$ on both the branching number

$d-1$ and the bias λ underscores a competition between graph expansion and drift.

The remainder of this paper is organized as follows. Section 2 presents our main theorems. Section 3 provides the detailed proofs, employing generating functions, Catalan number enumerations, and the Darboux method for asymptotic analysis. Section 4 presents numerical simulations that validate the asymptotic formulas and lists several open problems and conjectures motivated by our results.

2. Main results

We now briefly introduce the necessary notation. Let $G = (V(G), E(G))$ be a locally finite, connected infinite graph with a distinguished root vertex o. For a vertex x, write $|x|$ for the graph distance from x to o. For any edge $e = \{u, v\}$ of G, u and v are its endpoints. Define the graph distance from e to o as $|e| = \min\{|u|, |v|\}$, where $|u|$ and $|v|$ are the graph distances from u and v to o, respectively. Let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. For each $n \in \mathbb{Z}_+$, we define the ball of radius n centered at o as

$$B_G(n) = \{x \in V(G) : \ |x| \le n\};$$

The boundary of $B_G(n)$,

$$\partial B_G(n) = \{x \in V(G) : \ |x| = n\}.$$

Let $M_n = |\partial B_G(n)|$ be the cardinality of $\partial B_G(n)$ for any $n \in \mathbb{Z}_+$. Define the growth rate of G as

$$\mathrm{gr}(G) = \liminf_{n\to\infty} \sqrt[n]{M_n}.$$

Since the sequence $\{M_n\}_{n=0}^{\infty}$ is submultiplicative, the limit $\mathrm{gr}(G) = \lim_{n\to\infty} \sqrt[n]{M_n}$ exists indeed. It is easy to see that for $\mathbb{T}_d$, $M_n = |\partial B_{\mathbb{T}_d}(n)| = d * (d-1)^{n-1}$. Hence, $\mathrm{gr}(\mathbb{T}_d) = d - 1$.

The biased random walk $RW_\lambda = (X_n)_{n\ge 0}$ is the nearest-neighbor Markov chain whose transition probabilities are determined by assigning conductance λ^{-n} to every edge at distance n from o (see below for the explicit form).

$$p(v,u) := p_\lambda(v,u) = \begin{cases} 1/d_v & \text{if } v = o, \\ \frac{\lambda}{d_v + (\lambda-1)d_v^-} & \text{if } u \in \partial B_G(|v|-1) \text{ and } v \neq o, \\ \frac{1}{d_v + (\lambda-1)d_v^-} & \text{otherwise.} \end{cases}$$

Here d_v is the degree of vertex v; and d_v^-, d_v^0 and d_v^+ are the numbers of edges connecting v to $\partial B_G(|v|-1)$, $\partial B_G(|v|)$ and $\partial B_G(|v|+1)$ respectively. Note

$$d_v^+ + d_v^0 + d_v^- = d_v, \ d_v^- \ge 1, \ d_o^- = d_o^0 = 0;$$

Its n -step transition probability is denoted by

$$p_\lambda^{(n)}(x,y) = \mathbb{P}_x(X_n = y),$$

where $\mathbb{P}_x$ is the law of the walk started at x. The spectral radius is defined as

$$\rho_G(\lambda) = \limsup_{n\to\infty} p_\lambda^{(n)}(o,o)^{1/n}.$$

Define

$$\tau_y^+ = \tau_y^+(\lambda) = \inf\{n \geq 1 |\ X_n = y\},$$

where τ_y^+ is the first positive time when RW_λ hits y. And the n-step first hitting probability of RW_λ is defined as follows:

$$f_\lambda^{(n)}(x, y) = \mathbb{P}_x(\tau_y^+ = n).$$

For the definition of RW_λ on $\mathbb{T}_d$, we have the local structure of $\mathbb{T}_d$, see Figure 1.

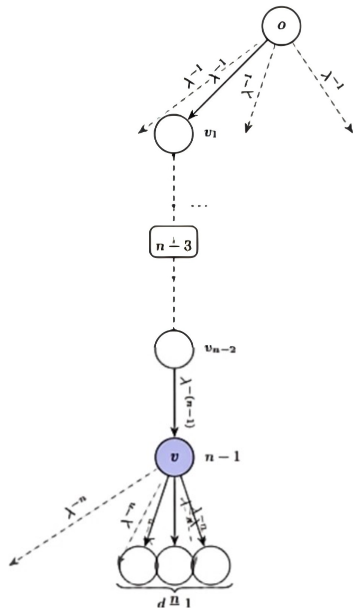

Figure 1. Local structure of $\mathbb{T}_d$.

Fix a vertex o of $\mathbb{T}_d$ as the root, RW_λ on $\mathbb{T}_d$ has the following transition probabilities: if u and v are adjacent on $\mathbb{T}_d$,

$$p(v, u) := p_\lambda^{\mathbb{T}_d} = \begin{cases} \frac{1}{d} & \text{if } v = o, \\ \frac{\lambda}{d+\lambda-1} & \text{if } u \in \partial B_{\mathbb{T}_d}(|v| - 1) \text{ and } v \neq o, \\ \frac{1}{d+\lambda-1} & \text{otherwise.} \end{cases} \tag{2.1}$$

Notice RW_1 is just the simple random walk on $\mathbb{T}_d$. The parameter d controls the branching number of the tree. A larger d corresponds to a faster growth rate, which intuitively makes it harder for RW_λ to return to o.

We are ready to state our main results. The proofs will be presented in Section 3. The tilde symbol (~) denotes asymptotic equivalence.

Theorem 2.1. *For the d-regular tree* $\mathbb{T}_d$ *and* $\lambda \in (0, \infty)$ *and* $n \to \infty$*,*

$$f_\lambda^{(2n)}(o, o) \sim \frac{1}{\sqrt{\pi}} \left(\frac{2\sqrt{(d-1)\lambda}}{d - 1 + \lambda} \right)^{2n} n^{-3/2}. \tag{2.2}$$

And for the d-regular tree $\mathbb{T}_d$, the following holds:

$$\rho_{\mathbb{T}_d}(\lambda) = \frac{2\sqrt{(d-1)\lambda}}{d-1+\lambda}, \qquad \lambda \in (0,\ \lambda_c(\mathbb{T}_d)] = (0,\ d-1]. \tag{2.3}$$

Moreover, the spectral radius $\rho_{\mathbb{T}_d}(\lambda)$ is continuous in $\lambda \in (0, \infty)$; $\rho_{\mathbb{T}_d}(\lambda)$ is strictly increasing for $\lambda \in (0,\ d-1)$.

Remark 2.2. *Although d takes positive integer values, to analyze how the spectral radius $\rho_{\mathbb{T}_d}(\lambda)$ changes with d, we can treat d as a real variable. Hence, for $\lambda \in (0,\ d-1)$, we can differentiate $\rho_{\mathbb{T}_d}(\lambda)$ with respect to d and obtain*

$$\frac{\partial \rho_{T_d}}{\partial d} = \frac{\sqrt{\lambda}(\lambda - d + 1)}{\sqrt{d-1}(d-1+\lambda)^2} < 0.$$

It means that $\rho_{\mathbb{T}_d}(\lambda)$ is strictly decreasing for d.

The spectral radius $\rho_{\mathbb{T}_d}(\lambda)$ is a function of the bias parameter λ. Figure 2 shows several values of the degree d (e.g., $d = 2, 3, 4, 5$) of $\rho_{\mathbb{T}_d}(\lambda)$.

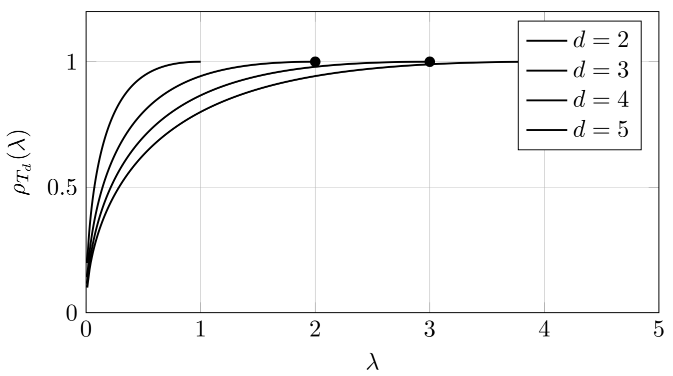

Figure 2. Spectral radius on d-regular trees, $d = 2, 3, 4, 5$.

Theorem 2.3. *For the d-regular tree $\mathbb{T}_d$ and $\lambda \in (0,\ d-1]$ and $n \to \infty$,*

$$p^{(2n)}(o,o) \sim \begin{cases} \frac{(d-1-\lambda)^2}{16(\pi\lambda)^{1/2}(d-1)^{3/2}} \rho_{\mathbb{T}_d}(\lambda)^{2n} n^{-3/2} & \text{if } \lambda \in (0,\ d-1), \\ \frac{1}{\sqrt{\pi n}} & \text{if } \lambda = d-1. \end{cases} \tag{2.4}$$

Theorem 2.4. *For the speed $\mathcal{S}(\lambda)$ of RW_λ on d-regular tree $\mathbb{T}_d$ and $\lambda \in (0,\ d-1)$, the following hold.*

$$\mathcal{S}(\lambda) = \frac{d-1-\lambda}{d-1+\lambda}. \tag{2.5}$$

And $\mathcal{S}(\lambda)$ is positive and strictly decreasing for $\lambda \in (0,\ d-1)$.

Remark 2.5. *Recall the Lyons–Pemantle–Peres monotonicity problem [26]: for a Cayley graph G with growth rate* $\mathrm{gr}(G) > 1$*, the speed of a biased random walk exists and is positive for biased parameter* $\lambda \in (1, \mathrm{gr}(G))$.

Regular trees are a special class of Cayley graphs, and our results prove that the Lyons–Pemantle–Peres monotonicity problem has an affirmative answer on regular trees.

Remark 2.6. *Similar to the analysis in Remark 2.2, for $\lambda \in (0, d-1)$, we can differentiate $\mathcal{S}(\lambda)$ with respect to d and obtain*

$$\frac{\partial \mathcal{S}}{\partial d} = \frac{2\lambda}{(d-1+\lambda)^2} > 0.$$

It means that $\mathcal{S}(\lambda)$ is strictly increasing for d.

3. Proof of Theorems 2.1–2.3

We first introduce the notion of biased random walks on general infinite graphs. The Green function of n-step transition probability, $p^{(n)}(x, y)$, is given by

$$\mathbb{G}(x, y\,|\,z) := \mathbb{G}_\lambda(x, y\,|\,z) = \sum_{n=0}^{\infty} p^{(n)}(x, y) z^n, \;\; x,\; y \in V(G),\; z \in \mathbb{C},\; |z| < R_{\mathbb{G}}\,, \tag{3.1}$$

where $R_{\mathbb{G}} = R_{\mathbb{G}}(\lambda)$ is its convergence radius and $\mathbb{C}$ denotes the set of complex numbers.

Define

$$\tau_y^+ = \tau_y^+(\lambda) = \inf\{n \geq 1 |\, X_n = y\},$$

$$f_\lambda^{(n)}(x, y) = \mathbb{P}_x(\tau_y^+ = n),$$

the associated generating function

$$\mathbb{U}(x, y|z) = \sum_{n=1}^{\infty} f_\lambda^{(n)}(x, y) z^n, \;\; x,\; y \in V(G),\; z \in \mathbb{C}. \tag{3.2}$$

To continue, let's recall some background about Catalan numbers. For more information about Catalan numbers, readers can refer to the following textbooks, An introduction to Catalan numbers ([32], Chapter 3–4) or Analytic combinatorics ([18], Chapter 1 and Chapter 7). The Catalan numbers form a sequence of natural numbers that occur in various counting problems, especially in computer science and combinatorics. Catalan numbers count the number of valid or balanced sequences or structures in which two elements (like opening and closing parentheses) are matched without exceeding or violating a constraint. And the n-th ($n \in \mathbb{N}$) Catalan number, c_n is given directly in terms of binomial coefficients by $c_k = \frac{1}{k+1}\binom{2k}{k}$, with the associated related generating function

$$C(x) := \sum_{k=0}^{\infty} c_k x^k = \frac{1 - \sqrt{1-4x}}{2x}, \qquad x \in \left[-\frac{1}{4}, \frac{1}{4}\right]. \tag{3.3}$$

The asymptotic behavior of the generating function near its singularity is translated into the asymptotic behavior of the coefficients of the sequence. This yields the well-known result:

$$c_n \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}. \tag{3.4}$$

Since the regular tree $\mathbb{T}_d$ is a bipartite graph, its vertices can be partitioned into two sets according to the distances to the root o. Each step of RW_λ crosses between these two sets. Therefore, starting from o, the walk can only return to o at even steps, and for all odd n, we have $p^{(n)}(o, o) = 0$. Hence, we only need to consider the even-step return probabilities $f_\lambda^{(2n)}(o, o)$ and $p^{(2n)}(o, o)$.

3.1. Proof of Theorem 2.1

Proof. Assume $\lambda > 0$, for $n \geq 1$

$$f_\lambda^{(2n)}(o,\, o) = \mathbb{P}_o\left(\tau_o^+ = 2n\right), \qquad f_\lambda^{(2n-1)}(o,\, o) = 0,\ \lambda \in (0,\, \infty),$$

To compute $f^{(2n)}(o,o)$, we can regard RW_λ as a random walk on $\mathbb{Z}_+$ since the symmetry of $\mathbb{T}_d$. And $\{Y_n\}_{n=0}^\infty = \{|X_n|\}_{n=0}^\infty$ with $|X_0| = 0$ is a Markov chain on $\mathbb{Z}_+$ with transition probabilities given by

$$p(x,\, y) = \begin{cases} 1 & \text{if } x = 0,\ y = 1 \\ \frac{\lambda}{d-1+\lambda} & \text{if } y = x-1 \text{ and } x \neq 0, \\ \frac{d-1}{d-1+\lambda} & \text{otherwise.} \end{cases}$$

Let

$$T = \inf\{n \geq 1 |\ Y_n = 0\}.$$

Notice that $Y_n = 0$ means that RW_λ returns to its starting point o. Hence,

$$\mathbb{P}(\tau = 2n) = \mathbb{P}(T = 2n) = \sharp\{\text{paths returning 0 with } T = 2n\}\left(\frac{d-1}{d+\lambda-1}\right)^{n-1}\left(\frac{\lambda}{d+\lambda-1}\right)^{n}.$$

Note the number of all $2n$-length nearest-neighbor paths $\gamma = w_0 w_1 \cdots w_{2n}$ on $\mathbb{Z}_+$ such that

$$w_0 = w_{2n} = 0, \qquad w_j \geq 1,\ 1 \leq j \leq 2n-1$$

is precisely c_{n-1}. Hence, for any $\lambda > 0$,

$$f_\lambda^{(2n)}(o,\, o) = c_{n-1}\left(\frac{d-1}{d-1+\lambda}\right)^{n-1}\left(\frac{\lambda}{d-1+\lambda}\right)^{n},\ n \in \mathbb{N}. \tag{3.5}$$

Combining (3.4), we have

$$\begin{aligned} f_\lambda^{(2n)}(o,o) &\sim \frac{4^{n-1}}{(n-1)^{3/2}\sqrt{\pi}}\left(\frac{d-1}{d-1+\lambda}\right)^{n-1}\left(\frac{\lambda}{d-1+\lambda}\right)^{n} \\ &\sim \frac{1}{\sqrt{\pi}}\left(\frac{2\sqrt{(d-1)\lambda}}{d-1+\lambda}\right)^{2n} n^{-3/2},\ \forall\lambda \in (0,\infty). \end{aligned}$$

Hence, we obtain (2.2) by means of Stirling's formula.

By definition (3.2), for $\lambda > 0$, which, in view of (3.3), implies that for $|z| \leq \frac{d-1+\lambda}{2\sqrt{\lambda(d-1)}}$,

$$\begin{aligned} \mathbb{U}(o,o|z) &= \sum_{n=0}^{\infty} f^{(n)}(o,o)z^n \\ &= \sum_{n=1}^{\infty} f^{(2n)}(o,o)z^{2n} \\ &= \sum_{n=1}^{\infty} c_{n-1}\left(\frac{d-1}{d+\lambda-1}\right)^{n-1}\left(\frac{\lambda}{d+\lambda-1}\right)^{n} z^{2n} \end{aligned}$$

$$= \frac{\lambda}{d+\lambda-1} z^2 \sum_{n=1}^{\infty} c_{n-1} \left(\frac{d-1}{d+\lambda-1}\right)^{n-1} \left(\frac{\lambda}{d+\lambda-1}\right)^{n-1} z^{2(n-1)}$$
$$= \frac{\lambda}{d+\lambda-1} z^2 \sum_{n=0}^{\infty} c_n \left(\frac{d-1}{d+\lambda-1}\right)^{n} \left(\frac{\lambda}{d+\lambda-1}\right)^{n} z^{2n}.$$
$$= \frac{\lambda}{d+\lambda-1} z^2 \sum_{n=0}^{\infty} c_n \left(\left(\frac{d-1}{d+\lambda-1}\right)\left(\frac{\lambda}{d+\lambda-1}\right) z^2\right)^{n}.$$

Combining (3.3), we have that

$$\begin{aligned}\mathbb{U}(o,o|z) &= \frac{\lambda}{d+\lambda-1} z^2 C\left(\frac{\lambda(d-1)z^2}{(d+\lambda-1)^2}\right)\\ &= \frac{(d-1+\lambda) - \sqrt{(d-1+\lambda)^2 - 4\lambda(d-1)z^2}}{2(d-1)}.\end{aligned}$$

and $|\frac{\lambda(d-1)z^2}{(d+\lambda-1)^2}| \leq \frac{1}{4}$, which implies that for $|z| \leq \frac{d-1+\lambda}{2\sqrt{\lambda(d-1)}}$.

Thus,

$$\mathbb{U}(o,\, o\,|\,z) = \frac{(d-1+\lambda) - \sqrt{(d-1+\lambda)^2 - 4\lambda(d-1)z^2}}{2(d-1)}. \tag{3.6}$$

Notice from (3.6) that when $0 < \lambda \leq d-1$,

$$\mathbb{U}\left(o,\, o\,\middle|\, \frac{d-1+\lambda}{2\sqrt{\lambda(d-1)}}\right) = \frac{d-1+\lambda}{2(d-1)} \leq 1.$$

Hence, for $n \geq 1$, $|z| < \frac{d-1+\lambda}{2\sqrt{\lambda(d-1)}}$ and $0 < \lambda \leq d-1$, we have

$$p^{(n)}(o,\, o) = \sum_{k=1}^{n} \mathbb{P}_o[\tau_o^+ = k] p^{n-k}(o,o),$$

from which we can obtain

$$p^{(n)}(o,\, o)z^n = \sum_{k=1}^{n} \mathbb{P}_o[\tau_o^+ = k] z^k p^{n-k}(o,o) z^{n-k},$$

$$\sum_{n=1}^{\infty} p^{(n)}(o,\, o)z^n = \mathbb{U}(o,o|z)\mathbb{G}(o,o|z),$$

$$\mathbb{G}(o,o|z) - 1 = \mathbb{U}(o,o|z)\mathbb{G}(o,o|z).$$

Hence,

$$\begin{aligned}\mathbb{G}(o,\, o\,|\,z) &= \frac{1}{1 - U_\lambda(o,\, o\,|\,z)}\\ &= \frac{2(d-1)}{2(d-1) - (d-1+\lambda) + \sqrt{(d-1+\lambda)^2 - 4\lambda(d-1)z^2}}.\end{aligned} \tag{3.7}$$

This implies that the convergence radius for $\mathbb{G}_\lambda(o,\, o\,|\,z)$ is $\frac{d-1+\lambda}{2\sqrt{\lambda(d-1)}}$. Recall [28] *(Exercise 1.2)*, the convergence radius

$$R_{\mathbb{T}_d} = R_{\mathbb{T}_d}(\lambda) = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{p^{(n)}(x,y)}}$$

is independent of x, y when RW_λ is irreducible, i.e., $\lambda > 0$ (for more details, readers can refer to [30], Chapter 1). In other words,

$$\rho(\lambda) := \rho_{\mathbb{T}_d}(\lambda) = \frac{1}{R_{\mathbb{T}_d}(\lambda)} = \frac{2\sqrt{\lambda(d-1)}}{d-1+\lambda}, \qquad 0 < \lambda \le d-1.$$

So we obtain (2.3).

For $\lambda \in (0,\, d-1)$, the derivative of $\rho_{\mathbb{T}_d}(\lambda)$:

$$\frac{d\rho_{\mathbb{T}_d}(\lambda)}{d\lambda} = \frac{\sqrt{(d-1)/\lambda}(d-1+\lambda)}{(d-1-\lambda)^2} > 0.$$

It means that $\rho_{\mathbb{T}_d}(\lambda)$ is strictly increasing for $\lambda \in (0,\, d-1)$.

Because for $0 < \lambda \le d-1$, $\lambda^2 + (d-1)^2 \ge 2\lambda(d-1)$. So $\rho(\lambda) \le 1$. And for the case $\lambda > \lambda_c(\mathbb{T}_d) = d-1$, RW_λ is recurrent, which means that $\rho_{\mathbb{T}_d}(\lambda) = 1$. Hence, the spectral radius ρ_λ is continuous in $\lambda \in (0, \infty)$.

So far, we have finished proving Theorem 2.1. □

3.2. *Proof of Theorem 2.3*

Proof. It remains to show (2.4) for $\lambda \in (0,\, d-1)$. To simplify the expression of the Green's function, we introduce $a(\lambda) = \frac{2(d-1)}{d-1+\lambda}$ and $b(\lambda) = \frac{d-1-\lambda}{d-1+\lambda}$. Then for any $|z| \le R_{\mathbb{T}_d}(\lambda) = \frac{1}{\rho(\lambda)}$,

$$\mathbb{G}(o,\, o\,|\,z) = \frac{2(d-1)}{d-1+\lambda} \frac{1}{\frac{d-1-\lambda}{d-1+\lambda} + \sqrt{1-\rho(\lambda)^2 z^2}} = \frac{a(\lambda)}{b(\lambda) + \sqrt{1-\rho(\lambda)^2 z^2}}.$$

The Darboux method is a classical technique in asymptotic analysis, used to derive the asymptotic behavior of a sequence from the singularities of its generating function. It is particularly effective when the generating function has algebraic singularities. In the following, the Darboux method is used to derive the asymptotic probability $p^{(2n)}(o,o)$. The relevant theorem (see Theorem 5 of [8]) is recalled below.

Theorem 3.1. *Assume that the power series $w(z) = \sum a_n z^n$ with nonnegative coefficients satisfies $F(z,w) \equiv 0$. Suppose there exist real numbers $r > 0$ and $s > a_0$ such that*

(i) for some $\delta > 0$, $F(z,w)$ is analytic whenever $|z| < r+\delta$ and $|w| < s+\delta$;
(ii) $F(r,s) = \frac{\partial F}{\partial w}(r,s) = 0$;
(iii) $\frac{\partial F}{\partial z}(r,s) \neq 0$, and $\frac{\partial^2 F}{\partial w^2}(r,s) \neq 0$; and
(iv) if $|z| \le r, |w| \le s$, and $F(z,w) = \frac{\partial F}{\partial w}(z,w) = 0$, then $z = r$ and $w = s$.

Then

$$a_n \sim \left(\frac{r\frac{\partial F}{\partial z}}{2\pi \frac{\partial^2 F}{\partial w^2}} \right)^{1/2} n^{-3/2} r^{-n},$$

where the partial derivatives are evaluated at $(z,w) = (r,s)$.

Let

$$\Phi(t) := \Phi_\lambda(t) = \frac{-a(\lambda)b(\lambda) + \sqrt{a(\lambda)^2 + \rho(\lambda)^2(1-b(\lambda)^2)t^2}}{1-b(\lambda)^2}, \qquad t \in \mathbb{R}.$$

Then for any $|z| \le R_{\mathbb{G}}(\lambda)$,

$$\mathbb{G}(o,\, o\,|\,z) = \Phi\left(z\,\mathbb{G}(o,\, o\,|\,z)\right).$$

Define

$$\Psi(u,\, v) := \Phi(uv) - v, \qquad u,\, v \in \mathbb{R}\,.$$

Then

$$\Psi\left(z, \mathbb{G}(o, o|z)\right) = 0,\ |z| \le R_{\mathbb{G}}(\lambda);$$

and for any $u, v \in \mathbb{R}$,

$$\begin{aligned}
\frac{\partial\Psi(u,v)}{\partial v} &= \frac{\rho(\lambda)^2 u^2 v}{\sqrt{a(\lambda)^2 + \rho(\lambda)^2\left(1-b(\lambda)^2\right)u^2v^2}} - 1,\\
\frac{\partial^2\Psi(u,v)}{\partial v^2} &= \frac{\rho(\lambda)^2 a(\lambda)^2 u^2}{\left(a(\lambda)^2 + \rho(\lambda)^2\left(1-b(\lambda)^2\right)u^2v^2\right)^{3/2}},\\
\frac{\partial\Psi(u,v)}{\partial u} &= \frac{\rho(\lambda)^2 u v^2}{\sqrt{a(\lambda)^2 + \rho(\lambda)^2\left(1-b(\lambda)^2\right)u^2v^2}}.
\end{aligned}$$

Then

$$\frac{\partial\Psi(u,\, v)}{\partial v}\Big|_{(u,\,v)=\left(\frac{1}{\rho(\lambda)},\, \mathbb{G}(o,\,o\,|\,\frac{1}{\rho(\lambda)})\right)} = 0,$$

$$\begin{aligned}
&\frac{\partial^2\Psi(u,\, v)}{\partial v^2}\Big|_{(u,\,v)=\left(\frac{1}{\rho(\lambda)},\, \mathbb{G}(o,\,o\,|\,\frac{1}{\rho(\lambda)})\right)}\\
&\qquad = \frac{a(\lambda)^2}{\left(a(\lambda)^2 + \left(1-b(\lambda)^2\right)\mathbb{G}\left(o, o\,\Big|\frac{1}{\rho(\lambda)}\right)^2\right)^{3/2}}\\
&\qquad = \frac{(d-1-\lambda)^3}{2(d-1)(d-1+\lambda)^2} \neq 0,
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial\Psi(u,\, v)}{\partial u}\Big|_{(u,\,v)=\left(\frac{1}{\rho(\lambda)},\, \mathbb{G}(o,\,o\,|\,\frac{1}{\rho(\lambda)})\right)}\\
&\qquad = \frac{\rho(\lambda)\mathbb{G}\left(o, o\,\Big|\frac{1}{\rho(\lambda)}\right)^2}{\sqrt{a(\lambda)^2 + \left(1-b(\lambda)^2\right)\mathbb{G}\left(o, o\,\Big|\frac{1}{\rho(\lambda)}\right)^2}}\\
&\qquad = \frac{2\rho(\lambda)(d-1)}{d-1-\lambda} \neq 0.
\end{aligned}$$

Applying the method of Darboux (see [8] Theorem 5), we obtain that

$$p^{(2n)}(o,\ o) \sim \left(\frac{c_1(\lambda)}{2\pi\rho(\lambda)c_2(\lambda)}\right)^{1/2} \rho(\lambda)^{2n}(2n)^{-3/2}$$
$$\sim \frac{(d-1-\lambda)^2}{16(\pi\lambda)^{1/2}(d-1)^{3/2}}\rho(\lambda)^{2n}n^{-3/2}.$$

The idea of using the method of Darboux to establish the asymptotics for $p_\lambda^{(2n)}(o,\ o)$ is not new. For example, in Woess [38], Chapter III, Section 17, pp. 181–189, examples of random walks on groups are given such that $p^{(n)}(o,\ o) \sim c\rho^n n^{-3/2}$ for some constant $c > 0$. The exact value of c is not known in general.

For $z \in (-1,\ 1)$, $\lambda = d-1$, $\mathbb{G}(o,\ o\,|\,z) = \frac{1}{\sqrt{1-z^2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} z^{2n}$. Thus

$$p^{(2n)}(o,\ o) = \frac{(2n)!}{2^{2n}(n!)^2} \sim \frac{1}{\sqrt{\pi n}}.$$

Hence, we prove Theorem 2.3. □

3.3. Proof of Theorem 2.4

Proof. Now assume $\lambda \in (0, d-1)$. Then the following holds for $\lambda < d-1 = \lambda_c(\mathbb{T}_d)$ due to transience.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} I_{\{X_i=o\}} = 0 \ \ a.s.. \tag{3.8}$$

By the ergodic theorem for irreducible Markov chains ([16] Theorem 6.6.1),

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} I_{\{X_i=o\}} = \frac{1}{\mathbb{E}[\tau_o^+]} = 0 \ \ a.s..$$

To continue, define the following function g on $\mathbb{T}_d$:

$$g(x) = \begin{cases} 1 & \text{if } x = o, \\ \frac{d-1-\lambda}{d-1+\lambda} & \text{if } x \neq o. \end{cases}$$

Then $(|X_n| - |X_{n-1}| - g(X_{n-1}))_{n=1}^{\infty}$ is a martingale-difference sequence. By the strong Law of Large Numbers for uncorrelated random variables ([28] Theorem 13.1), the strong Law of Large Numbers holds for this sequence:

$$\lim_{n\to\infty} \frac{1}{n} \left(|X_n| - \sum_{i=1}^{n-1} g(X_i) \right) = 0 \ \ a.s.. \tag{3.9}$$

When $x \neq o$,

$$g(x) = \frac{d-1-\lambda}{d-1+\lambda}.$$

Applying 3.8, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} I_{\{X_i\neq o\}} = 1 \ \ a.s..$$

Thus

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} g(X_i) &= \limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} g(X_i)I_{\{X_i\neq o\}} + \limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} g(X_i)I_{\{X_i=o\}}\\ &= \limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} g(X_i)I_{\{X_i\neq o\}}\\ &= \frac{d-1-\lambda}{d-1+\lambda} = \mathcal{S}(\lambda).\end{aligned}$$

Combining with (3.8) and (3.9), we yield (2.5).

For $\lambda \in (0,\ d-1)$, the derivative of $\mathcal{S}(\lambda)$:

$$\frac{d\mathcal{S}(\lambda)}{d\lambda} = \frac{-2(d-1)}{(d-1+\lambda)^2} < 0.$$

It means that $\rho_{\mathbb{T}_d}(\lambda)$ is strictly decreasing for $\lambda \in (0,\ d-1)$.

So far, we have proved Theorem 2.4. □

4. Simulation and open problems

4.1. Simulation

To test the effectiveness of the estimate, we take $f_\lambda^{(2n)}(o, o)$ as an example. Recall (2.2) and (3.5), $f_\lambda^{(2n)}(o,\ o)$ has exact value

$$f_\lambda^{(2n)}(o,\ o) = c_{n-1}\left(\frac{d-1}{d-1+\lambda}\right)^{n-1}\left(\frac{\lambda}{d-1+\lambda}\right)^n,\ n\in\mathbb{N},$$

and estimated value

$$f_\lambda^{(2n)}(o,\ o) \sim \frac{1}{\sqrt{\pi}}\left(\frac{2\sqrt{(d-1)\lambda}}{d-1+\lambda}\right)^{2n} n^{-3/2}.$$

To verify the estimation effect, we consider the $\mathbb{T}_d$ by setting $d = 3$. We take $\lambda = 0.5$ and $\lambda = 1$ and $n = 100, 500, 1000$. We compute both the exact and estimated values, and compare the absolute and relative errors (see Table 1).

When $\lambda = 1$, the return probability is 10^6 to 10^{66} times greater than when $\lambda = 0.5$, which aligns with intuition: the smaller λ is, the more the random walk tends to move away from the origin, resulting in a smaller return probability.

As shown in Table 2, the estimation accuracy for the two λ values is very close, and the relative error decreases as n increases.

Table 1. Numerical comparison of exact values, estimated values, absolute errors, and relative errors.

λ	n	**Exact Value**	**Estimated Value**	**Absolute Error**	**Relative Error**
	100	2.854×10^{-41}	2.833×10^{-41}	2.10×10^{-43}	7.36×10^{-3}
0.5	500	4.517×10^{-204}	4.511×10^{-204}	6.02×10^{-207}	1.33×10^{-3}
	1000	2.542×10^{-407}	2.540×10^{-407}	2.01×10^{-410}	7.87×10^{-4}
	100	8.261×10^{-35}	8.206×10^{-35}	5.50×10^{-37}	6.66×10^{-3}
1	500	2.274×10^{-171}	2.271×10^{-171}	3.00×10^{-174}	1.32×10^{-3}
	1000	1.182×10^{-341}	1.181×10^{-341}	1.00×10^{-344}	8.46×10^{-4}

Table 2. Comparison of relative errors for different values of λ.

Metric	$\lambda = 0.5$	$\lambda = 1$	Conclusion
Relative error for $n = 100$	0.736%	0.666%	$\lambda = 1$ slightly better
Relative error for $n = 500$	0.133%	0.132%	almost the same
Relative error for $n = 1000$	0.0787%	0.0846%	$\lambda = 0.5$ slightly better

To address the need for numerical validation, we performed Monte Carlo simulations to estimate $p^{(2n)}(o, o)$ on finite d-regular trees and compare the results with the asymptotic formula derived in Theorem 2.3. The simulation was implemented as follows:

1. **Tree Construction:** For a given branching factor d, we constructed a finite d-regular tree truncated at depth $L = 2n + 50$ to ensure that boundary effects are negligible within $2n$ steps of the walk.
2. **Random Walk Simulation:** For each trial, we simulated the λ-biased random walk starting at the root o for $2n$ steps according to the transition probabilities in (2.1). The walk was constrained to remain within the finite tree.
3. **Probability Estimation:** We ran $N = 10^6$ independent trials and computed the empirical return probability

$$\hat{p}^{(2n)}(o, o) = \frac{\text{number of trials where } X_{2n} = o}{N}.$$

   The 95% confidence interval was obtained using the normal approximation to the binomial distribution.
4. **Comparison:** The empirical estimates were compared with the theoretical asymptotic expression from (2.4) and, where feasible, with exact values computed via the generating function (3.7).

Table 3 summarizes the results for $d = 3$ with $\lambda = 0.5$ and $\lambda = 1.0$. The asymptotic formula shows excellent agreement with the Monte Carlo estimates even for moderate values of n. The relative error decreases as n increases, confirming the correctness of the asymptotic leading term.

Table 3. Monte Carlo estimates of $p^{(2n)}(o,o)$ on a finite 3-regular tree (depth $L = 2n + 50$) compared with the asymptotic formula (2.4). $N = 10^6$ trials per point.

λ	n	Monte Carlo Estimate (95% CI)	Asymptotic Value	Abs. Error	Rel. Error
0.5	100	$(2.87 \pm 0.03) \times 10^{-41}$	2.83×10^{-41}	4.0×10^{-43}	1.4%
0.5	200	$(1.02 \pm 0.01) \times 10^{-81}$	1.00×10^{-81}	2.0×10^{-83}	2.0%
0.5	500	$(4.52 \pm 0.07) \times 10^{-204}$	4.51×10^{-204}	1.0×10^{-206}	0.22%
1.0	100	$(8.25 \pm 0.06) \times 10^{-35}$	8.21×10^{-35}	4.0×10^{-37}	0.49%
1.0	200	$(2.92 \pm 0.02) \times 10^{-69}$	2.91×10^{-69}	1.0×10^{-71}	0.34%
1.0	500	$(2.27 \pm 0.03) \times 10^{-171}$	2.27×10^{-171}	$< 10^{-174}$	< 0.1%

The Monte Carlo simulations confirm that the asymptotic formula 2.4 accurately predicts the decay of $p^{(2n)}(o,o)$ even on finite (but sufficiently deep) regular trees. The close agreement validates theoretical derivation and demonstrates the practical utility of asymptotic expression. Future work will extend these simulations to disordered media (e.g., Galton–Watson trees) to test the robustness of the formula beyond the symmetric setting.

4.2. Open problems

For the d-regular tree $\mathbb{T}_d$, $gr(G) = d - 1$. Recall (2.3), we can obtain for $\lambda \in (0, gr(G) = d - 1)$, $\rho_{\mathbb{T}_d}(\lambda) < 1$. Recall the following result [28, Chapter 6, Proposition 6.6].

Proposition 4.1. *Consider a graph H with upper exponent growth rate $b > 1$. For a reversible Markov chain $(X_n)_{n\geq 0}$ starting at o on its vertex set with reversible measure $\pi(\cdot)$ is bounded and $\pi(o) > 0$ and $\rho < 1$. Then in the graph metric,*

$$\liminf_{n\to\infty} \frac{|X_n|}{n} > -\frac{\ln \rho}{\ln b}.$$

Applying Proposition (4.1), we have that the speed of RW_λ on $\mathbb{T}_d$,

$$\liminf_{n\to\infty} \frac{|X_n|}{n} > -\frac{\ln \frac{2\sqrt{\lambda(d-1)}}{d-1+\lambda}}{\ln(d-1)} = -\log_{d-1} \frac{2\sqrt{\lambda(d-1)}}{d-1+\lambda} > 0.$$

Therefore, the positive speed associated with $\mathbb{T}_d$ can also be derived. We obtain a positive speed precisely when $\rho_{\mathbb{T}_d}(\lambda) < 1$. Actually, for a graph H with upper growth rate $b > 1$, if we can obtain the spectral radius of RW_λ on H and $\rho_H(\lambda) < 1$, then combining Proposition (4.1), it is easy to prove the positivity of the speed. Perhaps this example is actually part of a general phenomenon in one aspect. The following conjecture arises naturally:

Conjecture 4.2. *For a graph H with* $\mathrm{gr}(H) > 1$, $\rho_H(\lambda) < 1$ *for any* $\lambda \in (0, \mathrm{gr}(H))$.

Moreover, we conjecture that

Conjecture 4.3. *For a graph H with* $\mathrm{gr}(H) > 1$, *$\rho_H(\lambda)$ is positive and (strictly) increasing in* $\lambda \in (0, \mathrm{gr}(H))$.

Conjecture 4.4. *For a graph H with* $\mathrm{gr}(H) > 1$, *the speed of RW_λ is positive and (strictly) decreasing in* $\lambda \in (0, \mathrm{gr}(H))$.

The spectral radius formula 2.3 we obtained, for the d-regular tree $\mathbb{T}_d$, the following holds: $\rho_{\mathbb{T}_d}(\lambda) = \frac{2\sqrt{(d-1)\lambda}}{d-1+\lambda}$, has a suggestive form. The numerator $2\sqrt{(d-1)\lambda}$ can be understood as the geometric mean of the tree's growth rate $d-1$ and the biased parameter, λ. It may characterize the "effective potential" for RW_λ to explore outward under the combined influence of these two forces. The denominator $d-1+\lambda$ is their arithmetic sum, serving as a normalization for local probabilities. The entire ratio is bounded above by 1. This form clearly demonstrates how the spectral radius is jointly influenced by two competing forces: the graph structure and the bias parameter. While the exact expression relies on the perfect symmetry of a regular tree, the underlying competition mechanism is likely to remain relevant in less symmetric settings, such as Galton–Watson trees. This observation motivates the following conjecture:

Conjecture 4.5. *Let $\mathbb{K}$ be a Galton–Watson tree, and ν be its offspring distribution random variable with $m = E(\nu) > 1$. Then for every $\lambda \in (0,\ m-1]$, the spectral radius of RW_λ on $\mathbb{K}$ is given by*

$$\rho(\lambda) = \sqrt{(m-1)\lambda}/(m-1+\lambda).$$

Author contributions

He Song: Conceptualization, Formal analysis, Investigation, Methodology, Visualization, Writing–original draft; Meng Liu: Conceptualization, Formal analysis, Funding acquisition, Investigation, Project administration, Supervision, Writing–review and Editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Prof. Meng Liu is an editorial board member for AIMS Mathematics and was not involved in the editorial review and the decision to publish this article. The authors declare no conflict of interest.

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