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Research article

Pessimistic multigranulation reduction of partially labeled generalized neighborhood decision information systems based on related family and matrix

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Abstract: The multigranulation rough set model is an important rough set model that approximates the target concept using a multigranularity structure. The multigranulation reductions of the generalized neighborhood decision information systems (GNDISs) based on multigranularity rough sets are general models for the multigranulation reductions of decision information systems (DISs) with no missing decision attribute values. In practical applications, missing labels exist in many datasets. Unfortunately, the theory of multigranulation reduction of GNDISs is not suitable for attribute reduction of partially labeled data. For this reason, the concept of partially labeled generalized neighborhood decision information systems (p-GNDISs) is proposed in this paper, and pessimistic multigranulation reduction of p-GNDISs is discussed. Moreover, the related family-based approach is provided for getting all the partially labeled, pessimistic reducts (PLP-reducts) of a p-GNDIS. Meanwhile, the matrix operations of the generalized neighborhood pessimistic lower approximation and the pessimistic multigranulation positive region on a p-GNDIS are presented. Relationships between the Boolean matrix of the related family and the matrices for computing pessimistic lower approximations are explored. Then, a logic algorithm to get a PLP-reduct of a p-GNDIS by matrix operations is presented. The pessimistic multigranulation reduction of p-GNDISs by related families method and matrix operations provides a theoretical foundation for designing algorithms of multigranulation reduction for partially labeled data.

Keywords: partially labeled generalized neighborhood information system; knowledge reduction; pessimistic multigranulation rough sets; related families
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1. Introduction

Rough set theory [9] is a useful mathematical tool to deal with uncertain or fuzzy information, which has been used in knowledge discovery and machine learning [8, 18], group decision-making [6, 24, 25], and so on. A multigranulation rough set model [10, 11] is a typical generalized form of the Pawlak rough set model which extends the single-granularity approximation mechanism to a multigranularity framework. By utilizing multiple granular relations to jointly approximate a target subset in the universe of discourse, the multigranulation rough set effectively depicts the uncertainty of concepts from multiple perspectives and granular levels, thereby overcoming the insufficiency of single-granulation rough sets in processing multisource data. A pessimistic multigranulation rough set model is one of the most important multigranulation rough set models, because it adopts a stringent pessimistic fusion strategy that ensures maximum certainty of lower approximations and minimum redundancy of upper approximations for high-confidence decision-making. Moreover, all-inclusive requirements for lower approximations of pessimistic multigranulation rough set models effectively filter out noisy or inconsistent granular information. As a powerful mathematical tool for uncertain information processing, multigranulation rough sets have gained extensive attention and applications in knowledge discovery, intelligent decision-making, and machine learning.

Feature selection, which is also called "attribute reduction" in rough set theory, is an important data preprocessing procedure. In rough set theory, datasets are represented as various information systems whose attribute reductions are discussed by different rough set models. The multigranulation rough set models describe the information systems via multiple granular structures [10, 11], which are excellent feature selection models and thus have become popular focuses of study. Attribute reduction of complete DISs and incomplete DISs based on multigranulation rough sets were explored by Qian et al. [10–12]. Zhang et al. constructed a generalized multigranulation fuzzy neighborhood rough set model for attribute reduction of a fuzzy decision system [27]. To present a general model for the multigranulation reduction of information systems and DISs by discernibility techniques, the notations of a generalized neighborhood information system (GNIS) and a generalized neighborhood decision information system (GNDIS) were introduced by Zhang and Li [28, 29], and discernibility matrices were constructed to compute the multigranulation reducts of GNISs and GNDISs.

The attribute reduction methods for fully labeled data are quite abundant. However, in practical applications, many data lack labels. Hence, the partially labeled data are commonly present in the real world, whose attribute reduction has become one of the research hot spots. Qian et al. defined local rough set for attribute reduction of limited labeled big data [13]. An et al. designed a labeling method based on an uncertainty measure and presented a semisupervised feature selection algorithm for a partially labeled decision table using a k-nearest neighbors-fuzzy rough set model [1]. By exploring connections between rough set theory and belief function theory, Campagner et al. discussed a weakly supervised feature selection for superset decision tables [3]. Semisupervised feature selection for partially labeled mixed-type data by multicriteria measure approach based on the dependency, information entropy, and information granulation was explored by Shu et al. [15]. Shu et al. discussed semisupervised feature selection of partially labeled DISs by leveraging hyperinterval granules labeling and local mixed neighborhood entropy [14]. Sun et al. introduced a generalized variable-precision neighborhood rough set model based on a three-way decision (TWD) model and designed a global feature selection algorithm based on the unified feature measures in semisupervised partially labeled heterogeneous data [16]. An attribute reduction algorithm for partially labeled heterogeneous DISs based on misclassification cost and self-information was designed in [19]. Xu et al. discussed accelerating semisupervised feature selection for ordered partially labeled data based on neighborhood discernibility degrees with pseudolabel granular balls and distance matrix updating methods [20]. A semisupervised attribute reduction method for semi-supervised DISs which considers attribute relevance, redundancy, and label irrelevance from the perspective of label distribution was proposed by Dai et al. [4]. All of these results discuss attribute reductions of partially labeled DISs.

The related family method is an efficient attribute reduction method. Yang et al. proposed the related family-based approach to perform attribute reduction of covering DISs [21]. Local attribute reduction for partially labeled data by local related family was discussed by Yang et al. [22]. The related families-based incremental approaches were used by Lang et al. to perform attribute reduction of dynamic covering DISs with object set or attribute set changes [7]. Cai et al. investigated incremental approaches to update reducts of dynamic covering DISs with dynamic covering granularity by updating related families [2]. A semisupervised feature selection method based on a fuzzy related family for partially labeled data was explored in [5].

A GNDIS is a general model of a DIS, and the multigranulation reduction of GNDISs by the discernibility technique provides a theoretical basis for the multigranulation reduction of DISs based on discernibility. However, the reduction theory of GNDISs is not applicable to partially labeled datasets. The purpose of this paper is to present a general model for the multigranulation reduction of partially labeled DISs by the related families and matrix operations. The notation of a partially labeled generalized neighborhood decision information system (p-GNDIS) is introduced in this paper, the pessimistic multigranulation reduction of p-GNDISs based on the pessimistic multigranulation rough sets are discussed, and the related family is constructed to compute the pessimistic multigranulation reducts of p-GNDISs.

The remaining structure of this paper is organized as follows. In Section 2, we introduce the definitions of the pessimistic multigranulation rough sets in a p-GNDIS. In Section 3, the pessimistic multigranulation reduction of p-GNDISs is presented and characterized by the related family. Section 4 computes the pessimistic multigranulation positive region based on matrix operations. The pessimistic multigranulation reduction based on matrix operations is presented in Section 5. Section 6 concludes the study.

2. Preliminary knowledge

In this section, we review some basic concepts about the definitions of multigranulation rough sets in GNDISs and introduce the concept of partially labeled generalized neighborhood decision information systems. Throughout this paper, the universe of discourse U is nonempty and finite. The family of all subsets of U is denoted by $\mathcal{P}(U)$. For $X \subseteq U$, X^C is the complementary set of X.

Several types of neighborhood operators were introduced by Yao in 1998 [23].

Definition 1. *[23] Let U be a universe. A mapping $N : U \to \mathcal{P}(U)$ is called a neighborhood operator. If $x \in N(x)$ for all $x \in U$, N is a reflexive neighborhood operator. If $x \in N(y) \Rightarrow y \in N(x)$ for all $x, y \in U$, N is a symmetric neighborhood operator. If $[y \in N(x), z \in N(y)] \Rightarrow z \in N(x)$ for all $x, y, z \in U$, N is a transitive neighborhood operator. If the neighborhood operator N is reflexive, symmetric and transitive, N is called a Pawlak neighborhood operator.*

Denote $\{N(x)|x \in U\}$ by C_N. Clearly, if N is a reflexive, then C_N is a covering of U. If N is a Pawlak neighborhood operator, then C_N forms a partition of U.

The generalized neighborhood multigranulation rough sets were presented in [28].

Definition 2. *[28] Let $N_1, N_2, \cdots, N_m (m \geq 2)$ be reflexive neighborhood operators on U and $\mathcal{N} = \{N_1, N_2, \cdots, N_m\}$, $N_d : U \to P(U)$ be a Pawlak neighborhood operator. Then, $(U, \mathcal{N}, N_d)$ is called a generalized neighborhood decision information system (GNDIS). For $X \subseteq U$, define the generalized neighborhood pessimistic lower approximation $\underline{\sum_{\mathcal{N}} N_k^P}(X)$ and pessimistic upper approximation $\overline{\sum_{\mathcal{N}} N_k^P}(X)$ by*

$$\underline{\sum_{\mathcal{N}} N_k^P}(X) = \{x \in U | (N_1(x) \subseteq X) \wedge (N_2(x) \subseteq X) \wedge \cdots \wedge (N_m(x) \subseteq X)\}, \tag{2.1}$$

$$\overline{\sum_{\mathcal{N}} N_k^P}(X) = \sim \underline{\sum_{\mathcal{N}} N_k^P}(\sim X). \tag{2.2}$$

$(\underline{\sum_{\mathcal{N}} N_k^P}(X), \overline{\sum_{\mathcal{N}} N_k^P}(X))$ *is the generalized neighborhood pessimistic multigranulation rough set of X.*

Properties of $(\underline{\sum_{\mathcal{N}} N_k^P}(X), \overline{\sum_{\mathcal{N}} N_k^P}(X))$ are presented as follows.

Proposition 1. *[28] Let $(U, \mathcal{N}, N_d)$ be a GNDIS, $X, Y \subseteq U$, $\mathcal{H} \subseteq \mathcal{N}$ and $\mathcal{H} \neq \emptyset$. Then,*

(1) $\underline{\sum_{\mathcal{N}} N_k^P}(\emptyset) = \emptyset$, $\underline{\sum_{\mathcal{N}} N_k^P}(U) = U$, $\overline{\sum_{\mathcal{N}} N_k^P}(\emptyset) = \emptyset$, $\overline{\sum_{\mathcal{N}} N_k^P}(U) = U$,

(2) $\underline{\sum_{\mathcal{N}} N_k^P}(X) \subseteq X \subseteq \overline{\sum_{\mathcal{N}} N_k^P}(X)$,

(3) $X \subseteq Y \Rightarrow \underline{\sum_{\mathcal{N}} N_k^P}(X) \subseteq \underline{\sum_{\mathcal{N}} N_k^P}(Y)$, $\overline{\sum_{\mathcal{N}} N_k^P}(X) \subseteq \overline{\sum_{\mathcal{N}} N_k^P}(Y)$,

(4) $\underline{\sum_{\mathcal{N}} N_k^P}(X \cap Y) = \underline{\sum_{\mathcal{N}} N_k^P}(X) \cap \underline{\sum_{\mathcal{N}} N_k^P}(Y)$, $\overline{\sum_{\mathcal{N}} N_k^P}(X \cup Y) = \overline{\sum_{\mathcal{N}} N_k^P}(X) \cup \overline{\sum_{\mathcal{N}} N_k^P}(Y)$,

(5) $\underline{\sum_{\mathcal{N}} N_k^P}(\underline{\sum_{\mathcal{N}} N_k^P}(X)) \subseteq \underline{\sum_{\mathcal{N}} N_k^P}(X)$, $\overline{\sum_{\mathcal{N}} N_k^P}(X) \subseteq \overline{\sum_{\mathcal{N}} N_k^P}(\overline{\sum_{\mathcal{N}} N_k^P}(X))$,

(6) $\underline{\sum_{\mathcal{N}} N_k^P}(X) \subseteq \underline{\sum_{\mathcal{H}} N_k^P}(X)$, $\overline{\sum_{\mathcal{H}} N_k^P}(X) \subseteq \overline{\sum_{\mathcal{N}} N_k^P}(X)$.

In practical applications, due to the difficulty of labeling samples, some samples lack labeling, so partially labeled data are common. This is manifested in some decision information systems as a lack of decision attribute values. Correspondingly, we provide the concept of partially labeled generalized neighborhood decision information systems.

Definition 3. *Let $\mathcal{N} = \{N_1, N_2, \cdots, N_m\}$ be a family of reflexive neighborhood operators on U and $N_d : U \to \mathcal{P}(U)$ a neighborhood operator. $TS = \{x \in U | N_d(x) \neq \emptyset\}$ is called a target set. If $TS \neq U$, and $N_d |_{TS}: TS \to \mathcal{P}(U)$ is a Pawlak neighborhood operator, then $(U, \mathcal{N}, N_d)$ is called a partially labeled generalized neighborhood decision information system (p-GNDIS).*

In a p-GNDIS, it is clear that $N_d(x) \cap TS = N_d(x)$ for all $x \in TS$. It is shown that a GNDIS is a general model of a DIS in [29]. The neighborhood $N_k(x)$ could be an equivalent class, a compatible class, or a neighborhood class, and so on. We present a p-GNDIS induced from a neighborhood DIS by employing Example 1.

Example 1. *A neighborhood DIS (U, A, d) is presented in Table 1, where U is the universe, A is a family of condition attributes that a mapping $a : U \to V_a$ for any $a \in A$, and V_a is the continuous value set of a. For any $A_k \subseteq A$, the neighborhood granule $\delta_{A_k}(x)$ of the object x with respect to A_k is defined as*

$$\delta_{A_k}(x) = \{y \in U | \triangle_{A_k}(x, y) \le \delta_k\},$$

where $\delta_k \ge 0$ is a parameter, $\triangle_{A_k}(x, y) = (\sum_{a \in A_k} |a(x) - a(y)|^2)^{\frac{1}{2}}$.

Let $\mathcal{A}^\delta = \{A_1 = \{a_1, a_2\}, A_2 = \{a_3, a_4, a_5\}, A_3 = \{a_6\}, A_4 = \{a_7, a_8\}\}$, and $\delta_1 = 0.3, \delta_2 = 0.3, \delta_3 = 0.2, \delta_4 = 0.15$. The neighborhood granule $\delta_{A_i}(x_j)(i = 1, 2, 3, 4; j = 1, 2, \cdots, 6)$ is shown in Table 2.

Table 1. A neighborhood DIS with missing decision attribute values.

	A_1		A_2			A_3	A_4		d
	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	
x_1	1.2	0.7	2.1	1.9	2.2	3.1	1.5	1.7	2
x_2	1.4	1.0	1.8	1.8	2.1	3.3	1.6	1.8	1
x_3	1.6	0.8	2.0	1.9	2.3	3.5	1.2	1.6	*
x_4	1.5	0.6	2.4	1.7	2.6	3.0	1.3	1.7	1
x_5	1.7	0.5	1.7	1.9	2.4	2.9	1.5	1.9	2
x_6	1.0	0.8	2.3	2.0	2.5	2.7	1.4	1.8	2

Table 2. The neighborhood granules of elements in Example 1.

*	x_1	x_2	x_3	x_4	x_5	x_6
$\delta_{A_1}(x_i)$	$\{x_1, x_6\}$	$\{x_2, x_3\}$	$\{x_2, x_3, x_4\}$	$\{x_3, x_4, x_5\}$	$\{x_4, x_5\}$	$\{x_1, x_6\}$
$\delta_{A_2}(x_i)$	$\{x_1, x_3\}$	$\{x_2, x_3\}$	$\{x_1, x_2, x_3\}$	$\{x_4\}$	$\{x_5\}$	$\{x_6\}$
$\delta_{A_3}(x_i)$	$\{x_1, x_2, x_4, x_5\}$	$\{x_1, x_2, x_3\}$	$\{x_2, x_3\}$	$\{x_1, x_4, x_5\}$	$\{x_1, x_4, x_5, x_6\}$	$\{x_5, x_6\}$
$\delta_{A_4}(x_i)$	$\{x_1, x_2, x_6\}$	$\{x_1, x_2, x_5\}$	$\{x_3, x_4\}$	$\{x_3, x_4, x_6\}$	$\{x_2, x_5, x_6\}$	$\{x_1, x_4, x_5, x_6\}$

Let $N_k(x) = \delta_{A_k}(x)(k = 1, 2, 3, 4)$ for all $x \in U$. Then, we get that $\mathcal{N} = \{N_k | k = 1, 2, 3, 4\}$ is a family of reflexive neighborhood operators. Let $N_d(x_1) = N_d(x_5) = N_d(x_6) = \{x_1, x_5, x_6\}$, $N_d(x_2) = N_d(x_4) = \{x_2, x_4\}$, and $N_d(x_3) = \emptyset$; then, N_d is a neighborhood operator. Thus, $TS = \{x_1, x_2, x_4, x_5, x_6\} \neq \emptyset$, and $N_d |_{TS} : TS \to \mathcal{P}(U)$ is a Pawlak neighborhood operator. By Definition 3, $(U, \mathcal{N}, N_d)$ is a p-GNDIS.

3. Pessimistic multigranulation reduction of p-GNDISs

To obtain important neighborhood operators and improve the computational efficiency, we discuss the reduction of p-GNDISs in this section.

Let $(U, \mathcal{N}, N_d)$ be a p-GNDIS with the target set TS. The partition of TS with respect to N_d is referred to as $TS/N_d = \{N_d(x)|x \in TS\} \triangleq \{V_1, V_2, \cdots, V_l\}$. The pessimistic multigranulation positive region of N_d on TS is defined as

$$Pos^P_{\mathcal{N}}(TS) = \bigcup_{j=1}^{l} (\underline{\sum_{\mathcal{N}} N^P_k(V_j)}). \tag{3.1}$$

Obviously, $Pos^P_{\mathcal{N}}(TS) \subseteq TS$. If $TS = Pos^P_{\mathcal{N}}(TS)$, we say that the p-GNDIS $(U, \mathcal{N}, N_d)$ is consistent. If $TS \neq Pos^P_{\mathcal{N}}(TS)$, we say that the p-GNDIS $(U, \mathcal{N}, N_d)$ is inconsistent. We present a reduction of a p-GNDIS based on the concept of the positive region of N_d on TS.

Definition 4. *Given a p-GNDIS $(U, \mathcal{N}, N_d)$ and a target set $TS \subseteq U$, let $\mathcal{H} \subseteq \mathcal{N}$ and $\mathcal{H} \neq \emptyset$. If $Pos^P_{\mathcal{N}}(TS) = Pos^P_{\mathcal{H}}(TS)$, then $\mathcal{H}$ is defined as a partially labeled, pessimistic consistent set (PLP-consistent set). Denote the set of all PLP-consistent sets by $CONS^P_{TS}(\mathcal{N})$. If $\mathcal{H} \in CONS^P_{TS}(\mathcal{N})$, and $\mathcal{H}' \notin CONS^P_{TS}(\mathcal{N})$ whenever $\mathcal{H}' \subset \mathcal{H}$, then $\mathcal{H}$ is said to be a partially labeled, pessimistic reduct (PLP-reduct). The set of all PLP-reducts is denoted by $RED^P_{TS}(\mathcal{N})$, and the core with respect to PLP-reducts is defined as $CORE^P_{TS}(\mathcal{N}) = \bigcap\{\mathcal{H}|\mathcal{H} \in RED^P_{TS}(\mathcal{N})\}$.*

A PLP-reduct is a minimal subset of $\mathcal{N}$ that preserves the positive region of N_d on TS. For a consistent p-GNDIS $(U, \mathcal{N}, N_d)$, $RED^P_{TS}(\mathcal{N}) = \{\{N_k\}|N_k \in \mathcal{N}\}$ and $CORE^P_{TS}(\mathcal{N}) = \emptyset$. Then, the PLP-reducts of a consistent p-GNDIS can be obtained easily. In this paper, we discuss the pessimistic multigranulation reduction of inconsistent p-GNDISs.

Example 2. *A p-GNDIS $(U, \mathcal{N}, N_d)$ is presented in Table 3, where $U = \{x_1, x_2, \cdots, x_6\}$ and $\mathcal{N} = \{N_1, N_2, \cdots, N_5\}$. $TS = \{x_1, x_2, x_4, x_5, x_6\}$, and $TS/N_d = \{\{x_1, x_5, x_6\}, \{x_2, x_4\}\}$ with $V_1 = \{x_1, x_5, x_6\}, V_2 = \{x_2, x_4\}$.*

Table 3. A p-GNDIS.

*	x_1	x_2	x_3	x_4	x_5	x_6
$N_1(x_i)$	$\{x_1, x_3\}$	$\{x_2, x_4\}$	$\{x_1, x_2, x_3\}$	$\{x_4\}$	$\{x_5\}$	$\{x_1, x_6\}$
$N_2(x_i)$	$\{x_1, x_2, x_3\}$	$\{x_2, x_5\}$	$\{x_1, x_3\}$	$\{x_2, x_4\}$	$\{x_1, x_5\}$	$\{x_3, x_6\}$
$N_3(x_i)$	$\{x_1, x_5\}$	$\{x_2, x_3\}$	$\{x_1, x_3\}$	$\{x_2, x_4\}$	$\{x_1, x_5\}$	$\{x_1, x_6\}$
$N_4(x_i)$	$\{x_1, x_2\}$	$\{x_2, x_3\}$	$\{x_3, x_4\}$	$\{x_2, x_4\}$	$\{x_5\}$	$\{x_4, x_6\}$
$N_5(x_i)$	$\{x_1, x_6\}$	$\{x_2, x_6\}$	$\{x_1, x_3\}$	$\{x_2, x_4\}$	$\{x_1, x_5\}$	$\{x_3, x_6\}$
$N_d(x_i)$	$\{x_1, x_5, x_6\}$	$\{x_2, x_4\}$	$\emptyset$	$\{x_2, x_4\}$	$\{x_1, x_5, x_6\}$	$\{x_1, x_5, x_6\}$

By Definition 2, we get that $\underline{\sum_{\mathcal{N}} N^P_k(V_1)} = \{x_5\}$, $\underline{\sum_{\mathcal{N}} N^P_k(V_2)} = \{x_4\}$. Then, $Pos_{\mathcal{N}}(TS) = \{x_4, x_5\}$.

Let $\mathcal{N}' = \{N_1, N_5\}$. We have that $Pos^P_{\mathcal{N}'}(TS) = \{x_4, x_5\}$. It follows that $\mathcal{N}'$ is a PLP-consistent set. Let $\mathcal{N}_1 = \{N_1\}$ and $\mathcal{N}_2 = \{N_5\}$. Then, $Pos^P_{\mathcal{N}_1}(TS) = \{x_2, x_4, x_5, x_6\} \neq Pos^P_{\mathcal{N}'}(TS)$ and $Pos^P_{\mathcal{N}_2}(TS) = \{x_1, x_4, x_5\} \neq Pos^P_{\mathcal{N}'}(TS)$. It follows from Definition 4 that $\mathcal{N}'$ is a PLP-reduct.

A related family of target set TS is constructed to compute all the PLP-reducts.

Definition 5. *Consider that* $(U, \mathcal{N}, N_d)$ *is a p-GNDIS, where* $\mathcal{N} = \{N_1, N_2, \cdots, N_m\}$ *and the target set is* TS. *Letting* $v \in TS$, *define*

$$\gamma(v) = \{N_k \in \mathcal{N} | N_k(v) \nsubseteq N_d(v)\}. \tag{3.2}$$

Define the related family of TS *as*

$$R^P(TS, \mathcal{N}, N_d) = \{\gamma(v) | v \in TS \setminus Pos^P_{\mathcal{N}}(TS)\}. \tag{3.3}$$

Proposition 2. *Let* $(U, \mathcal{N}, N_d)$ *be a p-GNDIS with* $\mathcal{N} = \{N_1, N_2, \cdots, N_m\}$ *and the target set* TS. *Then,*
(1) for any $v \in Pos^P_{\mathcal{N}}(TS)$, $\gamma(v) = \emptyset$,
(2) for any $v \in TS \setminus Pos^P_{\mathcal{N}}(TS)$, $\gamma(v) \neq \emptyset$.

Proof. (1) For any $v \in Pos^P_{\mathcal{N}}(TS)$, by Definition 2, $N_k(v) \subseteq N_d(v)$ for all $k \in \{1, \cdots, m\}$. Then, according to Definition 5, $\gamma(v) = \emptyset$.

(2) For any $v \in TS \setminus Pos^P_{\mathcal{N}}(TS)$, due to Definition 2, there exists a $k \in \{1, \cdots, m\}$ such that $N_k(v) \nsubseteq N_d(v)$. Then, $\gamma(v) \neq \emptyset$. □ □

Utilizing the related family of TS, the PLP-reducts can be characterized.

Theorem 1. *Suppose that* $(U, \mathcal{N}, N_d)$ *is a p-GNDIS. Letting* $\mathcal{H} \subseteq \mathcal{N}$ *and* $N_k \in \mathcal{N}$,

(1) $\mathcal{H} \in CONS^P_{TS}(\mathcal{N}) \Leftrightarrow \mathcal{H} \cap \gamma(v) \neq \emptyset$ *for each* $\gamma(v) \in R^P(TS, \mathcal{N}, N_d)$;

(2) $\mathcal{H} \in RED^P_{TS}(\mathcal{N}) \Leftrightarrow \mathcal{H} \cap \gamma(v) \neq \emptyset$ *for all* $\gamma(v) \in R^P(TS, \mathcal{N}, N_d)$, *and for any* $\mathcal{H}_0 \subset \mathcal{H}$, *there exists a* $\gamma(v) \in R^P(TS, \mathcal{N}, N_d)$ *such that* $\gamma(v) \cap \mathcal{H}_0 = \emptyset$;

(3) $N_k \in CORE^P_{TS}(\mathcal{N}) \Leftrightarrow \exists \gamma(v) \in R^P(TS, \mathcal{N}, N_d)$, $\gamma(v) = \{N_k\}$.

Proof. (1) "⇒". $\forall \gamma(v) \in R^P(TS, \mathcal{N}, N_d)$, $\gamma(v) \neq \emptyset$ and $v \in TS \setminus Pos^P_{\mathcal{N}}(TS)$. Because $\mathcal{H}$ is a PLP-consistent set, $Pos^P_{\mathcal{N}}(TS) = Pos^P_{\mathcal{H}}(TS)$. Then, $v \in TS \setminus Pos^P_{\mathcal{H}}(TS)$. It follows that $v \in TS$ and $v \notin \underline{\sum_{\mathcal{H}} N^P_k}(N_d(v))$. Hence, there exists an $N_k \in \mathcal{H}$ such that $N_k(v) \nsubseteq N_d(v)$. This means that $N_k \in \mathcal{H} \cap \gamma(v)$. Thus, $\mathcal{H} \cap \gamma(v) \neq \emptyset$.

"⇐". It is clear that $Pos^P_{\mathcal{N}}(TS) \subseteq Pos^P_{\mathcal{H}}(TS)$. Now, we prove $Pos^P_{\mathcal{H}}(TS) \subseteq Pos^P_{\mathcal{N}}(TS)$. $\forall v \notin Pos^P_{\mathcal{N}}(TS)$; if $v \notin TS$, then $v \notin Pos^P_{\mathcal{N}}(TS)$. If $v \in TS$, then $v \in TS \setminus Pos^P_{\mathcal{N}}(TS)$. This implies that $v \notin \underline{\sum_{\mathcal{N}} N^P_k}(N_d(v))$. Then, there exists an $N'_k \in \mathcal{N}$ such that $N'_k(v) \nsubseteq N_d(v)$. It follows that $\gamma(v) \neq \emptyset$. Thus, $\gamma(v) \in R^P(TS, \mathcal{N}, N_d)$, and $\mathcal{H} \cap \gamma(v) \neq \emptyset$. Let $N_h \in \mathcal{H} \cap \gamma(v)$. Hence, $N_h(v) \nsubseteq N_d(v)$. Therefore, $v \notin \underline{\sum_{\mathcal{H}} N^P_k}(N_d(v))$. It follows that $v \notin Pos^P_{\mathcal{H}}(TS)$.

(2) According to (1), it is easy to verify (2).

(3) "⇒". Because $N_k \in CORE^P_{TS}(\mathcal{N})$, $Pos^P_{\mathcal{N}}(TS) \supsetneq Pos^P_{\mathcal{N}-\{N_k\}}(TS)$. Otherwise, $Pos^P_{\mathcal{N}}(TS) = Pos^P_{\mathcal{N}-\{N_k\}}(TS)$. Then, there exists a PLP-reduct set $\mathcal{H} \subseteq \mathcal{N} - \{N_k\}$, which contradicts $N_k \in CORE^P_{TS}(\mathcal{N})$. Let $v_0 \in Pos^P_{\mathcal{N}-\{N_k\}}(TS) \setminus Pos^P_{\mathcal{N}}(TS)$. Then, $N_k(v_0) \nsubseteq N_d(v_0)$ and $N_h(v_0) \subseteq N_d(v_0)$ for all $N_h \in \mathcal{N} - \{N_k\}$. It follows that $\gamma(v_0) = \{N_k\}$ and $v_0 \in TS \setminus Pos^P_{\mathcal{N}}(TS)$.

"⇐". By Theorem 1, for each $\mathcal{H} \in RED^P_{TS}(\mathcal{N})$, $\gamma(v) \cap \mathcal{H} \neq \emptyset$. Hence, $N_k \in \gamma(v)$, which shows that $N_k \in CORE^P_{TS}(\mathcal{N})$. □

By Theorem 1, the PLP-reducts can be obtained by the related family $R^P(TS, \mathcal{N}, N_d)$. From Definition 5, $|R^P(TS, \mathcal{N}, N_d)| \leq |TS| \leq |U|$ and $\gamma(v)$ can be obtained directly by the neighborhoods $N_k(v)$ and $N_d(v)$. It can be seen that, due to the definition of the multiple granular structure of pessimistic multigranulation rough sets, the related families are easy to be obtained and small in number, thereby making the PLP-reducts easily computed.

Definition 6. *Let $(U, \mathcal{N}, N_d)$ be a p-GNDIS, whose related family of target set TS is $R^P(TS, \mathcal{N}, N_d)$. Define $f(R^P(TS, \mathcal{N}, N_d)) = \wedge\{\vee\gamma(v)|\ \gamma(v) \in R^P(TS, \mathcal{N}, N_d)\}$.*

$\vee\gamma(v)$ is the disjunction of all neighborhood operators in $\gamma(v)$, and $\wedge\{\vee\gamma(v)|\gamma(v) \in R^P(TS, \mathcal{N}, N_d)\}$ is the conjunction of $\vee\gamma(v)$.

Theorem 2. *Letting $\mathcal{H} = \{N_1, N_2, \cdots, N_k\} \subseteq \mathcal{N}$, $\mathcal{H} \in RED^P_{TS}(\mathcal{N}) \Leftrightarrow N_1 \wedge N_2 \wedge \cdots \wedge N_k$ is a prime implicant of $f(R^P(TS, \mathcal{N}, N_d))$.*

Proof. It is trivial based on Definition 6 and therefore omitted here. □

Example 3 below is employed to explain the related family method for computing all the PLP-reducts of a p-GNDIS.

Example 3. *Continued from Example 2, according to Definition 5, we obtain that $TS \backslash Pos^P_{\mathcal{N}}(TS) = \{x_1, x_2, x_6\}$ and*

$\gamma(x_1) = \{N_1, N_2, N_4\}$,
$\gamma(x_2) = \{N_2, N_3, N_4, N_5\}$,
$\gamma(x_6) = \{N_2, N_4, N_5\}$.

Due to Definition 6,

$$f(R^P(TS, \mathcal{N}, N_d)) = (N_1 \vee N_2 \vee N_4) \wedge (N_2 \vee N_3 \vee N_4 \vee N_5) \wedge (N_2 \vee N_4 \vee N_5)$$
$$= (N_2) \vee (N_4) \vee (N_1 \wedge N_5).$$

Thus, $\{N_2\}$, $\{N_4\}$, $\{N_1, N_5\}$ are PLP-reducts.

Based on the related family of TS, we present an algorithm (Algorithm 1) to calculate all the PLP-reducts of a p-GNDIS. In Algorithm 1, Steps 1–8 compute the related family of TS, whose time complexity is $O(|TS|^2|\mathcal{N}|)$. Steps 9–14 are to obtain all the PLP-reducts, whose time complexity is $O(\prod_{\gamma(v)\in R^P(TS,\mathcal{N},N_d)} |\gamma(v)|)$. Hence, the total time complexity of Algorithm 1 is $O(|TS|^2|\mathcal{N}| + \prod_{\gamma(v)\in R^P(TS,\mathcal{N},N_d)} |\gamma(v)|)$.

4. Computing the pessimistic multigranulation positive region based on matrix operations

In this section, we first present the matrix operations of the generalized neighborhood pessimistic lower approximation on a p-GNDIS. Then, the pessimistic multigranulation positive region can be calculated based on matrix operations.

Unless otherwise specified, for a matrix $A_{r\times s}(A_{r\times 1})$, we use $A[i, j](A[i])$ to represent the (i, j) element of A (the element of the ith row of A).

Let $U = \{x_i\,|i = 1, 2, ..., n\}$ and $X \subseteq U$. The characteristic function $g(X) = (g[1], g[2], ..., g[n])^T$ of X is defined as follows:

$$g[i] = \begin{cases} 1, & x_i \in X; \\ 0, & x_i \notin X, \end{cases} \quad i = 1, 2, .., n, \tag{4.1}$$

where T denotes the transpose operation.

It is clear that the characteristic function is a Boolean vector.

Algorithm 1 A logic algorithm for computing all the PLP-reducts of a p-GNDIS.

Input: A p-GNDIS $(U, \mathcal{N}, N_d)$ with $TS = \{x_1, x_2, \cdots, x_l\}$ and $\mathcal{N} = \{N_1, N_2, \cdots, N_m\}$.
Output: All the PLP-reducts $RED^P_{TS}(\mathcal{N})$

for $i = 1 : l$ **do**
  Initialize $\gamma(x_i) \leftarrow \emptyset$;
  for $k = 1 : m$ **do**
    if $N_k(x_i) \nsubseteq N_d(x_i)$ **then**
      $\gamma(x_i) \leftarrow \gamma(x_i) \cup \{N_k\}$
    end if
  end for
end for
Initialize $RED^P_{TS}(\mathcal{N}) \leftarrow \emptyset$;
for $i = 1 : l$ and $\gamma(x_i) \neq \emptyset$ **do**
  $RED^P_{TS}(\mathcal{N}) \leftarrow RED^P_{TS}(\mathcal{N}) \wedge (\vee\gamma(x_i))$
end for
Compute $RED^P_{TS}(\mathcal{N}) \leftarrow \vee_{l=1}^{t}(\wedge_{k=1}^{s_l} N_k)$;
Return $RED^P_{TS}(\mathcal{N})$.

Definition 7. *Consider that $(U, \mathcal{N}, N_d)$ is a p-GNDIS, where $U = \{x_i \,|\, i = 1, 2, ..., n\}$, and $\mathcal{N} = \{N_1, N_2, \cdots, N_m\}$. For $N_k \in \mathcal{N}$, define a matrix $M_{N_k} = \left(M_{N_k}[i, j]\right)_{n\times n}$ of N_k as follows:*

$$M_{N_k}[i, j] = \begin{cases} 1, & x_j \in N_k(x_i); \\ 0, & x_j \notin N_k(x_i), \end{cases} \quad i, j = 1, 2, .., n. \tag{4.2}$$

The matrix M_{N_k} provides a matrix representation of the neighborhood $N_k(x_i)(i = 1, 2, \cdots, n))$, with the ith row denoting the transpose of the characteristic function of $N_k(x_i)$.

Example 4. *Continued from Example 2, by Definition 7, we get*

$$M_{N_1} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_{N_2} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad M_{N_3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_{N_4} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}, \quad M_{N_5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Definition 8. *[17] Let $M = (M[i, j])_{s\times t}$ be a matrix. Define a matrix operation $\sim$ as follows:* $\sim M = (\sim M[i, j])_{s\times t}$*, where* $\sim M[i, j] = \begin{cases} 1, & M[i, j] = 0; \\ 0, & M[i, j] \neq 0. \end{cases}$

Lemma 1. *[17] Let $X, Y \subseteq U$. $X \subseteq Y \Leftrightarrow (g(X))^T g(Y^c) = 0$.*

Clearly, $g(Y^c) =\sim g(Y)$. In the following, we denote the matrix $(\sim g(V_1), \sim g(V_2), \cdots, \sim g(V_l)) = (g(V_1^c), g(V_2^c), \cdots, g(V_l^c))$ by $g(V^c)$.

Proposition 3. *Consider that $(U, \mathcal{N}, N_d)$ is a p-GNDIS, where $U = \{x_i | i = 1, 2, ..., n\}$, $\mathcal{N} = \{N_1, N_2, \cdots, N_m\}$, the target set is TS, and $TS/N_d = \{V_1, V_2, \cdots, V_l\}$. For any $k \in \{1, 2, ..., m\}$, $i \in \{1, 2, ..., n\}$, $j \in \{1, 2, ..., l\}$, we have*
(1) $(M_{N_k} \cdot g(V^c))[i, j] = 0 \Leftrightarrow N_k(x_i) \subseteq V_j$,
(2) $(M_{N_k} \cdot g(V^c))[i, j] \neq 0 \Leftrightarrow N_k(x_i) \nsubseteq V_j$.

Proof. We can obtain the conclusion by Definition 7 and Lemma 1. □

Example 5. *Consider Example 4. By Definition 7, we get*

$$M_{N_1} \cdot g(V^c) = \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ 2 & 2 \\ 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix}, \quad M_{N_2} \cdot g(V^c) = \begin{pmatrix} 2 & 2 \\ 1 & 1 \\ 1 & 2 \\ 2 & 0 \\ 0 & 2 \\ 1 & 2 \end{pmatrix}, \quad M_{N_3} \cdot g(V^c) = \begin{pmatrix} 0 & 2 \\ 2 & 1 \\ 1 & 2 \\ 2 & 0 \\ 0 & 2 \\ 0 & 2 \end{pmatrix},$$

$$M_{N_4} \cdot g(V^c) = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 2 & 1 \\ 2 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad M_{N_5} \cdot g(V^c) = \begin{pmatrix} 0 & 2 \\ 1 & 1 \\ 1 & 2 \\ 2 & 0 \\ 0 & 2 \\ 1 & 2 \end{pmatrix}.$$

Then,

$$\sim (M_{N_1} \cdot g(V^c)) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sim (M_{N_2} \cdot g(V^c)) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sim (M_{N_3} \cdot g(V^c)) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix},$$

$$\sim (M_{N_4} \cdot g(V^c)) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sim (M_{N_5} \cdot g(V^c)) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

We present two matrix operations to get the generalized neighborhood pessimistic lower approximations.

Definition 9. *[26] Let* $M_1 = (M_1[i, j])_{n_1\times n_2}$ *and* $M_2 = (M_2[i, j]))_{n_1\times n_2}$ *be two matrices. The maximum and minimum operations of* M_1 *and* M_2 *are defined by*
(1) Maximum operation " max " : $\max(M_1, M_2) = (\max(M_1[i, j], M_2[i, j]))_{n_1\times n_2}$*;*
(2) Minimum operation " min " : $\min(M_1, M_2) = (\min(M_1[i, j], M_2[i, j]))_{n_1\times n_2}$.
max(M_1) *represents a column vector by taking the maximum operation of all column vextor of* M_1.

Theorem 3. *Consider that* $(U, \mathcal{N}, N_d)$ *is a p-GNDIS, where* $U = \{x_i | i = 1, 2, ..., n\}$*,* $\mathcal{N} = \{N_1, N_2, \cdots, N_m\}$*, the target set is* TS*, and* $TS/N_d = \{V_1, V_2, \cdots, V_l\}$*. Let* $min_{k=1}^{m}(\sim (M_{N_k} \cdot g(V^c))) = (o[i, j])_{n\times l}$*. Then, for each* $i \in \{1, 2, ..., n\}$*,* $j \in \{1, 2, ..., l\}$*,*
(1) $o[i, j] = 1 \Leftrightarrow x_i \in \underline{\sum_{\mathcal{N}} N_k^P}(V_j)$*,*
(2) $o[i, j] = 0 \Leftrightarrow x_i \notin \underline{\sum_{\mathcal{N}} N_k^P}(V_j)$*.*

Proof. (1) $o[i, j] = 1 \Leftrightarrow \forall k \in \{1, 2, \cdots, m\}, (\sim (M_{N_k} \cdot g(V^c)))[i, j] = 1$ by Definition 9
$\Leftrightarrow \forall k \in \{1, 2, \cdots, m\}, (M_{N_k} \cdot g(V^c))[i, j] = 0$ due to Definition 8
$\Leftrightarrow \forall k \in \{1, 2, \cdots, m\}, N_k(x_i) \subseteq V_j$ according to Proposition 3
$\Leftrightarrow x_i \in \underline{\sum_{\mathcal{N}} N_k^P}(V_j)$ by Definition 2.
(2) $o[i, j] = 0 \Leftrightarrow \exists k \in \{1, 2, \cdots, m\}, (\sim (M_{N_k} \cdot g(V^c)))[i, j] = 0$ due to Definition 9
$\Leftrightarrow \exists k \in \{1, 2, \cdots, m\}, (M_{N_k} \cdot g(V^c))[i, j] \neq 0$ by Definition 8
$\Leftrightarrow \exists k \in \{1, 2, \cdots, m\}, N_k(x_i) \nsubseteq V_j$ by Proposition 3
$\Leftrightarrow x_i \notin \underline{\sum_{\mathcal{N}} N_k^P}(V_j)$ according to Definition 2. □

From Theorem 3, we can compute the the generalized neighborhood pessimistic lower approximations based on matrix operations.

Theorem 4. *Consider that* $(U, \mathcal{N}, N_d)$ *is a p-GNDIS, where* $U = \{x_i | i = 1, 2, ..., n\}$*,* $\mathcal{N} = \{N_1, N_2, \cdots, N_m\}$*, the target set is* TS*, and* $TS/N_d = \{V_1, V_2, \cdots, V_l\}$*. Let* $max(min_{k=1}^{m}(\sim M_{N_k}(g(V^c)))) = (h_i)_{n\times 1}$*. Then, for each* $i \in \{1, 2, ..., n\}$*,*
(1) $h_i = 1 \Leftrightarrow x_i \in Pos_{\mathcal{N}}(TS)$*,*
(2) $h_i = 0 \Leftrightarrow x_i \notin Pos_{\mathcal{N}}(TS)$*.*

Proof. (1) $h_i = 1 \Leftrightarrow \exists j \in \{1, 2, ..., l\}, o_{ij} = 1$ by Definition 9
$\Leftrightarrow \exists j \in \{1, 2, ..., l\}, x_i \in \underline{\sum_{\mathcal{N}} N_k^P}(V_j)$ based on Theorem 2
$\Leftrightarrow x_i \in Pos_{\mathcal{N}}(TS)$.
(2) $h_i = 0 \Leftrightarrow \forall j \in \{1, 2, ..., l\}, o_{ij} = 0$ according to Definition 9
$\Leftrightarrow \forall j \in \{1, 2, ..., l\}, x_i \notin \underline{\sum_{\mathcal{N}} N_k^P}(V_j)$ by Theorem 2
$\Leftrightarrow x_i \notin Pos_{\mathcal{N}}(TS)$ by Definition 9. □

According to Theorem 4, the $Pos_{\mathcal{N}}(TS)$ can be calculated by matrix operations. By Theorems 3 and 4, due to the multiple-granularity structure of multigranulation rough sets, we can obtain $\underline{\sum_{\mathcal{N}} N_k^P}(V_j)$ and $Pos_{\mathcal{N}}(TS)$ by performing separate calculations for each granule $N_k (k = 1, \cdots, m)$.

Example 6. *Consider Example 5. By Definition 9, we obtain*

$$min_{k=1}^{5}(\sim (M_{N_k} \cdot g(V^c))) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, max(min_{k=1}^{5}(\sim (M_{N_5} \cdot g(V^c)))) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

It follows that $\underline{\sum_{\mathcal{N}} N_k^P}(V_1) = \{x_5\}$, $\underline{\sum_{\mathcal{N}} N_k^P}(V_2) = \{x_4\}$, *and* $Pos_{\mathcal{N}}(TS) = \{x_4, x_5\}$.

5. Computing the pessimistic multigranulation reduction based on matrix operations

In this section, we present the Boolean matrix of the related family and give the matrix operations for a PLP-reduct. At the same time, we construct the relationships between the matrices for the positive region and the the Boolean matrix of the related family.

Definition 10. *Consider that* $(U, \mathcal{N}, N_d)$ *is a p-GNDIS, where* $U = \{x_i \,|i = 1, 2, ..., n\}$, $\mathcal{N} = \{N_1, N_2, \cdots, N_m\}$, *and the target set is* TS. *The related family of* TS *is* $R^P(TS, \mathcal{N}, N_d)$. *Define the Boolean matrix* $M(R^P) = (M(R^P)[i, j])_{n\times m}$ *of the related family* $R^P(TS, \mathcal{N}, N_d)$ *by: for each* $i \in \{1, 2, ..., n\}$, $j \in \{1, 2, ..., m\}$,

$$M(R^P)[i, j] = \begin{cases} 1, & \gamma(x_i) \in R^P(TS, \mathcal{N}, N_d), N_j \in \gamma(x_i); \\ 0, & otherwise. \end{cases} \tag{5.1}$$

The matrix $M(R^P)$ represents the related family in the form of a Boolean matrix. The ith row of $M(R^P)$ is the transpose of the characteristic function of $\gamma(x_i)$ with $\gamma(x_i) \in R^P(TS, \mathcal{N}, N_d)$.

Example 7. *Continued from Example 3, according to Definition 10, we obtain*

$$M(R^P) = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Theorem 5. *Consider that* $(U, \mathcal{N}, N_d)$ *is a p-GNDIS, where* $U = \{x_i \,|i = 1, 2, ..., n\}$, $\mathcal{N} = \{N_1, N_2, \cdots, N_m\}$, *the target set is* TS, *and* $TS/N_d = \{V_1, V_2, \cdots, V_l\}$. *Denote* $\mathbf{q}_k =\sim max(\sim (M_{N_k} \cdot g(V^c)))(k = 1, 2, \cdots, m)$, *and* $Q = (\mathbf{q}_1, \cdots, \mathbf{q}_m)$. *Then,*

$$diag(g(TS \setminus Pos_{\mathcal{N}}(TS))^T)Q = M(R^P). \tag{5.2}$$

Proof. For any $i \in \{1, 2, ..., n\}$, $k \in \{1, 2, \cdots, m\}$,

$g(TS \setminus Pos_{\mathcal{N}}(TS))[i] = 1$ and $\mathbf{q}_k[i] = 1$

$\Leftrightarrow x_i \in TS \setminus Pos_{\mathcal{N}}(TS)$, and $max(\sim (M_{N_k} \cdot g(V^c)))[i] = 0$ according to Definition 8

$\Leftrightarrow x_i \in TS \setminus Pos_{\mathcal{N}}(TS)$, and $(\sim (M_{N_k} \cdot g(V^c)))[i, j] = 0$ for all $j \in \{1, \cdots, m\}$ by Definition 9

$\Leftrightarrow x_i \in TS \setminus Pos_{\mathcal{N}}(TS)$, and $(M_{N_k} \cdot g(V^c))[i, j] \neq 0$ for all $j \in \{1, \cdots, m\}$ by Definition 8
$\Leftrightarrow x_i \in TS \setminus Pos_{\mathcal{N}}(TS)$, and $N_k(x_i) \nsubseteq V_j$ for all $j \in \{1, \cdots, m\}$ based on Proposition 3
$\Rightarrow x_i \in TS \setminus Pos_{\mathcal{N}}(TS)$, and $N_k(x_i) \nsubseteq N_d(x_i)$
$\Rightarrow N_k \in \gamma(x_i)$, and $\gamma(x_i) \in R^P(TS, \mathcal{N}, N_d)$ by Definition 5
$\Rightarrow M(R^P)[i, k] = 1$ according to Definition 10;
$g(TS \setminus Pos_{\mathcal{N}}(TS))[i] = 1$ and $\mathbf{q}_k[i] = 0$
$\Leftrightarrow x_i \in TS \setminus Pos_{\mathcal{N}}(TS)$, and $max(\sim (M_{N_k} \cdot g(V^c)))[i] = 1$ due to Definition 8
$\Leftrightarrow x_i \in TS \setminus Pos_{\mathcal{N}}(TS)$, and $\exists j \in \{1, 2, \cdots, m\}$ such that $(\sim (M_{N_k} \cdot g(V^c)))[i, j] = 1$ by Definition 9
$\Leftrightarrow x_i \in TS \setminus Pos_{\mathcal{N}}(TS)$, and $\exists j \in \{1, 2, \cdots, m\}$ such that $(M_{N_k} \cdot g(V^c))[i, j] = 0$ by Definition 8
$\Leftrightarrow x_i \in TS \setminus Pos_{\mathcal{N}}(TS)$, and $\exists j \in \{1, 2, \cdots, m\}$ such that $N_k(x_i) \subseteq V_j$ based on Proposition 3
$\Rightarrow x_i \in TS \setminus Pos_{\mathcal{N}}(TS)$, $N_k(x_i) \subseteq N_d(x_i)$
$\Rightarrow N_k \notin \gamma(x_i)$ by Definition 5
$\Rightarrow M(R^P)[i, k] = 0$ due to Definition 10;
$g(TS \setminus Pos_{\mathcal{N}}(TS))[i] = 0$ and $\mathbf{q}_k[i] = 1$, or $g(TS \setminus Pos_{\mathcal{N}}(TS))[i] = 0$ and $\mathbf{q}_k[i] = 0$
$\Rightarrow x_i \notin TS \setminus Pos_{\mathcal{N}}(TS)$
$\Rightarrow \gamma(x_i) \notin R^P(TS, \mathcal{N}, N_d)$ based on Definition 5
$\Rightarrow M(R^P)[i, k] = 0$ from Definition 10.

Because the elements of $diag(g(TS \setminus Pos_{\mathcal{N}}(TS))^T)Q$ and $M(R^P)$ are equal to 0 or 1, we get that $M(R^P) = diag(g(TS \setminus Pos_{\mathcal{N}}(TS))^T)Q$. □

According to Theorem 5, the Boolean matrix $M(R^P)$ of the related family $R^P(TS, \mathcal{N}, N_d)$ can be obtained from the matrix operations for $Pos_{\mathcal{N}}(TS)$.

Example 8. *Continued from Example 7, we get*

$$\mathbf{q}_1 = \begin{pmatrix}1\\0\\1\\0\\0\\0\end{pmatrix}, \mathbf{q}_2 = \begin{pmatrix}1\\1\\1\\0\\0\\1\end{pmatrix}, \mathbf{q}_3 = \begin{pmatrix}0\\1\\1\\0\\0\\0\end{pmatrix}, \mathbf{q}_4 = \begin{pmatrix}1\\1\\1\\0\\0\\1\end{pmatrix}, \mathbf{q}_5 = \begin{pmatrix}0\\1\\1\\0\\1\\1\end{pmatrix}.$$

Clearly, $g(TS \setminus Pos_{\mathcal{N}}(TS))^T = (1, 1, 0, 0, 0, 1)^T$. *Thus,*

$$\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\end{pmatrix}\begin{pmatrix}1&1&0&1&0\\0&1&1&1&1\\1&1&1&1&1\\0&0&0&0&0\\0&0&0&0&1\\0&1&0&1&1\end{pmatrix} = \begin{pmatrix}1&1&0&1&0\\0&1&1&1&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&1&1\end{pmatrix},$$

that is, $diag(g(TS \setminus Pos_{\mathcal{N}}(TS))^T)Q = M(R^P)$.

Theorem 6. *Consider that* $(U, \mathcal{N}, N_d)$ *is a p-GNDIS, where* $U = \{x_i | i = 1, 2, ..., n\}$, $\mathcal{N} = \{N_1, N_2, \cdots, N_m\}$. $M(R^P) = (M(R^P)[i, j])_{n \times m}$ *is the Boolean matrix of related family* $R^P(TS, \mathcal{N}, N_d)$. *Then,*

$$g(TS \setminus Pos_{\mathcal{N}}(TS)) = max(M(R^P)). \tag{5.3}$$

Proof. For any $i \in \{1, 2, ..., n\}$, if $g(TS \setminus Pos_{\mathcal{N}}(TS))[i] = 1$, then we have $x_i \in TS \setminus Pos_{\mathcal{N}}(TS)$. By Proposition 2, $\gamma(x_i) \neq \emptyset$. Thus, $\exists N_j \in \mathcal{N}$ such that $N_j \in \gamma(x_i)$. By Definition 10, $M(R^P)[i, j] = 1$, which follows that $(max(M(R^P))[i] = 1$.

If $g(TS \setminus Pos_{\mathcal{N}}(TS))[i] = 0$, then $x_i \notin TS \setminus Pos_{\mathcal{N}}(TS)$, which implies that $x_i \notin TS$ or $x_i \in Pos_{\mathcal{N}}(TS)$. If $x_i \notin TS$, by Definition 5 and Definition 10, $M(R^P)[i, j] = 0$ for all $j \in \{1, \cdots, m\}$. Hence, $(max(M(R^P))[i] = 0$. If $x_i \in Pos_{\mathcal{N}}(TS)$, by Proposition 2, $\gamma(x_i) = \emptyset$. According to Definition 10, $M(R^P)[i, j] = 0$ for all $j \in \{1, \cdots, m\}$. It implies that $(max(M(R^P))[i] = 0$.

Because the elements of $g(TS \setminus Pos_{\mathcal{N}}(TS))$ and $max(M(R^P))$ are equal to 0 or 1, we get that $max(M(R^P)) = g(TS \setminus Pos_{\mathcal{N}}(TS))$. □

To obtain a PLP-reduct quickly, we design a heuristic algorithm to get a suboptimal PLP-reduct based on matrix operations. In Algorithm 2, the time complexity of Steps 1–8 is $O(|U|^2|\mathcal{N}||TS/N_d|)$, and the time complexity of Steps 9–14 is $O(|U||\mathcal{N}|^2)$. Hence, the total time complexity of Algorithm 2 is $O(|U|^2|\mathcal{N}||TS/N_d| + |U||\mathcal{N}|^2)$. The suboptimal PLP-reduct obtained from Algorithm 1 may not be minimal, but it still preserves the positive region unchanged.

Algorithm 2 A logic algorithm for computing a suboptimal PLP-reduct of a p-GNDIS based on matrix operations.

Input: A p-GNDIS $(U, \mathcal{N}, N_d)$ with $U = \{x_1, x_2, \cdots, x_n\}$, $\mathcal{N} = \{N_1, N_2, \cdots, N_m\}$, $TS \subseteq U$ and $TS/N_d = \{V_1, V_2, \cdots, V_l\}$.
Output: A PLP-reduct RED.

1: Construct the matrix $M_{N_k}(k = 1, \cdots, m)$ and $g(V^c)$;
2: **for** $k = 1 : m$ **do**
3: Compute $\sim (M_{N_k} \cdot g(V^c))$;
4: Compute $\mathbf{q}_k = \sim max(\sim (M_{N_k} \cdot g(V^c)))$;
5: **end for**
6: Compute $g(Pos_{\mathcal{N}}(TS)) \leftarrow max(min_{k=1}^{m}(\sim M_{N_k}(g(V^c))))$;
7: Compute $M(R^P)$ by (5.2) in Theorem 5;
8: Initialize $RED \leftarrow \emptyset$, $H \leftarrow \emptyset$, $H_0 \leftarrow \emptyset$ and $T \leftarrow \emptyset$;
9: **while** $g(TS) - g(Pos_{\mathcal{N}}(TS)) \neq H$ **do**
10: Select ith column T from $M(R^P)$ such that T has the highest number of elements 1;
11: $H_0 \leftarrow max(H, T)$;
12: **if** $H \neq H_0$ **then**
13: $H \leftarrow H_0$;
14: $RED \leftarrow RED \cup \{N_i\}$;
15: **end if**
16: Remove T from $M(R^P)$;
17: **end while**
18: **Return** RED.

Example 9. *Continued from Example 8, by Algorithm 2, we can get the reduct* $\{N_2\}$.

6. Conclusions

The pessimistic multigranulation reduction of p-GNDISs has been discussed in this paper, and the related family method has been used for obtaining all the PLP-reducts of a p-GNDIS. The matrix operations of the pessimistic multigranulation positive region on a p-GNDIS have been presented, and relationships between the matrices for the pessimistic multigranulation positive region and the Boolean matrix of the related family have been explored. Hence, a logic algorithm to get a PLP-reduct of a p-GNDIS by matrix operations has been desinged. A p-GNDIS is a general model of a partially labeled decision information systems, a partially labeled incomplete decision information system, a partially labeled neighborhood decision information system, and so on. However, the p-GNDIS discussed in this paper satisfies that the labeled objects must be perfectly separable into equivalence classes. In our further work, we will explore the multigranulation reduction of different kinds of decision information systems, p-GNDISs with the operator N_d as a general neighborhood operator, and dynamic p-GNDISs.

Author contributions

Yanlan Zhang: Conceptualization, funding acquisition, formal analysis, writing – original draft; Changqing Li: Conceptualization, validation, funding acquisition, writing – review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interests.

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