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Research article

An innovative perspective on fractional inequalities through fractional operators and extended convexity

Muhammad Imran[1], Ahsan Mehmood[2], Shahid Mubeen[1,3], Muhammad Samraiz[1] and Ishtiaq Ali[4,*]

[1] Department of Mathematics, University of Sargodha, Sargodha 40100, Pakistan; imranmath84@gmail.com, muhammad.samraiz@uos.edu.pk

[2] School of Mathematical Sciences, Shanghai Key Laboratory of Pure Mathematics and Mathematical Practice, East China Normal University, 500 Dongchuan Road, Shanghai 200241, China; mehmoodahsan154@gmail.com

[3] Department of Mathematics, Baba Guru Nanak Univeristy, Nankana Sahib, Pakistan; smjhanda@gmail.com

[4] Department of Mathematics and Statistics, College of Science, King Faisal University, P.O. Box 400, Al-Ahsa 31982, Saudi Arabia; iamirzada@kfu.edu.sa

* **Correspondence:** Email: iamirzada@kfu.edu.sa.

Abstract: The theory of integral inequalities is significantly advanced by the relationship between fractional calculus and convexity. The current study used extended convex functions and Katugampola fractional operators to derive Hermite-Hadamard, Fejér-Hermite-Hadamard-type inequalities as well as a few other fractional integral inequalities. We used two extended-type convex functions: log-convex and exponentially trigonometric convex functions. Our conclusions were supported by tabular data and graphical representations, which offer numerical and visual validation of the explored results. This study improves mathematical analysis and expands the scope of these inequalities by emphasizing their applicability across different forms of convexity. The knowledge acquired is highly valuable for theoretical investigation as well as real-world applications in a variety of scientific fields.

Keywords: log-convex function; exponentially trigonometric convex function; Katugampola fractional integrals; Hölder's inequality; Hermite-Hadamard inequalities
Mathematics Subject Classification: 26A33, 26D10, 35J05

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1. Introduction

The study of inequalities is still expanding, and it has become a practical and effective tool for investigating a wide range of problems in numerous mathematical fields. This theory has attracted the attention of many academics in recent years, inspiring new research directions and influencing various facets of mathematical analysis and its applications. Inequalities such as those attributed to Jensen, Hadamard, Hilbert, Hardy, Opial, Sobolev, Levin, and Lyapunov are among the many classical varieties that have a long history and significant influence across diverse areas of mathematics. Key ideas in the study of inequalities include Minkowski's inequality, Hölder's inequality, and the arithmetic mean geometric mean inequality. Given the widespread use of these and many other fundamental inequalities, it is not surprising that numerous studies have been conducted, achieving a variety of significant results. The last few decades have witnessed rapid progress in inequality theory, leading to new results as well as simpler proofs of previously established findings. Although the concepts of convex sets and convex functions are relatively simple to explain, it is remarkable how many diverse ideas they inspire. Numerous important fields, including statistical mechanics, thermodynamics, mathematical economics, and statistics, rely fundamentally on convexity. Given how simple it is to explain the concepts of convex sets and functions, it might be surprising to discover how many different ideas they can inspire. Convexity turns out to be essential in many real-world disciplines, including statistical mechanics, thermodynamics, mathematical economics, and statistics. Fractional calculus is a subfield of mathematical analysis that studies integrals and derivatives of arbitrary order. The field has gained considerable prominence in recent years due to its wide range of applications across different disciplines. Over the past several decades, scientific activity in this area has increased significantly. Fractional derivatives and integrals, which are often more accurate and flexible than their classical counterparts, have attracted growing interest for modeling and simulating complex systems in numerous applications. In fact, fractional calculus provides a variety of powerful tools for solving problems involving special functions in mathematical physics, as well as their extensions and generalizations in one or more variables, including differential and integral equations. In [1–4], several researchers discussed applications of fractional calculus in physics, the theory and applications of fractional differential equations, bioengineering applications of fractional calculus, and introductory aspects of fractional derivatives, fractional differential equations, methods for their solutions, and related applications. In [5], the authors developed new fractional derivatives based on an extended Mittag-Leffler function, leading to the introduction of nonlocal and nonsingular kernels. One derivative is formulated within the Riemann-Liouville framework, while the other is defined in the Caputo sense. The corresponding fractional integral was obtained by applying the Laplace transform. These novel derivatives were employed to model heat flow in heterogeneous media with multiple scales. In [6], the authors proposed a new definition of a fractional derivative with a smooth kernel, assuming distinct roles for the temporal and spatial variables. In [7], the researchers investigated general fractional derivatives involving a nonsingular power law kernel. Furthermore, Losada et al. [8] studied related fractional differential equations and derived the fractional integral associated with the newly introduced Caputo-Fabrizio fractional derivative. In [9], Yang et al. introduced a new fractional derivative in the form of a normalized function, which employs a singular kernel. These works illustrate the development of generalized fractional calculus. A generalized system of fractional derivatives and integrals, together with the corresponding Laplace transforms, was discussed by Jarad and Abdeljawad [10]. In [11], Samraiz et al. demonstrated how such generalized operators can be applied to solve fractional partial differential equations arising in mathematical physics. In [12], the author extended this theory by defining new fractional operators whose kernels are based on the multivariate Mittag-Leffler function. The Laplace transform was also employed to obtain analytical solutions to anomalous heat diffusion problems. Abdeljawad et al. [13] characterized the right fractional derivative and its corresponding right fractional integral associated with a newly introduced nonlocal fractional derivative with a Mittag-Leffler kernel, and they also derived the corresponding integration-by-parts formula. In [14], a monotonicity result for the Caputo-type fractional difference operator was established, along with a variant of the mean value theorem that extends to fractional differences and compares them with the classical discrete fractional case. Furthermore, Abdeljawad et al. [15] presented an integration-by-parts formula, characterized the right fractional derivative and its associated right fractional integral with an exponential kernel, and validated the results using the q-operator.

In both pure and applied mathematics, fractional integral inequalities play a crucial role in the advancement and development of numerous mathematical methods. The precise formulation of these inequalities is key to ensuring the existence and uniqueness of fractional models. Convexity theory plays a foundational role in this area due to its distinctive properties and structure. Convex functions are also fundamental in optimization, economics, and machine learning, as they guarantee a unique global minimum and predictable behavior. Interested researchers can study the details of monotonicity, convexity, and inequalities involving generalized elliptic integrals in [16]. In [17], Sun et al. established new local fractional Hermite-Hadamard-type and Ostrowski-type inequalities with generalized Mittag-Leffler kernels for generalized h-preinvex functions. Furthermore, in [18], Hyder et al. proved a new class of fractional inequalities through the concept of convexity and extended Riemann-Liouville integrals. In recent years, several researchers have derived Hermite-Hadamard inequalities by combining convex function theory with Riemann-Liouville fractional integrals [19–21]. Building on this framework, Wu et al. [22] investigated Hermite-Hadamard inequalities for the k-fractional integral operator.

Today, numerous researchers are concerned with the developments of new results related to both classical and weak inequalities involving fractional integrals, such as Hermite-Hadamard, Hermite-Hadamard-Fejér, midpoint, trapezoidal-type, and many others. In [23], Sarikaya et al. obtained new fractional integral inequalities based on the Riemann-Liouville fractional integral. Their work generalized trapezoidal- and midpoint-type inequalities associated with the Eulerian Beta function. In [24], the authors introduced new fractional integral inequalities, including Hermite-Hadamard-type inequalities and their refinements, using classical convex functions and fractional integral operators. In the present work, we establish new inequalities involving extended notions of convexity. Moreover, the validity of the obtained results is illustrated through graphical representations and numerical tables.

2. Preliminaries

The key terms and ideas that aid in our comprehension of our primary findings are covered in this part.

Definition 2.1. *The function $\zeta\colon I \subseteq \mathbb{R} \to \mathbb{R}$ is called convex if it satisfies the following inequality:*

$$\zeta(ta_1 + (1-t)b_1) \leq t\zeta(a_1) + (1-t)\zeta(b_1), \tag{2.1}$$

where $a_1, b_1 \in I \subseteq \mathbb{R}$ and $t \in [0,1]$. The usefulness and richness of such a basic description is surprising. For more information, scholars can refer to [25].

Definition 2.2. *Let $\zeta : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a_1, b_1 \in I$ with $a_1 < b_1$. The inequality*

$$\zeta\Big(\frac{a_1+b_1}{2}\Big) \le \frac{1}{b_1-a_1}\int_{a_1}^{b_1} \zeta(t)dt \le \frac{\zeta(a_1)+\zeta(b_1)}{2} \tag{2.2}$$

is frequently referred to as Hermite-Hadamard's inequality in literature. For more details, see [26].

Definition 2.3. *A function ζ: $I \subseteq \mathbb{R} \to \mathbb{R}$ is called an exponential trigonometric convex function if*

$$\zeta(ta_1 + (1-t)b_1) \le \frac{\sin(\frac{\pi t}{2})}{e^{1-t}}\zeta(a_1) + \frac{\cos(\frac{\pi t}{2})}{e^{t}}\zeta(b_1), \tag{2.3}$$

for every $a_1, b_1 \in I \subseteq \mathbb{R}$, $t \in [0,1]$. The expressions $\frac{\sin(\frac{\pi t}{2})}{e^{1-t}} \le t$ and $\frac{\cos(\frac{\pi t}{2})}{e^{t}} \le 1-t$ hold for all $t \in [0,1]$. Consequently, every exponential trigonometric convex function is convex in the classical sense, but the converse is not generally true. This stronger condition often leads to more precise inequalities in analysis, particularly for establishing tighter bounds in fractional and integral operator theory. Introducing such a specialized class allows mathematicians to prove refined versions of classical results (like Hermite-Hadamard, Ostrowski, or trapezoidal inequalities) under a more restrictive, structured assumption. For further details, please visit [27].

Definition 2.4. *A function ζ: $I \to (0,\infty)$ is said to be log-convex if $\log \zeta$ is convex or, equivalently, if for every $a_1, b_1 \in I$ and $t \in [0,1]$, one has the inequality*

$$log\zeta(ta_1 + (1-t)b_1) \le tlog[\zeta(a_1)] + (1-t)log[\zeta(b_1)]. \tag{2.4}$$

Dragomir in [28, 29] developed some integral inequalities for log-convex functions. These inequalities are closely related to the standard Hermite-Hadamard inequality. Log-convex functions are frequently employed in statistics, optimization, and probability theory.

Definition 2.5. *The Riemann-Liouville left- and right-side fractional integrals of order β can be defined by [2]*

$$(I^{\beta}_{a+})\zeta(x) = \frac{1}{\Gamma(\beta)}\int_a^x (x-t)^{\beta-1}\zeta(t)dt, \tag{2.5}$$

where $x > a$, and

$$(I^{\beta}_{b-})\zeta(x) = \frac{1}{\Gamma(\beta)}\int_x^b (t-x)^{\beta-1}\zeta(t)dt, \tag{2.6}$$

where $x < b$.

Definition 2.6. *Let $[a,b] \subseteq \mathbb{R}$ be a finite interval. Then, the left- and right-side Katugampola fractional integrals of order $\beta > 0$ of $\zeta \in X^{\rho}_{c}(a,b)$ are defined by [2]*

$$({}^{\rho}I^{\beta}_{a+})\zeta(x) = \frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_a^x t^{\rho-1}(x^{\rho}-t^{\rho})^{\beta-1}\zeta(t)dt, \tag{2.7}$$

where $x > a$, and

$$({}^{\rho}I_{b-}^{\beta})\zeta(x) = \frac{\rho^{1-\beta}}{\Gamma(\beta)} \int_x^b t^{\rho-1}(t^{\rho} - x^{\rho})^{\beta-1}\zeta(t)dt, \tag{2.8}$$

where $x < b$.

Lemma 2.7. *[24] Suppose that $\zeta : [x_1^{\rho}, y_1^{\rho}] \subseteq \mathbb{R} \to \mathbb{R}$ is a differentiable mapping on (x_1^{ρ}, y_1^{ρ}) with $0 \leq x_1 < y_1$. Thus, if the fractional integrals exist, then the following equality is valid:*

$$\zeta\Big(\frac{x_1^{\rho} + y_1^{\rho}}{2}\Big) - \frac{\beta\rho^{\beta}\Gamma(\beta+1)}{2(x_1^{\rho} - y_1^{\rho})^{\beta}}[{}^{\rho}I_{x_1+}^{\beta} f(y_1^{\rho}) +^{\rho} I_{y_1-}^{\beta}\zeta(x_1^{\rho})]$$
$$= \frac{y_1^{\rho} - x_1^{\rho}}{2} \int_0^1 [(1-t^{\rho})^{\beta} - t^{\rho\beta}]t^{\rho-1}\zeta'(t^{\rho}x_1^{\rho} + (1-t^{\rho})y_1^{\rho})dt,$$

where $\rho > 0$, $\beta > 0$, and $t \in [0, 1]$.

Lemma 2.8. *[30] For $0 < \rho \leq 1$, we have $|a_1^{\rho} - b_1^{\rho}| \leq (b_1 - a_1)^{\rho}$.*

Lemma 2.9. *[30] If $h : [a_1, b_1] \to \mathbb{R}$ is a function such that it is non-negative, symmetric, and integrable to $\frac{a_1+b_1}{2}$, then the following result is valid:*

$$I_{a_1+}^{\mu}(h)(b_1) = I_{b_1-}^{\mu}(h)(a_1) = \frac{1}{2}[I_{a_1+}^{\mu}(h)(b_1) + I_{b_1-}^{\mu}(h)(a_1)].$$

3. Main results

This section uses exponential trigonometric and log-convex functions along with Katugampola fractional operators to prove some novel findings involving Hermite-Hadamard, Fejér, and a few other fractional integral inequalities. First, we use an exponentially trigonometric convex function to create a new Hermite-Hadamard-type inequality.

Theorem 3.1. *Suppose that ζ: $[a_1, b_1] \to \mathbb{R}$ is $L^1[a_1, b_1]$ and an exponentially trigonometric convex function, and then we have*

$$\zeta\Big(\frac{a_1 + b_1}{2}\Big) \leq \sqrt{\frac{2}{e}}\frac{1}{(b_1 - a_1)} \int_{a_1}^{b_1} \zeta(u)du \leq \sqrt{\frac{2}{e}}\Big(\frac{\zeta(a_1) + \zeta(b_1)}{2}\Big).$$

Proof. Let $[x_1, y_1] \subseteq [a_1, b_1] \subseteq \mathbb{R}$, and then by the exponential trigonometric convex function, we have

$$\zeta\Big(\frac{x_1 + y_1}{2}\Big) \leq \frac{1}{\sqrt{2e}}\zeta(x_1) + \frac{1}{\sqrt{2e}}\zeta(y_1).$$

Set $x_1 = ta_1 + (1-t)b_1$, $y_1 = (1-t)a_1 + tb_1$, and then we have

$$\zeta\Big(\frac{a_1 + b_1}{2}\Big) \leq \frac{1}{\sqrt{2e}}[\zeta(ta_1 + (1-t)b_1) + \zeta((1-t)a_1 + tb_1)].$$

Integrating both sides w.r.t t over $[0, 1]$:

$$\zeta\Big(\frac{a_1 + b_1}{2}\Big) \int_0^1 dt \leq \frac{1}{\sqrt{2e}} \int_0^1 \zeta(ta_1 + (1-t)b_1)dt + \frac{1}{\sqrt{2e}} \int_0^1 \zeta((1-t)a_1 + t(b_1))dt.$$

By selecting a feasible replacement, we obtain

$$\zeta\Big(\frac{a_1+b_1}{2}\Big) \leq \frac{1}{\sqrt{2e}}\frac{1}{(b_1-a_1)}\int_{a_1}^{b_1}\zeta(u)du + \frac{1}{\sqrt{2e}}\frac{1}{(b_1-a_1)}\int_{a_1}^{b_1}\zeta(w)dw.$$

Hence,

$$\zeta\Big(\frac{a_1+b_1}{2}\Big) \leq \sqrt{\frac{2}{e}}\frac{1}{(b_1-a_1)}\int_{a_1}^{b_1}\zeta(u)du. \tag{3.1}$$

In order to show the second part of the inequality, we proceed as

$$\zeta(ta_1+(1-t)b_1) \leq \frac{\sin\frac{\pi t}{2}}{e^{1-t}}\zeta(a_1) + \frac{\cos\frac{\pi t}{2}}{e^t}\zeta(b_1),$$

and

$$\zeta((1-t)a_1+(t)b_1) \leq \frac{\cos\frac{\pi t}{2}}{e^t}\zeta(a_1) + \frac{\sin\frac{\pi t}{2}}{e^{1-t}}\zeta(b_1).$$

Adding both of the inequalities above yields

$$\zeta(ta_1+(1-t)b_1)+\zeta((1-t)a_1+(t)b_1) \leq (\frac{\sin\frac{\pi t}{2}}{e^{1-t}} + \frac{\cos\frac{\pi t}{2}}{e^t})(\zeta(a_1)+\zeta(b_1)).$$

Since $(\frac{\sin\frac{\pi t}{2}}{e^{1-t}} + \frac{\cos\frac{\pi t}{2}}{e^t}) \leq 1$, for $t \in [0,1]$, then

$$\zeta(ta_1+(1-t)b_1)+\zeta((1-t)a_1+(t)b_1) \leq (\frac{\sin\frac{\pi t}{2}}{e^{1-t}} + \frac{\cos\frac{\pi t}{2}}{e^t})(\zeta(a_1)+\zeta(b_1)) \leq \zeta(a_1)+\zeta(b_1).$$

Integrating both sides w.r.t t over $[0,1]$:

$$\int_0^1 \zeta(ta_1+(1-t)b_1)dt + \int_0^1 \zeta\Big((1-t)a_1+t(b_1)\Big)dt \leq \Big(\zeta(a_1)+\zeta(b_1)\Big)\int_0^1 dt.$$

Following the same process as previously used, we have

$$\sqrt{\frac{2}{e}}\frac{1}{(b_1-a_1)}\int_{a_1}^{b_1}\zeta(u)du \leq \sqrt{\frac{2}{e}}(\zeta(a_1)+\zeta(b_1)). \tag{3.2}$$

When (3.1) and (3.2) are combined, we have

$$\zeta\Big(\frac{a_1+b_1}{2}\Big) \leq \sqrt{\frac{2}{e}}\frac{1}{(b_1-a_1)}\int_{a_1}^{b_1}\zeta(u)du \leq \sqrt{\frac{2}{e}}\Big(\frac{\zeta(a_1)+\zeta(b_1)}{2}\Big).$$

Now, we develop a Hermite-Hadamard-type fractional inequality with Katugampola-type fractional operators and an exponentially trigonometric convex function.

Theorem 3.2. *Suppose that $\zeta : [x_1^\rho, y_1^\rho] \subseteq \mathbb{R} \to \mathbb{R}$ is a positive function with $0 \leq a_1 < b_1$ and $\zeta \in X_c^p$. If ζ is also an exponential trigonometric convex function, then we have, for $\beta > 0$ and $\rho > 0$,*

$$\zeta\Big(\frac{a_1^\rho+b_1^\rho}{2}\Big) \leq \frac{\Gamma(\beta+1)}{\sqrt{2e}}\frac{\rho^\beta}{(b_1^\rho-a_1^\rho)^\beta}\Big({}^\rho I_{a_1+}^\beta\zeta(b_1^\rho) + {}^\rho I_{b_1-}^\beta\zeta(a_1^\rho)\Big) \leq \Big(\frac{\zeta(a_1^\rho)+\zeta(b_1^\rho)}{\sqrt{2e}}\Big).$$

Proof. We have the exponential trigonometric convexity function

$$\zeta\Big(\frac{x_1^\rho + y_1^\rho}{2}\Big) \leq \frac{1}{\sqrt{2e}}\zeta(x_1^\rho) + \frac{1}{\sqrt{2e}}\zeta(y_1^\rho).$$

Set $x_1^\rho = t^\rho a_1^\rho + (1-t^\rho)b_1^\rho$ and $y_1^\rho = (1-t^\rho)a_1^\rho + t^\rho b_1^\rho$, where $t \in [0,1]$, and then

$$\zeta\Big(\frac{a_1^\rho + b_1^\rho}{2}\Big) \leq \frac{1}{\sqrt{2e}}\zeta(t^\rho a_1^\rho + (1-t^\rho)b_1^\rho) + \zeta((1-t^\rho)a_1^\rho + t^\rho b_1^\rho). \tag{3.3}$$

By multiplying both sides of (3.3) by $t^{\beta\rho-1}$, then integrating w.r.t t over $[0,1]$, we have

$$\zeta\Big(\frac{a_1^\rho + b_1^\rho}{2}\Big)\int_0^1 t^{\beta\rho-1}dt \leq \frac{1}{\sqrt{2e}}\int_0^1 t^{\beta\rho-1}\zeta(t^\rho a_1^\rho + (1-t^\rho)b_1^\rho)dt + \frac{1}{\sqrt{2e}}\int_0^1 t^{\beta\rho-1}\zeta((1-t^\rho)a_1^\rho + t^\rho b_1^\rho)dt.$$

By using appropriate substitution, we get

$$\begin{aligned}
\zeta\Big(\frac{a_1^\rho + b_1^\rho}{2}\Big) &\leq \frac{\beta\rho}{\sqrt{2e}}\frac{1}{(b_1^\rho - a_1^\rho)}\int_{a_1}^{b_1}\Big(\frac{b_1^\rho - u^\rho}{b_1^\rho - a_1^\rho}\Big)^{\beta-1}u^{\rho-1}\zeta(u^\rho)du \\
&\quad + \frac{\beta\rho}{\sqrt{2e}}\frac{1}{(b_1^\rho - a_1^\rho)}\int_{a_1}^{b_1}\Big(\frac{v^\rho - a_1^\rho}{b_1^\rho - a_1^\rho}\Big)^{\beta-1}v^{\rho-1}\zeta(v^\rho)dv \\
&= \frac{\beta\rho}{\sqrt{2e}}\frac{1}{(b_1^\rho - a_1^\rho)}\Big(\int_{a_1}^{b_1}\Big(\frac{b_1^\rho - u^\rho}{b_1^\rho - a_1^\rho}\Big)^{\beta-1}u^{\rho-1}\zeta(u^\rho)du + \int_{a_1}^{b_1}\Big(\frac{v^\rho - a_1^\rho}{b_1^\rho - a_1^\rho}\Big)^{\beta-1}v^{\rho-1}\zeta(v^\rho)dv\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\zeta\Big(\frac{a_1^\rho + b_1^\rho}{2}\Big) \leq \frac{\Gamma(\beta+1)}{\sqrt{2e}}\frac{\rho^\beta}{(b_1^\rho - a_1^\rho)^\beta}\Big(&\frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_{a_1}^{b_1}(b_1^\rho - u^\rho)^{\beta-1}u^{\rho-1}\zeta(u^\rho)du \\
&+ \frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_{a_1}^{b_1}(v^\rho - a_1^\rho)^{\beta-1}v^{\rho-1}\zeta(v^\rho)dv\Big).
\end{aligned}$$

Hence,

$$\zeta\Big(\frac{a_1^\rho + b_1^\rho}{2}\Big) \leq \frac{\Gamma(\beta+1)}{\sqrt{2e}}\frac{\rho^\beta}{(b_1^\rho - a_1^\rho)^\beta}\Big({}^\rho I_{a_1+}^\beta\zeta(b_1^\rho) + {}^\rho I_{b_1-}^\beta\zeta(a_1^\rho)\Big). \tag{3.4}$$

To prove the inequality‘s second part, we have

$$\zeta(t^\rho a_1^\rho + (1-t^\rho)b_1^\rho) \leq \frac{\sin\frac{\pi t^\rho}{2}}{e^{1-t^\rho}}\zeta(a_1^\rho) + \frac{\cos\frac{\pi t^\rho}{2}}{e^{t^\rho}}\zeta(b_1^\rho),$$

and

$$\zeta((1-t^\rho)a_1^\rho + (t^\rho)b_1^\rho) \leq \frac{\cos\frac{\pi t^\rho}{2}}{e^{t^\rho}}\zeta(a_1^\rho) + \frac{\sin\frac{\pi t}{2}}{e^{1-t^\rho}}\zeta(b_1^\rho).$$

Adding both of the inequalities above yields

$$\zeta(t^\rho a_1^\rho + (1-t^\rho)b_1^\rho) + \zeta((1-t^\rho)a_1^\rho + (t^\rho)b_1^\rho) \leq \left(\frac{\sin\frac{\pi t^\rho}{2}}{e^{1-t^\rho}} + \frac{\cos\frac{\pi t^\rho}{2}}{e^{1-t^\rho}}\right)(\zeta(a_1^\rho) + \zeta(b_1^\rho)) \leq \zeta(a_1^\rho) + \zeta(b_1^\rho).$$

For $t \in [0,1]$, $\left(\frac{\sin\frac{\pi t^\rho}{2}}{e^{1-t^\rho}} + \frac{\cos\frac{\pi t^\rho}{2}}{e^{1-t^\rho}}\right) \leq 1$, we have

$$\zeta(t^\rho a_1^\rho + (1-t^\rho)b_1^\rho) + \zeta((1-t^\rho)a_1^\rho + (t^\rho)b_1^\rho) \leq \zeta(a_1^\rho) + \zeta(b_1^\rho). \tag{3.5}$$

Multiplying both sides of (3.5) by $t^{\rho\beta-1}$, then integrating w.r.t t over $[0,1]$, we have

$$\int_0^1 t^{\rho\beta-1}\zeta(t^\rho a_1^\rho + (1-t^\rho)b_1^\rho)dt + \int_0^1 t^{\rho\beta-1}\zeta((1-t^\rho)a_1^\rho + (t^\rho)b_1^\rho)dt \leq (\zeta(a_1^\rho) + \zeta(b_1^\rho))\int_0^1 t^{\rho\beta-1}dt.$$

By repeating the same criteria to prove the first part of the inequality, we get

$$\frac{\Gamma(\beta+1)}{\sqrt{2e}}\frac{\rho^\beta}{(b_1^\rho - a_1^\rho)^\beta}\left({}^\rho I^\beta_{a_1+}\zeta(b_1^\rho) + {}^\rho I^\beta_{b_1-}\zeta(a_1^\rho)\right) \leq \frac{\zeta(a_1^\rho) + \zeta(b_1^\rho)}{\sqrt{2e}}. \tag{3.6}$$

From (3.4) and (3.6), we get

$$\zeta\left(\frac{a_1^\rho + b_1^\rho}{2}\right) \leq \frac{\Gamma(\beta+1)}{\sqrt{2e}}\frac{\rho^\beta}{(b_1^\rho - a_1^\rho)^\beta}\left({}^\rho I^\beta_{a_1+}\zeta(b_1) + {}^\rho I^\beta_{b_1-}\zeta(a_1)\right) \leq \frac{\zeta(a_1^\rho) + \zeta(b_1^\rho)}{\sqrt{2e}}.$$

Corollary 3.3. *We obtain the result for the Riemann-Liouville fractional integrals if $\rho \to 1$ in Theorem 3.2:*

$$\zeta\left(\frac{a_1 + b_1}{2}\right) \leq \frac{\Gamma(\beta+1)}{\sqrt{2e}}\frac{1}{(b_1 - a_1)^\beta}\left({}^\rho I^\beta_{a_1+}\zeta(b_1^\rho) + {}^\rho I^\beta_{b_1-}\zeta(a_1^\rho)\right) \leq \sqrt{\frac{2}{e}}\left(\frac{\zeta(a_1) + \zeta(b_1)}{2}\right).$$

Corollary 3.4. *We obtain the Hermite-Hadamard-type inequality for the exponentially trigonometric convex function proved in Theorem 3.1 by replacing $\rho \to 1$ as well as $\beta \to 1$:*

$$\zeta\left(\frac{a_1 + b_1}{2}\right) \leq \sqrt{\frac{2}{e}}\frac{1}{(b_1 - a_1)}\int_{a_1}^{b_1}\zeta(u)du \leq \sqrt{\frac{2}{e}}\left(\frac{\zeta(a_1) + \zeta(b_1)}{2}\right).$$

Corollary 3.5. *For graphical representation, we choose $\rho = 2$, $a_1 = 0$, $b_1 = 3$, $\zeta(t) = e^{\sin t}$ in* (3.2) *to obtain the following inequality:*

$$e^{\sin\frac{9}{2}} \leq \frac{2\beta}{\sqrt{2e}9^\beta}\left(\int_0^3 t(9-t^2)^{\beta-1}e^{\sin t}dt + \int_0^3 t^{2\beta-1}e^{\sin t}dt\right) \leq \frac{1+e^{\sin 9}}{\sqrt{2e}}. \tag{3.7}$$

For a two-dimensional graphical representation in Figure 1, we fix β = 0.0001.

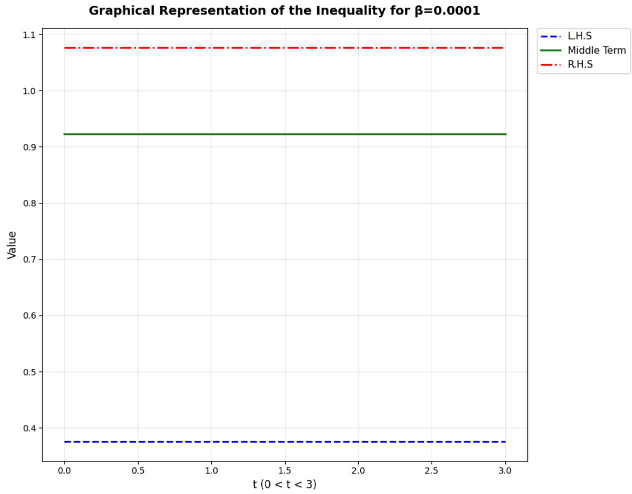

Figure 1. The figure presents a 2D comparison graph of the function given in (3.7) for $0 < t < 3$.

For a three-dimensional graphical representation in Figure 2, we choose $0 < \beta < 0.01$.

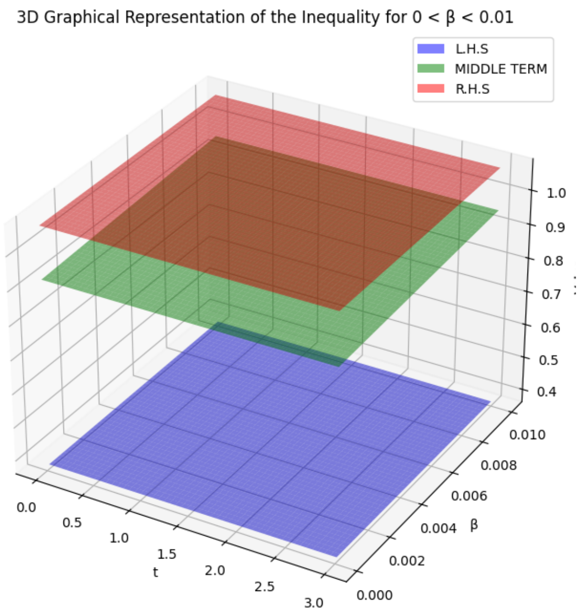

Figure 2. The figure illustrates a 3D comparison plot of the function defined in (3.7) over the interval $0 < t < 3$.

Table 1 numerically illustrates inequality (3.7).

Table 1. Numerical illustration of inequality (3.7).

β	Left	Middle	Right
0.0010	0.3762	0.9259	1.0765
0.0050	0.3762	0.9382	1.0765
0.0100	0.3762	0.9533	1.0765

Using an exponentially trigonometric convex function, we will now define a new Hermite-Hadamard fractional inequality involving the interval's midpoint as an upper and lower limit.

Theorem 3.6. *Suppose that $\zeta : [x_1^\rho, y_1^\rho] \subseteq \mathbb{R} \to \mathbb{R}$ is a positive function with $0 \le a_1 < b_1$ and $\zeta \in X_c^p$. If ζ is also an exponential trigonometric convex function, then we have, for $\beta > 0$ and $\rho > 0$,*

$$\zeta\Big(\frac{a_1^\rho + b_1^\rho}{2}\Big) \le \frac{\Gamma(\beta+1)}{\sqrt{2e}}\frac{(2\rho)^\beta}{(b_1^\rho - a_1^\rho)^\beta}\Big({}^\rho I^\beta_{(\frac{a_1^\rho+b_1^\rho}{2})+}\zeta(b_1^\rho) +{}^\rho I^\beta_{(\frac{a_1^\rho+b_1^\rho}{2})-}\zeta(a_1^\rho)\Big) \le \Big(\frac{\zeta(a_1^\rho)+\zeta(b_1^\rho)}{\sqrt{2e}}\Big).$$

Proof. By exponential trigonometric convexity,

$$\zeta\Big(\frac{x_1^\rho + y_1^\rho}{2}\Big) \le \frac{1}{\sqrt{2e}}\zeta(x_1^\rho) + \frac{1}{\sqrt{2e}}\zeta(y_1^\rho).$$

Set $x_1^\rho = \frac{t^\rho}{2}a_1^\rho + (\frac{2-t^\rho}{2})b_1^\rho$ and $y_1^\rho = (\frac{2-t^\rho}{2})a_1^\rho + \frac{t^\rho}{2}b_1^\rho$, and then

$$\zeta\Big(\frac{a_1^\rho + b_1^\rho}{2}\Big) \le \frac{1}{\sqrt{2e}}\zeta(\frac{t^\rho}{2}a_1^\rho + (\frac{2-t^\rho}{2})b_1^\rho) + \zeta((\frac{2-t^\rho}{2})a_1^\rho + \frac{t^\rho}{2}b_1^\rho). \tag{3.8}$$

Multiplying both sides of (3.8) by $t^{\beta\rho-1}$, then integrating w.r.t t from $[0, 1]$, we have

$$\begin{aligned}\zeta\Big(\frac{a_1^\rho + b_1^\rho}{2}\Big)\int_0^1 t^{\beta\rho-1}dt \le{}& \frac{1}{\sqrt{2e}}\int_0^1 t^{\beta\rho-1}\zeta\Big(\frac{t^\rho}{2}a_1^\rho + (\frac{2-t^\rho}{2})b_1^\rho\Big)dt\\ &+ \frac{1}{\sqrt{2e}}\int_0^1 t^{\beta\rho-1}\zeta((\frac{2-t^\rho}{2})a_1 + \frac{t^\rho}{2}(b_1^\rho))dt.\end{aligned}$$

By using appropriate substitution, we get

$$\begin{aligned}\zeta\Big(\frac{a_1^\rho + b_1^\rho}{2}\Big) &\le \frac{\beta\rho}{\sqrt{2e}}\frac{1}{(b_1^\rho - a_1^\rho)}\int_{(\frac{a_1^\rho+b_1^\rho}{2})}^{b_1} 2^\beta(\frac{b_1^\rho - u^\rho}{b_1^\rho - a_1^\rho})^{\beta-1}u^{\rho-1}\zeta(u^\rho)du\\ &\quad+\frac{\beta\rho}{\sqrt{2e}}\frac{1}{(b_1^\rho - a_1^\rho)}\int_{a_1}^{(\frac{a_1^\rho+b_1^\rho}{2})} 2^\beta(\frac{v^\rho - a_1^\rho}{b_1^\rho - a_1^\rho})^{\beta-1}v^{\rho-1}\zeta(v^\rho)dv\\ &= \frac{\beta\rho}{\sqrt{2e}}\frac{2^\beta}{(b_1^\rho - a_1^\rho)^\beta}\Big(\int_{(\frac{a_1^\rho+b_1^\rho}{2})}^{b_1}(b_1^\rho - u^\rho)^{\beta-1}u^{\rho-1}\zeta(u^\rho)du + \int_{a_1}^{(\frac{a_1^\rho+b_1^\rho}{2})}(v^\rho - a_1^\rho)^{\beta-1}v^{\rho-1}\zeta(v^\rho)dv\Big),\end{aligned}$$

$$\begin{aligned}\zeta\Big(\frac{a_1^\rho + b_1^\rho}{2}\Big) \le{}& \frac{\Gamma(\beta+1)}{\sqrt{2e}}\frac{(2\rho)^\beta}{(b_1^\rho - a_1^\rho)^\beta}\Big(\frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_{(\frac{a_1^\rho+b_1^\rho}{2})}^{b_1}(b_1^\rho - u^\rho)^{\beta-1}u^{\rho-1}\zeta(u^\rho)du\\ &+\frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_{a_1}^{(\frac{a_1^\rho+b_1^\rho}{2})}(v^\rho - a_1^\rho)^{\beta-1}v^{\rho-1}\zeta(v^\rho)dv\Big).\end{aligned}$$

Hence,

$$\zeta\Big(\frac{a_1^\rho + b_1^\rho}{2}\Big) \le \frac{\Gamma(\beta+1)}{\sqrt{2e}}\frac{(2\rho)^\beta}{(b_1^\rho - a_1^\rho)^\beta}\Big({}^\rho I^\beta_{(\frac{a_1^\rho+b_1^\rho}{2})+}\zeta(b_1^\rho) +{}^\rho I^\beta_{(\frac{a_1^\rho+b_1^\rho}{2})-}\zeta(a_1^\rho)\Big). \tag{3.9}$$

So to find the inequality's second part, we have

$$\zeta(\frac{t^{\rho}}{2}a_1^{\rho}+(\frac{2-t^{\rho}}{2})b_1^{\rho})\leq\frac{\sin\frac{\pi}{2}\frac{t^{\rho}}{2}}{e^{\frac{2-t^{\rho}}{2}}}\zeta(a_1^{\rho})+\frac{\cos\frac{\pi}{2}\frac{t^{\rho}}{2}}{e^{\frac{t^{\rho}}{2}}}\zeta(b_1^{\rho}),$$

and

$$\zeta((\frac{2-t^{\rho}}{2})a_1^{\rho}+(\frac{t^{\rho}}{2}))\leq\frac{\cos\frac{\pi}{2}\frac{t^{\rho}}{2}}{e^{\frac{t^{\rho}}{2}}}\zeta(a_1^{\rho})+\frac{\sin\frac{\pi}{2}\frac{t^{\rho}}{2}}{e^{\frac{2-t^{\rho}}{2}}}\zeta(b_1^{\rho}).$$

Adding both of the inequalities above yields

$$\begin{aligned}&\zeta(\frac{t^{\rho}}{2}a_1^{\rho}+(\frac{2-t^{\rho}}{2})b_1^{\rho})+\zeta((\frac{2-t^{\rho}}{2})a_1^{\rho}+(\frac{t^{\rho}}{2}))\\ \leq&\Big(\frac{\sin\frac{\pi}{2}\frac{t^{\rho}}{2}}{e^{\frac{2-t^{\rho}}{2}}}+\frac{\cos\frac{\pi}{2}\frac{t^{\rho}}{2}}{e^{\frac{t^{\rho}}{2}}}\Big)(\zeta(a_1^{\rho})+\zeta(b_1^{\rho}))\leq(\zeta(a_1^{\rho})+\zeta(b_1^{\rho})).\end{aligned}$$

Multiplying both sides of (3.10) by $t^{\beta\rho-1}$, then integrating w.r.t t from $t\in[0,1]$, we have

$$\int_0^1 t^{\beta\rho-1}\zeta(\frac{t^{\rho}}{2}a_1^{\rho}+(\frac{2-t^{\rho}}{2})b_1^{\rho})dt+\int_0^1 t^{\beta\rho-1}\zeta((\frac{2-t^{\rho}}{2})a_1^{\rho}+(\frac{t^{\rho}}{2}))dt\leq(\zeta(a_1^{\rho})+\zeta(b_1^{\rho}))\int_0^1 t^{\beta\rho-1}dt.$$

By repeating the same criteria to prove the first part of the inequality, we get

$$\frac{\Gamma(\beta+1)}{\sqrt{2e}}\frac{(2\rho)^{\beta}}{(b_1^{\rho}-a_1^{\rho})^{\beta}}\Big({}^{\rho}I^{\beta}_{(\frac{a_1^{\rho}+b_1^{\rho}}{2})+}\zeta(b_1^{\rho})+{}^{\rho}I^{\beta}_{(\frac{a_1^{\rho}+b_1^{\rho}}{2})-}\zeta(a_1^{\rho})\Big)\leq\Big(\frac{\zeta(a_1^{\rho})+\zeta(b_1^{\rho})}{\sqrt{2e}}\Big). \tag{3.10}$$

After combining (3.9) and (3.10), we get

$$\zeta\Big(\frac{a_1^{\rho}+b_1^{\rho}}{2}\Big)\leq\frac{\Gamma(\beta+1)}{\sqrt{2e}}\frac{(2\rho)^{\beta}}{(b_1^{\rho}-a_1^{\rho})^{\beta}}\Big({}^{\rho}I^{\beta}_{(\frac{a_1^{\rho}+b_1^{\rho}}{2})+}\zeta(b_1^{\rho})+{}^{\rho}I^{\beta}_{(\frac{a_1^{\rho}+b_1^{\rho}}{2})-}\zeta(a_1^{\rho})\Big)\leq\Big(\frac{\zeta(a_1^{\rho})+\zeta(b_1^{\rho})}{\sqrt{2e}}\Big).$$

Corollary 3.7. *We obtain the following inequality for Riemann-Liouville fractional integrals, which results from replacing* $\rho\to1$ *in Theorem 3.6:*

$$\zeta\Big(\frac{a_1+b_1}{2}\Big)\leq\frac{\Gamma(\beta+1)}{\sqrt{2e}}\frac{2^{\beta}}{(b_1-a_1)^{\beta}}\Big(I^{\beta}_{(\frac{a_1+b_1}{2})+}\zeta(b_1)+I^{\beta}_{(\frac{a_1+b_1}{2})-}\zeta(a_1)\Big)\leq\sqrt{\frac{2}{e}}\Big(\frac{\zeta(a_1)+\zeta(b_1)}{2}\Big).$$

Corollary 3.8. *We obtain the Hermite-Hadamard-type inequality for an exponentially trigonometric convex function proved in* (3.1) *after replacing* $\rho\to1$ *and* $\beta\to1$, *where* f *is symmetrical about the point* $(\frac{a_1+b_1}{2})$*:*

$$\zeta\Big(\frac{a_1+b_1}{2}\Big)\leq\sqrt{\frac{2}{e}}\frac{1}{(b_1-a_1)}\int_{a_1}^{b_1}\zeta(u)du\leq\sqrt{\frac{2}{e}}\Big(\frac{\zeta(a_1)+\zeta(b_1)}{2}\Big).$$

Corollary 3.9. *For graphical representation, we choose* $\rho = 2$, $a_1 = 0$, $b_1 = 3$, $\zeta(t) = e^{\sin t}$ *in* (3.6)*, and we have the following inequality:*

$$e^{\sin\frac{9}{2}} \le \frac{\beta 2^{\beta+1}}{\sqrt{2e}9^{\beta}}\left(\int_{4.5}^{9} t(81-t^2)^{\beta-1}e^{\sin t}dt + \int_0^{4.5} t(\frac{81}{4}-t^2)^{\beta-1}e^{\sin t}dt\right) \le \frac{1+e^{\sin 9}}{\sqrt{2e}}. \tag{3.11}$$

For a two-dimensional graphical representation of relation (3.11) *in Figure 3, we fix* $\beta = 0.0001$.

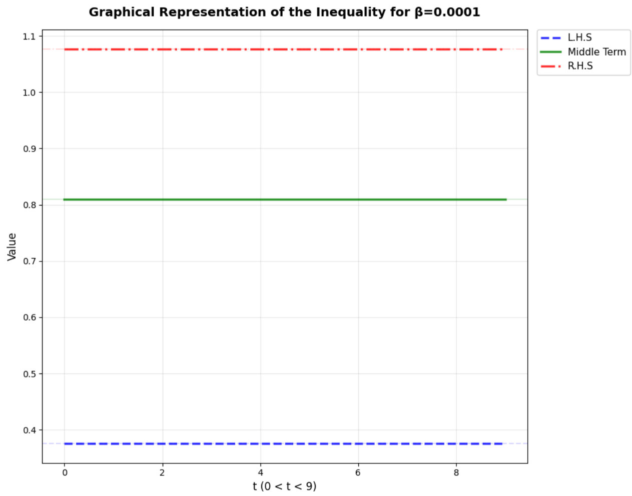

Figure 3. The figure displays a 2D visualization of the three sides of inequality (3.11) for $0 < t < 3$.

For a three-dimensional graphical representation of relation (3.11) *in Figure 4, we choose* $0 < \beta < 0.01$.

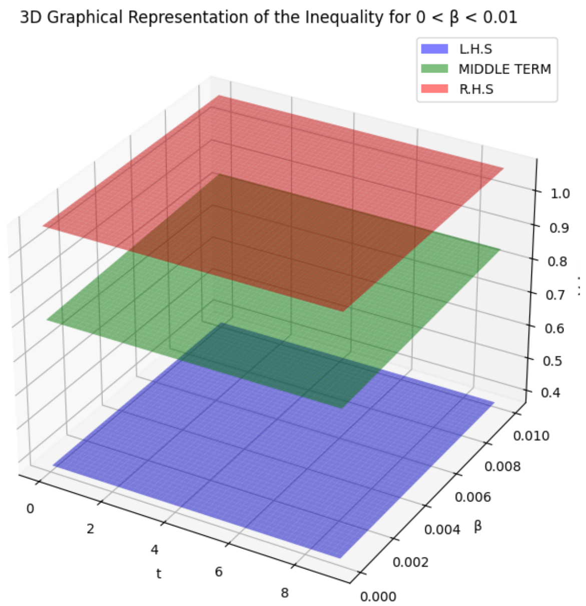

Figure 4. This figure showcases a 3D graph comparing the functions specified in (3.11) for $0 < t < 3$.

Table 2 illustrates the numerical interpretation of inequality (3.11).

Table 2. Further, we fix some more values for β in the inequality (3.11).

β	Left	Middle	Right
0.0010	0.3762	0.8122	1.0765
0.0050	0.3762	0.8249	1.0765
0.0100	0.3762	0.8411	1.0765

Here, we formulate a new Hermite-Hadamard-type inequality through a log-convex function.

Theorem 3.10. *Let $\zeta : I \subseteq \mathbb{R} \to (0,\infty)$ be a log-convex mapping on I and $a_1, b_1 \in I$ with $a < b$. Then we have*

$$\zeta\Big(\frac{a_1+b_1}{2}\Big) \leq \frac{1}{(b_1-a_1)}\int_{a_1}^{b_1}\zeta(u)du \leq \sqrt{\zeta(a_1)\zeta(b_1)}.$$

Proof. Let $x_1, y_1 \in [a_1, b_1] \subseteq \mathbb{R}$, and then for the log-convex function, we have

$$\zeta\Big(\frac{x_1+y_1}{2}\Big) \leq \sqrt{\zeta(x_1)}\sqrt{\zeta(y_1)}.$$

Set $x_1 = ta_1 + (1-t)b_1$, $y_1 = (1-t)a_1 + tb_1$, and then we have

$$\zeta\Big(\frac{a_1+b_1}{2}\Big) \leq \sqrt{\zeta(ta_1+(1-t)b_1)}\sqrt{\zeta((1-t)a_1+tb_1)}.$$

Integrate both sides w.r.t t over $[0, 1]$, we have

$$\zeta\Big(\frac{a_1+b_1}{2}\Big)\int_0^1 dt \leq \int_0^1 \sqrt{\zeta(ta_1+(1-t)b_1)}\sqrt{\zeta((1-t)a_1+t(b_1))}dt.$$

By using the Roger-Hölder inequality,

$$\zeta\Big(\frac{a_1+b_1}{2}\Big) \leq \sqrt{\int_0^1 \zeta(ta_1+(1-t)b_1)dt}\sqrt{\int_0^1 \zeta((1-t)a_1+tb_1)dt}.$$

By selecting a feasible replacement, we obtain

$$\zeta\Big(\frac{a_1+b_1}{2}\Big) \leq \frac{1}{(b_1-a_1)}[\sqrt{\int_{a_1}^{b_1}\zeta(u)du}][\sqrt{\int_{a_1}^{b_1}\zeta(v)dv}] = \frac{1}{(b_1-a_1)}\int_{a_1}^{b_1}\zeta(u)du.$$

This implies that

$$\zeta\Big(\frac{a_1+b_1}{2}\Big) \leq \frac{1}{(b_1-a_1)}\int_{a_1}^{b_1}\zeta(u)du. \tag{3.12}$$

On the other hand,

$$\zeta(ta_1+(1-t)b_1) \leq [\zeta(a_1)]^t[\zeta(b_1)]^{1-t},$$

and

$$\zeta((1-t)a_1+(t)b_1) \leq (\zeta(a_1))^{1-t}(\zeta(b_1))^{t}.$$

If the two previous inequalities are multiplied, we have

$$\zeta(ta_1+(1-t)b_1)\zeta((1-t)a_1+(t)b_1) \leq [\zeta(a_1)]^{t}[\zeta(b_1)]^{1-t}[\zeta(a_1)]^{1-t}[\zeta(b_1)]^{t} \leq \zeta(a_1)\zeta(b_1).$$

Now taking the square root on both sides, we have

$$\sqrt{\zeta(ta_1+(1-t)b_1)}\sqrt{\zeta((1-t)a_1+(t)b_1)} \leq \sqrt{\zeta(a_1)\zeta(b_1)}.$$

Integrating both sides with respect to $t \in [0,1]$ gives us

$$\int_0^1 \sqrt{\zeta(ta_1+(1-t)b_1}\sqrt{\zeta(1-t)a_1+tb_1}dt \leq \sqrt{\zeta(a_1)\zeta(b_1)}\int_0^1 dt.$$

Following the same process as previously used, we have

$$\frac{1}{(b_1-a_1)}\int_{a_1}^{b_1}\zeta(u)du \leq \sqrt{\zeta(a_1)\zeta(b_1)}. \tag{3.13}$$

When (3.12) and (3.13) are combined, we have

$$\zeta\Big(\frac{a_1+b_1}{2}\Big) \leq \frac{1}{(b_1-a_1)}\int_{a_1}^{b_1}\zeta(u)du \leq \sqrt{\zeta(a_1)\zeta(b_1)}.$$

We now have a novel Hermite-Hadamard-type fractional integral inequality via log-convex functions.

Theorem 3.11. *Suppose that $\zeta : [x_1^\rho, y_1^\rho] \subseteq \mathbb{R} \to (0,\infty)$ is a positive function with $0 \leq a_1 < b_1$ and $\zeta \in X_c^p$. If ζ is also a log-convex function, then we have, for $\beta > 0$ and $\rho > 0$,*

$$\zeta\Big(\frac{a_1^\rho+b_1^\rho}{2}\Big) \leq \frac{\Gamma(\beta+1)\rho^\beta}{(b_1^\rho-a_1^\rho)^\beta}\Big(\sqrt{{}^\rho I_{a_1+}^\beta\zeta(b_1^\rho)\,{}^\rho I_{b_1-}^\beta\zeta(a_1^\rho)}\Big) \leq \sqrt{\zeta(a_1^\rho)\zeta(b_1^\rho)}.$$

Proof. By log-convexity,

$$\zeta\Big(\frac{x_1^\rho+y_1^\rho}{2}\Big) \leq \sqrt{\zeta(x_1^\rho)}\sqrt{\zeta(y_1^\rho)}.$$

Set $x_1^\rho = t^\rho a_1^\rho + (1-t^\rho)b_1^\rho$ and $y_1^\rho = (1-t^\rho)a_1^\rho + t^\rho b_1^\rho$, and then

$$\zeta\Big(\frac{a_1^\rho+b_1^\rho}{2}\Big) \leq \sqrt{\zeta(t^\rho a_1^\rho+(1-t^\rho)b_1^\rho)}\sqrt{\zeta((1-t^\rho)a_1^\rho+t^\rho b_1^\rho)}. \tag{3.14}$$

Multiplying both sides of (3.14) by $t^{\beta\rho-1}$, then integrating w.r.t t from $[0,1]$ gives us

$$\zeta\Big(\frac{a_1^\rho+b_1^\rho}{2}\Big)\int_0^1 t^{\beta\rho-1}dt \leq \int_0^1 t^{\beta\rho-1}\sqrt{\zeta(t^\rho a_1^\rho+(1-t^\rho)b_1^\rho)}\sqrt{\zeta((1-t^\rho)a_1^\rho+t^\rho b_1^\rho})dt.$$

By the Roger-Hölder inequality, we get

$$\frac{1}{\beta\rho}\zeta\Big(\frac{a_1^\rho + b_1^\rho}{2}\Big) \leq \sqrt{\int_0^1 t^{\beta\rho-1}\zeta(t^\rho a_1^\rho + (1-t^\rho)b_1^\rho)dt} \times \sqrt{\int_0^1 t^{\beta\rho-1}\zeta((1-t^\rho)a_1^\rho + (t^\rho)b_1^\rho)dt}.$$

By using an appropriate substitution, we get

$$\begin{aligned}\zeta\Big(\frac{a_1^\rho + b_1^\rho}{2}\Big) &\leq \sqrt{\frac{\beta\rho}{(b_1^\rho - a_1^\rho)}\int_{a_1}^{b_1}(\frac{b_1^\rho - u^\rho}{b_1^\rho - a_1^\rho})^{\beta-1}u^{\rho-1}\zeta(u^\rho)du} \\ &\quad\times \sqrt{\frac{\beta\rho}{(b_1^\rho - a_1^\rho)}\int_{a_1}^{b_1}(\frac{v^\rho - a_1^\rho}{b_1^\rho - a_1^\rho})^{\beta-1}v^{\rho-1}\zeta(v^\rho)dv}, \\ &= \sqrt{\frac{\beta\rho}{(b_1^\rho - a_1^\rho)^\beta}}\sqrt{\int_{a_1}^{b_1}(b_1^\rho - u^\rho)^{\beta-1}u^{\rho-1}\zeta(u^\rho)du}\sqrt{\frac{\beta\rho}{(b_1^\rho - a_1^\rho)^\beta}}\sqrt{\int_{a_1}^{b_1}(v^\rho - a_1^\rho)^{\beta-1}v^{\rho-1}\zeta(v^\rho)dv}.\end{aligned}$$

So, we have

$$\zeta\Big(\frac{a_1^\rho + b_1^\rho}{2}\Big) \leq \frac{\Gamma(\beta+1)\rho^\beta}{(b_1^\rho - a_1^\rho)^\beta}\Big[\sqrt{\frac{\rho^{\beta-1}}{\Gamma(\beta)}\int_{a_1}^{b_1}(b_1^\rho - u^\rho)^{\beta-1}u^{\rho-1}\zeta(u^\rho)du} \times \sqrt{\frac{\rho^{\beta-1}}{\Gamma(\beta)}\int_{a_1}^{b_1}(v^\rho - a_1^\rho)^{\beta-1}v^{\rho-1}\zeta(v^\rho)dv}\Big].$$

Hence,

$$\zeta\Big(\frac{a_1^\rho + b_1^\rho}{2}\Big) \leq \frac{\Gamma(\beta+1)\rho^\beta}{(b_1^\rho - a_1^\rho)^\beta}\Big(\sqrt{{}^\rho I^\beta_{a_1+}\zeta(b_1^\rho)\,{}^\rho I^\beta_{b_1-}\zeta(a_1^\rho)}\Big). \tag{3.15}$$

Now, we prove the inequality's second part:

$$\zeta(t^\rho a_1^\rho + (1-t^\rho)b_1^\rho) \leq [\zeta(a_1^\rho)]^{t^\rho}[\zeta(b_1^\rho)]^{1-t^\rho},$$

and

$$\zeta((1-t^\rho)a_1^\rho + (t^\rho)b_1^\rho) \leq [\zeta(a_1^\rho)]^{1-t^\rho}[\zeta(b_1^\rho)]^{t^\rho}.$$

Multiplying both of the above inequalities, and simplifying, gives us

$$\zeta(t^\rho a_1^\rho + (1-t^\rho)b_1^\rho)\zeta((1-t^\rho)a_1^\rho + (t^\rho)b_1^\rho) \leq [\zeta(a_1^\rho)]^{t^\rho}[\zeta(b_1^\rho)]^{1-t^\rho}[\zeta(a_1^\rho)]^{1-t^\rho}[\zeta(b_1^\rho)]^{t^\rho} \leq \zeta(a_1^\rho)\zeta(b_1^\rho).$$

Applying a square root on both sides of the above inequality, we have

$$\sqrt{\zeta(t^\rho a_1^\rho + (1-t^\rho)b_1^\rho)}\sqrt{\zeta((1-t^\rho)a_1^\rho + (t^\rho)b_1^\rho)} \leq \sqrt{\zeta(a_1^\rho)\zeta(b_1^\rho)}. \tag{3.16}$$

Multiplying both sides of (3.16) by $t^{\beta\rho-1}$, then integrating w.r.t t from $t \in [0, 1]$ gives us

$$\int_0^1 t^{\beta\rho-1}\sqrt{\zeta(t^\rho a_1^\rho + (1-t^\rho)b_1^\rho)}\sqrt{\zeta((1-t^\rho)a_1^\rho + (t^\rho)b_1^\rho)} \leq \sqrt{\zeta(a_1^\rho)\zeta(b_1^\rho)}\int_0^1 t^{\beta\rho-1}dt.$$

By repeating the same process as to obtained the first part of the inequality, we have

$$\frac{\Gamma(\beta+1)\rho^{\beta}}{(b_1^{\rho}-a_1^{\rho})^{\beta}}\Big[\sqrt{{}^{\rho}I^{\beta}_{a_1+}\zeta(b_1^{\rho})^{\rho}I^{\beta}_{b_1-}\zeta(a_1^{\rho})}\Big] \le \sqrt{\zeta(a_1^{\rho})\zeta(b_1^{\rho})}. \tag{3.17}$$

By combining (3.15) and (3.17), we get

$$\zeta\Big(\frac{a_1^{\rho}+b_1^{\rho}}{2}\Big) \le \frac{\Gamma(\beta+1)\rho^{\beta}}{(b_1^{\rho}-a_1^{\rho})^{\beta}}\Big(\sqrt{{}^{\rho}I^{\beta}_{a_1+}\zeta(b_1^{\rho})^{\rho}I^{\beta}_{b_1-}\zeta(a_1^{\rho})}\Big) \le \sqrt{\zeta(a_1^{\rho})\zeta(b_1^{\rho})}.$$

Corollary 3.12. *We establish the Riemann-Liouville fractional integral result if* $\rho \to 1$ *in Theorem 3.11:*

$$\zeta\Big(\frac{a_1+b_1}{2}\Big) \le \frac{\Gamma(\beta+1)}{(b_1-a_1)^{\beta}}\Big(\sqrt{I^{\beta}_{a_1+}\zeta(b_1)I^{\beta}_{b_1-}\zeta(a_1)}\Big) \le \sqrt{\zeta(a_1)\zeta(b_1)}.$$

Corollary 3.13. *We establish the Hermite-Hadamard-type inequality proved in* (3.10) *if we replace* $\rho \to 1$ *and* $\beta \to 1$, *i.e.,*

$$\zeta\Big(\frac{a_1+b_1}{2}\Big) \le \frac{1}{(b_1-a_1)}\int_{a_1}^{b_1}\zeta(u)du \le \sqrt{\zeta(a_1)\zeta(b_1)}.$$

We now establish a new Hermite-Hadamard-type inequality where the midpoint of the interval is a lower and upper limit.

Theorem 3.14. *Suppose that* $\zeta : [x_1^{\rho}, y_1^{\rho}] \subseteq \mathbb{R} \to (0,\infty)$ *is a positive function with* $0 \le a_1 < b_1$ *and* $\zeta \in X_c^p$. *If* ζ *is also a log-convex function, then we have, for* $\beta > 0$ *and* $\rho > 0$,

$$\zeta\Big(\frac{a_1^{\rho}+b_1^{\rho}}{2}\Big) \le \frac{(2\rho)^{\beta}\Gamma(\beta+1)}{(b_1^{\rho}-a_1^{\rho})^{\beta}}\sqrt{{}^{\rho}I^{\beta}_{(\frac{a_1^{\rho}+b_1^{\rho}}{2})+}\zeta(b_1)^{\rho\rho}I^{\beta}_{(\frac{a_1^{\rho}+b_1^{\rho}}{2})-}\zeta(a_1)^{\rho}} \le \sqrt{\zeta(a_1^{\rho})\zeta(b_1^{\rho})}.$$

Proof. By the log-convex function, we have

$$\zeta\Big(\frac{x_1^{\rho}+y_1^{\rho}}{2}\Big) \le \sqrt{\zeta(x_1^{\rho})}\sqrt{\zeta(y_1^{\rho})}.$$

Set $x_1^{\rho} = \frac{t^{\rho}}{2}a_1^{\rho} + (\frac{2-t^{\rho}}{2})b_1^{\rho}$ and $y_1^{\rho} = (\frac{2-t^{\rho}}{2})a_1^{\rho} + (\frac{t^{\rho}}{2})b_1^{\rho}$, and then

$$\zeta\Big(\frac{a_1^{\rho}+b_1^{\rho}}{2}\Big) \le \sqrt{\zeta(\frac{t^{\rho}}{2}a_1^{\rho}+(\frac{2-t^{\rho}}{2})b_1^{\rho})}\sqrt{\zeta((\frac{2-t^{\rho}}{2})a_1^{\rho}+(\frac{t^{\rho}}{2})b_1^{\rho})]}. \tag{3.18}$$

Multiplying both sides of (3.18) by $t^{\beta\rho-1}$, then integrating w.r.t t from $[0,1]$ gives us

$$\zeta\Big(\frac{a_1^{\rho}+b_1^{\rho}}{2}\Big)\int_0^1 t^{\beta\rho-1}dt \le \int_0^1 t^{\beta\rho-1}\sqrt{\zeta(\frac{t^{\rho}}{2}a_1^{\rho}+(\frac{2-t^{\rho}}{2})b_1^{\rho})}\sqrt{\zeta((\frac{2-t^{\rho}}{2})a_1^{\rho}+(\frac{t^{\rho}}{2})b_1^{\rho})}dt.$$

After using the Roger-Hölder inequality, we get

$$\zeta\Big(\frac{a_1^{\rho}+b_1^{\rho}}{2}\Big)\frac{1}{\beta\rho} \le \sqrt{\int_0^1 t^{\beta\rho-1}\zeta(\frac{t^{\rho}}{2}a_1^{\rho}+(\frac{2-t^{\rho}}{2})b_1^{\rho})dt} \times \sqrt{\int_0^1 t^{\beta\rho-1}\zeta((\frac{2-t^{\rho}}{2})a_1^{\rho}+(\frac{t^{\rho}}{2})b_1^{\rho}]dt}.$$

Defining a suitable substitution, we get

$$\begin{aligned}\zeta\Big(\frac{a_1^\rho+b_1^\rho}{2}\Big) &\leq \sqrt{\int_{(\frac{a_1^\rho+b_1^\rho}{2})}^{b_1}(\frac{\beta\rho}{(b_1^\rho-a_1^\rho)})2^\beta(\frac{b_1^\rho-u^\rho}{b_1^\rho-a_1^\rho})^{\beta-1}u^{\rho-1}\zeta(u^\rho)du}\\ &\quad\times\sqrt{\int_{(\frac{a_1^\rho+b_1^\rho}{2})}^{a_1}(\frac{\beta\rho}{(b_1^\rho-a_1^\rho)})2^\beta(\frac{v^\rho-a_1^\rho}{b_1^\rho-a_1^\rho})^{\beta-1}v^{\rho-1}\zeta(v^\rho)dv}\\ &= \frac{2^\beta\cdot\beta\rho}{(b_1^\rho-a_1^\rho)^\beta}\sqrt{\int_{(\frac{a_1^\rho+b_1^\rho}{2})}^{b_1}(b_1^\rho-u^\rho)^{\beta-1}u^{\rho-1}\zeta(u^\rho)du}\times\sqrt{\int_{(\frac{a_1^\rho+b_1^\rho}{2})}^{a_1}(v^\rho-a_1^\rho)^{\beta-1}v^{\rho-1}\zeta(v^\rho)dv}.\end{aligned}$$

So, we have

$$\begin{aligned}\zeta\Big(\frac{a_1^\rho+b_1^\rho}{2}\Big) &\leq \frac{(2\rho)^\beta\Gamma(\beta+1)}{(b_1^\rho-a_1^\rho)^\beta}\sqrt{\int_{(\frac{a_1^\rho+b_1^\rho}{2})}^{b-1}\frac{\rho^{1-\beta}}{\Gamma(\beta)}(b_1^\rho-u^\rho)^{\beta-1}u^{\rho-1}\zeta(u^\rho)du}\\ &\quad\times\sqrt{\int_{(\frac{a_1^\rho+b_1^\rho}{2})}^{a_1}\frac{\rho^{1-\beta}}{\Gamma(\beta)}(v^\rho-a_1^\rho)^{\beta-1}v^{\rho-1}\zeta(v^\rho)dv}.\end{aligned}$$

Hence,

$$\zeta\Big(\frac{a_1^\rho+b_1^\rho}{2}\Big)\leq\frac{(2\rho)^\beta\Gamma(\beta+1)}{(b_1^\rho-a_1^\rho)^\beta}\sqrt{{}^\rho I^\beta_{(\frac{a_1^\rho+b_1^\rho}{2})+}\zeta(b_1^\rho){}^\rho I^\beta_{(\frac{a_1^\rho+b_1^\rho}{2})-}\zeta(a_1^\rho)}. \tag{3.19}$$

For the inequality's second part, we have

$$\zeta\Big(\frac{t^\rho}{2}a_1^\rho+(\frac{2-t^\rho}{2})b_1^\rho\Big)\leq[\zeta(a_1^\rho)]^{\frac{t^\rho}{2}}[\zeta(b_1^\rho)]^{\frac{2-t^\rho}{2}},$$

and

$$\zeta((\frac{2-t^\rho}{2})a_1^\rho+(\frac{t^\rho}{2})b_1^\rho)\leq[\zeta(a_1^\rho)]^{\frac{2-t^\rho}{2}}[\zeta(b_1^\rho)]^{\frac{t^\rho}{2}}).$$

Multiplying both of the above inequalities, after simplification, we get

$$\zeta\Big(\frac{t^\rho}{2}a_1^\rho+(\frac{2-t^\rho}{2})b_1^\rho\Big)\zeta((\frac{2-t^\rho}{2})a_1^\rho+(\frac{t^\rho}{2})b_1^\rho)\leq\zeta(a_1^\rho)\zeta(b_1^\rho).$$

Taking the square root on both sides, we have

$$\sqrt{\zeta\Big(\frac{t^\rho}{2}a_1^\rho+(\frac{2-t^\rho}{2})b_1^\rho\Big)}\sqrt{\zeta((\frac{2-t^\rho}{2})a_1^\rho+(\frac{t^\rho}{2})b_1^\rho)}\leq\sqrt{\zeta(a_1^\rho)\zeta(b_1^\rho)}. \tag{3.20}$$

Multiplying both sides of (3.20) by $t^{\beta\rho-1}$, then integrating w.r.t t from $t\in[0,1]$ gives us

$$\int_0^1 t^{\beta\rho-1}\sqrt{\zeta\Big(\frac{t^\rho}{2}a_1^\rho+(\frac{2-t^\rho}{2})b_1^\rho\Big)}\sqrt{\zeta((\frac{2-t^\rho}{2})a_1^\rho+(\frac{t^\rho}{2})b_1^\rho)}\leq\sqrt{\zeta(a_1^\rho)\zeta(b_1^\rho)}\int_0^1 t^{\beta\rho-1}dt.$$

By repeating the same criteria as already used in the above inequalities, we get

$$\frac{(2\rho)^{\beta}\Gamma(\beta+1)}{(b_1^{\rho}-a_1^{\rho})^{\beta}}\sqrt{{}^{\rho}I^{\beta}_{(\frac{a_1^{\rho}+b_1^{\rho}}{2})+}\zeta(b_1^{\rho})\,{}^{\rho}I^{\beta}_{(\frac{a_1^{\rho}+b_1^{\rho}}{2})-}\zeta(a_1^{\rho})} \leq \sqrt{\zeta(a_1^{\rho})\zeta(b_1^{\rho})}. \tag{3.21}$$

After combining (3.19) and (3.21), we get

$$\zeta\Big(\frac{a_1^{\rho}+b_1^{\rho}}{2}\Big) \leq \frac{(2\rho)^{\beta}\Gamma(\beta+1)}{(b_1^{\rho}-a_1^{\rho})^{\beta}}\sqrt{{}^{\rho}I^{\beta}_{(\frac{a_1^{\rho}+b_1^{\rho}}{2})+}\zeta(b_1^{\rho})\,{}^{\rho}I^{\beta}_{(\frac{a_1+b_1}{2})-}\zeta(a_1^{\rho})} \leq \sqrt{\zeta(a_1^{\rho})\zeta(b_1^{\rho})}.$$

Corollary 3.15. *For the inequality involving Riemann-Liouville fractional integrals, we use $\rho \to 1$ in Theorem 3.14:*

$$\zeta\Big(\frac{a_1+b_1}{2}\Big) \leq \frac{\Gamma(\beta+1)2^{\beta}}{(b_1-a_1)^{\beta}}\sqrt{I^{\beta}_{(\frac{a_1+b_1}{2})+}\zeta(b_1)I^{\beta}_{(\frac{a_1+b_1}{2})-}\zeta(a_1)} \leq \sqrt{\zeta(a_1)\zeta(b_1)}.$$

Corollary 3.16. *We can find the Hermite-Hadamard-type inequality proved in* (3.10) *if we replace $\rho \to 1$, and ζ is symmetrical about the point $\frac{(a_1+b_1)}{2}$:*

$$\zeta\Big(\frac{a_1+b_1}{2}\Big) \leq \frac{1}{(b_1-a_1)}\int_{a_1}^{b_1}\zeta(u)du \leq \sqrt{\zeta(a_1)\zeta(b_1)}.$$

We now have a mid point-type inequality with a log-convex function.

Theorem 3.17. *Suppose that $\zeta : [a_1^{\rho}, b_1^{\rho}] \subseteq \mathbb{R} \to (0,\infty)$ is a differentiable mapping on (a_1^{ρ}, b_1^{ρ}) with $0 \leq a_1 < b_1$. If $|\zeta'|$ is log-convex on $[a_1^{\rho}, b_1^{\rho}]$, then the following inequality holds:*

$$\left|\frac{\zeta(a^{\rho})+\zeta(b^{\rho})}{2} - \frac{\rho^{\beta-1}\Gamma(\beta+1)}{(b^{\rho}-a^{\rho})^{\beta}}[{}^{\rho}I^{\beta}_{a+}\zeta(b) + {}^{\rho}I^{\beta}_{b-}\zeta(a)]\right| \leq \frac{(b_1^{\rho}-a_1^{\rho})}{\beta\rho(\beta+1)}\left|\zeta'(a_1^{\rho})+\zeta'(b_1^{\rho})\right|,$$

under the conditions $|\zeta'(a_1^{\rho})| \neq 0$ and $|\zeta'(b_1^{\rho})| \neq 0$.

Proof. From [24], we have

$$\begin{aligned}
&\left|\frac{\zeta(a_1^{\rho})+\zeta(b_1^{\rho})}{2} - \frac{\rho^{\beta-1}\Gamma(\beta+1)}{(b_1^{\rho}-a_1^{\rho})^{\beta}}[{}^{\rho}I^{\beta}_{a_1+}\zeta(b_1) + {}^{\rho}I^{\beta}_{b_1-}\zeta(a_1)]\right| \\
\leq\ & \frac{(b_1^{\rho}-a_1^{\rho})}{\beta}\int_0^1 t^{\rho(\beta+1)-1}\left|\zeta'((1-t^{\rho})a_1^{\rho}+t^{\rho}b_1^{\rho}) - \zeta'(t^{\rho}a_1^{\rho}+(1-t^{\rho})b_1^{\rho})\right|dt \\
=\ & \frac{(b_1^{\rho}-a_1^{\rho})}{\beta}\int_0^1 t^{\rho(\beta+1)-1}\Big(|\zeta'((1-t^{\rho})a_1^{\rho}+t^{\rho}b_1^{\rho})| + |\zeta'(t^{\rho}a_1^{\rho}+(1-t^{\rho})b_1^{\rho})|\Big)dt.
\end{aligned}$$

Since ζ' is log-convex, then

$$\begin{aligned}
&\left|\frac{\zeta(a_1^{\rho})+\zeta(b_1^{\rho})}{2} - \frac{\rho^{\beta-1}\Gamma(\beta+1)}{(b_1^{\rho}-a_1^{\rho})^{\beta}}[{}^{\rho}I^{\beta}_{a_1+}\zeta(b_1) + {}^{\rho}I^{\beta}_{b_1-}\zeta(a_1)]\right| \\
\leq\ & \frac{(b_1^{\rho}-a_1^{\rho})}{\beta}\int_0^1 t^{\rho(\beta+1)-1}[|(\zeta'(a_1^{\rho}))|^{(1-t^{\rho})}|(\zeta'(b_1^{\rho}))|^{t^{\rho}} + |\zeta'(a_1^{\rho})|^{t^{\rho}}|\zeta'(b_1^{\rho})|^{(1-t^{\rho})}|]dt.
\end{aligned}$$

For $|\zeta'(a_1^\rho)| > 0$, $|\zeta'(b_1^\rho)| > 0$, $t^\rho > 0$, $1 - t^\rho > 0$ with $t \in [0, 1]$, use the inequality $a^\alpha b^\beta + c^\alpha d^\beta \leq (a+c)^{\alpha+\beta}$ with $c = b$ and $a = d$:

$$\begin{aligned} &\left|\frac{\zeta(a_1^\rho) + \zeta(b_1^\rho)}{2} - \frac{\rho^{\beta-1}\Gamma(\beta+1)}{(b_1^\rho - a_1^\rho)^\beta}[{}^\rho I^\beta_{a_1+}\zeta(b_1) +^\rho I^\beta_{b_1-}\zeta(a_1)]\right| \\ \leq\ & \frac{(b_1^\rho - a_1^\rho)}{\beta}\int_0^1 t^{\rho(\beta+1)-1}|\zeta'(a_1^\rho) + \zeta'(b_1^\rho)|dt \\ =\ & \frac{(b_1^\rho - a_1^\rho)}{\beta\rho(\beta+1)}|\zeta'(a_1^\rho) + \zeta'(b_1^\rho)|. \end{aligned}$$

Hence,

$$\left|\frac{\zeta(a^\rho) + \zeta(b^\rho)}{2} - \frac{\rho^{\beta-1}\Gamma(\beta+1)}{(b^\rho - a^\rho)^\beta}[{}^\rho I^\beta_{a+}\zeta(b) +^\rho I^\beta_{b-}\zeta(a)]\right| \leq \frac{(b_1^\rho - a_1^\rho)}{\beta\rho(\beta+1)}\left|\zeta'(a_1^\rho) + \zeta'(b_1^\rho)\right|.$$

Corollary 3.18. *For graphical representation, we choose $\rho = 2$, $a_1 = 1$, $b_1 = 4$, $\zeta(t) = e^t$ in* (3.17)*, and we have the following inequality:*

$$\left|\left(\frac{e + e^1 6}{2} - \frac{\beta}{15^\beta}\right)\left(\int_1^4 t(16 - t^2)^{\beta-1}e^t dt + \int_1^4 t(t^2-1)^{\beta-1}e^t dt\right)\right| \leq \frac{15(e + e^1 6)}{2\beta(\beta-1)}. \tag{3.22}$$

For the two-dimensional graphical representation of inequality (3.22) *given in Figure 5, we fix* $\beta = 0.0001$.

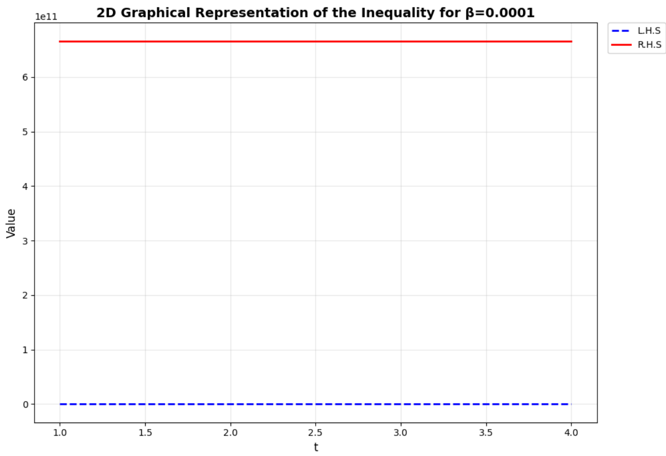

Figure 5. This figure features a 2D representation of the two components of the inequality given in (3.22) for $1 < t < 4$.

For the three-dimensional graphical representation of inequality (3.22) *given in Figure 6, we choose* $0 < \beta < 0.01$.

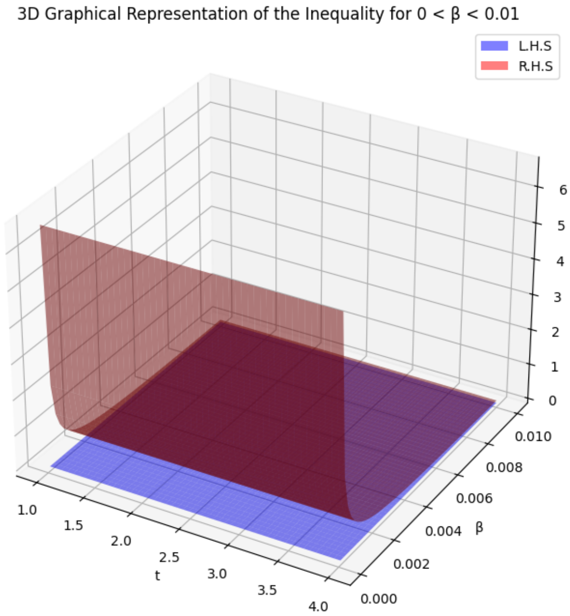

Figure 6. The figure illustrates a 3D plot capturing the two sides of the inequality given in (3.22) for $1 < t < 4$.

Table 3 illustrates the numerical interpretation of inequality (3.22).

Table 3. Further, we fix additional values of β in inequality (3.22).

β	Left-hand side	Right-hand side
0.0010	4443027.9816	66579270020.9018
0.0050	4443028.0628	13262855580.2831
0.0100	4443028.1629	6598598939.6953

We now build up a mid point-type inequality that is exponentially trigonometric convex.

Theorem 3.19. *Suppose that $\zeta : [a_1^\rho, b_1^\rho] \subseteq \mathbb{R} \to \mathbb{R}$ is a differentiable mapping on (a_1^ρ, b_1^ρ) with $0 \leq a_1 < b_1$. If $|\zeta'|$ is exponential trigonometric convex on $[a_1^\rho, b_1^\rho]$, then the following inequality holds:*

$$\left|\frac{\zeta(a_1^\rho)+\zeta(b_1^\rho)}{2} - \frac{\rho^{\beta-1}\Gamma(\beta+1)}{(b_1^\rho - a_1^\rho)^\beta}[{}^\rho I^\beta_{a_1+}\zeta(b_1) +^\rho I^\beta_{b_1-}\zeta(a_1)]\right| \leq \frac{(b_1^\rho - a_1^\rho)}{\beta\rho(\beta+1)}[|\zeta'(a_1^\rho) + \zeta'(b_1^\rho)|],$$

under the condition that $\left(\frac{\sin\frac{\pi t^\rho}{2}}{e^{1-t^\rho}} + \frac{\cos\frac{\pi t^\rho}{2}}{e^{t^\rho}}\right) \leq 1$, *for* $t \in [0, 1]$.

Proof. From [24], we have

$$\begin{aligned}
&\left|\frac{\zeta(a_1^\rho)+\zeta(b_1^\rho)}{2} - \frac{\rho^{\beta-1}\Gamma(\beta+1)}{(b_1^\rho - a_1^\rho)^\beta}[{}^\rho I^\beta_{a_1+}\zeta_1(b_1) +^\rho I^\beta_{b_1-}\zeta(a_1)]\right| \\
\leq\ & \frac{(b_1^\rho - a_1^\rho)}{\beta}\int_0^1 t^{\rho(\beta+1)-1}|\zeta'(t^\rho a_1^\rho + (1-t^\rho)b_1^\rho) - \zeta'(t^\rho b_1^\rho + (1-t^\rho)a_1^\rho)|dt \\
=\ & \frac{(b_1^\rho - a_1^\rho)}{\beta}\int_0^1 t^{\rho(\beta+1)-1}|\zeta'(t^\rho a_1^\rho + (1-t^\rho)b_1^\rho)| + |\zeta'(t^\rho b_1^\rho + (1-t^\rho)a_1^\rho)|dt.
\end{aligned}$$

Using an exponentially trigonometric convex function, we have

$$\begin{aligned}&\left|\frac{\zeta(a_1^\rho)+\zeta(b_1^\rho)}{2}-\frac{\rho^{\beta-1}\Gamma(\beta+1)}{(b_1^\rho-a_1^\rho)^\beta}[{}^\rho I^\beta_{a_1+}\zeta_1(b_1)+{}^\rho I^\beta_{b_1-}\zeta(a_1)]\right|\\ \leq\ &\frac{(b_1^\rho-a_1^\rho)}{\beta}\int_0^1 t^{\rho(\beta+1)-1}\Big(\frac{\sin\frac{\pi t^\rho}{2}}{e^{1-t^\rho}}|\zeta'(a_1^\rho)|+\frac{\cos\frac{\pi t^\rho}{2}}{e^{t^\rho}}|\zeta'(b_1^\rho)|+\frac{\cos\frac{\pi t^\rho}{2}}{e^{t^\rho}}|\zeta'(a_1^\rho)|+\frac{\sin\frac{\pi t^\rho}{2}}{e^{1-t^\rho}}\zeta'(b_1^\rho)\Big)dt\\ =\ &\frac{(b_1^\rho-a_1^\rho)}{\beta}|\zeta'(a_1^\rho)+\zeta'(b_1^\rho)|\int_0^1 t^{\rho(\beta+1)-1}(\frac{\sin\frac{\pi t^\rho}{2}}{e^{1-t^\rho}}+\frac{\cos\frac{\pi t^\rho}{2}}{e^{t^\rho}})dt.\end{aligned}$$

Hence,

$$\left|\frac{\zeta(a_1^\rho)+\zeta(b_1^\rho)}{2}-\frac{\rho^{\beta-1}\Gamma(\beta+1)}{(b_1^\rho-a_1^\rho)^\beta}[{}^\rho I^\beta_{a_1+}\zeta_1(b_1)+{}^\rho I^\beta_{b_1-}\zeta(a_1)]\right|\leq\frac{(b_1^\rho-a_1^\rho)}{\beta}|\zeta'(a_1^\rho)+\zeta'(b_1^\rho)|\int_0^1 t^{\rho(\beta+1)-1}dt.$$

Finally,

$$\left|\frac{\zeta(a_1^\rho)+\zeta(b_1^\rho)}{2}-\frac{\rho^{\beta-1}\Gamma(\beta+1)}{(b_1^\rho-a_1^\rho)^\beta}[{}^\rho I^\beta_{a_1+}\zeta(b_1)+{}^\rho I^\beta_{b_1-}\zeta(a_1)]\right|\leq\frac{(b_1^\rho-a_1^\rho)}{\beta\rho(\beta+1)}[|\zeta'(a_1^\rho)+\zeta'(b_1^\rho)|].$$

We now have a Fejèr type inequality involving a log-convex function.

Theorem 3.20. *Let $\zeta:[a_1^\rho,b_1^\rho]\to(0,\infty)$, $a_1^\rho<b_1^\rho$ be a log-convex function with $a_1^\rho<b_1^\rho$ and $\zeta\in X_c^p$. If $g:[a_1^\rho,b_1^\rho]\to\mathbb{R}$ is a function such that it is non-negative, symmetric and integrable to $\frac{a_1^\rho+b_1^\rho}{2}$, then we have the following inequality:*

$$\begin{aligned}&\zeta\Big(\frac{a_1^\rho+b_1^\rho}{2}\Big)\Big(\frac{{}^\rho I^\beta_{a_1+}(g)(b_1^\rho)+{}^\rho I^\beta_{b_1-}(g)(a_1^\rho)}{2}\Big)\\ \leq\ &\sqrt{{}^\rho I^\beta_{a_1+}(\zeta g)(b_1^\rho){}^\rho I^\beta_{b_1-}(\zeta g)(a_1^\rho)}\leq\sqrt{\zeta(a_1^\rho)\zeta(b_1^\rho)}\Big(\frac{{}^\rho I^\beta_{a_1+}(g)(b_1^\rho)+{}^\rho I^\beta_{b_1-}(g)(a_1^\rho)}{2}\Big),\end{aligned}$$

where $[a_1^\rho,b_1^\rho]\subseteq\mathbb{R}$.

Proof. Let $x_1^\rho,y_1^\rho\in[a_1^\rho,b_1^\rho]$, and then by log-convexity, we have

$$\zeta\Big(\frac{x_1^\rho+y_1^\rho}{2}\Big)\leq\sqrt{\zeta(x_1^\rho)}\sqrt{\zeta(y_1^\rho)}.$$

Set $x_1^\rho=t^\rho a_1^\rho+(1-t^\rho)b_1^\rho$, $y_1^\rho=(1-t^\rho)a_1^\rho+t^\rho b_1^\rho$, and then we have

$$\zeta\Big(\frac{a_1^\rho+b_1^\rho}{2}\Big)\leq\sqrt{\zeta(t^\rho a_1^\rho+(1-t^\rho)b_1^\rho)}\sqrt{\zeta((1-t^\rho)a_1^\rho+t^\rho b_1^\rho)}.\tag{3.23}$$

Multiplying both sides of (3.23) by $t^{\rho\beta-1}g(t^\rho b_1^\rho+(1-t^\rho)a_1^\rho)$, then integrating w.r.t t from $[0,1]$, we have

$$\begin{aligned}&\zeta\Big(\frac{a_1^\rho+b_1^\rho}{2}\Big)\int_0^1[t^{\rho\beta-1}g(t^\rho b_1^\rho+(1-t^\rho)a_1^\rho]dt\\ \leq\ &\int_0^1[t^{\rho\beta-1}g(t^\rho b_1^\rho+(1-t^\rho)a_1^\rho]\times\sqrt{\zeta(t^\rho a_1^\rho+(1-t^\rho)b_1^\rho)}\sqrt{\zeta((1-t^\rho)a_1^\rho+t^\rho(b_1^\rho))}dt.\end{aligned}$$

By applying the Roger-Hölder inequality, we get

$$
\begin{aligned}
&\zeta\Big(\frac{a_1^\rho+b_1^\rho}{2}\Big)\int_0^1[t^{\rho\beta-1}g\Big(t^\rho b_1^\rho+(1-t^\rho)a_1^\rho\Big)dt\\
\leq\ & \sqrt{\int_0^1 t^{\rho\beta-1}g\Big(t^\rho b_1^\rho+(1-t^\rho)a_1^\rho\Big)\zeta(t^\rho a_1^\rho+(1-t^\rho)b_1^\rho)dt)}\\
&\times\sqrt{\int_0^1 t^{\rho\beta-1}g\Big(t^\rho b_1^\rho+(1-t^\rho)a_1^\rho\Big)\zeta\Big((1-t^\rho)a_1^\rho+(t^\rho)b_1^\rho dt\Big)}.
\end{aligned}
$$

By using substations, we get

$$
\begin{aligned}
&\frac{1}{(b_1^\rho-a_1^\rho)^\beta}\zeta\Big(\frac{a_1^\rho+b_1^\rho}{2}\Big)\int_{a_1}^{b_1}[(u^\rho-a_1^\rho)^{\beta-1}g(u^\rho)du\\
\leq\ & \sqrt{\frac{1}{(b_1^\rho-a_1^\rho)^\beta}\int_{a_1}^{b_1}(u^\rho-a_1^\rho)^{\beta-1}g(u^\rho)\zeta(a_1^\rho+b_1^\rho-u^\rho)du}\\
&\times\sqrt{\frac{1}{(b_1^\rho-a_1^\rho)^\beta}\int_{a_1}^{b_1}(u^\rho-a_1^\rho)^{\beta-1}\zeta(u^\rho)g(u^\rho)du}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\frac{1}{(b_1^\rho-a_1^\rho)^\beta}\zeta\Big(\frac{a_1^\rho+b_1^\rho}{2}\Big)\int_{a_1}^{b_1}[(u^\rho-a_1^\rho)^{\beta-1}g(u^\rho)du\\
\leq\ & \sqrt{\frac{1}{(b_1^\rho-a_1^\rho)^\beta}}\sqrt{\int_{a_1}^{b_1}(u^\rho-a_1^\rho)^{\beta}g(u^\rho)\zeta(a_1^\rho+b_1^\rho-u^\rho)du^\rho}\\
&\times\sqrt{\frac{1}{(b_1^\rho-a_1^\rho)^\beta}}\sqrt{\int_{a_1}^{b_1}(u^\rho-a_1^\rho)^{\beta-1}\zeta(u^\rho)g(u^\rho)du}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\frac{1}{(b_1^\rho-a_1^\rho)^\beta}\zeta\Big(\frac{a_1^\rho+b_1^\rho}{2}\Big)\int_{a_1}^{b_1}[(u^\rho-a_1^\rho)^{\beta-1}g(u^\rho)du\\
\leq\ & \frac{1}{(b_1^\rho-a_1^\rho)^\beta}[\sqrt{\frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_{a_1}^{b_1}[(u^\rho-a_1^\rho)^{\beta-1}g(u^\rho)\zeta(a_1^\rho+b_1^\rho-u^\rho)du}\\
&\times\sqrt{\frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_{a_1}^{b_1}(u^\rho-a_1^\rho)^{\beta-1}\zeta(u^\rho)g(u^\rho)du}]\\
=\ & \frac{1}{(b_1^\rho-a_1^\rho)^\beta}[\sqrt{\frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_{a_1}^{b_1}[(b_1^\rho-u^\rho)^{\beta-1}g(a_1^\rho+b_1^\rho-u^\rho)\zeta(u^\rho)du}\\
&\times\sqrt{\frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_{a_1}^{b_1}(u^\rho-a_1^\rho)^{\beta-1}\zeta(u^\rho)g(u^\rho)du}]
\end{aligned}
$$

$$= \frac{1}{(b_1^\rho - a_1^\rho)^\beta}\Big[\sqrt{\frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_{a_1}^{b_1}[(b_1^\rho - u^\rho)^{\beta-1}g(u^\rho)\zeta(u^\rho)du}$$
$$\times\sqrt{\frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_{a_1}^{b_1}(u^\rho - a_1^\rho)^{\beta-1}\zeta(u^\rho)g(u^\rho)du}\Big].$$

By using Lemma 2.8, we have

$$\zeta\Big(\frac{a_1^\rho + b_1^\rho}{2}\Big)\Big(\frac{{}^\rho I_{a_1+}^\beta(g)(b_1^\rho) + {}^\rho I_{b_1-}^\beta(g)(a_1^\rho)}{2}\Big) \le \sqrt{{}^\rho I_{a_1+}^\beta(\zeta g)(b_1^\rho){}^\rho I_{b_1-}^\beta(\zeta g)(a_1^\rho)}. \tag{3.24}$$

To prove the inequality's second part, we have

$$\zeta(t^\rho a_1^\rho + (1 - t^\rho)b_1^\rho) \le [\zeta(a_1^\rho)]^{t^\rho}[\zeta(b_1^\rho)]^{1-t^\rho},$$

and

$$\zeta((1 - t^\rho)a_1^\rho + t^\rho b_1^\rho) \le [\zeta(a_1^\rho)]^{1-t^\rho}[\zeta(b_1^\rho)]^{t^\rho}.$$

By multiplying both of the above inequalities, we get

$$\zeta(t^\rho a_1^\rho + (1 - t^\rho)b_1^\rho)\zeta((1 - t^\rho)a_1^\rho + t^\rho b_1^\rho) \le [\zeta(a_1^\rho)]^{t^\rho}[\zeta(b_1^\rho)]^{1-t^\rho}[\zeta(a_1^\rho)]^{1-t^\rho}[\zeta(b_1^\rho)]^{t^\rho} \le \zeta(a_1^\rho)\zeta(b_1^\rho).$$

Applying a square root on both sides gives us

$$\sqrt{\zeta(t^\rho a_1^\rho + (1 - t^\rho)b_1^\rho)}\sqrt{(1 - t^\rho)a_1^\rho + t^\rho b_1^\rho} \le \sqrt{\zeta(a_1^\rho)\zeta(b_1^\rho)}. \tag{3.25}$$

Multiplying both sides of (3.25) by $t^{\rho\beta-1}g(t^\rho b_1^\rho + (1-t^\rho)a_1^\rho$, then integrating w.r.t t over $t \in [0, 1]$ gives us

$$\int_0^1 t^{\rho\beta-1}g(t^\rho b_1^\rho + (1 - t^\rho)a_1^\rho)\sqrt{\zeta(t^\rho a_1^\rho + (1 - t^\rho)b_1^\rho}\sqrt{\zeta(1 - t^\rho)a_1^\rho + t^\rho b_1^\rho}dt$$
$$\le \sqrt{\zeta(a_1^\rho)\zeta(b_1^\rho)}\int_0^1 t^{\rho\beta-1}g(t^\rho b_1^\rho + (1 - t^\rho)a_1^\rho)dt.$$

By repeating the same process as we used in the above inequality, we get

$$\sqrt{{}^\rho I_{a_1+}^\beta(\zeta g)(b_1^\rho){}^\rho I_{b_1-}^\beta(\zeta g)(a_1^\rho)} \le \sqrt{\zeta(a_1^\rho)\zeta(b_1^\rho)}\Big(\frac{{}^\rho I_{a_1+}^\beta(g)(b_1^\rho) + {}^\rho I_{b_1-}^\beta(g)(a_1^\rho)}{2}\Big). \tag{3.26}$$

From (3.24) and (3.26), we get

$$\zeta\Big(\frac{a_1^\rho + b_1^\rho}{2}\Big)\Big(\frac{{}^\rho I_{a_1+}^\beta(g)(b_1^\rho) + {}^\rho I_{b_1-}^\beta(g)(a_1^\rho)}{2}\Big)$$
$$\le \sqrt{{}^\rho I_{a_1+}^\beta(\zeta * g)(b_1^\rho){}^\rho I_{b_1-}^\beta(\zeta * g)(a_1^\rho)} \le \sqrt{\zeta(a_1^\rho)\zeta(b_1^\rho)}\Big(\frac{{}^\rho I_{a_1+}^\beta(g)(b_1^\rho) + {}^\rho I_{b_1-}^\beta(g)(a_1^\rho)}{2}\Big).$$

Corollary 3.21. *We gain the inequality involving the Riemann-Liouville fractional integral result if $\rho \to 1$ in Theorem 3.20:*

$$\xi(\frac{a_1+b_1}{2})\Big(\frac{I^{\beta}_{a_1+}(g)(b_1)+I^{\beta}_{b_1-}(g)(a_1)}{2}\Big)$$
$$\leq \sqrt{I^{\beta}_{a_1+}(\xi g)(b_1)I^{\beta}_{b_1-}(\xi g)(a_1)} \leq \sqrt{\xi(a_1)\xi(b_1)}\Big(\frac{I^{\beta}_{a_1+}(g)(b_1)+I^{\beta}_{b_1-}(g)(a_1)}{2}\Big).$$

Corollary 3.22. *For graphical representation, we choose $\rho = 2$, $a_1 = 1$, $b_1 = 4$, $\zeta(t) = e^{\sin t}$ in* (3.17), *and we have the following inequality:*

$$\Big|\frac{e^{\sin 1}+e^{\sin 16}}{2} - \frac{\beta}{15^{\beta}}\Big(\int_1^4 t(16-t^2)^{\beta-1}e^t dt + \int_1^4 t(t^2-1)^{\beta-1}e^t dt\Big)\Big|$$
$$\leq \frac{|15(\cos 1 e^{\sin 1} + \cos 16 e^{\sin 16})|}{2\beta(\beta-1)}. \tag{3.27}$$

For the two-dimensional graphical representation of inequality (3.27) *in Figure 7, we fix $\beta = 0.0001$.*

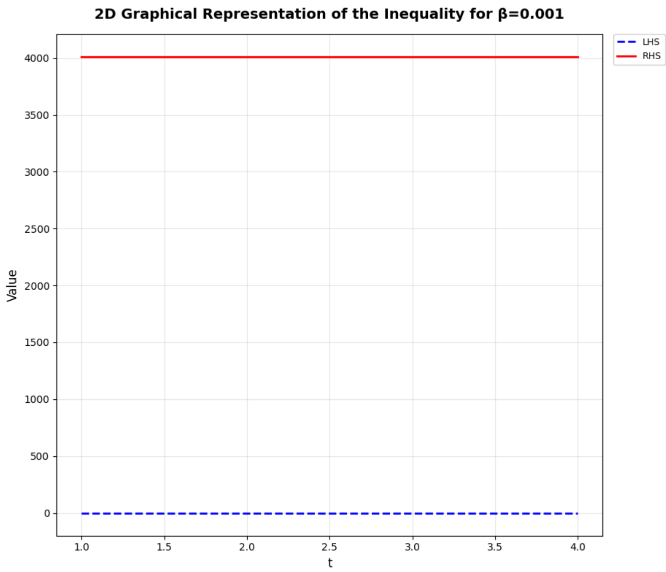

Figure 7. This figure shows a 2D plot depicting the three components of the inequality given in (3.27) for $1 < t < 4$.

For the three-dimensional graphical representation of inequality (3.27) *in Figure 8, we choose* $0 < \beta < 0.01$.

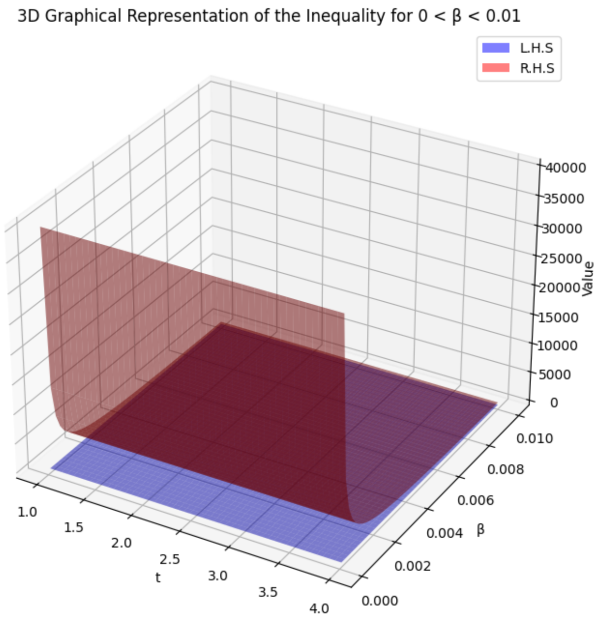

Figure 8. The figure displays a 3D visualization of the function from (3.27), comparing its behavior for $1 < t < 4$.

Table 4 illustrates the numerical interpretation of inequality (3.27).

Table 4. Numerical results of (3.27) for fixed values of β.

β	Left-hand side	Right-hand side
0.0010	0.1398	4010.7022
0.0050	0.1377	798.9478
0.0100	0.1352	397.4963

We now have a Fejér-type inequality involving an exponentially trigonometric convex function.

Theorem 3.23. *Let $\zeta : [a_1^\rho, b_1^\rho] \to \mathbb{R}$, $a_1^\rho < b_1^\rho$ be a exponential trigonometric convex function with $a_1^\rho < b_1^\rho$ and $\zeta \in L[a_1^\rho, b_1^\rho]$. If $g : [a_1^\rho, b_1^\rho] \to \mathbb{R}$ is a function such that it is non-negative, integrable, and symmetric to $\frac{a_1^\rho + b_1^\rho}{2}$, then the following inequality for fractional integrals holds:*

$$\zeta\Big(\frac{a_1^\rho + b_1^\rho}{2}\Big)\Big[\frac{{}^\rho I^\mu_{a_1+}(g)(b_1^\rho) + {}^\rho I^\mu_{b_1-}(g)(a_1^\rho)}{2}\Big]$$
$$\leq \sqrt{\frac{2}{e}}\Big[\frac{{}^\rho I^\mu_{a_1+}(\zeta * g)(b_1^\rho) + {}^\rho I^\mu_{b_1-}(\zeta * g)(a_1^\rho)}{2}\Big] \leq \sqrt{\frac{2}{e}}\frac{[\zeta(a_1^\rho) + \zeta(b_1^\rho)]}{2}\Big[\frac{{}^\rho I^\mu_{a_1+}(g)(b_1^\rho) + {}^\rho I^\mu_{b_1-}(g)(a_1^\rho)}{2}\Big],$$

where $[a_1^\rho, b_1^\rho] \subseteq \mathbb{R}$.

Proof. Let $x_1^\rho, y_1^\rho \in [a_1^\rho, b_1^\rho]$, and then by exponential trigonometric convexity, we have

$$\zeta\Big(\frac{x_1^\rho + y_1^\rho}{2}\Big) \leq \frac{1}{\sqrt{2e}}\zeta(x_1^\rho) + \frac{1}{\sqrt{2e}}\zeta(y_1^\rho).$$

Set $x_1^\rho = t^\rho a_1^\rho + (1-t^\rho)b_1^\rho$, $y_1^\rho = (1-t^\rho)a_1^\rho + t^\rho b_1^\rho$, and then we have

$$\zeta\Big(\frac{a_1^\rho + b_1^\rho}{2}\Big) \le \frac{1}{\sqrt{2e}}[\zeta(t^\rho a_1^\rho + (1-t^\rho)b_1^\rho) + \zeta((1-t^\rho)a_1^\rho + t^\rho b_1^\rho)]. \tag{3.28}$$

Multiplying both sides of (3.28) by $t^{\beta\rho-1}g(t^\rho b_1^\rho + (1-t^\rho)a_1^\rho$, then integrating w.r.t t over $[0,1]$, we have

$$\begin{aligned}
&\zeta\Big(\frac{a_1^\rho + b_1^\rho}{2}\Big)\int_0^1 t^{\beta\rho-1}g\Big(t^\rho b_1^\rho + (1-t^\rho)a_1^\rho\Big)dt \\
\le\ & \frac{1}{\sqrt{2e}}\int_0^1 t^{\beta\rho-1}g\Big(t^\rho b_1^\rho + (1-t^\rho)a_1^\rho\Big)\zeta(ta_1^\rho + (1-t^\rho)b_1^\rho)dt \\
&+\frac{1}{\sqrt{2e}}\int_0^1 t^{\beta\rho-1}g\Big(t^\rho b_1^\rho + (1-t^\rho)a_1^\rho\Big)\zeta\Big((1-t^\rho)a_1^\rho + t^\rho b_1^\rho\Big)dt.
\end{aligned}$$

By using appropriate substitution, we get

$$\begin{aligned}
&\frac{1}{(b_1^\rho - a_1^\rho)^\beta}\zeta\Big(\frac{a_1^\rho + b_1^\rho}{2}\Big)\int_{a_1}^{b_1}\Big(u^\rho - a_1^\rho\Big)^{\beta-1}g(u^\rho)du \\
\le\ & \frac{1}{\sqrt{2e}}\frac{1}{(b_1^\rho - a_1^\rho)^\beta}\int_{a_1}^{b_1}\Big(u^\rho - a_1^\rho\Big)^{\beta-1}g(u)\zeta(a_1^\rho + b_1^\rho - u^\rho)du \\
&+\frac{1}{\sqrt{2e}}\frac{1}{(b_1^\rho - a_1^\rho)^\beta}\int_{a_1}^{b_1}\Big(u^\rho - a_1^\rho\Big)^{\beta-1}\zeta(u^\rho)g(u^\rho)du.
\end{aligned}$$

This can be written as

$$\begin{aligned}
&\zeta\Big(\frac{a_1^\rho + b_1^\rho}{2}\Big)\int_{a_1}^{b_1}(u^\rho - a_1^\rho)^{\beta-1}g(u^\rho)du \\
\le\ & \frac{1}{\sqrt{2e}}\int_{a_1}^{b_1}(u^\rho - a_1^\rho)^{\beta-1}g(u^\rho)\zeta(a_1^\rho + b_1^\rho - u^\rho)du + \frac{1}{\sqrt{2e}}\int_{a_1}^{b_1}(u^\rho - a_1^\rho)^{\beta-1}\zeta(u^\rho)g(u^\rho)du \\
=\ & \frac{1}{\sqrt{2e}}\int_{a_1}^{b_1}\Big(b_1^\rho - u^\rho\Big)^{\beta-1}g(a_1^\rho + b_1^\rho - u^\rho)\zeta(u^\rho)du + \frac{1}{\sqrt{2e}}\int_{a_1}^{b_1}(u^\rho - a_1^\rho)^{\beta-1}\zeta(u^\rho)g(u^\rho)du \\
=\ & \frac{1}{\sqrt{2e}}\int_{a_1}^{b_1}\Big(b_1^\rho - u^\rho\Big)^{\beta-1}g(u^\rho)\zeta(u^\rho)du + \frac{1}{\sqrt{2e}}\int_{a_1}^{b_1}(u^\rho - a_1^\rho)^{\beta-1}\zeta(u^\rho)g(u^\rho)du.
\end{aligned}$$

Finally, we have

$$\zeta\Big(\frac{a_1^\rho + b_1^\rho}{2}\Big)\Big[\frac{{}^\rho I^\mu_{a_1+}(g)(b_1^\rho) + {}^\rho I^\mu_{b_1-}(g)(a_1^\rho)}{2}\Big] \le \sqrt{\frac{2}{e}}\Big[\frac{{}^\rho I^\mu_{a_1+}(\zeta * g)(b_1^\rho) + {}^\rho I^\mu_{b_1-}(\zeta * g)(a_1^\rho)}{2}\Big]. \tag{3.29}$$

Now to prove the inequality's second part, we have

$$\zeta(t^\rho a_1^\rho + (1-t^\rho)b_1^\rho) \le \frac{\sin\frac{\pi t^\rho}{2}}{e^{1-t^\rho}}\zeta(a_1^\rho) + \frac{\cos\frac{\pi t^\rho}{2}}{e^{t^\rho}}\zeta(b_1^\rho),$$

and

$$\zeta((1-t^\rho)a_1^\rho + (t^\rho)b_1^\rho) \le \frac{\cos\frac{\pi t^\rho}{2}}{e^{t^\rho}}\zeta(a_1^\rho) + \frac{\sin\frac{\pi t^\rho}{2}}{e^{1-t^\rho}}\zeta(b_1^\rho).$$

After adding, we have

$$\zeta(t^\rho a_1^\rho + (1-t^\rho)b_1^\rho) + \zeta((1-t^\rho)a_1^\rho + (t^\rho)b_1^\rho) \le (\frac{\sin\frac{\pi t^\rho}{2}}{e^{1-t^\rho}} + \frac{\cos\frac{\pi t^\rho}{2}}{e^{t^\rho}})(\zeta(a_1^\rho) + \zeta(b_1^\rho)).$$

Since $t \in [0,1]$, then $(\frac{\sin\frac{\pi t^\rho}{2}}{e^{1-t^\rho}} + \frac{\cos\frac{\pi t^\rho}{2}}{e^{t^\rho}}) \le 1$ and

$$\begin{aligned}
&\zeta(t^\rho a_1^\rho + (1-t^\rho)b_1^\rho) + \zeta((1-t^\rho)a_1^\rho + (t)b_1^\rho)\\
&\le (\frac{\sin\frac{\pi t^\rho}{2}}{e^{1-t^\rho}} + \frac{\cos\frac{\pi t^\rho}{2}}{e^{t^\rho}})(\zeta(a_1^\rho) + \zeta(b_1^\rho)) \le \zeta(a_1^\rho) + \zeta(b_1^\rho)\\
\Rightarrow\ &\zeta(t^\rho a_1^\rho + (1-t^\rho)b_1^\rho) + \zeta((1-t^\rho)a_1^\rho + (t^\rho)b_1^\rho) \le \zeta(a_1^\rho) + \zeta(b_1^\rho).
\end{aligned} \tag{3.30}$$

Multiplying both sides of (3.30) by $t^{\beta\rho-1}g(t^\rho b_1^\rho + (1-t^\rho)a_1^\rho)$, then integrating w.r.t t over $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 [t^{\beta\rho-1}g(t^\rho b_1^\rho + (1-t^\rho)a_1^\rho\zeta(t^\rho a_1^\rho + (1-t^\rho)b_1^\rho dt]\\
&+\int_0^1 [t^{\beta\rho-1}g(t^\rho b_1^\rho + (1-t^\rho)a_1^\rho(\zeta(1-t^\rho)a_1^\rho + t^\rho b_1^\rho)]dt\\
\le\ & [\zeta(a_1^\rho) + \zeta(b_1^\rho)]\int_0^1 t^{\beta\rho-1}g(t^\rho b_1^\rho + (1-t^\rho)a_1^\rho)dt.
\end{aligned}$$

By repeating the same process as we use to prove the first part of the inequality, we get

$$\sqrt{\frac{2}{e}}\frac{{}^\rho I^\mu_{a_1+}(\zeta * g)(b_1^\rho) + {}^\rho I^\mu_{b_1-}(\zeta * g)(a_1^\rho)}{2} \le \sqrt{\frac{2}{e}}\frac{\zeta(a_1^\rho) + \zeta(b_1^\rho)}{2}\frac{{}^\rho I^\mu_{a_1+}(g)(b_1^\rho) + {}^\rho I^\mu_{b_1-}(g)(a_1^\rho)}{2}.$$

From (3.29) and (3.31), we get

$$\begin{aligned}
\zeta\Big(\frac{a_1^\rho + b_1^\rho}{2}\frac{{}^\rho I^\mu_{a_1+}(g)(b_1^\rho) + {}^\rho I^\mu_{b_1-}(g)(a_1^\rho)}{2} &\le \sqrt{\frac{2}{e}}\frac{{}^\rho I^\mu_{a_1+}(\zeta * g)(b_1^\rho) + {}^\rho I^\mu_{b_1-}(\zeta * g)(a_1^\rho)}{2}\\
&\le \sqrt{\frac{2}{e}}\frac{\zeta(a_1^\rho) + \zeta(b_1^\rho)}{2}\frac{{}^\rho I^\mu_{a_1+}(g)(b_1^\rho) + {}^\rho I^\mu_{b_1-}(g)(a_1^\rho)}{2}.
\end{aligned}$$

Corollary 3.24. *If $\rho \to 1$ in Theorem 3.23, then we get the result for the Riemann-Liouville fractional integrals:*

$$\begin{aligned}
\zeta\Big(\frac{a_1 + b_1}{2}\Big)\frac{I^\beta_{a_1+}(g)(b_1) + I^\beta_{b_1-}(g)(a_1)}{2} &\le \frac{1}{\sqrt{2e}}\Big(I^\beta_{a_1+}(\zeta * g)(b_1) + I^\beta_{b_1-}(\zeta * g)(a_1)\Big)\\
&\le \frac{1}{\sqrt{2e}}\Big(\frac{\zeta(a_1) + \zeta(b_1)}{2}\Big)\Big(I^\beta_{a_1+}(g)(b_1) + I^\beta_{b_1-}(g)(a_1)\Big).
\end{aligned}$$

4. Further inequalities by exponentially trigonometric convex functions

The findings of Jleli et al. [26] are further generalized in this section. Let $\zeta : [a_1, b_1] \to \mathbb{R}$ be a given function, where a_1 and b_1 are positive real numbers. Define $\Psi(x) := \zeta(x) + \zeta(a_1 + b_1 - x)$. Then it is easy to show that if $\zeta(x)$ is convex on $[a_1, b_1]$, and then $\Psi(x)$ is also convex. The function Ψ has a number of unique characteristics, particularly,

(i) $\Psi(x)$ is symmetric to $\frac{(a_1+b_1)}{2}$;
(ii) $\Psi(a_1) = \Psi(b_1) = \zeta(a_1) + \zeta(b_1)$;
(iii) $\Psi(\frac{a_1+b_1}{2}) = 2\zeta\Big(\frac{a_1+b_1}{2}\Big)$.

Theorem 4.1. *If ζ is an exponentially trigonometric convex function on $[a_1, b_1]$ and $\zeta \in L[a_1, b_1]$, then Ψ is also integrable, and the following inequalities hold:*

$$\Psi(\frac{a_1+b_1}{2}) \leq \sqrt{\frac{2}{e}}\frac{\Gamma(\beta+1)\rho^{\beta}}{(b_1^{\rho}-a_1^{\rho})^{\beta}}[\frac{{}^{\rho}I^{\beta}_{a_1+}\Psi(b_1)+{}^{\rho}I^{\beta}_{b_1-}\Psi(a_1)}{2}] \leq \frac{\Psi(a_1)+\Psi(b_1)}{\sqrt{2e}},$$

with $\beta > 0$ and $\rho > 0$.

Proof. Since ζ is an exponentially trigonometric convex function and $x_1, y_1 \in [a_1, b_1]$, then by definition,

$$\zeta\Big(\frac{x_1+y_1}{2}\Big) \leq \frac{1}{\sqrt{2e}}\zeta(x_1) + \frac{1}{\sqrt{2e}}\zeta(y_1).$$

Set $x_1 = ta_1 + (1-t)b_1$, $y_1 = (1-t)a_1 + tb_1$, and then we have

$$\zeta\Big(\frac{a_1+b_1}{2}\Big) \leq \frac{1}{\sqrt{2e}}[\zeta(ta_1+(1-t)b_1) + \zeta((1-t)a_1+tb_1)],$$

$$2\zeta\Big(\frac{a_1+b_1}{2}\Big) \leq \sqrt{\frac{2}{e}}[\zeta(ta_1+(1-t)b_1) + \zeta((1-t)a_1+tb_1)].$$

We convert the previous expression into Ψ:

$$\Psi(\frac{a_1+b_1}{2}) \leq \sqrt{\frac{2}{e}}\Psi((1-t)a_1+tb_1). \tag{4.1}$$

Multiplying both sides of (4.1) by $\frac{((1-t)a_1+tb_1)^{\rho-1}}{[b_1^{\rho}-((1-t)a_1+tb_1)^{\rho}]^{1-\beta}}$, then integrating w.r.t t over [0,1], we have

$$\Psi(\frac{a_1+b_1}{2})\frac{(b_1^{\rho}-a_1^{\rho})^{\beta}}{\beta\rho(b_1-a_1)} \leq \sqrt{\frac{2}{e}}\int_0^1 \frac{((1-t)a_1+tb_1)^{\rho-1}}{[b_1^{\rho}-((1-t)a_1+tb_1)^{\rho}]^{1-\beta}}\Psi((1-t)a_1+tb_1)dt.$$

Using some suitable substitutions, we get

$$\Psi(\frac{a_1+b_1}{2})\frac{(b_1^{\rho}-a_1^{\rho})^{\beta}}{\beta\rho(b_1-a_1)} \leq \sqrt{\frac{2}{e}}\int_{a_1}^{b_1}\frac{u^{\rho-1}}{(b_1^{\rho}-u^{\rho})^{1-\beta}}\Psi(u)\frac{du}{b_1-a_1} = \sqrt{\frac{2}{e}}\frac{\Gamma(\beta)\rho^{\beta-1}}{b_1-a_1}{}^{\rho}I^{\beta}_{a_1+}\Psi(b_1).$$

Hence,

$$\Psi(\frac{a_1+b_1}{2}) \leq \sqrt{\frac{2}{e}}\frac{\Gamma(\beta+1)\rho^{\beta}}{(b_1^{\rho}-a_1^{\rho})^{\beta}}{}^{\rho}I^{\beta}_{a_1+}\Psi(b_1). \tag{4.2}$$

Similarly, if we multiply both sides of (4.1) by $\frac{((1-t)a_1+tb_1)^{\rho-1}}{[((1-t)a_1+tb_1)^{\rho}-a_1^{\rho}]^{1-\beta}}$ and integrate w.r.t t over [0,1], we get

$$\Psi(\frac{a_1+b_1}{2}) \leq \sqrt{\frac{2}{e}}\frac{\Gamma(\beta+1)\rho^{\beta}}{(b_1^{\rho}-a_1^{\rho})^{\beta}}{}^{\rho}I^{\beta}_{b_1-}\Psi(a_1). \tag{4.3}$$

Adding (4.2) and (4.3), we get

$$\Psi(\frac{a_1+b_1}{2}) \leq \sqrt{\frac{2}{e}}\frac{\Gamma(\beta+1)\rho^{\beta}}{(b_1^{\rho}-a_1^{\rho})^{\beta}}[\frac{{}^{\rho}I^{\beta}_{a_1+}\Psi(b_1)+{}^{\rho}I^{\beta}_{b_1-}\Psi(a_1)}{2}]. \tag{4.4}$$

To prove the second part of the inequality, we have

$$\zeta(ta_1+(1-t)b_1) \leq \frac{\sin\frac{\pi t}{2}}{e^{1-t}}\zeta(a_1)+\frac{\cos\frac{\pi t}{2}}{e^{t}}\zeta(b_1),$$

and

$$\zeta((1-t)a_1+tb_1) \leq \frac{\cos\frac{\pi t}{2}}{e^{t}}\zeta(a_1)+\frac{\sin\frac{\pi t}{2}}{e^{1-t}}\zeta(b_1).$$

Adding both of the above inequalities and using the result for $t\in[0,1]$, $\frac{\sin\frac{\pi t}{2}}{e^{1-t}}+\frac{\cos\frac{\pi t}{2}}{e^{t}} \leq 1$, we have

$$\zeta(ta_1+(1-t)b_1)+\zeta((1-t)a_1+tb_1) \leq \zeta(a_1)+\zeta(b_1).$$

Using the $\Psi(x)$ relation, we have

$$\Psi((1-t)a_1+tb_1) \leq \frac{\Psi(a_1)+\Psi(b_1)}{2}. \tag{4.5}$$

Multiplying both sides of (4.5) by $\frac{((1-t)a_1+tb_1)^{\rho-1}}{[b_1^{\rho}-((1-t)a_1+tb_1)^{\rho}]^{1-\beta}}$, then integrating w.r.t t over [0,1], we have

$$\frac{\Gamma(\beta)\rho^{\beta-1}}{b_1-a_1}{}^{\rho}I^{\beta}_{a_1+}\Psi(b_1) \leq \frac{(b_1^{\rho}-a_1^{\rho})^{\beta}}{\beta\rho(b_1-a_1)}\frac{\Psi(a_1)+F(b_1)}{2}.$$

Hence,

$$\frac{\Gamma(\beta+1)\rho^{\beta}}{(b_1^{\rho}-a_1^{\rho})^{\beta}}{}^{\rho}I^{\beta}_{a_1+}\Psi(b_1) \leq \frac{\Psi(a_1)+\Psi(b_1)}{2}. \tag{4.6}$$

Similarly, if we multiply both sides of (4.5) by $\frac{((1-t)a_1+tb_1)^{\rho-1}}{[((1-t)a_1+tb_1)^{\rho}-a_1^{\rho}]^{1-\beta}}$ and integrate w.r.t t over [0,1], we get

$$\frac{\Gamma(\beta+1)\rho^{\beta}}{(b_1^{\rho}-a_1^{\rho})^{\beta}}{}^{\rho}I^{\beta}_{b_1-}\Psi(a_1) \leq \frac{\Psi(a_1)+\Psi(b_1)}{2}. \tag{4.7}$$

Adding (4.6) and (4.7), we get

$$\sqrt{\frac{2}{e}}\frac{\Gamma(\beta+1)\rho^{\beta}}{(b_1^{\rho}-a_1^{\rho})^{\beta}}[\frac{{}^{\rho}I^{\beta}_{a_1+}\Psi(b_1)+{}^{\rho}I^{\beta}_{b_1-}\Psi(a_1)}{2}] \leq \frac{\Psi(a_1)+\Psi(b_1)}{\sqrt{2e}}. \tag{4.8}$$

From (4.4) and (4.8), we have

$$\Psi(\frac{a_1+b_1}{2}) \leq \sqrt{\frac{2}{e}}\frac{\Gamma(\beta+1)\rho^{\beta}}{(b_1^{\rho}-a_1^{\rho})^{\beta}}[\frac{{}^{\rho}I^{\beta}_{a_1+}\Psi(b_1)+{}^{\rho}I^{\beta}_{b_1-}\Psi(a_1)}{2}] \leq \frac{\Psi(a_1)+\Psi(b_1)}{\sqrt{2e}}.$$

Theorem 4.2. *If ζ is an exponentially trigonometric convex function on $[a_1, b_1]$ and $\zeta \in L[a_1, b_1]$, then $\Psi(x)$ is also exponentially trigonometric convex, and $\Psi \in L[a_1, b_1]$. If $g; [a_1, b_1] \to \mathbb{R}$ is a function which is non-negative and integrable, then the following fractional integral inequality is valid:*

$$\Psi(\frac{a_1+b_1}{2})\sqrt{\frac{2}{e}}[{}^{\rho}I^{\beta}_{a_1+}(g\Psi)(b_1){}^{\rho}I^{\beta}_{b_1-}(g\Psi)(a_1) \leq \frac{\Psi(a_1)+\Psi(b_1)}{2}\frac{{}^{\rho}I^{\beta}_{a_1+}g(b_1)+{}^{\rho}I^{\beta}_{b_1-}g(a_1)}{\sqrt{2e}},$$

with $\beta > 0$ and $\rho > 0$.

Proof. Let $x_1, y_1 \in [a_1, b_1]$ and $t \in [0, 1]$, and we have

$$\begin{aligned}\Psi(tx_1+(1-t)y_1) = \quad & \zeta(tx_1+(1-t)y_1)+\zeta(a_1+b_1-(tx_1+(1-t)y_1)) \\ = \quad & \zeta(tx_1+(1-t)y_1)+\zeta((a_1+b_1-x_1)t+(1-t)(a_1+b_1-y_1)).\end{aligned}$$

Since ζ is an exponentially trigonometric convex function, then

$$\begin{aligned}\Psi(tx_1+(1-t)y_1) &\leq \frac{sin\frac{\pi}{2}t}{e^{1-t}}\zeta(x_1)+\frac{cos\frac{\pi}{2}t}{e^{t}}\zeta(y_1)+\frac{sin\frac{\pi}{2}t}{e^{1-t}}\zeta(a_1+b_1-x_1)+\frac{cos\frac{\pi}{2}t}{e^{t}}\zeta(a_1+b_1-y_1) \\ &= \frac{sin\frac{\pi}{2}t}{e^{1-t}}(\zeta(x_1)+\zeta(a+b-x_1))+\frac{cos\frac{\pi}{2}t}{e^{t}}(\zeta(y_1)+\zeta(a_1+b_1-y_1)).\end{aligned}$$

So,

$$\Psi(tx_1+(1-t)y_1) \leq \frac{sin\frac{\pi}{2}t}{e^{1-t}}\Psi(x_1)+\frac{cos\frac{\pi}{2}t}{e^{t}}\Psi(y_1).$$

Hence, Ψ is an exponential trigonometric convex function.

Since $\zeta(x_1)$ is an exponentially trigonometric convex function and $x_1, y_1 \in [a_1, b_1]$, then by definition,

$$\zeta\Big(\frac{x_1+y_1}{2}\Big) \leq \frac{1}{\sqrt{2e}}\zeta(x_1)+\frac{1}{\sqrt{2e}}\zeta(y_1).$$

Set $x_1 = ta_1+(1-t)b_1$, $y_1 = (1-t)a_1+tb_1$, and then we have

$$\zeta\Big(\frac{a_1+b_1}{2}\Big) \leq \frac{1}{\sqrt{2e}}[\zeta(ta_1+(1-t)b_1)+\zeta((1-t)a_1+tb_1)],$$

$$2\zeta\Big(\frac{a_1+b_1}{2}\Big) \leq \sqrt{\frac{2}{e}}[\zeta(ta_1+(1-t)b_1)+\zeta((1-t)a_1+tb_1)].$$

Convert the previous expression into $\Psi(x_1)$:

$$\Psi(\frac{a_1+b_1}{2}) \leq \sqrt{\frac{2}{e}}\Psi((1-t)a_1+tb_1). \tag{4.9}$$

Multiplying both sides of (4.9) by $\frac{((1-t)a_1+tb_1)^{\rho-1}}{[b_1^{\rho}-((1-t)a_1+tb_1)^{\rho}]^{1-\beta}}g((1-t)a+tb)$ and integrating w.r.t t over [0,1], we have

$$\begin{aligned}&\frac{\rho^{1-\beta}\Gamma(\beta)}{b-a}{}^{\rho}I^{\beta}_{a_1+}g(b_1)\Psi(\frac{a_1+b_1}{2}) \\ &\leq \sqrt{\frac{2}{e}}\int_0^1 \frac{((1-t)a_1+tb_1)^{\rho-1}}{[b_1^{\rho}-((1-t)a_1+tb_1)^{\rho}]^{1-\beta}}g((1-t)a_1+tb_1)\Psi((1-t)a_1+tb_1)dt.\end{aligned}$$

By using substitution, we get

$$\frac{\rho^{1-\beta}\Gamma(\beta)}{b-a}{}^{\rho}I^{\beta}_{a_1+}g(b_1)\Psi(\frac{a_1+b_1}{2}) \leq \sqrt{\frac{2}{e}}\int_{a_1}^{b_1}\frac{u^{\rho-1}}{(b_1^{\rho}-u^{\rho})^{1-\beta}}(g\Psi)(u)\frac{du}{b_1-a_1} = \sqrt{\frac{2}{e}}\frac{\Gamma(\beta)\rho^{\beta-1}}{b_1-a_1}{}^{\rho}I^{\beta}_{a_1+}(g\Psi)(b_1).$$

Hence,

$$^{\rho}I^{\beta}_{a_1+}g(b_1)\Psi(\frac{a_1+b_1}{2}) \leq \sqrt{\frac{2}{e}}[{}^{\rho}I^{\beta}_{a_1+}(g\Psi)(b_1)]. \tag{4.10}$$

Similarly, if we multiply both sides of (4.9) by $\frac{((1-t)a_1+tb_1)^{\rho-1}}{[((1-t)a_1+tb_1)^{\rho}-a_1^{\rho}]^{1-\beta}}g((1-t)a_1+tb_1)$ and integrate w.r.t t over [0,1], we get

$$^{\rho}I^{\beta}_{b_1-}g(a_1)\Psi(\frac{a_1+b_1}{2}) \leq \sqrt{\frac{2}{e}}[{}^{\rho}I^{\beta}_{b_1-}(g\Psi)(a_1)]. \tag{4.11}$$

Adding (4.10) and (4.11), we get

$$\Psi(\frac{a_1+b_1}{2})[{}^{\rho}I^{\beta}_{a_1+}g(b_1)+{}^{\rho}I^{\beta}_{b_1+}g(a_1)] \leq \sqrt{\frac{2}{e}}[{}^{\rho}I^{\beta}_{a_1+}(g\Psi)(b_1)+{}^{\rho}I^{\beta}_{b_1-}(g\Psi)(a_1)]. \tag{4.12}$$

For the inequality's second part, we use

$$\zeta(ta_1+(1-t)b_1) \leq \frac{\sin\frac{\pi t}{2}}{e^{1-t}}\zeta(a_1)+\frac{\cos\frac{\pi t}{2}}{e^{t}}\zeta(b_1),$$

and

$$\zeta((1-t)a_1+tb_1) \leq \frac{\cos\frac{\pi t}{2}}{e^{t}}\zeta(a_1)+\frac{\sin\frac{\pi t}{2}}{e^{1-t}}\zeta(b_1).$$

Adding both of the above inequalities and using the result for $t \in [0,1]$, $\frac{\sin\frac{\pi t}{2}}{e^{1-t}}+\frac{\cos\frac{\pi t}{2}}{e^{t}} \leq 1$, we have

$$\zeta(ta_1+(1-t)b_1)+\zeta((1-t)a_1+tb_1) \leq \zeta(a_1)+\zeta(b_1).$$

Using the $\Psi(x)$ relation, we have

$$\Psi((1-t)a_1+tb_1) \leq \frac{\Psi(a_1)+\Psi(b_1)}{2}. \tag{4.13}$$

Multiplying both sides of (4.13) by $\frac{((1-t)a_1+tb_1)^{\rho-1}}{[b_1^{\rho}-((1-t)a_1+tb_1)^{\rho}]^{1-\beta}}g((1-t)a_1+tb_1)$ and integrating w.r.t t over [0,1], we have

$$\frac{\Gamma(\beta)\rho^{\beta-1}}{b_1-a_1}{}^{\rho}I^{\beta}_{a_1+}(g\Psi)(b_1) \leq \frac{\Gamma(\beta)\rho^{\beta-1}}{b_1-a_1}{}^{\rho}I^{\beta}_{a_1+}g(b_1)\frac{\Psi(a_1)+\Psi(b_1)}{2}.$$

Hence,

$$^{\rho}I^{\beta}_{a_1+}(g\Psi)(b_1) \leq {}^{\rho}I^{\beta}_{a_1+}g(b_1)[\frac{\Psi(a_1)+\Psi(b_1)}{2}]. \tag{4.14}$$

Similarly, we can get

$$^{\rho}I^{\beta}_{b_1-}(g\Psi)(a_1) \leq {}^{\rho}I^{\beta}_{b_1-}g(a_1)[\frac{\Psi(a_1)+\Psi(b_1)}{2}]. \tag{4.15}$$

Adding (4.14) and (4.15), we get

$$\sqrt{\frac{2}{e}}[{}^{\rho}I^{\beta}_{a_1+}(g\Psi)(b_1)^{\rho}I^{\beta}_{b_1-}(g\Psi)(a_1) \leq \frac{\Psi(a_1)+\Psi(b_1)}{2}[\frac{{}^{\rho}I^{\beta}_{a_1+}g(b_1)+{}^{\rho}I^{\beta}_{b_1-}g(a_1)}{\sqrt{2e}}]. \tag{4.16}$$

From (4.12) and (4.16),

$$\Psi(\frac{a_1+b_1}{2})\sqrt{\frac{2}{e}}[{}^{\rho}I^{\beta}_{a_1+}(g\Psi)(b_1)^{\rho}I^{\beta}_{b_1-}(g\Psi)(a_1) \leq \frac{\Psi(a_1)+\Psi(b_1)}{2}\Big(\frac{{}^{\rho}I^{\beta}_{a_1+}g(b_1)+{}^{\rho}I^{\beta}_{b_1-}g(a_1)}{\sqrt{2e}}\Big).$$

Remark 4.3. *If $\rho \to 1$ in Theorem 4.2, we get the result for the Riemann-Liouville fractional integrals.*

5. Discussion

Figures 1 and 2 illustrate the 2D and 3D visualizations of the left, middle, and right-hand terms in inequality (3.7). For specific parameter values, the corresponding numerical results of these terms are provided in Table 1. The left, middle, and right expressions of inequality (3.11) are graphically represented in Figure 3 (2D) and Figure 4 (3D). Numerical evaluations of these expressions for selected values are summarized in Table 2. Graphical representations of the left-hand side and right-hand side of (3.22) are presented in Figure 5 with 2D plots, while Figure 6 shows 3D plots. Furthermore, the quantitative comparison of these three components for assigned parameter values is tabulated in Table 3. To support the analytical findings, inequality (3.27) is examined both visually and numerically. Figures 7 and 8 provide 2D and 3D depictions of its two components (left and right). A corresponding numerical summary for specific input values is presented in Table 4. In each case, the left-hand side is less than the right-hand side, thus confirming the existence of the proposed inequality.

6. Conclusions

Convex functions are closely related to several important inequalities that have important ramifications for many different fields. Among these, Jensen's inequality is a fundamental concept in probability theory and statistics, offering crucial information about distributions and expected values. In a similar vein, the Hermite-Hadamard inequality improves our capacity to examine the behavior of convex functions by providing useful bounds for their integrals. These inequalities are useful tools in mathematical analysis, optimization, and other areas of mathematics in addition to shedding light on the inherent characteristics of convex functions. Understanding and using these inequalities can open up important viewpoints and approaches for dealing with challenging issues in these domains.

A flexible and reliable tool for simulating complex systems and phenomena in a variety of fields is the Katugampola fractional integral. It offers a more thorough framework for capturing nonlocal and memory-dependent effects by expanding and generalizing conventional fractional integrals. These effects are crucial in systems where past states affect future behavior. This integral is used in the study of fractional quantum mechanics, viscoelastic materials, and anomalous diffusion processes in physics, where it aids in the description of phenomena that depart from classical models. By simulating memory effects in recurrent networks and enhancing learning efficiency and long-term dependency handling, it facilitates the development of sophisticated neural network architectures. In the larger field of science, it helps with biological modeling by simulating intricate physiological processes, improves signal processing by making it possible to analyze non-stationary signals more effectively, and advances control theory by optimizing systems with fractional dynamics. Because of its broad applicability, the Katugampola fractional integral is a fascinating and quickly developing field of study with significant potential to improve our comprehension and analysis of complex systems in physics, neural networks, and other scientific fields.

With an emphasis on log-convex and exponentially trigonometric convex functions, this study has successfully explored novel approaches to investigate some fractional inequalities using Katugamapola fractional operators. Novel connections between inequalities, such as Hermite-Hadamard, Fejér, and Hölder inequalities, are established with Katugamapola fractional operators using exponentially trigonometric and log-convex functions. We also used $\rho \to 1$ to develop several corollaries in almost all theorems. Additionally, we verified that our findings reduce to standard Hermite-Hadamard and Fejér-type inequalities. This inequality may give a strict estimate of the mean square displacement in a fractional diffusion equation model, which has a theoretical limitation on the extent of spread of the particles in a complicated medium, such as porous rock or biological tissues. In gradient-based algorithm functions that fulfill our convexity property, the inequality can be exploited to obtain a new error bound on the new iteration of the algorithm, which may then be used to obtain a more efficient convergence criterion. In a filter design problem where the system is described by a fractional-order transfer function, the inequality may be used to provide conditions that determine stability in the frequency domain. Future research directions include constructing more generalized inequalities using advanced fractional operators and extending trigonometric and log-convexity to interval-valued functions. These concepts can also be applied in quantum calculus and multivariate calculus.

Author contributions

Muhammad Imran: Writing original draft, Methodology, Conceptualization, Data curation; Ahsan Mehmood: Visualization, Methodology, Formal analysis, Project administration, Writing review and editing; Shahid Mubeen: Validation, Formal analysis, Investigation, Writing review and editing; Muhammad Samraiz: Supervision, Software, Data curation, Investigation; Ishtiaq Ali: Visualization, Formal analysis, Project administration, Validation. All authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this paper.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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