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Research article

Topological analysis with some techniques for solving a fractional tsunami shallow water mathematical model-based on Hausdorff–locally compact structures and their analytical implications

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Abstract: This paper provides a comprehensive analytical study of the fractional Whitham–Broer–Kaup equations (WBKEs), formulated using the Atangana–Baleanu–Caputo (ABC) fractional derivative to model nonlinearoscillatory behavior and the dynamics of tsunami shallow water waves. Within the framework of Banach spaces endowed with the compact–open topology, we rigorously establish the existence, uniqueness, and Hyers–Ulam stability of solutions by applying fixed–point theorems. To construct approximate analytical solutions, we develop a hybrid approach that integrates fractional power series expansions (FPSEs) with the new iterative method (NIM), referred to as the expansion new iterative method (ENIM). This methodology efficiently handles nonlinearities and fractional–order effects, yielding rapidly convergent series solutions that remain consistent with exact analytical results. Overall, the study demonstrates the robustness of the fractional WBKE model under small perturbations and confirms its effectiveness in capturing the complex dynamics of tsunami shallow water wave propagation.

Keywords: fractional WBKEs; power series; ABC operator; stability
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1. Introduction

Fractional calculus (FC) extends the classical ideas of differentiation and integration by permitting operators of arbitrary, non–integer order. Although the concept originated in the discussions between Leibniz and L'Hopital during the 17th century, it was not until the 19th century that Riemann, Liouville, and Grunwald formalized fractional integrals and derivatives [1, 2]. Their contributions established a rigorous analytical basis for studying systems governed by nonlocality and hereditary effects. Since then, FC has evolved into a comprehensive mathematical framework capable of capturing long range temporal and spatial interactions. In recent decades, FC has attracted substantial attention due to its capacity to model complex dynamical processes. Advancements in numerical techniques have expanded the applicability of fractional differential equations (FDEs) in engineering and scientific research [3, 4]. Several fractional operators, including those of Caputo, Riemann–Liouville, Atangana–Baleanu, and Caputo–Fabrizio, have been introduced to describe anomalous transport, memory-driven diffusion, and multi–scale dynamics. These formulations have enhanced the modeling accuracy in viscoelasticity, electromagnetic phenomena, biological systems, and diffusion processes with memory effects [5]. Partial differential equations (PDEs) remain fundamental tools for describing processes in physics, fluid dynamics, quantum mechanics, electromagnetism, and related sciences. Incorporating fractional derivatives into PDEs leads to fractional PDEs (FPDEs), which provide greater flexibility for modeling physical behaviors influenced by history dependent and nonlocal phenomena [6]. FPDEs are capable of describing anomalous diffusion, viscoelastic materials, chaotic evolutions, and systems exhibiting spatial heterogeneity [7]. Considerable research has investigated theoretical aspects of FPDEs such as existence, uniqueness, stability, and convergence to ensure their mathematical and physical reliability [8, 9]. FPDE systems have wide applications in geophysics, biology, finance, and engineering. They effectively describe processes governed by hereditary laws, including groundwater flow through porous structures, charge transport in irregular media, and heat conduction in heterogeneous materials [10]. Motivated by these applications, many analytical and numerical methods have been developed. For instance, Kumar [11] proposed a method for nonlinear singular boundary value problems; Yang [6] formulated an integral technique for steady heat transfer; Mohyud Din et al. [7] utilized perturbation approaches; Xie et al. [12] developed hyperbolic techniques; Biazar and Aminikhah [10] studied coupled Burgers-type models; and Ahmad et al. [13] applied the Adomian decomposition method and He's polynomials to the WBKEs. Additional techniques include homotopy perturbation [14], parabolic convection–diffusion solvers [15], the residual power series method [16], the redefined quintic B–spline approach with Von Neumann analysis [17], the Adomian decomposition method [18], the Yang transform combined with decomposition techniques [19], the q–homotopy analysis transform [20], the Yang residue power series method [21], Sumudu-based decomposition [22], Laplace Adomian hybrid schemes [23], the natural decomposition method [20], scaling-based transformations [24], and the optimal homotopy asymptotic method [25]. These approaches demonstrate the adaptability and the precision of semi–analytical frameworks for nonlinear FPDEs, and perturbation-based studies of the Caputo–Fabrizio operator [26]. These efforts highlight ongoing advances in obtaining accurate approximations to nonlinear and FPDEs. The coupled WBKEs are effective in modeling nonlinear wave propagation of tsunami shallow water and related physical phenomena [27]. Their classical form is given by

$$\begin{cases}\omega_{\xi}(\kappa,\xi)+\omega(\kappa,\xi)\,\omega_{\kappa}(\kappa,\xi)+\psi_{\kappa}(\kappa,\xi)+\beta\,\omega_{\kappa\kappa}(\kappa,\xi)=0,\\ \psi_{\xi}(\kappa,\xi)+\left[\omega(\kappa,\xi)\psi(\kappa,\xi)\right]_{\kappa}-\beta\,\psi_{\kappa\kappa}(\kappa,\xi)+\upsilon\beta'\,\omega_{\kappa\kappa\kappa}(\kappa,\xi)=0,\end{cases} \tag{1.1}$$

where ω represents the horizontal velocity, υ is the maximum wave height, ψ the surface elevation, and the constants β and β' measure dispersive and dissipative effects. These equations capture nonlinear convection and dispersive wave behavior, and have been applied to shallow water models, plasma oscillations, and acoustics. A significant number of contributions address nonlinear and fractional versions of the WBKEs (1.1). Many fractional operators have been constructed to incorporate memory effects into physical models, motivating analytical and numerical developments such as the fractional Newton method [28], extended algebraic mapping techniques [29], homotopy perturbation methods [30], sine-Gordon expansions for Wu–Zhang models [31], Laplace-based residual power series techniques [32], Aboodh decomposition transform method [33], reproducing kernel Hilbert space approaches [34], variational iteration schemes [35], numerical studies of telegraph equations [36], monotone iterative methods [37], expansion methods [38], and modified Adams–Bashforth algorithms [39]. Nadeem et al. [40] further developed the Yang residual power series scheme for fractional heat transfer models. A fractional version of the WBKEs-based on the Atangana–Baleanu–Caputo (ABC) derivative is given in [41] for tsunami shallow water applications:

$$\begin{cases}{}^{ABC}\mathcal{D}^{\alpha}_{0,\xi}\,\omega(\kappa,\xi)+\omega(\kappa,\xi)\,\omega_{\kappa}(\kappa,\xi)+\psi_{\kappa}(\kappa,\xi)+\beta\,\omega_{\kappa\kappa}(\kappa,\xi)=0,\\ {}^{ABC}\mathcal{D}^{\alpha}_{0,\xi}\,\psi(\kappa,\xi)+\left[\omega(\kappa,\xi)\psi(\kappa,\xi)\right]_{\kappa}-\beta\,\psi_{\kappa\kappa}(\kappa,\xi)+\upsilon\beta'\,\omega_{\kappa\kappa\kappa}(\kappa,\xi)=0,\\ \omega(\kappa,0)=h_0(\kappa),\ \ \psi(\kappa,0)=\hat{h}_0(\kappa),\end{cases} \tag{1.2}$$

where ${}^{ABC}\mathcal{D}^{\alpha}_{0,\xi}$ is the ABC fractional derivative of order $0<\alpha<1$. Here, κ denotes the spatial coordinate, ξ the temporal variable, ω the depth-averaged horizontal velocity and ψ the free surface elevation. The parameters β and β' regulate dissipation and dispersion. This fractional model incorporates memory-driven behavior into nonlinear wave evolution. To solve the fractional WBKEs (1.2), numerous analytical and numerical methods have been introduced. Kumar et al. [41] presented a dynamical and computational investigation of the fractional WBKEs, highlighting how fractional-order effects influence shallow water wave evolution. Prakash and Kaur [42] introduced an efficient numerical simulation scheme for the coupled fractional WBKEs using the ABC derivative, demonstrating improved accuracy for modeling nonlocal shallow water dynamics. Jeelani et al. [43] analyzed an ABC fractional system with a generalized Mittag–Leffler kernel, establishing key mathematical properties and illustrating the impact of generalized nonlocal memory kernels on fractional dynamical behavior.

Motivated by these challenges, the present study formulates the fractional WBKEs using the ABC fractional derivative, which incorporates nonlocal memory effects through a nonsingular kernel. The analysis is carried out in Banach spaces endowed with the compact-open topology over locally compact Hausdorff domains. Within this setting, we establish the existence, uniqueness, continuity, and Hyers–Ulam stability of solutions by applying fixed–point theorems and continuity arguments. In addition to the theoretical analysis, we develop a new hybrid semi–analytical technique, referred to as the expansion new iterative method (ENIM). This method combines fractional power series expansions (FPSEs) with the new iterative method (NIM), enabling efficient handling of nonlinearities and fractional-order operators. The resulting series solutions exhibit rapid convergence and remain consistent with known exact solutions in the classical case. The effectiveness of the proposed approach is demonstrated through an application to a tsunami shallow water problem, highlighting the physical relevance and robustness of the fractional WBKE model. The main contributions of this work can be summarized as follows: Establishing rigorous existence and uniqueness results for the fractional WBKEs within a compact-open Banach space framework; proving Hyers–Ulam stability of the fractional WBKE system, ensuring robustness under small perturbations; developing a novel hybrid semi-analytical ENIM scheme based on fractional power series and the NIM and demonstrating the applicability and accuracy of the proposed method through a physically relevant tsunami shallow water example. Section 1 introduces fractional calculus and the motivation for studying the fractional WBKE model. Section 2 presents the definitions and properties of the ABC operators. Section 3 proves well-posedness and stability using continuity arguments and fixed-point theorems. Section 4 constructs fractional power series formulas and develops the ENIM procedure for obtaining convergent approximate solutions. Section 5 applies the ENIM to a tsunami wave problem, computes approximate solution terms, interprets the physical parameters, and compares the results with the classical exact solution for $\alpha = 1$.

2. Preliminaries

It is worth noting that the definitions and results presented about ABC fractional operators in this section were originally developed in the single variable setting [44–46]. We extend these findings to the two-variable case and, for clarity and to avoid unnecessary repetition, we cite only the works that contain the corresponding proofs in the one-variable framework, with the understanding that our contributions provide a natural and straightforward generalization to the two-variable setting.

Definition 2.1. [47] Let ω be any map on $\Omega \times [0, \xi_0]$, where $\xi_0 > 0$. The ABC integral operator of $\omega(\kappa, \xi)$ for order $\alpha > 0$ is given by

$$ {}^{ABC}\mathcal{I}^{\alpha}_{0,\xi}\, \omega(\kappa, \xi) = \frac{1-\alpha}{A^{\alpha}}\, \omega(\kappa, \xi) + \frac{\alpha}{\Gamma(\alpha) A^{\alpha}} \int_0^{\xi} (\xi - \sigma)^{\alpha - 1} \omega(\kappa, \sigma) d\sigma, \quad 0 < \alpha < 1, \tag{2.1} $$

where $A^{\cdot}$ is a condition referred to as the normalization function with $A^0 = A^1 = 1$.

Definition 2.2. [47] Let ω be any map on $\Omega \times [0, \xi_0]$. The ABC derivative operator of $\omega(\kappa, \xi)$ for order $\alpha > 0$ is given by

$$ {}^{ABC}\mathcal{D}^{\alpha}_{0,\xi}\, \omega(\kappa, \xi) = \begin{cases} \dfrac{A^{\alpha}}{1-\alpha} \displaystyle\int_0^{\xi} E_{\alpha}\left(\dfrac{-\alpha(\xi - \mu)^{\alpha}}{1-\alpha} \right) \omega^{(n)}_{\mu^n}(\kappa, \mu) d\mu, & n-1 < \alpha < n, \\ \omega^{(n)}_{\xi^n}(\kappa, \xi), \quad \alpha := n \in \mathbb{N}, \end{cases} \tag{2.2} $$

where $E_{\alpha}(\cdot)$ is is the Mittag–Leffler function

$$ E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \qquad \alpha > 0. $$

Recall [45]. For $\xi \geq 0$ and for $n - 1 < \alpha < n$, we have ${}^{ABC}\mathcal{D}^{\alpha}_{0,\xi}\; {}^{ABC}\mathcal{I}^{\alpha}_{0,\xi}\, \omega(\kappa, \xi) = \omega(\kappa, \xi)$.

Theorem 2.3. [44] Let ω be any map on $\Omega \times [0, \xi_0]$. Then

$$ {}^{ABC}\mathcal{I}^{\alpha}_{0,\xi}\ {}^{ABC}\mathcal{D}^{\alpha}_{0,\xi}\ \omega(\kappa, \xi) = \omega(\kappa, \xi) - \sum_{k=0}^{n-1} \frac{\xi^k}{k!} \omega^{(k)}_{\xi}(\kappa, \xi)\Big|_{\xi=0}. \tag{2.3} $$

3. Well-posedness and stability analysis of WBKEs (1.2)

In this section, we establish a mathematical setting for studying the existence, uniqueness, and stability of solutions to the fractional WBKEs (1.2). Working in Banach spaces endowed with the compact-open topology, we apply fixed-point principles to verify the well-posedness of the system. Under appropriate assumptions, this framework guarantees that the fractional WBKEs admit continuous, unique, and Hyers–Ulam stable solutions.

3.1. Locally compact Hausdorff structure of continuity conditions

Here, the domain is assumed to be a locally compact Hausdorff space, which ensures that the compact-open topology is well defined on the space of continuous functions. This setting provides uniform convergence on compact subsets and secures the continuity of the involved operators, a key requirement for applying fixed-point techniques to the fractional WBKEs (1.2). Define the operators $\overline{\omega}, \overline{\psi} : [\Omega \times [0, \xi_0]] \times \mathbb{R}^{\Omega\times[0,\xi_0]} \times \mathbb{R}^{\Omega\times[0,\xi_0]} \to \mathbb{R}$ by

$$ \overline{\omega}(\kappa, \xi, \omega, \psi) = -\omega(\kappa, \xi)\omega_{\kappa}(\kappa, \xi) - \psi_{\kappa}(\kappa, \xi) - \beta\, \omega_{\kappa\kappa}(\kappa, \xi) \tag{3.1} $$

and

$$ \overline{\psi}(\kappa, \xi, \omega, \psi) = -[\omega(\kappa, \xi)\psi(\kappa, \xi)]_{\kappa} + \beta\psi_{\kappa\kappa}(\kappa, \xi) - \upsilon\beta'\omega_{\kappa\kappa\kappa}(\kappa, \xi), \tag{3.2} $$

where $\mathbb{R}^{\Omega\times[0,\xi_0]}$ denotes the family of all functions from $\Omega \times [0, \xi_0]$ into $\mathbb{R}$. Apply the ABC integral operator of the system (1.2) and use (2.1) to get that

$$ \omega(\kappa, \xi) = h_0(\kappa) + \frac{1-\alpha}{A^{\alpha}}\, \overline{\omega}(\kappa, \xi, \omega, \psi) + \frac{\alpha}{\Gamma(\alpha)A^{\alpha}} \int_0^{\xi} (\xi - \sigma)^{\alpha-1}\, \overline{\omega}(\kappa, \sigma, \omega, \psi)d\sigma \tag{3.3} $$

and

$$ \psi(\kappa, \xi) = \hat{h}_0(\kappa) + \frac{1-\alpha}{A^{\alpha}}\, \overline{\psi}(\kappa, \xi, \omega, \psi) + \frac{\alpha}{\Gamma(\alpha)A^{\alpha}} \int_0^{\xi} (\xi - \sigma)^{\alpha-1}\, \overline{\psi}(\kappa, \sigma, \omega, \psi)d\sigma. \tag{3.4} $$

Define the operators $\mathcal{J}_{\omega}, \mathcal{J}_{\psi} : \mathbb{R}^{\Omega\times[0,\xi_0]} \times \mathbb{R}^{\Omega\times[0,\xi_0]} \to \mathbb{R}^{\Omega\times[0,\xi_0]}$ by

$$ \begin{aligned} \mathcal{J}_{\omega}(\omega, \psi)(\kappa, \xi) &= h_0(\kappa) + \frac{1-\alpha}{A^{\alpha}}\, \overline{\psi}(\kappa, \xi, \omega, \psi) \\ &+ \frac{\alpha}{\Gamma(\alpha)A^{\alpha}} \int_0^{\xi} (\xi - \sigma)^{\alpha-1}\, \overline{\omega}(\kappa, \sigma, \omega, \psi)d\sigma \end{aligned} \tag{3.5} $$

and

$$ \begin{aligned} \mathcal{J}_{\psi}(\omega, \psi)(\kappa, \xi) &= \hat{h}_0(\kappa) + \frac{1-\alpha}{A^{\alpha}}\, \overline{\psi}(\kappa, \xi, \omega, \psi) \\ &+ \frac{\alpha}{\Gamma(\alpha)A^{\alpha}} \int_0^{\xi} (\xi - \sigma)^{\alpha-1}\, \overline{\psi}(\kappa, \sigma, \omega, \psi)d\sigma. \end{aligned} \tag{3.6} $$

Consider the Banach space $\mathcal{B} := C_{\Omega_0} \times C_{\Omega_0}$ with the norm

$$\|(\omega, \psi)\|_{\mathcal{B}} = \max_{(\kappa,\xi)\in\Omega\times[0,\xi_0]} \{|\omega(\kappa, \xi)|, |\psi(\kappa, \xi)|\}.$$

We say $\overline{\omega}$ has Lipschitz property if there is a constant $\lambda_\omega > 0$ such that

$$|\overline{\omega}(\kappa, \xi, \omega, \psi) - \overline{\omega}(\kappa, \xi, \omega_1, \psi_1)| \leq \lambda_\omega \|(\omega, \psi) - (\omega_1, \psi_1)\|_{\mathcal{B}}$$

for all $(\kappa, \xi) \in \Omega \times [0, \xi_0]$.

Theorem 3.1. If $\overline{\omega}$ and $\overline{\psi}$ satisfy Lipschitz conditions, then $\mathcal{J}_\omega$ and $\mathcal{J}_\psi$ satisfy Lipschitz conditions, respectively.

Proof. By the Lipschitz property of $\overline{\omega}$ and $\overline{\psi}$, there are $\lambda_\omega, \lambda_\psi > 0$ such that

$$|\overline{\omega}(\kappa, \xi, \omega, \psi) - \overline{\omega}(\kappa, \xi, \omega_1, \psi_1)| \leq \lambda_\omega \|(\omega, \psi) - (\omega_1, \psi_1)\|_{\mathcal{B}}$$

and

$$\left|\overline{\psi}(\kappa, \xi, \omega, \psi) - \overline{\psi}(\kappa, \xi, \omega_1, \psi_1)\right| \leq \lambda_\psi \|(\omega, \psi) - (\omega_1, \psi_1)\|_{\mathcal{B}}.$$

Hence,

$$\begin{aligned}
&|\mathcal{J}_\omega(\omega, \psi)(\kappa, \xi) - \mathcal{J}_\omega(\omega_1, \psi_1)(\kappa, \xi)| \leq \frac{1-\alpha}{A^\alpha} |\overline{\omega}(\kappa, \xi, \omega, \psi) - \overline{\omega}(\kappa, \xi, \omega_1, \psi_1)| \\
&+ \frac{\alpha}{\Gamma(\alpha)A^\alpha} \left|\int_0^\xi (\xi - \sigma)^{\alpha-1}\, \overline{\omega}(\kappa, \sigma, \omega, \psi) d\sigma - \int_0^\xi (\xi - \sigma)^{\alpha-1}\, \overline{\omega}(\kappa, \sigma, \omega_1, \psi_1) d\sigma\right| \\
&\leq \frac{1-\alpha}{A^\alpha} |\overline{\omega}(\kappa, \xi, \omega, \psi) - \overline{\omega}(\kappa, \xi, \omega_1, \psi_1)| \\
&+ \frac{\alpha}{\Gamma(\alpha)A^\alpha} \int_0^\xi |\xi - \sigma|^{\alpha-1}\; |\overline{\omega}(\kappa, \xi, \omega, \psi) - \overline{\omega}(\kappa, \xi, \omega_1, \psi_1)|\, d\sigma \\
&\leq \frac{(1-\alpha)\lambda_\omega}{A^\alpha} \|(\omega, \psi) - (\omega_1, \psi_1)\|_{\mathcal{B}} + \frac{\xi_0^\alpha \lambda_\omega}{\Gamma(\alpha)A^\alpha} \|(\omega, \psi) - (\omega_1, \psi_1)\|_{\mathcal{B}}.
\end{aligned}$$

Then we have

$$\|\mathcal{J}_\omega(\omega, \psi) - \mathcal{J}_\omega(\omega_1, \psi_1)\| \leq \left[\frac{(1-\alpha)\lambda_\omega}{A^\alpha} + \frac{\xi_0^\alpha \lambda_\omega}{\Gamma(\alpha)A^\alpha}\right] \|(\omega, \psi) - (\omega_1, \psi_1)\|_{\mathcal{B}}. \tag{3.7}$$

Similarly, we have

$$\left\|\mathcal{J}_\psi(\omega, \psi) - \mathcal{J}_\psi(\omega_1, \psi_1)\right\| \leq \left[\frac{(1-\alpha)\lambda_\psi}{A^\alpha} + \frac{\xi_0^\alpha \lambda_\psi}{\Gamma(\alpha)A^\alpha}\right] \|(\omega, \psi) - (\omega_1, \psi_1)\|_{\mathcal{B}}. \tag{3.8}$$

□

Define the operators $O_\omega, O_\psi : \mathcal{B} \to \mathbb{R}^{\Omega\times[0,\xi_0]}$ as

$$O_\omega(\omega, \psi) = -\omega\, \omega_\kappa - \psi_\kappa - \beta\, \omega_{\kappa\kappa} \tag{3.9}$$

and

$$O_\psi(\omega,\psi) = -[\omega\psi]_\kappa + \beta\,\psi_{\kappa\kappa} - \upsilon\beta'\,\omega_{\kappa\kappa\kappa} \tag{3.10}$$

respectively, where $C_{\Omega_0} := C\,(\Omega\times[0,\xi_0]) \subseteq \mathbb{R}^{\Omega\times[0,\xi_0]}$ denotes the family of all continuous functions from $\Omega\times[0,\xi_0]$ into $\mathbb{R}$. For the Lipschitz continuity of the operators O_ω and O_ψ, note that if $\overline{\omega}$ and $\overline{\psi}$ satisfy Lipschitz conditions, then we have

$$|\langle O_\omega(\omega,\psi) - O_\omega(\omega_1,\psi_1), (\omega,\psi)-(\omega_1,\psi_1)\rangle| \le \lambda_\omega\,\|(\omega,\psi)-(\omega_1,\psi_1)\|^2_{\mathcal{B}}$$

and

$$|\langle O_\psi(\omega,\psi) - O_\psi(\omega_1,\psi_1), (\omega,\psi)-(\omega_1,\psi_1)\rangle| \le \lambda_\psi\,\|(\omega,\psi)-(\omega_1,\psi_1)\|^2_{\mathcal{B}}\,.$$

It is known that in any metrizable topological space, the collection of open balls forms a subbasis for its topology. Thus, in the metric space $\mathbb{R}$ endowed with the standard metric $d(\kappa,\xi) = |\kappa-\xi|$, the family of open balls $B(\kappa_0, r)$ generates the usual topology on $\mathbb{R}$. Since $\Omega\times[0,\xi_0]$ is a Hausdorff subspace of the Euclidean space $\mathbb{R}^2$, the family

$$CO = \{\langle H, B(\kappa_0,r)\rangle : H \text{ is a compact subset of } \Omega\times[0,\xi_0],\ \kappa_0\in\mathbb{R},\ r>0\}$$

constitutes a subbasis for the compact-open topology on C_{Ω_0}, where

$$\langle H, B(\kappa_0,r)\rangle = \{\omega\in C_{\Omega_0} : \omega[H]\subset B(\kappa_0,r)\}.$$

In what follows, the function space $\mathbb{R}^{\Omega\times[0,\xi_0]}$ is considered with this compact-open topology.

Theorem 3.2. If $\overline{\omega}$ is continuous, then the operator O_ω is continuous.

Proof. We will show that $O_\omega^{-1}(\langle H, B(\kappa_0,r)\rangle)$ is open in $\mathcal{B}$ for every subbasic open $\langle H, B(\kappa_0,r)\rangle$ of $\mathbb{R}^{\Omega\times[0,\xi_0]}$ for any compact set $H\subset\Omega\times[0,\xi_0]$, $\kappa_0\in\mathbb{R}$ and $r>0$. Note that

$$\begin{aligned} O_\omega^{-1}(\langle H, B(\kappa_0,r)\rangle) &= \{\,(\varsigma,\varsigma')\in\mathcal{B}:\ O_\omega(\varsigma,\varsigma')[H]\subset B(\kappa_0,r)\,\}\\ &= \{\,(\varsigma,\varsigma')\in\mathcal{B}:\ \overline{\omega}(\{(\varsigma,\varsigma')\}\times H)\subset B(\kappa_0,r)\,\}.\end{aligned}$$

Fix $(\varsigma_0,\varsigma_0')\in O_\omega^{-1}(\langle H, B(\kappa_0,r)\rangle)$. Then $\overline{\omega}(\varsigma_0,\varsigma_0',h)\in B(\kappa_0,r)$ for all $h\in H$, i.e., $\{(\varsigma_0,\varsigma_0')\}\times H\subset \overline{\omega}^{-1}(B(\kappa_0,r))$. Since

$$\overline{\omega}^{-1}(B(\kappa_0,r))\subset[\Omega\times[0,\xi_0]]\times C_{\Omega_0}\times C_{\Omega_0}$$

is open and $\overline{\omega}$ is continuous, then for each $h\in H$, there exist open neighborhoods $O_h\subset C_{\Omega_0}\times C_{\Omega_0}$ of $(\varsigma_0,\varsigma_0')$ and $V_h\subset\mathbb{R}^2$ of h such that

$$O_h\times V_h\ \subset\ \overline{\omega}^{-1}(B(\kappa_0,r)).$$

The family $\{V_h\}_{h\in H}$ is an open cover of the compact set H, so choose $h_1,\ldots,h_n\in H$ with $H\subset\bigcup_{i=1}^n V_{h_i}$, and set

$$O := \bigcap_{i=1}^n O_{h_i}.$$

Then O is an open neighborhood of $(\varsigma_0,\varsigma_0')$ in $\mathcal{B}$ and hence for any $(\varsigma,\varsigma')\in O$ and any $k\in H$, there exists i with $k\in V_{h_i}$ such that

$$(h,k)\in O_{h_i}\times V_{h_i}\subset\overline{\omega}^{-1}(B(\kappa_0,r)).$$

Then $\overline{\omega}(\varsigma,\varsigma',H)\subset B(\kappa_0,r)$; equivalently, $O_\omega(\varsigma,\varsigma')\in\langle H, B(\kappa_0,r)\rangle$. Hence, $O\subset O_\omega^{-1}(\langle H, B(\kappa_0,r)\rangle)$, that is, $O_\omega^{-1}(\langle H, B(\kappa_0,r)\rangle)$ is open. Therefore, O_ω is continuous. □

Theorem 3.3. If the operator $\mathcal{O}_\omega$ is continuous and Ω is locally compact set in $\mathbb{R}$, then the restriction $\overline{\omega}$ on $\Omega \times [0, \xi_0] \times \mathcal{B}$ is continuous.

Proof. We prove directly that $\overline{\omega}^{-1}(B(\kappa, r))$ is open in $[\Omega \times [0, \xi_0]] \times \mathcal{B}$ for each open ball $B(\kappa, r) \subset \mathbb{R}$. Fix an arbitrary open ball $B(\kappa_0, r) \subset \mathbb{R}$, and take a point $(\xi_0', \xi_0'', \varsigma_0, \varsigma_0') \in \overline{\omega}^{-1}(B(\kappa_0, r))$; that is, $\mathcal{O}_\omega(\varsigma_0, \varsigma_0')(\xi_0', \xi_0'') \in B(\kappa_0, r)$. Since Ω is locally compact in $\mathbb{R}$ and so it is Hausdorff space, then $\Omega \times [0, \xi_0]$ is a locally compact Hausdorff space. Hence, there exists an open neighborhood V of (ξ_0', ξ_0'') such that $\overline{V}$ is compact containing (ξ_0', ξ_0''), where $\overline{V}$ is the closer set of V. Set $H := \overline{V}$; then H is compact and $(\xi_0', \xi_0'') \in V \subset H$. Since $\mathcal{O}_\omega(\varsigma_0, \varsigma_0')(\xi_0', \xi_0'') \in B(\kappa_0, r)$ and $(\xi_0', \xi_0'') \in H$, it follows that $\mathcal{O}_\omega(\varsigma_0, \varsigma_0')[H] \subset \mathbb{R}$ intersects $B(\kappa_0, r)$ at least at (ξ_0', ξ_0''). Define the subbasic open set

$$\langle H, B(\kappa_0, r)\rangle = \{(\varsigma, \varsigma') \in \mathbb{R}^{\Omega\times[0,\xi_0]} \times \mathbb{R}^{\Omega\times[0,\xi_0]} : \mathcal{O}_\omega(\varsigma, \varsigma')[H] \subset B(\kappa_0, r)\}.$$

Since $\mathcal{O}_\omega$ is continuous and $\langle H, B(\kappa_0, r)\rangle$ is an open set in $\mathbb{R}^{\Omega\times[0,\xi_0]}$, then there exists an open neighborhood $O \subset \mathcal{B}$ of $(\varsigma_0, \varsigma_0')$ such that $\mathcal{O}_\omega(O) \subset \langle H, B(\kappa_0, r)\rangle$. Equivalently, for every $x \in O$, we have $\mathcal{O}_\omega(x)[H] \subset B(\kappa_0, r)$. Now consider the product open set

$$V \times O \subset [\Omega \times [0, \xi_0]] \times \mathcal{B}.$$

If $(\xi, \xi', \varsigma, \varsigma') \in V \times O$ and since $(\xi, \xi') \in V \subset H$ and $\mathcal{O}_\omega(\varsigma, \varsigma')[H] \subset B(\kappa_0, r)$, then we get $\overline{\omega}(\xi, \xi', \varsigma, \varsigma') = \mathcal{O}_\omega(\varsigma, \varsigma')(\xi, \xi') \in B(\kappa_0, r)$. Therefore, $O \times V \subset \overline{\omega}^{-1}(B(\kappa_0, r))$. We have found, for the arbitrary $(\xi_0', \xi_0'', \varsigma_0, \varsigma_0') \in \overline{\omega}^{-1}(B(\kappa_0, r))$, an open neighborhood $V \times O$ of $(\xi_0', \xi_0'', \varsigma_0, \varsigma_0')$ with $O \times V \subset \overline{\omega}^{-1}(B(\kappa_0, r))$; Hence, $\overline{\omega}^{-1}(B(\kappa_0, r))$ is open. Since $B(\kappa_0, r) \subset \mathbb{R}$ is arbitrary, then $\overline{\omega}$ is continuous. □

If $\overline{\psi\psi\psi\psi}$ is continuous, then the operator $\mathcal{O}_{\psi\psi\psi}$ is continuous. The proof follows similarly to Theorem 3.2. Furthermore, if the operator $\mathcal{O}_{\psi\psi}$ is continuous and Ω is a locally compact subset of $\mathbb{R}$, then the restriction of $\overline{\psi}$ to $\Omega \times [0, \xi_0] \times \mathcal{B}$ is continuous. The proof proceeds analogously to that of Theorem 3.3.

3.2. Well-posedness of fractional WBKEs (1.2)

Here, we analyze the well-posedness of the fractional WBKEs (1.2) by establishing the existence and uniqueness of their solutions in a Banach space endowed with the compact-open topology. Using fixed-point techniques, we identify the conditions that ensure the system possesses a unique and stable solution. For this purpose, we introduce the operator $\mathcal{J} : \mathcal{B} \to \mathcal{B}$ given by

$$\mathcal{J}(\omega, \psi)(\kappa, \xi) = (\mathcal{J}_\omega(\omega, \psi)(\kappa, \xi),\ \mathcal{J}_\psi(\omega, \psi)(\kappa, \xi))$$

for all $(\kappa, \xi) \in \Omega \times [0, \xi_0]$. To ensure the existence and uniqueness of solutions to (1.2), we impose the following assumptions:

- C1: The domain of (1.2) is the space $C^3\left[\Omega \times [0, \xi_0]\right]$ endowed with the compact-open topology, where Ω is a locally compact and bounded subset of $\mathbb{R}$.
- C2: There exist constants $\mathcal{R}_{\overline{\omega}}, \mathcal{R}_{\overline{\psi}} > 0$ such that $|\overline{\omega}(\kappa, \xi, \omega)| \le \mathcal{R}_{\overline{\omega}}$ and $\left|\overline{\psi}(\kappa, \xi, \omega, \psi)\right| \le \mathcal{R}_{\overline{\psi}}$.
- C3: The functions $\overline{\omega}$ and $\overline{\psi}$ satisfy Lipschitz conditions with constants $\lambda_\omega, \lambda_\psi > 0$, respectively.

Lemma 3.4. If C1 is satisfied, then the restrictions of the operators $\mathcal{O}_\omega$ and $\mathcal{O}_\psi$ are continuous.

Proof. Since the operators $\partial_\kappa, \partial_\kappa^2, \partial_\kappa^3 : C^3[\Omega \times [0,\xi_0]] \to C[\Omega \times [0,\xi_0]]$ are continuous, then easily to see that

$$O_\omega(\omega,\psi) = -\omega\,\omega_\kappa - \psi_\kappa - \beta\,\omega_{\kappa\kappa} \text{ and } O_\psi(\omega,\psi) = -[\omega\psi]_\kappa + \beta\,\psi_{\kappa\kappa} - \upsilon\beta'\,\omega_{\kappa\kappa\kappa}$$

are continuous. □

We now clarify why the operator $\mathcal{J} = (\mathcal{J}_\omega, \mathcal{J}_\psi)$ is compact in the Banach space $B = C(\Omega\times[0,\xi_0])\times C(\Omega \times [0,\xi_0])$ endowed with the compact-open topology. Recall that $\Omega \times [0,\xi_0]$ is a locally compact Hausdorff space; therefore, the compact-open topology on $C(\Omega\times[0,\xi_0])$ coincides with the topology of uniform convergence on compact subsets. Let $D \subset B$ be a bounded set. From the growth assumptions on the nonlinear operators and the boundedness of the initial data, it follows that $\mathcal{J}(D)$ is uniformly bounded in B. Moreover, the continuity of the nonlinear terms, together with the nonsingular and integrable kernel of the ABC fractional integral, ensures that $\mathcal{J}(D)$ is equicontinuous on every compact subset of $\Omega \times [0,\xi_0]$. By the Arzelà–Ascoli theorem, uniform boundedness and equicontinuity imply that $\mathcal{J}(D)$ is relatively compact in the compact-open topology. Hence, $\mathcal{J}$ is a compact operator on B. This compactness, together with the continuity of $\mathcal{J}$ and the boundedness of the set $\{u \in B : u = \lambda\mathcal{J}u,\ \lambda \in [0,1]\}$, allows the application of Schauder's fixed-point theorem to establish the existence of solutions to the fractional WBKEs (1.2).

Theorem 3.5. If C1 and C2 hold, then the system (1.2) has solutions.

Proof. To get this solution we will use Schaefer's fixed-point theorem. For any $\epsilon > 0$, let

$$D_\epsilon = \{(\omega,\psi) \in \mathcal{B} : \|(\omega,\psi)\|_{\mathcal{B}} \le \epsilon\}.$$

First, for the continuity property of $\mathcal{J}$, we will prove that $\|\mathcal{J}(\omega_n,\psi_n) - \mathcal{J}(\omega,\psi)\|_{\mathcal{B}} \to 0$ for any convergent sequence $(\omega_n,\psi_n) \to (\omega,\psi)$ in $\mathcal{B}$. Let $(\omega_n,\psi_n) \to (\omega,\psi)$ in $\mathcal{B}$. For any $(\kappa,\xi) \in \Omega \times [0,\xi_0]$, we have

$$\begin{aligned}
&|\mathcal{J}_\omega(\omega_n,\psi_n)(\kappa,\xi) - \mathcal{J}_\omega(\omega,\psi)(\kappa,\xi)| \\
&\le \frac{1-\alpha}{A^\alpha}\,|\overline{\omega}(\kappa,\xi,\omega_n,\psi_n) - \overline{\omega}(\kappa,\xi,\omega,\psi)| \\
&+ \frac{\alpha}{\Gamma(\alpha)A^\alpha}\left|\int_0^\xi (\xi-\sigma)^{\alpha-1}\ [\overline{\omega}(\kappa,\sigma,\omega_n,\psi_n) - \overline{\omega}(\kappa,\sigma,\omega,\psi)]\,d\sigma\right| \\
&\le \frac{1-\alpha}{A^\alpha}\,|\overline{\omega}(\kappa,\xi,\omega_n,\psi_n) - \overline{\omega}(\kappa,\xi,\omega,\psi)| \\
&+ \frac{\alpha}{\Gamma(\alpha)A^\alpha}\int_0^\xi (\xi-\sigma)^{\alpha-1}\ |\overline{\omega}(\kappa,\sigma,\omega_n,\psi_n) - \overline{\omega}(\kappa,\sigma,\omega,\psi)|\,d\sigma.
\end{aligned} \tag{3.11}$$

By Lemma 3.4, O_ω is continuous. Since Ω is a locally compact Hausdorff, then by Theorem 3.3, $\overline{\omega}$ is continuous. Hence, for any $(\kappa,\xi) \in \Omega \times [0,\xi_0]$, we have

$$|\overline{\omega}(\kappa,\xi,\omega_n,\psi_n) - \overline{\omega}(\kappa,\xi,\omega,\psi)| \to 0.$$

By the boundedness of Ω, we have $\omega_n \to \omega$ uniformly in D_ϵ. Hence, for all $n \in \mathbb{N}$ and for any $(\kappa,\xi) \in \Omega \times [0,\xi_0]$, $|\omega_n| < C_\omega$ and $|\omega| < C_\omega$ for some constants C_ω. So, by using the dominated convergence theorem, we get

$$\sup_{(\kappa,\xi)\in\Omega\times[0,\xi_0]} \|\mathcal{J}_\omega(\omega_n,\psi_n)(\kappa,\xi) - \mathcal{J}_\omega(\omega,\psi)(\kappa,\xi)\| \to 0.$$

Similarly,

$$\sup_{(\kappa,\xi)\in\Omega\times[0,\xi_0]}\left\|\mathcal{J}_\psi(\omega_n,\psi_n)(\kappa,\xi)-\mathcal{J}_\psi(\omega,\psi)(\kappa,\xi)\right\|\to 0.$$

That is, $\|\mathcal{J}(\omega_n,\psi_n)-\mathcal{J}(\omega,\psi)\|_{\mathcal{B}}\to 0$, and hence $\mathcal{J}$ is continuous.

Second, we prove that the boundedness of sets is topological property under the continuous map $\mathcal{J}$. Let $(\omega,\psi)\in D_\epsilon$. Then

$$\begin{aligned}
&|\mathcal{J}_\omega(\omega,\psi)(\kappa,\xi)|\\
&\leq |h_0(\kappa)|+\frac{1-\alpha}{A^\alpha}\,|\overline{\omega}(\kappa,\xi,\omega,\psi)|+\frac{\alpha}{\Gamma(\alpha)A^\alpha}\int_0^\xi(\xi-\sigma)^{\alpha-1}\;|\overline{\omega}(\kappa,\sigma,\omega,\psi)|\,d\sigma\\
&\leq |h_0(\kappa)|+\frac{(1-\alpha)\mathcal{R}_{\overline{\omega}}}{A^\alpha}+\frac{\xi_0^\alpha\mathcal{R}_{\overline{\omega}}}{\Gamma(\alpha)A^\alpha}
\end{aligned}\tag{3.12}$$

and

$$\begin{aligned}
&\left|\mathcal{J}_\psi(\omega,\psi)(\kappa,\xi)\right|\\
&\leq \left|\hat{h}_0(\kappa)\right|+\frac{1-\alpha}{A^\alpha}\left|\overline{\psi}(\kappa,\xi,\omega,\psi)\right|+\frac{\alpha}{\Gamma(\alpha)A^\alpha}\int_0^\xi(\xi-\sigma)^{\alpha-1}\;\left|\overline{\psi}(\kappa,\sigma,\omega,\psi)\right|d\sigma\\
&\leq \left|\hat{h}_0(\kappa)\right|+\frac{(1-\alpha)\mathcal{R}_{\overline{\psi}}}{A^\alpha}+\frac{\xi_0^\alpha\mathcal{R}_{\overline{\psi}}}{\Gamma(\alpha)A^\alpha}.
\end{aligned}\tag{3.13}$$

Hence, $\mathcal{J}(D_\epsilon)\subseteq D_{\epsilon'}$, where

$$\epsilon'=\max\left\{|h_0(\kappa)|+\frac{(1-\alpha)\mathcal{R}_{\overline{\omega}}}{A^\alpha}+\frac{\xi_0^\alpha\mathcal{R}_{\overline{\omega}}}{\Gamma(\alpha)A^\alpha},\left|\hat{h}_0(\kappa)\right|+\frac{(1-\alpha)\mathcal{R}_{\overline{\psi}}}{A^\alpha}+\frac{\xi_0^\alpha\mathcal{R}_{\overline{\psi}}}{\Gamma(\alpha)A^\alpha}\right\}.$$

Since ϵ' is independent on (ω,ψ), $\mathcal{J}(D_\epsilon)$ is uniformly bounded set.

Third, for the equicontinuity property of $\mathcal{J}$, let $(\kappa_1,\xi_1),(\kappa,\xi)\in\Omega\times[0,\xi_0]$ be arbitrary points. Note that

$$\begin{aligned}
&|\mathcal{J}_\omega(\omega,\psi)(\kappa,\xi)-\mathcal{J}_\omega(\omega,\psi)(\kappa_1,\xi_1)|\leq\frac{1-\alpha}{A^\alpha}\,|\overline{\omega}(\kappa,\xi,\omega,\psi)-\overline{\omega}(\kappa_1,\xi_1,\omega,\psi)|\\
&+\frac{\alpha}{\Gamma(\alpha)A^\alpha}\int_0^\xi\left[(\xi-\sigma)^{\alpha-1}-(\xi_1-\sigma)^{\alpha-1}\right]\;|\overline{\omega}(\kappa,\sigma,\omega,\psi)-\overline{\omega}(\kappa_1,\sigma,\omega,\psi)|\,d\sigma\\
&\leq\frac{1-\alpha}{A^\alpha}\,|\overline{\omega}(\kappa,\xi,\omega,\psi)-\overline{\omega}(\kappa_1,\xi_1,\omega,\psi)|\\
&+\frac{\alpha}{\Gamma(\alpha)A^\alpha}\int_0^\xi\left[(\xi-\sigma)^{\alpha-1}-(\xi_1-\sigma)^{\alpha-1}\right]\;[|\overline{\omega}(\kappa,\sigma,\omega,\psi)|+|\overline{\omega}(\kappa_1,\sigma,\omega,\psi)|]\,d\sigma\\
&\leq\frac{1-\alpha}{A^\alpha}\,|\overline{\omega}(\kappa,\xi,\omega,\psi)-\overline{\omega}(\kappa_1,\xi_1,\omega,\psi)|\\
&+\frac{2\epsilon'}{\Gamma(\alpha)A^\alpha}\int_0^\xi\left|(\xi-\sigma)^{\alpha-1}-(\xi_1-\sigma)^{\alpha-1}\right|d\sigma.
\end{aligned}\tag{3.14}$$

Since the kernel functions are integrable, it follows that $|\mathcal{J}_\omega(\omega,\psi)(\kappa,\xi)-\mathcal{J}_\omega(\omega,\psi)(\kappa_1,\xi_1)|\to 0$ as $(\kappa,\xi)\to(\kappa_1,\xi_1)$. Similarly, $\left|\mathcal{J}_\psi(\omega,\psi)(\kappa,\xi)-\mathcal{J}_\psi(\omega,\psi)(\kappa_1,\xi_1)\right|\to 0$ as $(\kappa,\xi)\to(\kappa_1,\xi_1)$. Therefore, $\|\mathcal{J}(\kappa,\xi)-\mathcal{J}(\kappa_1,\xi_1)\|_{\mathcal{B}}\to 0$ as $(\kappa,\xi)\to(\kappa_1,\xi_1)$, which implies that $\mathcal{J}$ possesses the equicontinuity property. Since boundedness and uniform boundedness are preserved under continuous mappings, the Arzelà–Ascoli theorem guarantees that $\mathcal{J}$ is a compact operator.

Finally, we prove that the set $\Delta = \{(\omega, \psi) \in \mathcal{B} : (\omega, \psi) = \delta\mathcal{J}(\omega, \psi),\ \ \delta \in [0, 1]\}$ is bounded in $\mathcal{B}$. Let $(\omega, \psi) = \delta\mathcal{J}(\omega, \psi)$ for some $\delta \in [0, 1]$. Then we have

$$\begin{aligned} |\omega(\kappa, \xi)| &= |\delta\mathcal{J}_\omega(\omega, \psi)(\kappa, \xi)| = \delta\, |\mathcal{J}_\omega(\omega, \psi)(\kappa, \xi)| \\ &\leq \delta \left[|h_0(\kappa)| + \frac{(1-\alpha)\mathcal{R}_{\overline{\omega}}}{A^\alpha} + \frac{\xi_0^\alpha \mathcal{R}_{\overline{\omega}}}{\Gamma(\alpha)A^\alpha} \right] \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} |\psi(\kappa, \xi)| &= \left|\delta\mathcal{J}_\psi(\omega, \psi)(\kappa, \xi)\right| = \delta \left|\mathcal{J}_\psi(\omega, \psi)(\kappa, \xi)\right| \\ &\leq \delta \left[\left|\hat{h}_0(\kappa)\right| + \frac{(1-\alpha)\mathcal{R}_{\overline{\psi}}}{A^\alpha} + \frac{\xi_0^\alpha \mathcal{R}_{\overline{\psi}}}{\Gamma(\alpha)A^\alpha} \right]. \end{aligned} \tag{3.16}$$

That is, Δ is bounded in $\mathcal{B}$. Therefore, by applying Schaefer's fixed-point theorem, it follows that $\mathcal{J}$ possesses at least one fixed-point in D_ϵ, which represents a solution of the system (1.2). □

Theorem 3.6. If C1–C3 hold and

$$(1-\alpha) + \frac{\xi_0^\alpha}{\Gamma(\alpha)} < \frac{A^\alpha}{\lambda_\omega + \lambda_\psi},$$

then the system (1.2) has a unique solution.

Proof. From C3 and Theorem 3.1, we get that for $(\omega, \psi), (\omega_1, \psi_1) \in \mathcal{B}$,

$$\|\overline{\omega}(\omega, \psi) - \overline{\omega}(\omega_1, \psi_1)\| \leq \left[\frac{(1-\alpha)\lambda_\omega}{A^\alpha} + \frac{\xi_0^\alpha \lambda_\omega}{\Gamma(\alpha)A^\alpha} \right] \|(\omega, \psi) - (\omega_1, \psi_1)\|_{\mathcal{B}}$$

and

$$\left\|\overline{\psi}(\omega, \psi) - \overline{\psi}(\omega_1, \psi_1)\right\| \leq \left[\frac{(1-\alpha)\lambda_\psi}{A^\alpha} + \frac{\xi_0^\alpha \lambda_\psi}{\Gamma(\alpha)A^\alpha} \right] \|(\omega, \psi) - (\omega_1, \psi_1)\|_{\mathcal{B}}.$$

Hence,

$$\begin{aligned} \|\mathcal{J}(\omega, \psi) - \mathcal{J}(\omega_1, \psi_1)\| &\leq \left[\frac{(1-\alpha)\lambda_\omega}{A^\alpha} + \frac{\xi_0^\alpha \lambda_\omega}{\Gamma(\alpha)A^\alpha} \right] \|(\omega, \psi) - (\omega_1, \psi_1)\|_{\mathcal{B}} \\ &+ \left[\frac{(1-\alpha)\lambda_\psi}{A^\alpha} + \frac{\xi_0^\alpha \lambda_\psi}{\Gamma(\alpha)A^\alpha} \right] \|(\omega, \psi) - (\omega_1, \psi_1)\|_{\mathcal{B}} \\ &= \left[(1-\alpha) + \frac{\xi_0^\alpha}{\Gamma(\alpha)} \right] \frac{\lambda_\omega + \lambda_\psi}{A^\alpha} \|(\omega, \psi) - (\omega_1, \psi_1)\|_{\mathcal{B}}. \end{aligned}$$

Therefore, when the inequality

$$\left[(1-\alpha) + \frac{\xi_0^\alpha}{\Gamma(\alpha)} \right] \frac{\lambda_\omega + \lambda_\psi}{A^\alpha} < 1 \ \left(\text{i.e., } (1-\alpha) + \frac{\xi_0^\alpha}{\Gamma(\alpha)} < \frac{A^\alpha}{\lambda_\omega + \lambda_\psi} \right)$$

is satisfied, the operator $\mathcal{J}$ qualifies as a contraction mapping. Accordingly, by invoking the Banach fixed-point theorem, we conclude that the system has a unique solution. □

3.3. The stability property

In this subsection, we investigate the Hyers–Ulam stability of the fractional WBKEs (1.2), assessing how small perturbations in the initial data or model parameters influence the behavior of their solutions. This stability concept guarantees that any approximate solution remains close to an exact one within a controlled bound, thereby reinforcing the reliability of the analytical results. By applying integral inequalities in conjunction with the previously established existence and uniqueness properties, we derive sufficient conditions ensuring the Hyers–Ulam stability of the fractional WBKEs (1.2). Let $(\hat{\omega}, \hat{\psi})$ denote an approximate solution of (1.2) satisfying

$$\begin{cases} \left| {}^{ABC}\mathcal{D}^{\alpha}_{0,\xi}\, \hat{\omega}(\kappa,\xi) - \overline{\omega}(\kappa,\xi,\hat{\omega},\hat{\psi}) \right| \leq \mu_1, \\ \left| {}^{ABC}\mathcal{D}^{\alpha}_{0,\xi}\, \hat{\psi}(\kappa,\xi) - \overline{\psi}(\kappa,\xi,\hat{\omega},\hat{\psi}) \right| \leq \mu_2, \end{cases} \tag{3.17}$$

for all $(\kappa,\xi) \in \Omega \times [0,\xi_0]$, where $\mu_1, \mu_2 > 0$. Following [48], the nonlinear system (1.2) is said to be Hyers–Ulam stable if there exists a unique solution (ω, ψ) of (1.2) such that

$$\|\hat{\omega} - \omega\| \leq \mu_1' \mu_1 \quad \text{and} \quad \left\|\hat{\psi} - \psi\right\| \leq \mu_2' \mu_2,$$

where $\mu_1', \mu_2' > 0$ are constants independent of μ_1, μ_2, and $\|\cdot\|$ represents the supremum norm on $C[\Omega \times [0,\xi_0]]$.

Theorem 3.7. If C1–C3 hold and

$$(1-\alpha) + \frac{\xi_0^{\alpha}}{\Gamma(\alpha)} < \frac{A^{\alpha}}{\lambda_{\omega} + \lambda_{\psi}},$$

then the system (1.2) is Hyers–Ulam stable.

Proof. Let $(\hat{\omega}, \hat{\psi})$ be an approximate solution and (ω, ψ) be a unique solution of (1.2). For the approximate solution $(\hat{\omega}, \hat{\psi})$ with satisfying the inequalities (3.17), there are two continuous functions g and g' on $\Omega \times [0,\xi_0]$ such that

$$|g(\kappa,\xi)| \leq \varepsilon_1, \ |g'(\kappa,\xi)| \leq \varepsilon_2,$$

$${}^{ABC}\mathcal{D}^{\alpha}_{0,\xi}\, \hat{\omega}(\kappa,\xi) = \overline{\omega}(\kappa,\xi,\hat{\omega},\hat{\psi}) + g(\kappa,\xi),$$

and

$${}^{ABC}\mathcal{D}^{\alpha}_{0,\xi}\, \hat{\psi}(\kappa,\xi) = \overline{\psi}(\kappa,\xi,\hat{\omega},\hat{\psi}) + g'(\kappa,\xi),$$

for all $(\kappa,\xi) \in \Omega \times [0,\xi_0]$. Here, the approximate solution $(\hat{\omega}, \hat{\psi})$ will be as

$$\begin{aligned} \hat{\omega}(\kappa,\xi) &= g_0(\kappa) + \frac{1-\alpha}{A^{\alpha}} \left[\overline{\omega}(\kappa,\xi,\hat{\omega},\hat{\psi}) + g(\kappa,\xi) \right] \\ &+ \frac{\alpha}{\Gamma(\alpha)A^{\alpha}} \int_0^{\xi} (\xi-\sigma)^{\alpha-1}\ \overline{\omega}(\kappa,\sigma,\hat{\omega},\hat{\psi}) d\sigma + \frac{\alpha}{\Gamma(\alpha)A^{\alpha}} \int_0^{\xi} (\xi-\sigma)^{\alpha-1}\ g(\kappa,\sigma) d\sigma \end{aligned}$$

and

$$\begin{aligned} \hat{\psi}(\kappa,\xi) &= \hat{g}_0(\kappa) + \frac{1-\alpha}{A^{\alpha}} \left[\overline{\psi}(\kappa,\xi,\hat{\omega},\hat{\psi}) + g(\kappa,\xi) \right] \\ &+ \frac{\alpha}{\Gamma(\alpha)A^{\alpha}} \int_0^{\xi} (\xi-\sigma)^{\alpha-1}\ \overline{\psi}(\kappa,\sigma,\hat{\omega},\hat{\psi}) d\sigma + \frac{\alpha}{\Gamma(\alpha)A^{\alpha}} \int_0^{\xi} (\xi-\sigma)^{\alpha-1}\ g'(\kappa,\sigma) d\sigma \end{aligned}$$

for all $(\kappa, \xi) \in \Omega \times [0, \xi_0]$. Then by Theorem 3.6, for all $(\kappa, \xi) \in \Omega \times [0, \xi_0]$ we have

$$\begin{aligned}
|\hat{\omega}(\kappa, \xi) - \omega(\kappa, \xi)| &\leq \frac{1-\alpha}{A^\alpha}\left[\left|\overline{\omega}(\kappa, \sigma, \hat{\omega}, \hat{\psi}) - \overline{\omega}(\kappa, \sigma, \omega, \psi)\right| + |g(\kappa, \sigma)|\right] \\
&+ \frac{\alpha}{\Gamma(\alpha)A^\alpha}\int_0^\xi (\xi - \sigma)^{\alpha-1}\ \left[\left|\overline{\omega}(\kappa, \sigma, \hat{\omega}, \hat{\psi}) - \overline{\omega}(\kappa, \sigma, \omega, \psi)\right| + |g(\kappa, \sigma)|\right] d\sigma \\
&\leq \frac{1-\alpha}{A^\alpha}\left[\lambda_\omega \left\|(\hat{\omega}, \hat{\psi}) - (\omega, \psi)\right\|_{\mathcal{B}} + \varepsilon_1\right] \\
&+ \frac{\alpha}{\Gamma(\alpha)A^\alpha}\int_0^\xi (\xi - \sigma)^{\alpha-1}\ \left[\lambda_\omega \left\|(\hat{\omega}, \hat{\psi}) - (\omega, \psi)\right\|_{\mathcal{B}} + \varepsilon_1\right] d\sigma \\
&\leq \left[\frac{1-\alpha}{A^\alpha} + \frac{\xi_0^\alpha}{\Gamma(\alpha)A^\alpha}\right]\left[\lambda_\omega \left\|(\hat{\omega}, \hat{\psi}) - (\omega, \psi)\right\|_{\mathcal{B}} + \varepsilon_1\right].
\end{aligned}$$

That is,

$$\|\hat{\omega} - \omega\| \leq \mathbb{C}_\alpha^A\left[\lambda_\omega \left\|(\hat{\omega}, \hat{\psi}) - (\omega, \psi)\right\|_{\mathcal{B}} + \varepsilon_1\right],$$

where $\mathbb{C}_\alpha^A := \dfrac{1-\alpha}{A^\alpha} + \dfrac{\xi_0^\alpha}{\Gamma(\alpha)A^\alpha}$. Similarly,

$$\left\|\hat{\psi} - \psi\right\| \leq \mathbb{C}_\alpha^A\left[\lambda_\psi \left\|(\hat{\omega}, \hat{\psi}) - (\omega, \psi)\right\|_{\mathcal{B}} + \varepsilon_2\right].$$

Hence,

$$\begin{aligned}
\|\hat{\omega} - \omega\| &\leq \mathbb{C}_\alpha^A\left[\lambda_\omega \left\|(\hat{\omega}, \hat{\psi}) - (\omega, \psi)\right\|_{\mathcal{B}} + \varepsilon_1\right] \\
&\leq \mathbb{C}_\alpha^A\left[\frac{\mathbb{C}_\alpha^A\lambda_\omega}{1 - \left(\mathbb{C}_\alpha^A(\lambda_\omega + \lambda_\psi)\right)}\max\{\varepsilon_1, \varepsilon_2\}\right] + \mathbb{C}_\alpha^A\varepsilon_1
\end{aligned}$$

and

$$\begin{aligned}
\left\|\hat{\psi} - \psi\right\| &\leq \mathbb{C}_\alpha^A\left[\lambda_\psi \left\|(\hat{\omega}, \hat{\psi}) - (\omega, \psi)\right\|_{\mathcal{B}} + \varepsilon_2\right] \\
&\leq \mathbb{C}_\alpha^A\left[\frac{\mathbb{C}_\alpha^A\lambda_\psi}{1 - \left(\mathbb{C}_\alpha^A(\lambda_\omega + \lambda_\psi)\right)}\max\{\varepsilon_1, \varepsilon_2\}\right] + \mathbb{C}_\alpha^A\varepsilon_2.
\end{aligned}$$

In particular, if we set $\varepsilon := \max\{\varepsilon_1, \varepsilon_2\}$, then

$$\|\hat{\omega} - \omega\| \leq \varepsilon\ \mathbb{C}_\alpha^A\left[\frac{\mathbb{C}_\alpha^A\lambda_\omega}{1 - \left(\mathbb{C}_\alpha^A(\lambda_\omega + \lambda_\psi)\right)} + 1\right]$$

and

$$\left\|\hat{\psi} - \psi\right\| \leq \varepsilon\ \mathbb{C}_\alpha^A\left[\frac{\mathbb{C}_\alpha^A\lambda_\psi}{1 - \left(\mathbb{C}_\alpha^A(\lambda_\omega + \lambda_\psi)\right)} + 1\right].$$

Take

$$\varepsilon_1' := \mathbb{C}_\alpha^A\left[\frac{\mathbb{C}_\alpha^A\lambda_\omega}{1 - \left(\mathbb{C}_\alpha^A(\lambda_\omega + \lambda_\psi)\right)} + 1\right] \text{ and } \varepsilon_2' := \mathbb{C}_\alpha^A\left[\frac{\mathbb{C}_\alpha^A\lambda_\psi}{1 - \left(\mathbb{C}_\alpha^A(\lambda_\omega + \lambda_\psi)\right)} + 1\right].$$

The Hyers–Ulam stability constants ε_1' and ε_2' measure how closely the approximate solution remains to the exact one. These constants are inversely related to the denominator

$$1 - \mathbb{C}_{\alpha}^{A}(\lambda_{\omega} + \lambda_{\psi}),$$

which characterizes the contraction gap of the operator. A smaller value of $\mathbb{C}_{\alpha}^{A}$, λ_{ω}, or λ_{ψ} increases this denominator, thereby reducing the values of ε_1' and ε_2' and yielding a stronger form of stability. To reinforce the stability of the system, it is therefore desirable to ensure that

$$\mathbb{C}_{\alpha}^{A}(\lambda_{\omega} + \lambda_{\psi}) \leq 1, \quad \text{i.e.,} \quad (1-\alpha) + \frac{\xi_0^{\alpha}}{\Gamma(\alpha)} < \frac{A^{\alpha}}{\lambda_{\omega} + \lambda_{\psi}}.$$

This condition may be fulfilled by choosing a smaller domain length ξ_0, reducing the Lipschitz constants, or increasing the fractional order α so that $\Gamma(\alpha)$ becomes larger. Any of these adjustments increase the denominator, thereby strengthening the Hyers–Ulam stability of the fractional nonlinear system. Consequently, the system (1.2) is Hyers–Ulam stable. □

Remark 3.8. The Hyers–Ulam stability established in Theorem 3.7 guarantees that the fractional WBKEs (1.2) are stable under small perturbations in the initial data and system parameters. This type of stability is local in nature, which is standard within the classical Hyers–Ulam stability framework for nonlinear FDEs. Extensions of the present analysis to global or generalized Hyers–Ulam–Rassias stability will be considered in future work.

Remark 3.9. The condition $C_{\alpha}^{A}(\lambda_{\omega} + \lambda_{\psi}) < 1$ ensures that nonlinear and fractional memory effects do not amplify perturbations in the system, guaranteeing stable wave evolution. Physically, this corresponds to controlled tsunami wave propagation, while numerically, it ensures convergence and robustness of the solution method. For realistic tsunami parameters, this condition is readily satisfied on bounded time intervals, which is standard in shallow water modeling.

4. Semi-analytical development for solving the WBKEs (1.2)

In this section, we develop a semi-analytical framework for solving the fractional WBKEs. Subsection 4.1 introduces the FPSEs and establishes the operator formulas needed for computing their coefficients. These results serve as the analytical foundation for Subsection 4.2, where the ENIM procedure is constructed. Thus, the methodology in 4.2 directly relies on the power series structures derived in 4.1.

4.1. On power series expansions

Lemma 4.1. Let $0 < \alpha < 1$. Then

$$ {}^{ABC}\mathcal{D}_{0,\xi}^{\alpha}\,(\xi^{n\alpha}) = \frac{A^{\alpha}\Gamma(n\alpha+1)}{(1-\alpha)\Gamma(n\alpha-\alpha+1)}\,\xi^{n\alpha-\alpha}\; H_n(\xi), \tag{4.1}$$

where

$$H_n(\xi) := \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{\alpha}{1-\alpha}\right)^m \xi^{m\alpha}}{\Gamma((n+m)\alpha+1)}. \tag{4.2}$$

Proof. By the definition of ${}^{ABC}\mathcal{D}^{\alpha}_{0,\xi}$ with $0 \leq \alpha < 1$, we have

$$\begin{aligned}
{}^{ABC}\mathcal{D}^{\alpha}_{0,\xi}\,(\xi^{n\alpha}) &= \frac{A^{\alpha}}{1-\alpha}\int_0^{\xi} E_{\alpha}\left(\frac{-\alpha(\xi-\sigma)^{\alpha}}{1-\alpha}\right)\frac{\partial}{\partial\sigma}(\sigma^{n\alpha})d\sigma \\
&= \frac{A^{\alpha}}{1-\alpha}n\alpha\int_0^{\xi} E_{\alpha}\left(\frac{-\alpha(\xi-\sigma)^{\alpha}}{1-\alpha}\right)\sigma^{n\alpha-1}d\sigma \\
&= \frac{A^{\alpha}}{1-\alpha}n\alpha\int_0^{\xi}\sigma^{n\alpha-1}\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{\alpha}{1-\alpha}\right)^m(\xi-\sigma)^{m\alpha}}{\Gamma(m\alpha+1)}d\sigma \\
&= \frac{A^{\alpha}}{1-\alpha}n\alpha\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{\alpha}{1-\alpha}\right)^m}{\Gamma(m\alpha+1)}\int_0^{\xi}(\xi-\sigma)^{m\alpha}\sigma^{n\alpha-1}d\sigma.
\end{aligned} \tag{4.3}$$

Note that

$$\int_0^{\xi}(\xi-\sigma)^{m\alpha}\sigma^{n\alpha-1}d\sigma = \xi^{n\alpha+m\alpha}\frac{\Gamma(n\alpha)\Gamma(m\alpha+1)}{\Gamma(n\alpha+m\alpha+1)}.$$

Hence, this is proof. □

Consider a power series

$$\sum_{n=0}^{\infty}c_n(\xi-a)^{n\alpha} = C_0 + C_1(\xi-a)^{\alpha} + C_2(\xi-a)^{2\alpha} + \cdots, \tag{4.4}$$

to be called an FPSE about a such that ξ is a variable, c_n are called the coefficients of the series, where $0 \leq n-1 < \alpha \leq n$, $n \in \mathbb{N}$, and $\xi \geq a$.

Theorem 4.2. Consider that the FPSE notation of Q about $a = 0$ has the form

$$Q(\xi) = \sum_{n=0}^{\infty}c_n\xi^{n\alpha}, \tag{4.5}$$

where $0 < \alpha < 1$ and $0 \leq \xi \leq R$. If $Q(\xi)$, ${}^{ABC}D^{n\alpha}_{0,\xi}Q(\xi) \in C[0,R]$ for $n = 1,2,3,\ldots$, then the terms c_n are given by

$$c_n = \frac{{}^{ABC}D^{n\alpha}_{0,\xi}Q(\xi)}{\Gamma(A^{\alpha},\alpha,n)}\Big|_{\xi=0}, \tag{4.6}$$

where ${}^{ABC}D^{n\alpha}_{0,\xi} = {}^{ABC}D^{\alpha}_{0,\xi}\circ{}^{ABC}D^{\alpha}_{0,\xi}\circ\cdots\circ{}^{ABC}D^{\alpha}_{0,\xi}$ (n times), R is the convergence radius, and

$$\Gamma(A^{\alpha},\alpha,n) := \frac{A^{\alpha}}{1-\alpha}\frac{\Gamma(n\alpha+1)}{\Gamma(2n\alpha-\alpha+1)\Gamma(1-n\alpha+\alpha)}\left(\frac{\alpha}{\alpha-1}\right)^{n-1}.$$

Proof. By Lemma 4.1 above, we have

$$\begin{aligned}
{}^{ABC}D^{\alpha}_{0,\xi}Q(\xi) &= \sum_{n=1}^{\infty} c_n\, {}^{ABC}D^{\alpha}_{0,\xi}(\xi^{n\alpha}) \\
&= \sum_{n=1}^{\infty} c_n \frac{A^{\alpha}\Gamma(n\alpha+1)}{(1-\alpha)\Gamma(n\alpha-\alpha+1)}\xi^{n\alpha-\alpha}\ H_n(\xi) \\
&= \sum_{n=1}^{\infty} c_n \frac{A^{\alpha}\Gamma(n\alpha+1)}{(1-\alpha)\Gamma(n\alpha-\alpha+1)}\xi^{n\alpha-\alpha}\ \sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{\alpha}{1-\alpha}\right)^m\xi^{m\alpha}}{\Gamma((n+m)\alpha+1)} \\
&= \sum_{n=1}^{\infty} c_n \frac{A^{\alpha}\Gamma(n\alpha+1)}{(1-\alpha)\Gamma(n\alpha-\alpha+1)}\ \sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{\alpha}{1-\alpha}\right)^m}{\Gamma((n+m)\alpha+1)}\xi^{(n-1)\alpha+m\alpha}.
\end{aligned}$$

Reindex with $k = n-1$:

$${}^{ABC}D^{\alpha}_{0,\xi}Q(\xi) = \sum_{k=0}^{\infty} c_{k+1}\frac{A^{\alpha}\Gamma((k+1)\alpha+1)}{(1-\alpha)\Gamma(k\alpha+1)}\ \sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{\alpha}{1-\alpha}\right)^m}{\Gamma((k+m+1)\alpha+1)}\xi^{k\alpha+m\alpha}.$$

Evaluating at $\xi = 0$ kills all $k \geq 1$, since $\xi^{k\alpha+m\alpha} = 0$, giving

$${}^{ABC}D^{\alpha}_{0,\xi}Q(\xi)\Big|_{\xi=0} = c_1\,\frac{A^{\alpha}}{1-\alpha}.$$

Define recursively $Q_0(\xi) = Q(\xi)$ and

$$Q_k(\xi) = {}^{C}D^{\alpha}_{a\xi}Q_{k-1}(\xi) = {}^{C}D^{k\alpha}_{a\xi}Q(\xi).$$

Using (2.2) repeatedly, by induction we get

$${}^{C}D^{k\alpha}_{a\xi}Q(\xi) = Q_k(\xi) = \sum_{k=n}^{\infty} c_n\,\frac{A^{\alpha}\Gamma(n\alpha+1)}{(1-\alpha)\Gamma(n\alpha-k\alpha+1)}\xi^{n\alpha-k\alpha}\ \sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{\alpha}{1-\alpha}\right)^m\xi^{m\alpha}}{\Gamma((n+m)\alpha+1)}.$$

Set $k = n$ in (4.1) to get

$${}^{ABC}D^{n\alpha}_{0,\xi}Q(\xi)\Big|_{\xi=0} = c_n\frac{A^{\alpha}}{1-\alpha}\frac{\Gamma(n\alpha+1)}{\Gamma(2n\alpha-\alpha+1)\Gamma(1-n\alpha+\alpha)}\left(\frac{\alpha}{\alpha-1}\right)^{n-1}.$$

Hence,

$$c_n = \frac{{}^{ABC}D^{n\alpha}_{0,\xi}Q(\xi)}{\Gamma(A^{\alpha},\alpha,n)}\Big|_{\xi=0}.$$

The proof is completed. □

For two-variables, a power series

$$\omega(\kappa,\xi) = \sum_{n=0}^{\infty} Q_n(\kappa)\,\xi^{n\alpha}, \kappa \in \Omega,\ 0 \leq \xi \leq R, \tag{4.7}$$

is called a multiple FPSE of ω about 0.

Theorem 4.3. If ${}^{ABC}D^{n\alpha}_{0,\xi}\omega(\kappa,\xi), n = 0,1,2,3,\ldots,$ is continuous on $\Omega \times [0,R]$, in the series (4.7) then

$$Q_n(\kappa) = \frac{{}^{ABC}D^{n\alpha}_{0,\xi}\omega(\kappa,\xi)}{\Gamma(A^\alpha,\alpha,n)}\Big|_{\xi=0}, \tag{4.8}$$

where $R = \min_{\kappa\in\Omega} R_\kappa$ with R_κ as the convergence radius of the FPSE (4.7).

Proof. Fix an arbitrary κ, and define the function Q_κ by $Q_\kappa(\xi) = \omega(\kappa,\xi)$. Then, for this fixed κ, the FPS (4.7) becomes

$$Q_\kappa(\xi) = \sum_{n=0}^{\infty} Q_n(\kappa)\xi^{n\alpha}.$$

By the continuity assumptions, $Q_\kappa(\xi)$ and ${}^{ABC}D^{n\alpha}_{a\,\xi}Q_\kappa(\xi) = {}^{ABC}D^{n\alpha}_{a\,\xi}\omega(\kappa,\xi)$ satisfy the hypotheses of Lemma 4.2, we get

$$Q_n(\kappa) = \frac{{}^{ABC}D^{n\alpha}_{a\,\xi}Q_\kappa(\xi)}{\Gamma(A^\alpha,\alpha,n)}\Big|_{\xi=0} = \frac{{}^{ABC}D^{n\alpha}_{a\,\xi}\omega(\kappa,\xi)}{\Gamma(A^\alpha,\alpha,n)}\Big|_{\xi=0}.$$

This is the proof. □

Theorem 4.4. Consider that the multiple FPSE notation of ω about 0 has the form

$$\omega(\kappa,\xi) = \sum_{n=0}^{\infty} Q_n(\kappa)\,\xi^{n\alpha}, \tag{4.9}$$

where $\kappa \in \Omega_1 \times \Omega_2 \times \cdots \Omega_m$ and $0 \le \xi \le R$. If ${}^{ABC}D^{n\alpha}_{0,\xi}\omega(\kappa,\xi), n = 0,1,2,3,\ldots,$ are continuous on $\Omega_1 \times \Omega_2 \times \cdots \Omega_m \times [0,R]$, then

$$Q_n(\kappa) = \frac{{}^{ABC}D^{n\alpha}_{0,\xi}\omega(\kappa,\xi)}{\Gamma(A^\alpha,\alpha,n)}\Big|_{\xi=0}, \tag{4.10}$$

where $R = \min_{\kappa\in\Omega_1\times\Omega_2\times\cdots\Omega_m} R_\kappa$ with R_κ as the convergence radius of the FPSE

$$\sum_{n=0}^{\infty} Q_n(\kappa)\xi^{n\alpha}. \tag{4.11}$$

Proof. The proof is directly from Theorem 4.3 and the Cartesian product properties. □

4.2. *Methodology of ENIM*

Consider that the following Caputo fractional system

$$\begin{cases} {}^{ABC}\mathcal{D}^{\alpha}_{0,\xi}\,\omega(\kappa,\xi) = \mathcal{N}_\omega(\kappa,\xi,\omega,\psi) + \mathcal{R}_\omega(\kappa,\xi,\omega,\psi), \\ {}^{ABC}\mathcal{D}^{\alpha}_{0,\xi}\,\psi(\kappa,\xi) = \mathcal{N}_\psi(\kappa,\xi,\omega,\psi) + \mathcal{R}_\psi(\kappa,\xi,\omega,\psi), \end{cases} \tag{4.12}$$

where $0 < \alpha \le 1$, $\mathcal{N}_\omega, \mathcal{N}_\psi$ denote the nonlinear parts, and $\mathcal{R}_\omega, \mathcal{R}_\psi$ denote any known functions. The initial conditions are met by ω and ψ, allowing them to be rewritten as:

$$\{\omega(\kappa,0) = h_0(\kappa) \text{ and } \psi(\kappa,0) = \hat{h}_0(\kappa). \tag{4.13}$$

Consider that the approximate series solution of (4.12) can be expressed as:

$$\omega(\kappa,\xi)=\sum_{n=0}^{\infty}h_n(\kappa)\frac{\xi^{n\alpha}}{\Gamma(A^{\alpha},\alpha,n)}\text{ and }\psi(\kappa,\xi)=\sum_{n=0}^{\infty}\hat{h}_n(\kappa)\frac{\xi^{n\alpha}}{\Gamma(A^{\alpha},\alpha,n)}. \tag{4.14}$$

To find the terms of $\langle h_n\rangle$ and $\langle\hat{h}_n\rangle$, we apply the NIM to system (4.12). The procedure begins by rewriting the FDEs in an equivalent integral form using the inverse operator of the ABC fractional derivative. Accordingly, system (4.12) can be expressed schematically as

$$\begin{cases}\omega(\kappa,\xi)=h_0(\kappa)+{}^{ABC}\mathcal{I}^{\alpha}_{0,\xi}\{\mathcal{N}_{\omega}(\kappa,\xi,\omega,\psi)+\mathcal{R}_{\omega}(\kappa,\xi,\omega,\psi)\},\\ \psi(\kappa,\xi)=\hat{h}_0(\kappa)+{}^{ABC}\mathcal{I}^{\alpha}_{0,\xi}\{\mathcal{N}_{\psi}(\kappa,\xi,\omega,\psi)+\mathcal{R}_{\psi}(\kappa,\xi,\omega,\psi)\}.\end{cases} \tag{4.15}$$

We separate the known (linear or source) parts from the nonlinear operators by introducing

$$\begin{cases}F_{\omega}(\kappa,\xi)=h_0(\kappa)+{}^{ABC}\mathcal{I}^{\alpha}_{0,\xi}\{\mathcal{R}_{\omega}(\kappa,\xi,\omega,\psi)\},\\ F_{\psi}(\kappa,\xi)=\hat{h}_0(\kappa)+{}^{ABC}\mathcal{I}^{\alpha}_{0,\xi}\{\mathcal{R}_{\psi}(\kappa,\xi,\omega,\psi)\}.\end{cases} \tag{4.16}$$

and the nonlinear operators

$$\begin{cases}\left[G_{\omega}(\omega,\psi)\right](\kappa,\xi)={}^{ABC}\mathcal{I}^{\alpha}_{0,\xi}\{\mathcal{N}_{\omega}(\kappa,\xi,\omega,\psi)\}\\ \left[G_{\psi}(\omega,\psi)\right](\kappa,\xi)={}^{ABC}\mathcal{I}^{\alpha}_{0,\xi}\{\mathcal{N}_{\psi}(\kappa,\xi,\omega,\psi)\}.\end{cases} \tag{4.17}$$

Thus, (4.12) can be written as

$$\omega=F_{\omega}+G_{\omega}(\omega,\psi)\text{ and }\psi=F_{\psi}+G_{\psi}(\omega,\psi). \tag{4.18}$$

Let $\langle S^m_{\omega}\rangle$ and $\langle S^m_{\psi}\rangle$ be the partial sum sequences of the series (4.14), that is,

$$S^m_{\omega}(\kappa,\xi)=\sum_{n=0}^{m}\omega_n(\kappa,\xi)\text{ and }S^m_{\psi}(\kappa,\xi)=\sum_{n=0}^{m}\psi_n(\kappa,\xi)$$

where

$$\omega_n(\kappa,\xi)=h_n(\kappa)\frac{\xi^{n\alpha}}{\Gamma(A^{\alpha},\alpha,n)}\text{ and }\psi_n(\kappa,\xi)=\hat{h}_n(\kappa)\frac{\xi^{n\alpha}}{\Gamma(A^{\alpha},\alpha,n)}.$$

Find the terms of $\langle h_n\rangle$ and $\langle\hat{h}_n\rangle$ by solving the following:

$$\omega_{m+1}(\kappa,\xi)=G_{\omega}(S^m_{\omega}(\kappa,\xi),S^m_{\psi}(\kappa,\xi))-G_{\omega}(S^{m-1}_{\omega}(\kappa,\xi),S^{m-1}_{\psi}(\kappa,\xi)), \tag{4.19}$$

and

$$\psi_{m+1}(\kappa,\xi)=G_{\psi}(S^m_{\omega}(\kappa,\xi),S^m_{\psi}(\kappa,\xi))-G_{\psi}(S^{m-1}_{\omega}(\kappa,\xi),S^{m-1}_{\psi}(\kappa,\xi)) \tag{4.20}$$

for $m=0,1,2,3,\cdots$, where

$$G_{\omega}(S^{-1}_{\omega}(\kappa,\xi),S^{-1}_{\psi}(\kappa,\xi))=0\text{ and }G_{\psi}(S^{-1}_{\omega}(\kappa,\xi),S^{-1}_{\psi}(\kappa,\xi))=0.$$

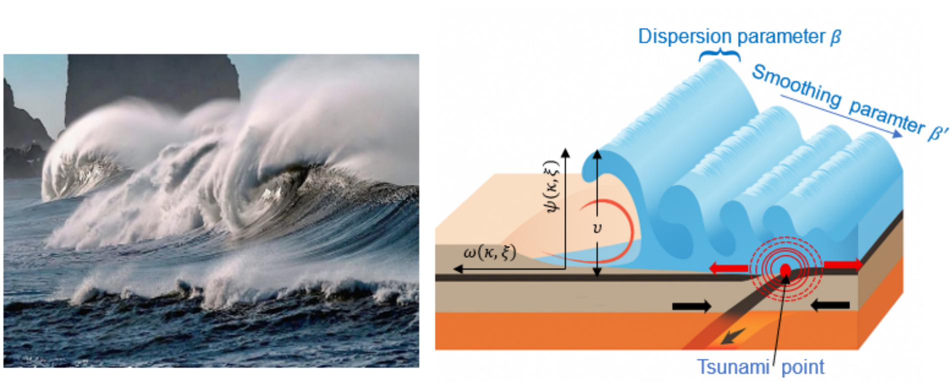
This procedure generates the successive approximations above, which furnish the m-th order approximate solutions of system (4.12). Under appropriate conditions on the nonlinear operators, the series $\sum_{n=0}^{\infty}\omega_n$ and $\sum_{n=0}^{\infty}\psi_n$ converge to the exact solutions ω and ψ, respectively.

5. Some applications

In this section, we apply the ENIM to obtain approximate analytical solutions for the ABC fractional WBKEs (1.2). This hybrid methodology combines the decomposition of nonlinear terms with the transform properties inherent in the Laplace operator, resulting in a powerful and efficient framework for handling nonlinear FPDEs. The method achieves rapid convergence, high accuracy, and computational simplicity.

Figure 1 illustrates tsunami shallow water dynamics from both a physical and mathematical perspective. In Figure 1(a), the propagation of tsunami waves toward a shoreline is shown, representing the real scenario that the WBKEs (1.2) are designed to model. Figure 1(b) displays a schematic graphical representation of the corresponding solutions, highlighting the effects of the parameters β, β', and υ, where υ denotes the maximum wave height in the model. The parameter β regulates the horizontal spreading and smoothing of the wave profile. Physically, β plays the role of an effective viscosity or dispersion coefficient: For $\beta > 0$, wave crests broaden and amplitudes decrease as energy spreads spatially, whereas for $\beta < 0$, the profile sharpens and may develop shock-like features. In Figure 1(b), this effect appears as the horizontal spreading of successive peaks. The higher-order dispersion parameter β' governs finer oscillatory structures, such as small ripples superimposed on the main wave. It enters through the term $\upsilon\beta'\,\omega_{\kappa\kappa\kappa}$ and contributes to oscillatory tails behind the primary wave crest. In Figure 1(b), these high-frequency ripples reflect the influence of β', with larger values producing more pronounced oscillations. This physical interpretation also informs the numerical example discussed later, where the wave maintains its dispersive character without dissipative smoothing under the parameter choice $\beta = 0, \beta' = 0.10, \upsilon = 10m$, and $A^{\alpha} = 1$. We now proceed to analyze the following fractional WBKEs relevant to tsunami shallow water phenomena:

$$\begin{cases} {}^{ABC}\mathcal{D}^{\alpha}_{0,\xi}\omega(\kappa,\xi) + \omega(\kappa,\xi)\omega_{\kappa}(\kappa,\xi) + \psi_{\kappa}(\kappa,\xi) = 0, \\ {}^{ABC}\mathcal{D}^{\alpha}_{0,\xi}\psi(\kappa,\xi) + [\omega(\kappa,\xi)\psi(\kappa,\xi)]_{\kappa} - \omega_{\kappa\kappa\kappa}(\kappa,\xi) = 0, \\ \omega(\kappa,0) = 5 - 0.2\coth(0.1\kappa + 1), \\ \psi(\kappa,0) = -0.02\ \mathrm{csch}(0.1\kappa + 1). \end{cases} \tag{5.1}$$



(a) Tsunami waves (b) Graphical representation

Figure 1. Tsunami shallow water phenomena with the system (1.2).

If $\alpha = 1$, then the exact solution of (5.1) is

$$\begin{aligned} \omega(\kappa,\xi) &= 5 - 0.2\coth(0.1\kappa - 5\xi + 1), \\ \psi(\kappa,\xi) &= -0.02\ \operatorname{csch}(0.1\kappa - 5\xi + 1). \end{aligned} \tag{5.2}$$

In this subsection, we will use the ENIM in solving (5.1). The initial conditions are met by ω and ψ, allowing them to be rewritten as:

$$\begin{cases} h_0(\kappa) = 5 - 0.2\coth(0.1\kappa + 1), \\ \hat{h}_0(\kappa) = -0.02\ \operatorname{csch}(0.1\kappa + 1). \end{cases} \tag{5.3}$$

Consider that the approximate series solution of (4.12) can be expressed as:

$$\begin{aligned} \omega(\kappa,\xi) &= \sum_{n=0}^{\infty} h_n(\kappa)\frac{\xi^{n\alpha}}{\Gamma(1,\alpha,n)}, \\ \psi(\kappa,\xi) &= \sum_{n=0}^{\infty} \hat{h}_n(\kappa)\frac{\xi^{n\alpha}}{\Gamma(1,\alpha,n)}. \end{aligned} \tag{5.4}$$

Apply ${}^{ABC}J^{\alpha}_{0,\xi}$ to each equation in (5.1), and we obtain

$$\begin{aligned} \omega(\kappa,\xi) &= 5 - 0.2\coth(0.1\kappa + 1) - {}^{ABC}J^{\alpha}_{0,\xi}[\omega\,\omega_\kappa + \psi_\kappa](\kappa,\xi), \\ \psi(\kappa,\xi) &= -0.02\ \operatorname{csch}(0.1\kappa + 1) - {}^{ABC}J^{\alpha}_{0,\xi}[\omega_\kappa\psi + \omega\psi_\kappa - \omega_{\kappa\kappa\kappa}](\kappa,\xi). \end{aligned} \tag{5.5}$$

Hence,

$$\begin{aligned} \omega(\kappa,\xi) &= 5 - 0.2\coth(0.1\kappa + 1) + \mathcal{N}_\omega(\omega,\psi)(\kappa,\xi), \\ \psi(\kappa,\xi) &= -0.02\ \operatorname{csch}(0.1\kappa + 1) + \mathcal{N}_\psi(\omega,\psi)(\kappa,\xi), \end{aligned} \tag{5.6}$$

where

$$\begin{aligned} \mathcal{N}_\omega(\omega,\psi)(\kappa,\xi) &:= -{}^{ABC}J^{\alpha}_{0,\xi}[\omega\,\omega_\kappa + \psi_\kappa](\kappa,\xi), \\ \mathcal{N}_\psi(\omega,\psi)(\kappa,\xi) &:= -{}^{ABC}J^{\alpha}_{0,\xi}[\omega_\kappa\psi + \omega\psi_\kappa - \omega_{\kappa\kappa\kappa}](\kappa,\xi). \end{aligned} \tag{5.7}$$

Let $\langle S^m_\omega\rangle$ and $\langle S^m_\psi\rangle$ be the partial sum sequences of the series (5.4), that is,

$$S^m_\omega(\kappa,\xi) = \sum_{n=0}^{m} \omega_n(\kappa,\xi) \text{ and } S^m_\psi(\kappa,\xi) = \sum_{n=0}^{m} \psi_n(\kappa,\xi),$$

where

$$\omega_n(\kappa,\xi) = h_n(\kappa)\frac{\xi^{n\alpha}}{\Gamma(1,\alpha,n)} \text{ and } \psi_n(\kappa,\xi) = \hat{h}_n(\kappa)\frac{\xi^{n\alpha}}{\Gamma(1,\alpha,n)}.$$

Solve the following:

$$\begin{aligned} \omega_0(\kappa,\xi) &= 5 - 0.2\coth(0.1\kappa + 1), \\ \omega_1(\kappa,\xi) &= \mathcal{N}_\omega(\omega_0,\psi_0)(\kappa,\xi), \\ \omega_2(\kappa,\xi) &= \mathcal{N}_\omega(\omega_0 + \omega_1, \psi_0 + \psi_1)(\kappa,\xi) - \mathcal{N}_\omega(\omega_0,\psi_0)(\kappa,\xi), \\ &\vdots \end{aligned} \tag{5.8}$$

and

$$
\begin{aligned}
&\psi_0(\kappa,\xi) = -0.02\ \text{csch}(0.1\kappa+1),\\
&\psi_1(\kappa,\xi) = \mathcal{N}_\psi(\omega_0,\psi_0)(\kappa,\xi),\\
&\psi_2(\kappa,\xi) = \mathcal{N}_\psi(\omega_0+\omega_1,\psi_0+\psi_1)(\kappa,\xi) - \mathcal{N}_\psi(\omega_0,\psi_0)(\kappa,\xi)\\
&\vdots
\end{aligned}
\tag{5.9}
$$

to get that

$$
\begin{aligned}
&h_1(\kappa) = \Big[0.004\ \coth(0.1\kappa+1) - 0.1\Big]\text{csch}^2(0.1\kappa+1)\\
&-0.002\,\text{csch}(0.1\kappa+1)\coth(0.1\kappa+1)\\
&\hat{h}_1(\kappa) = \Big[0.0004\coth(0.1\kappa+1) - 0.01\Big]\text{csch}(0.1\kappa+1)\coth(0.1\kappa+1)\\
&+0.0008\,\text{csch}^2(0.1\kappa+1)\coth^2(0.1\kappa+1) + 0.0004\text{csch}^4(0.1\kappa+1)\\
&+0.0004\,\text{csch}^3(0.1\kappa+1);\\
&h_2(\kappa) = (0.2\ \coth(0.1\kappa+1)-5)h_1' - 0.02\,\text{csch}^2(0.1\kappa+1)h_1 - \hat{h}_1';\\
&\hat{h}_2(\kappa) = 0.02\,\text{csch}(0.1\kappa+1)h_1' - 0.02\,\text{csch}^2(0.1\kappa+1)\hat{h}_1\\
&-0.008\,\text{csch}(0.1\kappa+1)\coth(0.1\kappa+1)h_1 - (5-0.02\coth(0.1\kappa+1))\hat{h}_1'\\
&\vdots
\end{aligned}
\tag{5.10}
$$

Hence, the approximate solution of (5.1) for three terms is given by

$$
\begin{aligned}
&\omega(\kappa,\xi) = \frac{1}{\Gamma(1,\alpha,0)}[5-0.2\coth(0.1\kappa+1)] + \frac{\xi^\alpha}{\Gamma(1,\alpha,1)}\\
&\times\Big[[0.004\ \coth(0.1\kappa+1)-0.1]\,\text{csch}^2(0.1\kappa+1) - 0.002\,\text{csch}(0.1\kappa+1)\coth(0.1\kappa+1)\Big]\\
&+\frac{\xi^{2\alpha}}{\Gamma(1,\alpha,2)}\Big[(0.2\ \coth(0.1\kappa+1)-5)h_1' - 0.02\,\text{csch}^2(0.1\kappa+1)h_1 - \hat{h}_1'\Big] + \cdots
\end{aligned}
$$

and

$$
\begin{aligned}
&\psi(\kappa,\xi) = \frac{-1}{\Gamma(1,\alpha,0)}0.02\ \text{csch}(0.1\kappa+1) + \frac{\xi^\alpha}{\Gamma(1,\alpha,1)}\\
&\times\Big[[0.0004\coth(0.1\kappa+1)-0.01]\,\text{csch}(0.1\kappa+1)\coth(0.1\kappa+1)\\
&+0.0008\,\text{csch}^2(0.1\kappa+1)\coth^2(0.1\kappa+1) + 0.0004\text{csch}^4(0.1\kappa+1)\\
&+0.0004\,\text{csch}^3(0.1\kappa+1)\Big] + \frac{\xi^{2\alpha}}{\Gamma(1,\alpha,2)}\Big[0.02\,\text{csch}(0.1\kappa+1)h_1' - 0.02\,\text{csch}^2(0.1\kappa+1)\hat{h}_1\\
&-0.008\,\text{csch}(0.1\kappa+1)\coth(0.1\kappa+1)h_1 - (5-0.02\coth(0.1\kappa+1))\hat{h}_1'\Big] + \cdots
\end{aligned}
$$

An overlay of the exact solution for $\alpha = 1$ has been added to Figures 3 and 4 at the same fixed κ values. A brief discussion has also been included in Section 5, immediately after these figures, confirming that the ENIM solutions exhibit the expected dispersive behavior of tsunami shallow water waves and show excellent agreement with the exact profiles.

6. Numerical discussion

In this section, we present a numerical investigation aimed at validating the efficiency and accuracy of the ENIM when applied to the ABC fractional WBKEs, which model tsunami-type shallow water wave dynamics. The objective of this analysis is to demonstrate that the ENIM approximations closely match the corresponding exact solutions when the fractional order is fixed at $\alpha = 1$ and to show that the resulting approximate solutions behave consistently with the tsunami-related physical wave features discussed earlier. The numerical evidence is provided through Table 1 and Figures 2-4, which collectively confirm both the quantitative accuracy and qualitative reliability of the ENIM scheme. Table 1 compares the exact solutions $\omega(\kappa,\xi)$ and $\psi(\kappa,\xi)$ with the ENIM generated approximations at several spatial and temporal points (κ,ξ). The table lists the exact values, the ENIM solutions, and the associated absolute errors, AE_ω and AE_ψ. A detailed examination of the entries shows that the ENIM results are in excellent agreement with the exact solutions: The absolute error in $\omega(\kappa,\xi)$ consistently remains of order 10^{-3}, while the error in $\psi(\kappa,\xi)$ is even smaller, typically between 10^{-6} and 10^{-5}. These error magnitudes remain stable across the tested ranges of κ and ξ, confirming the robustness, accuracy, and convergence of the ENIM approach in the classical case $\alpha = 1$. Figure 2 provides essential physical and mathematical background related to tsunami modeling. Figure $2_{\kappa=2.00}$ shows a real-world depiction of tsunami waves propagating across a shallow water region toward the coast. This visual context motivates the use of the fractional WBKEs by illustrating the type of long-crested, large-scale wave structures the model aims to capture. The figure also highlights how tsunami waves preserve their coherence over large distances while interacting with seabed topography, thus connecting the mathematical formulation to realistic oceanic behavior. Figure $2_{\kappa=4.00}$ offers a graphical representation of the analytical solution form employed in the model, illustrating the influence of the principal parameters. The parameter υ sets the maximum wave height, while the dispersion coefficient β controls the spreading or sharpening of the wave. Positive values of β lead to smoother, more broadened wave shapes, whereas negative values produce steeper, more peaked profiles. Meanwhile, the higher-order dispersion parameter β_0 generates fine oscillatory ripples trailing the primary crest. This subfigure clarifies how each parameter modulates the wave profile and deepens the physical interpretation of the model.

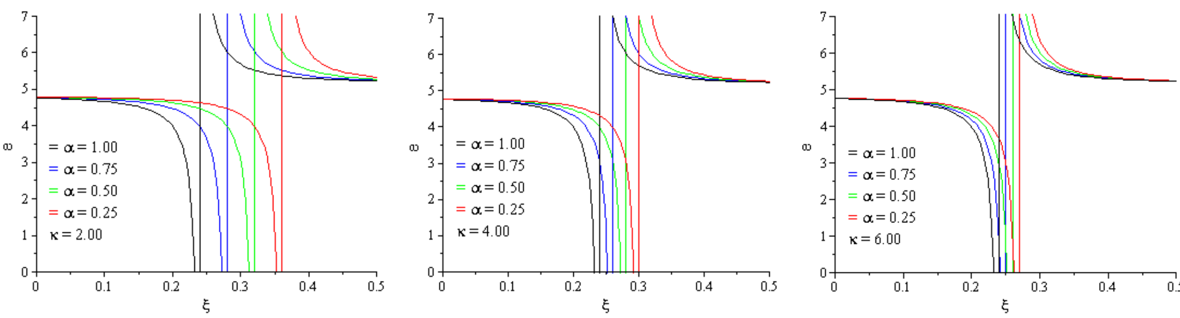


Figure 2. ENIM curves for the solutions $\omega(\kappa,\xi)$ of (5.1).

Table 1. Comparing AEs of exact solution and ENIM solution of (5.1) at $\alpha = 1$.

κ	ξ	$\omega(\kappa,\xi)_{\text{Exact}}$	$\psi(\kappa,\xi)_{\text{Exact}}$	$\omega(\kappa,\xi)_{\text{ENIM}}$	$\psi(\kappa,\xi)_{\text{ENIM}}$	AE_ω	AE_ψ
2.10	0.00	4.76095984	-0.01309206	4.76000765	-0.01309075	0.00095219	0.00000131
	0.20	4.03366004	-0.09454168	4.03285330	-0.09453223	0.00080673	0.00000945
	0.40	5.30376254	0.02286300	5.30270179	0.02286071	0.00106075	0.00000229
	0.60	5.21147001	0.00686991	5.21042772	0.00686922	0.00104229	0.00000069
	0.80	5.20151474	0.00246615	5.20047444	0.00246591	0.00104030	0.00000025
	0.10	4.67249454	-0.02593450	4.67156004	-0.02593191	0.00093450	0.00000259
2.20	0.00	4.76180670	-0.01293679	4.76085434	-0.01293549	0.00095236	0.00000129
	0.20	4.07628953	-0.09017988	4.07547427	-0.09017086	0.00081526	0.00000902
	0.40	5.30641634	0.02321443	5.30535506	0.02321211	0.00106128	0.00000232
	0.60	5.21170851	0.00694298	5.21066616	0.00694228	0.00104234	0.00000069
	0.80	5.20154546	0.00249113	5.20050515	0.00249088	0.00104031	0.00000025
	0.10	4.67580324	-0.02551540	4.67486807	-0.02551284	0.00093516	0.00000255
2.40	0.00	4.76344115	-0.01263332	4.76248847	-0.01263206	0.00095269	0.00000126
	0.20	4.15072777	-0.08253868	4.14989763	-0.08253042	0.00083015	0.00000825
	0.40	5.31197502	0.02394335	5.31091263	0.02394096	0.00106240	0.00000239
	0.60	5.21220091	0.00709171	5.21115847	0.00709100	0.00104244	0.00000071
	0.80	5.20160878	0.00254185	5.20056846	0.00254160	0.00104032	0.00000025
	0.10	4.68210834	-0.02470933	4.68117192	-0.02470686	0.00093642	0.00000247
2.60	0.00	4.76500008	-0.01233895	4.76404708	-0.01233771	0.00095300	0.00000123
	0.20	4.21351351	-0.07606320	4.21267081	-0.07605559	0.00084270	0.00000761
	0.40	5.31789166	0.02470933	5.31682808	0.02470686	0.00106358	0.00000247
	0.60	5.21271467	0.00724398	5.21167213	0.00724326	0.00104254	0.00000072
	0.80	5.20167471	0.00259363	5.20063438	0.00259337	0.00104033	0.00000026
	0.10	4.68802498	-0.02394335	4.68708737	-0.02394096	0.00093760	0.00000239
2.80	0.00	4.76648742	-0.01205327	4.76553413	-0.01205206	0.00095330	0.00000121
	0.20	4.26714446	-0.07050370	4.26629103	-0.07049665	0.00085343	0.00000705
	0.40	5.32419676	0.02551540	5.32313193	0.02551284	0.00106484	0.00000255
	0.60	5.21325076	0.00739992	5.21220811	0.00739918	0.00104265	0.00000074
	0.80	5.20174336	0.00264647	5.20070301	0.00264621	0.00104035	0.00000026
	0.10	4.69358366	-0.02321443	4.69264494	-0.02321211	0.00093872	0.00000232

Figure 3 presents the ENIM-generated temporal curves of the horizontal velocity $\omega(\kappa,\xi)$ at three different spatial positions. Each subfigure corresponds to a fixed value of κ and depicts how ω evolves over time. Figure $3_{\kappa=2.00}$ shows the curve at $\kappa = 2.00$. Here, the velocity exhibits its largest amplitude, consistent with the region near the main body of the wave. The temporal evolution is smooth, representing stable and physically meaningful horizontal motion. Figure $3_{\kappa=4.00}$ illustrates the solution at $\kappa = 4.00$. A clear reduction in amplitude is observed compared to Fig. $3_{\kappa=2.00}$, reflecting the natural attenuation of horizontal velocity as the wave propagates. The profile remains free of numerical artifacts, verifying the stability of the ENIM. Figure $3_{\kappa=6.00}$ shows the curve at $\kappa = 6.00$, farther from the wave core. The amplitude is significantly diminished, capturing the expected spatial decay of ω. The smoothness of the curve confirms the ENIM's ability to model the dispersive weakening of tsunami like waves without introducing spurious oscillations.

Figure 4 displays the ENIM-generated surface elevation $\psi(\kappa,\xi)$ at two spatial positions. Each subfigure represents the time evolution for a fixed κ. Figure $4_{\kappa=2.00}$ corresponds to $\kappa = 2.00$, where the elevation exhibits a relatively larger amplitude, reflecting the influence of the primary wave region. The solution evolves smoothly in time, capturing characteristic shallow water displacement behavior. Figure $4_{\kappa=4.00}$ corresponds to $\kappa = 4.00$. Here, the amplitude decreases, illustrating the expected attenuation of the wave with increasing distance. The curve remains stable and smooth throughout, further demonstrating the numerical reliability of the ENIM. Together, Figures $4_{\kappa=2.00}$ and $4_{\kappa=4.00}$ reveal the progressive spatial decay in surface elevation, consistent with the physical characteristics of long-crested tsunami waves.

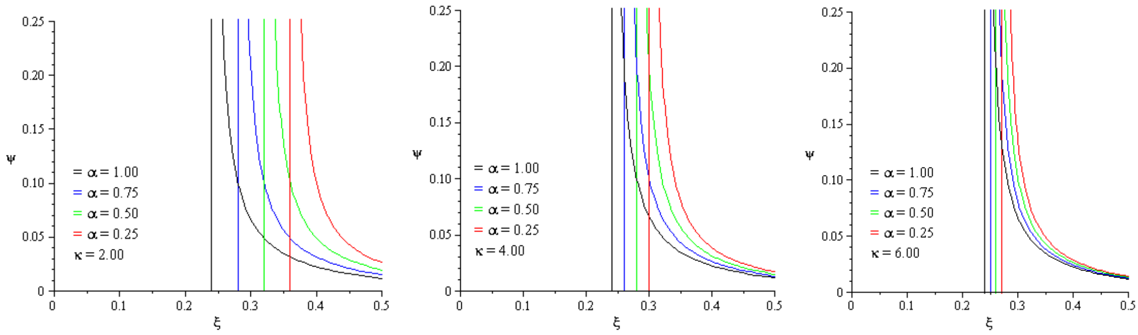


Figure 3. ENIM curves for the solutions $\psi(\kappa, \xi)$ of (5.1).

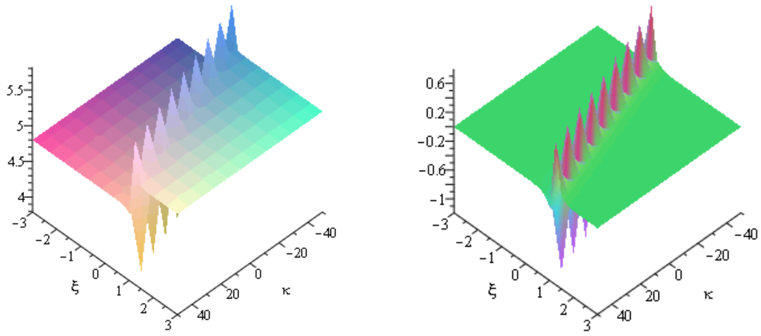


Figure 4. The ENIM surfaces $\omega(\kappa, \xi)$ and $\psi(\kappa, \xi)$, respectively, of (5.1).

Taken as a whole, Table 1 and Figures 2–4 demonstrate that the ENIM-based fractional power series approach is highly accurate, numerically stable, and capable of capturing essential tsunami related shallow water wave dynamics. The extremely small absolute errors confirm the quantitative performance of the method, while the smooth and realistic solution curves validate its qualitative effectiveness. Thus, strong evidence is provided that the ENIM is a reliable and efficient technique for solving the ABC fractional WBKEs. Compared with residual power series and Adomian decomposition methods, the ENIM avoids the computation of Adomian polynomials, leading to lower computational cost. The ENIM also requires fewer iterative terms to achieve accurate results, indicating faster convergence. The close agreement with exact solutions demonstrates that the ENIM maintains high accuracy while being computationally more efficient for the fractional WBKEs.

All operators and their corresponding details used in this work are presented in Table 2.

Table 2. Definitions of functions and operators used in the analysis.

Symbol	Definition
$\omega(\kappa,\xi)$	Exact solution of the fractional WBKEs
$\psi(\kappa,\xi)$	Exact solution of the fractional WBKEs
$\overline{\omega}(\kappa,\xi)$	$-\omega(\kappa,\xi)\omega_\kappa(\kappa,\xi)-\psi_\kappa(\kappa,\xi)-\beta\,\omega_{\kappa\kappa}(\kappa,\xi)$
$\overline{\psi}(\kappa,\xi)$	$-[\omega(\kappa,\xi)\psi(\kappa,\xi)]_\kappa+\beta\psi_{\kappa\kappa}(\kappa,\xi)-\upsilon\beta'\omega_{\kappa\kappa\kappa}(\kappa,\xi)$
$O_\omega(\omega,\psi)$	$-\omega\,\omega_\kappa-\psi_\kappa-\beta\,\omega_{\kappa\kappa}$
$O_\psi(\omega,\psi)$	$-(\omega\psi)_\kappa+\beta\,\psi_{\kappa\kappa}-\nu\beta_0\,\omega_{\kappa\kappa\kappa}$
$\mathcal{J}_\omega(\omega,\psi)$	$h_0(\kappa)+\frac{1-\alpha}{A^\alpha}\overline{\psi}(\kappa,\xi,\omega,\psi)+\frac{\alpha}{\Gamma(\alpha)A^\alpha}\int_0^\xi(\xi-\sigma)^{\alpha-1}\,\overline{\omega}(\kappa,\sigma,\omega,\psi)d\sigma$
$\mathcal{J}_\psi(\omega,\psi)$	$\hat{h}_0(\kappa)+\frac{1-\alpha}{A^\alpha}\overline{\psi}(\kappa,\xi,\omega,\psi)+\frac{\alpha}{\Gamma(\alpha)A^\alpha}\int_0^\xi(\xi-\sigma)^{\alpha-1}\,\overline{\psi}(\kappa,\sigma,\omega,\psi)d\sigma$

7. Conclusions

This work conducted a rigorous analytical and semi-analytical study of the ABC fractional WKBEs formulated with the ABC derivative to model tsunami shallow water dynamics. By employing Banach spaces endowed with the compact-open topology, we established the existence, uniqueness, continuity, and Hyers–Ulam stability of solutions, thereby confirming the mathematical reliability of the fractional WBKE model. Building on this theoretical foundation, we developed the ENIM, a hybrid approach that integrates FPSEs with the NIM to efficiently handle nonlinearities and fractional order effects. The application to tsunami wave propagation demonstrated that the ENIM yields rapidly convergent and highly accurate approximate solutions that remain consistent with classical results when $\alpha = 1$. Overall, the study highlights both the robustness of the ABC fractional WBKEs and the effectiveness of the ENIM as a powerful tool for analyzing nonlinear FPDES.

Author contributions

F.H.D., F.A., A.K.: Formal analysis; F.A., M.F.A.: Funding acquisition; F.H.D., K.M.S., A.S.: Investigation; F.H.D.: Project administration; A.S., A.K. N.A.: Resources; F.H.D., A.S.: Writing-original draft; F.A., M.F.A., K.M.S., A.K., N.A.: Writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

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References

1. R. Ashraf, R. Nawaz, O. Alabdali, N. Fewster–Young, A. H. Ali, F. Ghanim, et al., A new hybrid optimal auxiliary function method for approximate solutions of non–linear fractional partial differential equations, *Fractal Fract.*, **7** (2023), 673. http://dx.doi.org/10.3390/fractalfract7090673
2. F. H. Damag, On comparing analytical and numerical solutions of time Caputo fractional Kawahara equations via some techniques, *Mathematics*, **13** (2025), 2995. http://dx.doi.org/10.3390/math13182995
3. H. Qu, X. Liu, Z. She, Neural network method for fractional–order partial differential equations, *Neurocomputing*, **414** (2020), 225–237. http://dx.doi.org/10.1016/j.neucom.2020.07.025
4. Y. Li, F. Liu, I. W. Turner, T. Li, Time–fractional diffusion equation for signal smoothing, *Appl. Math. Comput.*, **326** (2018), 108–116. http://dx.doi.org/10.1016/j.amc.2017.12.037
5. K. S. Miller, B. Ross, *An introduction to fractional calculus and fractional differential equations*, Wiley, 1993.
6. X. J. Yang, A new integral transform method for solving steady heat–transfer problem, *Therm. Sci.*, **20** (2016), 639–642. http://dx.doi.org/10.2298/TSCI16S3639Y
7. S. T. Mohyud–Din, A. Yildirim, G. Demirli, Traveling wave solutions of Whitham–Broer–Kaup equations by homotopy perturbation method, *J. King Saud Univ. Sci.*, **22** (2010), 173–176. http://dx.doi.org/10.1016/j.jksus.2010.02.001
8. I. Podlubny, *Fractional differential equations*, Academic Press, 1999.
9. F. H. Damag, A. Saif, A. Kilicman, ϕ–Hilfer fractional Cauchy problems with almost sectorial and Lie bracket operators in Banach algebras, *Fractal Fract.*, **8** (2024), 741. http://dx.doi.org/10.3390/fractalfract8120741
10. J. Biazar, H. Aminikhah, Study of convergence of homotopy perturbation method for systems of partial differential equations, *Comput. Math. Appl.*, **58** (2009), 2221–2230. http://dx.doi.org/10.1016/j.camwa.2009.03.034
11. M. Kumar, Numerical solution of singular boundary value problems using advanced Adomian decomposition method, *Eng. Comput.*, **37** (2021), 2853–2863. http://dx.doi.org/10.1108/EC–02–2020–0063
12. F. Xie, Z. Yan, H. Zhang, Explicit and exact traveling wave solutions of Whitham–Broer–Kaup shallow water equations, *Phys. Lett. A*, **285** (2001), 76–80. http://dx.doi.org/10.1016/S0375–9601(01)00086–5
13. J. Ahmad, M. Mushtaq, N. Sajjad, Exact solution of Whitham–Broer–Kaup shallow water wave equations, *J. Sci. Arts*, **15** (2015), 5.

14. S. T. Mohyud–Din, M. A. Noor, Homotopy perturbation method for solving partial differential equations, *Z. Naturforsch. A*, **64** (2009), 157–170.

15. S. Yuzbasi, N. Sahin, Numerical solutions of singularly perturbed one–dimensional parabolic convection–diffusion problems by the Bessel collocation method, *Appl. Math. Comput.*, **220** (2013), 305–315.

16. L. Wang, X. Chen, Approximate analytical solutions of time fractional Whitham–Broer–Kaup equations by a residual power series method, *Entropy*, **17** (2015), 6519–6533.

17. R. Adel, A. M. Abdulqawi, Redefined quintic B–spline collocation method to solve the time–fractional Whitham–Broer–Kaup equations, *Comput. Math. Methods*, 2024. http://dx.doi.org/10.1155/2024/7326616

18. S. M. El–Sayed, D. Kaya, Exact and numerical traveling wave solutions of Whitham–Broer–Kaup equations, *Appl. Math. Comput.*, **167** (2005), 1339–1349. http://dx.doi.org/10.1016/j.amc.2004.08.012

19. H. Yasmin, Numerical analysis of time–fractional Whitham–Broer–Kaup equations with exponential–decay kernel, *Fractal Fract.*, **6** (2022), 142. http://dx.doi.org/10.3390/fractalfract6030142

20. R. Shah, H. Khan, D. Baleanu, Fractional Whitham–Broer–Kaup equations within modified analytical approaches, *Axioms*, **8** (2019), 125. http://dx.doi.org/10.3390/axioms8040125

21. S. Azzh, U. Roman, A. Nehad, S. Rasool, N. Kamsing, Implementation of Yang residual power series method to solve fractional non–linear systems, *AIMS Math.*, **8** (2023), 8294–8309. http://dx.doi.org/10.3934/math.2023418

22. A. Shams, E. Mohamed, A. Abdelgabar, H. Walid, Numerical solutions of time–fractional Whitham–Broer–Kaup equations via Sumudu decomposition method, *J. Math.*, 2023. http://dx.doi.org/10.1155/2023/4664866

23. A. Ali, K. Shah, R. A. Khan, Numerical treatment for traveling wave solutions of fractional Whitham–Broer–Kaup equations, *Alex. Eng. J.*, **57** (2018), 1991–1998.

24. H. Xu, W. Cheng, J. Cui, Multiple–soliton and periodic solutions to space–time fractional Whitham–Broer–Kaup equations, *Eur. Phys. J. Spec. Top.*, **231** (2022), 2353–2357.

25. H. Sirajul, M. Ishaq, Solution of coupled Whitham–Broer–Kaup equations using optimal homotopy asymptotic method, *Ocean Eng.*, **84** (2014), 81–88.

26. S. Ahmad, A. Ullah, A. Akgul, M. De la Sen, A novel homotopy perturbation method with applications to nonlinear fractional–order KdV and Burger equation with exponential–decay kernel, *J. Funct. Space.*, 2021.

27. B. A. Kupershmidt, Mathematics of dispersive water waves, *Commun. Math. Phys.*, **99** (1985), 51–73. http://dx.doi.org/10.1007/BF01204133

28. A. Akgul, A. Cordero, J. R. Torregrosa, A fractional Newton method with 2α–order of convergence and its stability, *Appl. Math. Lett.*, **98** (2019), 344–351.

29. A. R. Seadawy, M. Iqbal, D. Lu, Propagation of kink and anti–kink wave solitons for the nonlinear damped modified KdV equation arising in ion–acoustic wave in an unmagnetized collisional dusty plasma, *Physica A*, **544** (2020), 123560.

30. J. H. He, Homotopy perturbation technique, *Comput. Method. Appl. Mech. Eng.*, **178** (1999), 257–262. http://dx.doi.org/10.1016/S0045–7825(98)00390–9

31. T. Yazgan, E. Ilhan, E. Çelik, H. Bulut, On the new hyperbolic wave solutions to Wu–Zhang system models, *Opt. Quantum Electron.*, **54** (2022), 298.

32. F. H. Damag, A. Saif, On solving modified time Caputo fractional Kawahara equations in the framework of Hilbert algebras using the Laplace residual power series method, *Mathematics*, **9** (2025), 301.

33. I. Kadri, H. Gundogdu, Y. M. Modawy, A. Imam, K. A. A. Helal, I. Elshamy, et al., A comparative analysis of the non–linear time fractional Whitham–Broer–Kaup equations under Aboodh decomposition transform, *Eur. J. Pure Appl. Math.*, **18** (2025), 6626–6636.

34. B. Boutarfa, A. Akgul, M. Inc, New approach for the Fornberg–Whitham type equations, *J. Comput. Appl. Math.*, **312** (2017), 13–26.

35. B. R. Sontakke, A. S. Shelke, A. S. Shaikh, Solution of non–linear fractional differential equations by variational iteration method and applications, *Far East J. Math. Sci.*, **1** (2019), 113–129.

36. M. Inc, A. Akgul, A. Kilicman, Explicit solution of telegraph equation-based on reproducing kernel method, *J. Funct. Space. Appl.*, 2012.

37. D. B. Dhaigude, S. B. Kiwne, R. M. Dhaigude, Monotone iterative scheme for weakly coupled system of finite difference reaction–diffusion equations, *Commun. Appl. Anal.*, **12** (2008), 161.

38. S. Kilic, E. Celik, Complex solutions to the higher–order nonlinear Boussinesq type wave equation transform, *Ric. Mat.*, 2022.

39. M. U. Rahman, M. Arfan, Z. Shah, E. Alzahrani, Evolution of fractional mathematical model for drinking under Atangana–Baleanu Caputo derivatives, *Phys. Scr.*, **96** (2021), 115203.

40. M. Nadeem, S. H. Alotaibi, F. M. Alotaibi, Y. Alsayaad, Numerical investigation of two–dimensional fuzzy fractional heat problem with an external source variable, *PLoS One*, **19** (2024), e0304871. http://dx.doi.org/10.1371/journal.pone.0304871

41. A. Kumar, D. Kumar, D. Baleanu, Dynamical and computational analysis of fractional Whitham–Broer–Kaup equations arising in shallow–water waves, *Results Eng.*, 2025, 105392.

42. A. Prakash, H. Kaur, Numerical simulation of coupled fractional–order Whitham–Broer–Kaup equations arising in shallow water with Atangana–Baleanu derivative, *Math. Methods Appl. Sci.*, **46** (2023), 11583–11602.

43. M. Jeelani, A. S. Alnahdi, M. A. Almalahi, M. S. Abdo, H. A. Wahash, M. A. Abdelkawy, Study of the Atangana–Baleanu–Caputo type fractional system with a generalized Mittag–Leffler kernel, *AIMS Math.*, **7** (2022), 2001–2018.

44. S. G. Samko, *Fractional integrals and derivatives: Theory and applications*, 1993.

45. M. Awadalla, Fractional Hermite functions associated with the Atangana–Baleanu Caputo derivative power series solutions, Rodrigues representation, and orthogonality analysis, *AIMS Math.*, **10** (2025), 20586–20605.

46. Gulalai, S. Ahmad, F.A. Rihan, A. Ullah, Q. M. Al–Mdallal, A. Akgul, Nonlinear analysis of a nonlinear modified KdV equation under Atangana–Baleanu Caputo derivative, *AIMS Math.*, **7** (2022), 7847–7865.

47. W. Alfwzan, S. W. Yao, F. M. Allehiany, S. Ahmad, S. Saifullah, M. Inc, Analysis of fractional non–linear tsunami shallow water mathematical model with singular and non singular kernels, *Results Phys.*, **52** (2023), 106707.

48. I. A. Rus, Ulam stabilities of ordinary differential equations in a Banach space, *Carpath. J. Math.*, **26** (2010), 103–107.