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Research article

Non-uniform bounds in normal approximation for descent and inversion

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Abstract: The statistics defined on symmetric permutation groups, particularly descent and inversion, have been extensively investigated due to their wide-ranging applications in areas such as card shuffling and sorting algorithms. Descent and inversion were shown to satisfy the central limit theorem by Tanny (1973) and Bender (1973), respectively. Since then, many mathematicians studied the error bounds associated with these approximations. Uniform bounds were first established by Fulman in 2004, while Chuntee and Neammanee (2013) and Sumritnorrapong et al. (2018) derived the non-uniform bounds. The latest work of non-uniform bounds from Sumritnorrapong et al. (2018) was not practical since their main theorems are valid for large n and z ($n \geq 7.07 \times 10^6$ and $|z| \geq 8\sqrt{3}$ for descent and $n \geq 1.9 \times 10^8$ and $|z| \geq 24$ for inversion). In this paper, we extended the theorem to hold for arbitrary $n \in \mathbb{N}$ and $z \in \mathbb{R}$. Moreover, our constants were sharper than previously seen. The approach in this work was done by combining Stein's method with the exchangeable pair technique.

Keywords: Stein's method; exchangeable pairs technique; descent; inversion; normal approximation; uniform and non-uniform bounds
Mathematics Subject Classification: 60C05, 60F05

1. Introduction

Let S_n denote the symmetric permutation group on $\{1, 2, \ldots, n\}$. The statistics, namely descent, Des, and inversion, Inv, of S_n are defined for a random permutation π in S_n as follows:

$$\mathrm{Des}(\pi) = |\{(i, i+1) \in \{1, \ldots, n-1\} \times \{2, \ldots, n\} : \pi(i) > \pi(i+1)\}|, \text{ and}$$
$$\mathrm{Inv}(\pi) = |\{(i, j) \in \{1, \ldots, n\} \times \{1, \ldots, n\} : \pi(i) > \pi(j)\}|.$$

That is, the descent of π counts the number of adjacent index pairs $(i, i+1)$ such that $\pi(i) > \pi(i+1)$, while the inversion of π counts the number of index pairs (i, j) such that $1 \leq i < j \leq n$ and $\pi(i) > \pi(j)$.

We also note that

$$\mathbb{E}(\text{Des}) = \frac{n-1}{2}, \qquad \text{Var}(\text{Des}) = \frac{n+1}{12},$$
$$\mathbb{E}(\text{Inv}) = \frac{n(n-1)}{4}, \qquad \text{Var}(\text{Inv}) = \frac{n(n-1)(2n+5)}{72}$$

([1], p. 74). We then define the standardized forms of these statistics by

$$U := \frac{\text{Des} - \mathbb{E}(\text{Des})}{\sqrt{\text{Var}(\text{Des})}}, \tag{1.1}$$

$$\text{and } V := \frac{\text{Inv} - \mathbb{E}(\text{Inv})}{\sqrt{\text{Var}(\text{Inv})}}. \tag{1.2}$$

Descents and inversions are also used to study other areas such as the Coxeter group [2], Eulerian number [3], and combinatorics [4–6]. It is necessary to make a normal approximation for these two random variables due to various applications including sorting algorithm analysis.

The remainder of this paper is organized as follows: Section 2 provides a literature review and states the main result. Section 3 reviews Stein's method within the framework of the exchangeable pair approach and describes in detail the construction of the required pair for our analysis. Section 4 presents the proof of the main result, which are refined non-uniform bounds, extending previous results to hold for all $n \geq 4$ and any $z \in \mathbb{R}$. Section 5 illustrates the applicability of our results through application in a sorting algorithm. Section 6 provides a discussion on the applicability of the main result.

2. Literature review and main result

These two statistics were shown to be asymptotically normal in 1973 by Tanny [7] and Bender [8]. In 2004, Fulman [1] applied a theorem of Rinott and Rotar [9] to obtain uniform bounds for these two statistics, achieving an optimal rate of convergence of order $O\left(\frac{1}{\sqrt{n}}\right)$. Following Fulman's approach, Chuntee and Neammanee [10] derived explicit constants of the error bounds, which are 1096 and 5421, and further improved them to 13.42 and 14.24, respectively, by using a technique developed by Neammanee and Rattanawong [11] in 2008. Their results are summarized in Theorem 2.1.

Theorem 2.1. *[10] For $n \geq 2$,*

1) $\sup_{z\in\mathbb{R}} |\mathbb{P}(U \leq z) - \Phi(z)| \leq \frac{13.42}{\sqrt{n}}$,

2) $\sup_{z\in\mathbb{R}} |\mathbb{P}(V \leq z) - \Phi(z)| \leq \frac{14.24}{\sqrt{n}}$,

where Φ is a standard normal distribution function, i.e., $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx$ for real number z.

Moreover, they provided a first version of the non-uniform bound for these approximations, as stated below.

Theorem 2.2. *[10] For $z \in \mathbb{R}$ and $n \geq 2$,*

1) $|\mathbb{P}(U \leq z) - \Phi(z)| \leq \dfrac{C_1}{\sqrt{n}(1+|z|)^3}$,

2) $|\mathbb{P}(V \leq z) - \Phi(z)| \leq \dfrac{C_2}{\sqrt{n}(1+|z|)^3}$,

where C_1 and C_2 are unknown constants independent of n and z.

Subsequently, the unknown constants C_1 and C_2 in Theorem 2.2 were determined by Chuntee and Neammanee [12] in 2017 to be 101,066 and 546,952, respectively. Exponential non-uniform bounds were first established in the same paper as presented in Theorem 2.3 below.

Theorem 2.3. *[12]*

1) $|\mathbb{P}(U \leq z) - \Phi(z)| \leq \dfrac{51.25}{\sqrt{n}} e^{-|z|/4}$ for $z \in \mathbb{R}$ and $n > 10^7$,

2) $|\mathbb{P}(V \leq z) - \Phi(z)| \leq \dfrac{792.71}{\sqrt{n}} e^{-|z|/4}$ for $z \in \mathbb{R}$ and $n > 2.4 \times 10^8$.

Theorem 2.3 was further improved by Sumritnorrapong et al. [13] in 2018, resulting in Theorem 2.4.

Theorem 2.4. *[13]*

1) $|\mathbb{P}(U \leq z) - \Phi(z)| \leq \dfrac{10.980}{\sqrt{n}} e^{-z^2/32}$ for $|z| > 8\sqrt{3}$ and $n > z^6 > 7.07 \times 10^6$,

2) $|\mathbb{P}(V \leq z) - \Phi(z)| \leq \dfrac{93.467}{\sqrt{n}} e^{-z^2/96}$ for $|z| > 24$ and $n > z^6 > 1.91 \times 10^8$.

We observe that, we need a very large n ($n > 10^7$) to apply Theorems 2.3 and 2.4. This limitation motivates us to improve the error bounds and extend their validity to smaller values of n. Since we can calculate the values of $\mathrm{Des}(\pi)$ and $\mathrm{Inv}(\pi)$ for any $\pi \in S_n$, where $n \in \{1, 2, 3\}$, easily, the result in this work is based on the assumption that $n \geq 4$.

Theorem 2.5. *For $z \in \mathbb{R}$ and $n \geq 4$,*

1) $|\mathbb{P}(U \leq z) - \Phi(z)| \leq \dfrac{C_3(z)}{\sqrt{n}} e^{-z^2/30}$,

2) $|\mathbb{P}(V \leq z) - \Phi(z)| \leq \dfrac{C_4(z)}{\sqrt{n}} e^{-z^2/60}$,

where

$$C_3(z) = 2.17 + \frac{19.62}{e^{7z^2/290}} + \frac{3\sqrt{3}}{e^{89z^2/480}}\left(1 + \frac{9|z|}{16}\right) + \frac{3\sqrt{3}}{\sqrt{2\pi}e^{7z^2/15}} + \frac{14.39}{e^{13z^2/480}}(1+|z|),$$
$$C_4(z) = 3.23 + \frac{30.38}{e^{139z^2/27760}} + \frac{3\sqrt{3}}{e^{97z^2/480}}\left(1 + \frac{9|z|}{16}\right) + \frac{3\sqrt{3}}{\sqrt{2\pi}e^{29z^2/60}} + \frac{14.61}{e^{7z^2/780}}(1+|z|).$$

Note that the constants $C_3(z)$ and $C_4(z)$ in Theorem 2.5 are all rounded up. To compare our results with Theorem 2.4, we note that

$$|\mathbb{P}(U \le z) - \Phi(z)| \le \frac{3.54}{\sqrt{n}} e^{-z^2/30} \text{ for } |z| \ge 8\sqrt{3} \text{ for } n \ge 4,$$
$$|\mathbb{P}(V \le z) - \Phi(z)| \le \frac{7}{\sqrt{n}} e^{-z^2/60} \text{ for } |z| \ge 24 \text{ for } n \ge 4.$$

Hence, our results yield sharper constants and are applicable for any $n \ge 4$.

3. Stein's method and exchangeable pairs

Stein's method is another approach for obtaining normal approximations. In this section, we review its key ideas and outline how the exchangeable pair technique can be applied to derive non-uniform bounds for the descent and inversion statistics.

3.1. Stein's method with the exchangeable pairs technique

The central limit theorem provides conditions under which a suitably standardized distribution function can be approximated by the standard normal distribution function Φ. Moreover, the Berry-Esseen theorem gives an explicit error bound for this approximation. We classify such bounds into two types: the uniform and non-uniform bounds. Suppose F is a distribution function. A uniform bound does not depend on x, that is,

$$\sup_{x\in\mathbb{R}} |F(x) - \Phi(x)| \le C,$$

where C is independent of x. In contrast, a non-uniform bound depends on x, that is,

$$|F(x) - \Phi(x)| \le C(x).$$

A non-uniform bound is more suitable when the range of x is known, although it might not provide a uniform guarantee over the entire set of real numbers. In this work, we aim to improve non-uniform bounds on the normal approximation of descent and inversion by Stein's method which was first introduced by Stein [14] in 1972. This method provides a framework for normal approximation without relying on the Fourier technique [15] and can be extended to various distributions, including Poisson [16], gamma [17], beta [18], and Laplace distributions [19]. The method replaces the problem of comparing characteristic functions with the analysis of a differential equation involving an operator that characterizes the target distribution. This equation is commonly referred to as Stein's equation. For a standard normal random variable Z, Stein's equation takes the form

$$f'(w) - wf(w) = h(w) - \mathbb{E}h(Z), \tag{3.1}$$

where h is a real-valued measurable function satisfying $\mathbb{E}|h(Z)| < \infty$, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous piecewise-differentiable function. To investigate the Kolmogorov distance, we fix $z \in \mathbb{R}$ and choose h to be an indicator function $\mathbf{1}_{(-\infty,z]}$, where

$$\mathbf{1}_{(-\infty,z]}(w) = \begin{cases} 1, & \text{if } w \le z, \\ 0, & \text{if } w > z. \end{cases}$$

Consequently, Stein's equation (3.1) becomes

$$f'(w) - wf(w) = \mathbf{1}_{(-\infty,z]}(w) - \Phi(z). \tag{3.2}$$

The solution f_z of (3.2) is given by

$$f_z(w) = \begin{cases} \sqrt{2\pi}\, e^{w^2/2}\Phi(w)[1-\Phi(z)], & \text{if } w \le z, \\ \sqrt{2\pi}\, e^{w^2/2}\Phi(z)[1-\Phi(w)], & \text{if } w > z \end{cases} \tag{3.3}$$

([20], p. 14). In particular,

$$f_z'(w) = \begin{cases} (1-\Phi(z))\left[1 + \sqrt{2\pi} w e^{w^2/2}\Phi(w)\right], & \text{if } w \le z, \\ \Phi(z)\left[-1 + \sqrt{2\pi} w e^{w^2/2}(1-\Phi(w))\right], & \text{if } w > z. \end{cases} \tag{3.4}$$

This solution has many properties that will be used in this work, such as

$$|f_z'(w_1) - f_z'(w_2)| \le 1, \text{ for all } w_1, w_2 \in \mathbb{R}, \tag{3.5}$$

$$|f_z'(w)| \le 1, \text{ for all } w \in \mathbb{R} \tag{3.6}$$

([20], p. 16). In particular, for $z, \epsilon > 0$, Sumritnorrapong et al. ([13], p. 284) showed that

$$|f_z'(w)|^2 \le \begin{cases} \dfrac{e^{-z^2}}{2\pi z^2}, & \text{if } w \le 0, \\ \dfrac{e^{-z^2}}{\pi z^2} + 2e^{z^2(1/(1+\epsilon)^2-1)}, & \text{if } 0 < w \le \dfrac{z}{1+\epsilon}, \\ 1, & \text{if } w > \dfrac{z}{1+\epsilon}. \end{cases} \tag{3.7}$$

By taking an expectation of (3.2), we obtain

$$|\mathbb{P}(W \le z) - \Phi(z)| = |\mathbb{E}f_z'(W) - \mathbb{E}Wf_z(W)|. \tag{3.8}$$

This allows us to bound $|\mathbb{E}f_z'(W) - \mathbb{E}Wf_z(W)|$ instead of $|\mathbb{P}(W \le z) - \Phi(z)|$, which is the main idea of Stein's method.

In order to bound $|\mathbb{E}f_z'(W) - \mathbb{E}Wf_z(W)|$, there are three main coupling techniques including exchangeable pair ([20], p. 21), zero bias transformation ([20], p. 26), and size biasing ([20], p. 31). In this work, we employ an exchangeable pair technique. A pair (W, W') of random variables is called an exchangeable pair if $(W, W') \overset{d}{=} (W', W)$, where $X \overset{d}{=} Y$ denotes that X and Y have the same distribution. This also implies that W and W' are identically distributed. An exchangeable pair (W, W') is said to be a λ-Stein if it satisfies the linear regression condition

$$\mathbb{E}(W - W'|W) = \lambda W \tag{3.9}$$

for some $\lambda \in (0, 1)$. Moreover, if (W, W') is a $\lambda-$Stein, then it follows that

$$\mathbb{E}Wf(W) = \mathbb{E}\int_{-\infty}^{\infty} f'(W+t)K(t)dt, \tag{3.10}$$

where

$$K(t) = \frac{1}{2\lambda}(W' - W)\{\mathbf{1}(0 \le t \le W' - W) - \mathbf{1}(W' - W \le t \le 0)\} \tag{3.11}$$

([20], p. 22). From (3.11), it should be noted that

$$\int_{-\infty}^{\infty} K(t)\,dt = \frac{|W - W'|^2}{2\lambda}. \tag{3.12}$$

In the case where we can construct a random variable W' such that the pair (W, W') forms a λ-Stein exchangeable pair, then by using (3.8) and (3.10), we have

$$\begin{aligned} |\mathbb{P}(W \le z) - \Phi(z)| &= \left|\mathbb{E}f_z'(W) - \mathbb{E}\int_{-\infty}^{\infty} f_z'(W+t)K(t)dt\right| \\ &\le \left|\mathbb{E}f_z'(W) - \mathbb{E}f_z'(W)\int_{-\infty}^{\infty} K(t)dt\right| \\ &\quad + \left|\mathbb{E}\int_{-\infty}^{\infty}\{f_z'(W) - f_z'(W+t)\}K(t)dt\right| \\ &:= J_1 + J_2. \end{aligned} \tag{3.13}$$

Hence, we can bound J_1 and J_2 instead of $|\mathbb{E}f_z'(W) - \mathbb{E}Wf(W)|$. This technique is applied by many researchers (for examples, see [9, 10, 12, 13, 21]). From (3.12), we see that

$$\begin{aligned} J_1 &= \left|\mathbb{E}f_z'(W)\left(1 - \int_{-\infty}^{\infty} K(t)\,dt\right)\right| \\ &= \left|\mathbb{E}f_z'(W)\left(1 - \frac{1}{2\lambda}\mathbb{E}\left[(W - W')^2|W\right]\right)\right|. \end{aligned} \tag{3.14}$$

3.2. Exchangeable pairs for descent and inversion

In this section, we review exchangeable pairs for U and V that were constructed by Fulman [1] in 2004. Let I be a uniform random variable on $\{1, 2, \dots, n\}$; i.e., $\mathbb{P}(I = i) = \frac{1}{n}$ for $i = 1, 2, \dots, n$. For random permutation π on S_n, we define

$$\pi'(i) = \begin{cases} \pi(i), & \text{if } i \notin \{I, I+1, \dots, n\}, \\ \pi(i+1), & \text{if } i \in \{I, I+1, \dots, n-1\}, \\ \pi(I), & \text{if } i = n, \end{cases}$$

$U'(\pi) = U(\pi')$, and $V'(\pi) = V(\pi')$. Then, it follows that (U, U') and (V, V') are exchangeable pairs ([1] p. 71). Fulman [1] also showed that (U, U') and (V, V') are λ-Stein pairs with the same value of $\lambda = \frac{2}{n}$. Consequently, (3.10)–(3.14) are valid when $W = U$ or $W = V$, with $K = K_1$ or K_2, respectively, where

$$K_1(t) = \frac{n}{4}(U' - U)\{\mathbf{1}(0 \le t \le U' - U) - \mathbf{1}(U' - U \le t \le 0\}, \tag{3.15}$$

$$K_2(t) = \frac{n}{4}(V' - V)\{\mathbf{1}(0 \le t \le V' - V) - \mathbf{1}(V' - V \le t \le 0\}, \tag{3.16}$$

respectively. In the proof of the main results in Section 4, we define $\delta_U = |U - U'|$ and $\delta_V = |V - V'|$. We also know that

$$\delta_U \leq \frac{2\sqrt{3}}{\sqrt{n}}, \tag{3.17}$$

$$\delta_V \leq \frac{6}{\sqrt{n}} \tag{3.18}$$

([1], p. 75).

4. Proof of the main result

In this section, we give the proof of our main results for descent and inversion. Almost all of them are done for descent while inversion can be proved similarly. The main ideas that make Theorem 2.5 hold for all $n \geq 4$ and have a sharper constant come from a sharper bound for the moment in Lemma 4.1 and the fact that we make a bound on four separate cases.

4.1. Auxiliary results

For the proof of the main theorem, we will make a bound only for the case of $z \geq 0$. This can be done by the symmetry of Des and Inv, which can be seen in the following lemma.

Proposition 4.1. *$U \stackrel{d}{=} -U$ and $V \stackrel{d}{=} -V$.*

Proof. First, we will show that

$$\mathbb{P}(\text{Des} = k) = \mathbb{P}(\text{Des} = n - 1 - k) \text{ for } k = 0, 1, 2, \ldots, n - 1. \tag{4.1}$$

To prove (4.1), it suffices to show that there exists a bijection between $A = \{\pi \in S_n | \text{Des}(\pi) = k\}$ and $B = \{\pi \in S_n | \text{Des}(\pi) = n - 1 - k\}$. For $\pi \in S_n$, let $\tilde{\pi}$ be defined by $\tilde{\pi}(i) = n - \pi(i) + 1$ for any $i = 1, 2, \ldots, n$. Then, $\tilde{\pi} \in S_n$ and $\text{Des}(\pi) = k$ if and only if $\text{Des}(\tilde{\pi}) = n-1-k$. Let $g : A \to B$ be defined by $g(\pi) = \tilde{\pi}$. Then, g is a bijection from A onto B. Hence, (4.1) is true. Note that $\text{Im}\left(\text{Des} - \frac{n-1}{2}\right) = \text{Im}\left(\frac{n-1}{2} - \text{Des}\right) = \left\{k - \frac{n-1}{2} \middle| k = 0, 1, 2, \ldots, n - 1\right\}$ and for $k = 0, 1, 2, \ldots, n - 1$,

$$\begin{aligned}
\mathbb{P}\left(\text{Des} - \frac{n-1}{2} = k - \frac{n-1}{2}\right) &= \mathbb{P}(\text{Des} = k) \\
&= \mathbb{P}(\text{Des} = n - 1 - k) \\
&= \mathbb{P}(-\text{Des} = k - (n - 1)) \\
&= \mathbb{P}\left(-\text{Des} + \frac{n-1}{2} = k - \frac{n-1}{2}\right),
\end{aligned}$$

where we have used (4.1) in the second equality. Hence, $U \stackrel{d}{=} -U$. In the case of V, we use the same argument with the facts that $\text{Im}(\text{Inv}) = \left\{0, 1, 2, \ldots, \frac{n(n-1)}{2}\right\}$ and $\mathbb{E}(\text{Inv}) = \frac{n(n-1)}{4}$. □

To prove the main theorem, we utilize the Markov inequality and obtain the terms $\mathbb{E}e^{sU}$ and $\mathbb{E}e^{sV}$ for $s > 0$. To achieve an exponential bound, bounding $\mathbb{E}e^{sU}$ and $\mathbb{E}e^{sV}$ in the following lemma plays a crucial role. In fact, in 2017, Chuntee and Neammanee (Lemma 3.1 in [12]) give bounds for $\mathbb{E}e^{sU}$ and $\mathbb{E}e^{sV}$ for $n \geq 12$ and $n \geq 36$, respectively. The main theorem of [13] also required the condition $n \geq z^6$. In Lemma 4.1, we improve Lemma 3.1 of [12] by providing upper bounds for $\mathbb{E}e^{sU}$ and $\mathbb{E}e^{sV}$ that hold for all positive numbers s and $n \in \mathbb{N}$. The properties of an exchangeable pair with the K function are helpful to prove the result.

Lemma 4.2. *For $n \in \mathbb{N}$ and $s > 0$,*

1) $\mathbb{E}e^{sU} \leq e^{\frac{3}{2}e^{2\sqrt{3}s/\sqrt{n}}s^2}$,

2) $\mathbb{E}e^{sV} \leq e^{\frac{9}{2}e^{6s/\sqrt{n}}s^2}$.

Proof. (1) Let $s > 0$ and $h : (0, \infty) \to \mathbb{R}$ be defined by $h(\omega) = \mathbb{E}e^{\omega U}$ for $\omega > 0$. In [12], p. 1223, Chuntee and Neammanee showed that $h'(\omega) \leq \omega\mathbb{E}\int_{-\infty}^{\infty} e^{\omega(U+\delta_U)}K_1(t)\,dt$. From this fact, (3.17), and (3.12), we have

$$\begin{aligned} h'(\omega) &\leq \omega e^{2\sqrt{3}\omega/\sqrt{n}}\mathbb{E}e^{\omega U}\int_{-\infty}^{\infty} K_1(t)\,dt \\ &= \frac{\omega}{2\lambda}e^{2\sqrt{3}\omega/\sqrt{n}}\mathbb{E}e^{\omega U}\delta_U^2 \\ &\leq 3e^{2\sqrt{3}\omega/\sqrt{n}}\omega h(\omega). \end{aligned}$$

Hence, $\dfrac{h'(\omega)}{h(\omega)} \leq 3\omega e^{2\sqrt{3}\omega/\sqrt{n}}$. This implies

$$\begin{aligned} \int_0^s \frac{h'(\omega)}{h(\omega)}\,d\omega &\leq 3\int_0^s e^{2\sqrt{3}\omega/\sqrt{n}}\omega\,d\omega \\ \ln h(s) &\leq 3e^{2\sqrt{3}s/\sqrt{n}}\int_0^s \omega\,d\omega \\ h(s) &\leq e^{\frac{3}{2}e^{2\sqrt{3}s/\sqrt{n}}s^2}. \end{aligned}$$

(2) We can follow the same argument as in (1) by using (3.18) instead of (3.17). □

It should be noted that this bound for the moment generating function can be improved by making an explicit integration. Moreover, the bound of the moment generating function is optimal in the form of $e^{\alpha_1 e^{\alpha_2/\sqrt{n}}}$ by the fact that an absolute difference of descent and inversion with their exchangeable pairs are bounded by $\dfrac{\beta}{\sqrt{n}}$. In addition, we need bounds for $\mathbb{E}|f_z'(U)|^2$ and $\mathbb{E}|f_z'(V)|^2$ to prove the main theorem. The main techniques in the proof are the Markov inequality and a truncation technique.

Lemma 4.3. *For $n \geq 4$ and $1 \leq |z| \leq 0.5\sqrt{n}$, we have*

1) $\mathbb{E}|f_z'(U)|^2 \leq \dfrac{2.93}{\sqrt{n}}e^{-z^2/15}$,

2) $\mathbb{E}|f_z'(V)|^2 \leq \frac{2.89}{\sqrt{n}} e^{-z^2/30}$.

Proof. (1) For $\alpha > 0$, by applying Markov's inequality, Lemma 3.2 (1) with $s = \alpha|z|$, and the fact that $|z| \leq 0.5\sqrt{n}$, we obtain

$$\begin{aligned}\mathbb{P}(U > 0.9|z|) &= \mathbb{P}\left(e^{\alpha|z|U} > e^{0.9\alpha z^2}\right)\\ &\leq e^{-0.9\alpha z^2}\mathbb{E}e^{\alpha|z|U}\\ &\leq e^{z^2\left(\frac{3}{2}\alpha^2 e^{\sqrt{3}\alpha} - 0.9\alpha\right)}.\end{aligned}$$

By numerical optimization, we choose $\alpha = 0.186832$, and so, $\frac{3}{2}\alpha^2 e^{\sqrt{3}\alpha} - 0.9\alpha$ attains its minimum value of $-\frac{1}{10.5}$, which yields

$$\mathbb{P}(U > 0.9|z|) \leq e^{-z^2/10.5}. \tag{4.2}$$

It should be noted that this type of optimization will occur frequently in the proof. Now, we apply (3.7) with $\epsilon = \frac{1}{9}$ together with (4.2) and obtain that

$$\begin{aligned}\mathbb{E}|f_z'(U)|^2 &\leq \mathbb{E}|f_z'(U)|^2\mathbf{1}(U \leq 0) + \mathbb{E}|f_z'(U)|^2\mathbf{1}(0 < U \leq 0.9|z|) + \mathbb{E}|f_z'(U)|^2\mathbf{1}(U > 0.9|z|)\\ &\leq \frac{3e^{-z^2}}{2\pi z^2} + 2e^{-z^2/5.3} + e^{-z^2/10.5}.\end{aligned}$$

For $|z| \geq 1$, it follows that

$$\begin{aligned}\frac{3e^{-z^2}}{2\pi z^2} + 2e^{-z^2/5.3} + e^{-z^2/10.5} &\leq \frac{3e^{-z^2/15}}{2\pi z^2 e^{14z^2/15}} + \frac{2}{e^{97z^2/795}}e^{-z^2/15} + \frac{1}{e^{z^2/35}}e^{-z^2/15}\\ &\leq 2.93e^{-z^2/15}.\end{aligned}$$

Hence, $\mathbb{E}|f_z'(U)|^2 \leq 2.93e^{-z^2/15}$ for $1 \leq z \leq 0.5\sqrt{n}$.
(2) This can be completed by a similar argument as in (1). □

4.2. *Proof of the main theorem*

From the symmetry properties of U and V described in Proposition 4.1, it suffices to prove the main theorem for the case $z \geq 0$. In [13], Sumritnorrapong et al. obtained an upper bound for $\mathbb{E}e^{sU}$ under the condition $\frac{2\sqrt{3}}{\sqrt{n}} s \leq 1.256$. As a consequence, in the proof of their main theorem, they require both $n^{1/6} \geq z$ and $|z| \geq 8\sqrt{3}$ in order to achieve an error term of order $O(n^{-1/2})$. These constraints imply $n \geq z^6 \geq (8\sqrt{3})^6 = 7{,}077{,}888$. In contrast, our work relaxes the condition $\frac{2\sqrt{3}}{\sqrt{n}} s \leq 1.256$ (Lemma 4.2), which removes the restriction $z \geq 8\sqrt{3}$ and allows us to prove the main theorem for the regime $z \leq c\sqrt{n}$. Since the descent statistic satisfies $|U| \leq \sqrt{3n}$, we further partition the interval $0 \leq z \leq \sqrt{3n}$ into three subranges. For Case 1 ($0 \leq z \leq 1$), this avoids the singular term $1/z$ appearing in the bound. For $1 \leq z \leq \sqrt{3n}$, we use Lemmas 4.2 and 4.3 as key tools. To achieve smaller constants in the bound, we further separate the range into two cases (Case 2 for $1 \leq z \leq 0.5\sqrt{n}$ and Case 3 for $0.5\sqrt{n} \leq z \leq \sqrt{3n}$). For the remaining $z \geq \sqrt{3n}$, the proof proceeds by a different approach, as detailed in Case 4.

Case 1. $0 \leq z \leq 1$. By Theorem 2.1 (1), it follows that

$$|P(U \leq z) - \Phi(z)| \leq \frac{13.42e^{1/14.4}}{\sqrt{n}} e^{-z^2/14.4} = \frac{14.39}{\sqrt{n}} e^{-z^2/14.4}. \tag{4.3}$$

Case 2. $1 \leq z \leq 0.5\sqrt{n}$. In this case, we will improve the results from Sumritnorrapong et al. [13] by using Lemma 4.1. From (3.13), it suffices to bound J_1 and J_2. In 2015, Chuntee and Neammanee ([10], p. 2317) showed that

$$\mathbb{E}\left(1 - \frac{1}{2\tau}\mathbb{E}[\delta_U^2|U]\right)^2 = \frac{1.6}{n+1}. \tag{4.4}$$

From this fact, Lemma 4.1 (1), (3.6), and (3.14), the term J_1 can be bounded as

$$J_1 \leq \sqrt{\mathbb{E}|f_z'(U)|^2}\sqrt{\mathbb{E}\left(1 - \frac{1}{2\tau}\mathbb{E}[\delta_U^2|U]\right)^2} \leq \frac{2.17}{\sqrt{n}} e^{-z^2/30}. \tag{4.5}$$

In 2001, Chen ([22], p. 250) showed that

$$\left| f_z'(w) - f_z'(w+t) - \int_t^0 h(w+u)\,du \right| \leq \mathbf{1}(z - \max(0,t) < w < z - \min(0,t)), \tag{4.6}$$

where

$$h(w) = (wf_z(w))' = \begin{cases} (\sqrt{2\pi}(1+w^2)e^{w^2/2}(1-\Phi(w)) - w)\Phi(z), & \text{if } w \geq z, \\ (\sqrt{2\pi}(1+w^2)e^{w^2/2}\Phi(w) + w)(1-\Phi(z)), & \text{if } w < z. \end{cases}$$

Hence,

$$J_2 \leq J_{21} + J_{22},$$

where

$$J_{21} = \mathbb{E}\int_{-\infty}^{\infty} \mathbf{1}\,(z - \max(0,t) < U < z + \min(0,t))\, K_1(t)\,dt, \quad \text{and}$$
$$J_{22} = \mathbb{E}\int_{-\infty}^{\infty}\int_t^0 h(U+u)K_1(t)\,du\,dt.$$

Sumritnorrapong ((32) and (35) of [13], p. 285) showed that

$$J_{21} \leq \frac{4e^{4\sqrt{3}\alpha z/\sqrt{n}}}{e^{\alpha z^2}}\mathbb{E}|U|\delta_U e^{\alpha z U}. \tag{4.7}$$

From Lemma 4.1 (1) and the fact that $|z| \leq 0.5\sqrt{n}$, we have

$$\mathbb{E}e^{2\alpha z U} \leq e^{6\alpha^2 e^{2\sqrt{3}\alpha} z^2}.$$

Hence,

$$
\begin{aligned}
J_{21} &\le \frac{8\sqrt{3}}{\sqrt{n}}\frac{e^{2\sqrt{3}\alpha}}{e^{\alpha z^2}}\sqrt{\mathbb{E}U^2}\sqrt{\mathbb{E}e^{2\alpha z U}}\\
&\le \frac{8\sqrt{3}}{\sqrt{n}}\frac{e^{2\sqrt{3}\alpha}}{e^{\alpha z^2}}\sqrt{\mathbb{E}e^{2\alpha z U}}\\
&\le \frac{8\sqrt{3}}{\sqrt{n}}e^{(3\alpha^2 e^{2\sqrt{3}\alpha}-\alpha)z^2+2\sqrt{3}\alpha}.
\end{aligned}
$$

By choosing $\alpha^* = 0.100313$, which minimizes $3\alpha^2 e^{2\sqrt{3}\alpha} - \alpha$, we obtain

$$
J_{21} \le \frac{19.62}{\sqrt{n}}e^{-z^2/17.4}. \tag{4.8}
$$

By a similar argument as in pp. 285–286 of [13], we can show that

$$
J_{22} \le \frac{3\sqrt{3}}{\sqrt{n}}\left[\frac{1}{z}\left(1+\frac{9z^2}{16}\right)e^{-7z^2/32}+\frac{e^{-z^2/2}}{\sqrt{2\pi}}\right]+\frac{3.003\sqrt{3}}{\sqrt{n}}(1+z)\mathbb{P}\left(U>\frac{3z}{4}-\frac{2\sqrt{3}}{\sqrt{n}}\right) \tag{4.9}
$$

for all $\alpha > 0$. To complete the proof in this case, it remains to bound $\mathbb{P}\left(U>\frac{3z}{4}-\frac{2\sqrt{3}}{\sqrt{n}}\right)$. Using the fact that $z \le 0.5\sqrt{n}$ and applying an argument similar to that used in (4.2), we obtain

$$
\mathbb{P}\left(U>\frac{3z}{4}-\frac{2\sqrt{3}}{\sqrt{n}}\right) \le 1.33e^{-z^2/14.4}. \tag{4.10}
$$

By (4.9) and (4.10), it follows that

$$
\begin{aligned}
J_{22} &\le \frac{3\sqrt{3}}{\sqrt{n}}\left[\frac{1}{z}\left(1+\frac{9z^2}{16}\right)e^{-7z^2/32}+\frac{e^{-z^2/2}}{\sqrt{2\pi}}\right]+\frac{6.92}{\sqrt{n}}(1+z)e^{-z^2/14.4}\\
&\le \frac{3\sqrt{3}}{\sqrt{n}}\left[\left(1+\frac{9z}{16}\right)e^{-7z^2/32}+\frac{e^{-z^2/2}}{\sqrt{2\pi}}\right]+\frac{6.92}{\sqrt{n}}(1+z)e^{-z^2/14.4}.
\end{aligned} \tag{4.11}
$$

Therefore, by (4.5), (4.8), and (4.11),

$$
\begin{aligned}
|\mathbb{P}(U\le z)-\Phi(z)| \le{}& \frac{2.17}{\sqrt{n}}e^{-z^2/30}+\frac{19.62}{\sqrt{n}}e^{-z^2/17.4}+\frac{3\sqrt{3}}{\sqrt{n}}\left(1+\frac{9z}{16}\right)e^{-7z^2/32}\\
&+\frac{3\sqrt{3}}{\sqrt{2\pi n}}e^{-z^2/2}+\frac{14.39}{\sqrt{n}}(1+z)e^{-z^2/14.4}
\end{aligned} \tag{4.12}
$$

for any $1 \le z \le 0.5\sqrt{n}$.

Case 3. $0.5\sqrt{n} \le z \le \sqrt{3n}$**.** Note that

$$
|P(U\le z)-\Phi(z)| = |1-P(U>z)-(1-\Phi(-z))|
$$

$$\leq \max\{\Phi(-z), P(U > z)\}. \tag{4.13}$$

By using the Gaussian tail bound inequality ([20], p. 38), we have

$$\Phi(-z) = 1 - \Phi(z) \leq \frac{1}{\sqrt{2\pi}z} e^{-z^2/2} \leq \frac{2}{\sqrt{2\pi n}} e^{-z^2/2}. \tag{4.14}$$

By applying Markov's inequality, Lemma 4.1 (1) with $s = \alpha z$ for $\alpha > 0$, and the fact that $z \leq \sqrt{3n}$ in the last inequality, we obtain that

$$\mathbb{P}(U > z) = \mathbb{P}(e^{sU} > e^{sz}) \leq \frac{\mathbb{E}e^{sU}}{e^{sz}} \leq e^{\frac{3}{2}\alpha^2 z^2 e^{6\alpha} - \alpha z^2} =: e^{g(\alpha) z^2},$$

where $g(\alpha) = \frac{3}{2}\alpha^2 e^{6\alpha} - \alpha$. Since $g(\alpha)$ attains its minimum at $\alpha^* = 0.119648$, we have

$$\mathbb{P}(U > z) \leq e^{g(\alpha^*) z^2} = e^{-z^2/13.2} \leq \frac{6.34}{\sqrt{n}} e^{-z^2/17.4}. \tag{4.15}$$

Thus,

$$|\mathbb{P}(U \leq z) - \Phi(z)| \leq \frac{6.34}{\sqrt{n}} e^{-z^2/17.4}. \tag{4.16}$$

Case 4. $z > \sqrt{3n}$**.** Since $0 \leq \mathrm{Des}(\pi) \leq n-1$ for any $\pi \in S_n$ and by (1.1), it follows that $\mathbb{P}(U \leq z) = 1$. Hence,

$$|\mathbb{P}(U \leq z) - \Phi(z)| = 1 - \Phi(z) \leq \frac{1}{\sqrt{2\pi}z} e^{-z^2/2} \leq \frac{1}{\sqrt{6\pi n}} e^{-z^2/2}. \tag{4.17}$$

By comparing (4.3), (4.12), (4.16), and (4.17), for any $z \geq 0$,

$$\begin{aligned}
|P(U \leq z) - \Phi(z)| &\leq \frac{2.17}{\sqrt{n}} e^{-z^2/30} + \frac{19.62}{\sqrt{n}} e^{-z^2/17.4} + \frac{3\sqrt{3}}{\sqrt{n}} \left[\left(1 + \frac{9z}{16}\right) e^{-7z^2/32} + \frac{e^{-z^2/2}}{\sqrt{2\pi}} \right] \\
&\quad + \frac{14.39}{\sqrt{n}} (1 + |z|) e^{-z^2/14.4} \\
&\leq \frac{c_1(z)}{\sqrt{n}} e^{-z^2/30},
\end{aligned} \tag{4.18}$$

where $c_1(z) = 2.17 + \frac{19.62}{e^{7z^2/290}} + \frac{3\sqrt{3}}{e^{89z^2/480}}\left(1 + \frac{9|z|}{16}\right) + \frac{3\sqrt{3}}{\sqrt{2\pi} e^{7z^2/15}} + \frac{14.39}{e^{13z^2/480}}(1 + z)$.

Next, we will give the proof for $z < 0$. By the symmetry of descent as can be seen from Proposition 4.1 (1), we have

$$\begin{aligned}
|\mathbb{P}(U \leq z) - \Phi(z)| &= |\mathbb{P}(-U \geq -z) - \Phi(z)| \\
&= |1 - \mathbb{P}(-U < -z) - \Phi(z)| \\
&= |\mathbb{P}(U < -z) - \Phi(-z)|.
\end{aligned} \tag{4.19}$$

If $-z \notin \mathrm{Im}(\mathrm{Des})$, then $\mathbb{P}(U < -z) = \mathbb{P}(U \le -z)$ and we can apply (4.18) immediately. Suppose that $-z \in \mathrm{Im}(\mathrm{Des})$. Thus, $-z = \dfrac{k - \binom{n-1}{2}}{\sqrt{\frac{n+1}{12}}}$ for some $k \in \{0, 1, \ldots, n-1\}$. It follows that

$$\mathbb{P}(U < -z) = \mathbb{P}(U \le -z - \epsilon)$$

for any $\epsilon \in \left(0, \sqrt{\dfrac{12}{n+1}}\right)$. By a triangle inequality and (4.18), we have

$$\begin{aligned}
|\mathbb{P}(U \le z) - \Phi(z)| &\le |\mathbb{P}(U \le -z - \epsilon) - \Phi(-z-\epsilon)| + |\Phi(-z-\epsilon) - \Phi(-z)| \\
&\le \frac{C_3(-z-\epsilon)}{\sqrt{n}} e^{-(z+\epsilon)^2/30} + \frac{1}{\sqrt{2\pi}} \int_{-z-\epsilon}^{-z} e^{-t^2/2}\, dt \\
&\le \frac{C_3(-z-\epsilon)}{\sqrt{n}} e^{-(z+\epsilon)^2/30} + \frac{\epsilon}{\sqrt{2\pi}} e^{-(z+\epsilon)^2/2}.
\end{aligned}$$

Since $\epsilon \in (0, 1)$ is arbitrary, by taking $\epsilon \to 0^+$, it follows that

$$|\mathbb{P}(U \le z) - \Phi(z)| = \frac{C_3(-z)}{\sqrt{n}} e^{-z^2/30}. \tag{4.20}$$

From (4.18) and (4.20), the theorem is proved.

Furthermore, if $|z| \ge 8\sqrt{3}$,

$$\begin{aligned}
|P(U \le z) - \Phi(z)| &\le \frac{2.17}{\sqrt{n}} e^{-z^2/30} + \frac{0.19}{\sqrt{n}} e^{-z^2/30} + \frac{1.58 \times 10^{-14}}{\sqrt{n}} e^{-z^2/30} \\
&\quad + \frac{2.54 \times 10^{-39}}{\sqrt{n}} e^{-z^2/30} + \frac{1.18}{\sqrt{n}} e^{-z^2/30} \\
&\le \frac{3.54}{\sqrt{n}} e^{-z^2/30}.
\end{aligned}$$

For Theorem 2.5 (2), we can use the same argument as in the proof of Theorem 2.5 (1).

5. Application

The inversion makes a strong relationship with a sorting algorithm that operates via adjacent swaps. Classic examples include bubble sorting and selection sorting. In this section, we will consider bubble sorting since it repeatedly swaps adjacent elements whenever they are in the wrong order. Each swap removes exactly one inversion. From this reason, its number of swaps is equal to the inversion as can be seen in Section 5.2.2 of [23]. By Theorem 2.5, we can approximate the probability for the number of swaps, which may be used to quantify the risk that an algorithm's runtime exceeds a given threshold. For example, the probability that a bubble sort will sort the permutation with $n = 500$ within $k = 100{,}000$ swap counts is

$$\mathbb{P}(\mathrm{Inv} \le 100{,}000) = \mathbb{P}\left(\frac{\mathrm{Inv} - \mathbb{E}(\mathrm{Inv})}{\sqrt{\mathrm{Var}(\mathrm{Inv})}} \le \frac{100{,}000 - \mathbb{E}(\mathrm{Inv})}{\sqrt{\mathrm{Var}(\mathrm{Inv})}}\right) = \mathbb{P}(V \le 20.162).$$

Note that $\mathbb{E}(\text{Inv}) = \frac{500(500-1)}{4}$ and $\text{Var}(\text{Inv}) = \frac{500(500-1)(2(500)+5)}{72}$. By Theorem 2.5 (2), it follows that $|\mathbb{P}(V \leq 20.162) - \Phi(20.162)| \leq \frac{C_4(20.162)e^{-20.162^2/60}}{\sqrt{500}} = 7.787 \times 10^{-4}$. Hence, the probability of the inversion count is approximated by $\Phi(20.162)$ with an error bound of 7.787×10^{-4}. For further background on the inversion's role in sorting algorithm analysis, see [23].

Figure 1 shows the comparison between the uniform bound (Theorem 2.1 (2)) and non-uniform bound (Theorem 2.5 (2)) for an inversion count. Observe that the error bound is decreasing for $|z| \geq 1.886$. In particular, the value of z such that the error bound of a non-uniform bound is better than the uniform bound (Theorem 2.1) is around $|z| \geq 10.3$ regardless of the choice of n, and we cannot apply Theorem 2.4 in this case.

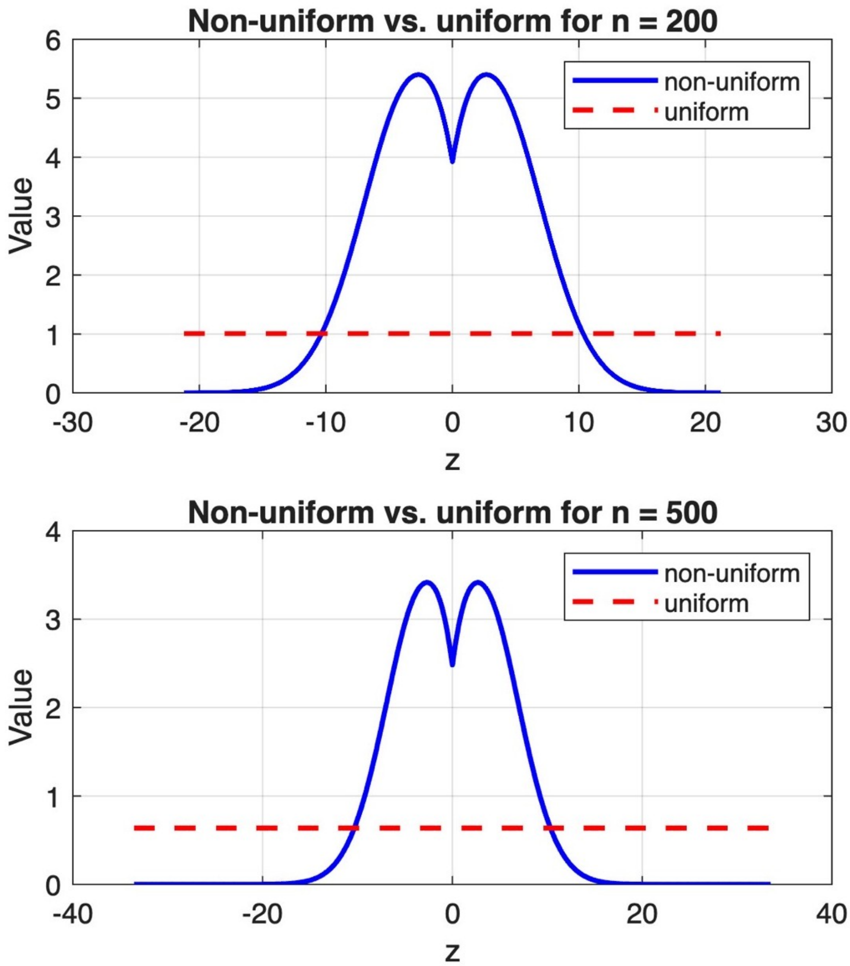

Figure 1. Comparison of non-uniform and uniform bounds for $n = 200$ and $n = 500$.

6. Discussion

For small values of n, we can compute $\mathbb{P}(\text{Des} = k)$ and $\mathbb{P}(\text{Inv} = k)$ exactly by direct numerical simulation, so there is no need to approximate it. But if we wanted to make an approximation, by Theorem 2.5, the results turn out to be worse, as can be seen in Tables 1 and 2.

Table 1. Comparison between an exact error and the approximate error bound from Theorem 2.5 (1) when $k = n - 2$.

n	$\left\|\mathbb{P}\left(U \leq \frac{k-\frac{n-1}{2}}{\sqrt{\frac{n+1}{12}}}\right) - \Phi\left(\frac{k-\frac{n-1}{2}}{\sqrt{\frac{n+1}{12}}}\right)\right\|$	Bound from Theorem 2.5 (1)
5	0.0703	25.53
6	0.0234	23.03
7	0.0070	20.28
8	0.0019	17.62
9	5.05×10^{-4}	15.19
10	1.28×10^{-4}	13.04
11	3.16×10^{-5}	11.15

Table 2. Comparison between an exact error and the approximate error bound from Theorem 2.5 (2) when $k = \dfrac{n(n-1)}{2} - 1$.

n	$\left\|\mathbb{P}\left(V \leq \frac{k-\frac{n(n-1)}{4}}{\sqrt{\frac{n(n-1)(2n+5)}{72}}}\right) - \Phi\left(\frac{k-\frac{n(n-1)}{4}}{\sqrt{\frac{n(n-1)(2n+5)}{72}}}\right)\right\|$	Bound from Theorem 2.5 (2)
5	0.0250	33.62
6	0.0073	31.12
7	0.0022	28.83
8	6.48×10^{-4}	26.79
9	1.96×10^{-4}	24.95
10	6.00×10^{-5}	23.29
11	1.84×10^{-5}	21.78

Observe that if the value of $|z|$ is greater than 6.14 for descents and 10.34 for inversions, as the non-uniform bound is better than the uniform bound, then, by comparing the values, the last term in both $C_3(z)$ and $C_4(z)$ is the most dominant.

Meanwhile, for large z, $C_3(z)$ and $C_4(z)$ are close to 2.17 and 3.23, respectively. For example, $C_3(z) \leq 2.1701$ for $|z| \geq 30$ and $C_4(z) \leq 3.2404$ for $|z| \geq 40$. For $z \geq 15$, the bound satisfies $|\mathbb{P}(U \leq z) - \Phi(z)| \leq \frac{2.78}{\sqrt{n}} e^{-z^2/30} \leq \frac{0.0015}{\sqrt{n}}$ and $|\mathbb{P}(V \leq z) - \Phi(z)| \leq \frac{44.11}{\sqrt{n}} e^{-z^2/60} \leq \frac{1.03}{\sqrt{n}}$. So the users can choose n, which makes the error bounds as small as they need. At the same error bound, Theorem 2.5 provides a smaller n than the previous results.

Author contributions

Natthapol Dejtrakulwongse: Methodology, validation, visualization, writing–original draft, writing–review and editing; Kritsana Neammanee: Conceptualization, methodology, supervision, validation, writing–original draft, writing–review and editing; Suporn Jongpreechaharn: Conceptualization, methodology, supervision, validation, writing–original draft, writing–review and editing. All authors have read and approved the final version of the manuscritp for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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