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Research article

Periodic solutions and asymptotic properties of first order linear nonhomogeneous neutral delay differential equations

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Abstract: This article concerns first-order linear nonhomogeneous neutral delay differential equations with periodic coefficients and constant delays, where the coefficients share a common period and the delays are multiples of this period. First, we obtain periodic solutions of linear nonhomogeneous neutral delay differential equations using the variation of parameters method. These periodic solutions are expressed analytically. Two examples demonstrating the applicability of our results are also included. Second, we investigate the asymptotic behavior and estimation of solutions to linear nonhomogeneous neutral delay differential equations. The results are obtained using an appropriate real root of the relevant characteristic equation. Three examples are given to illustrate our results. Finally, we present the special case of first-order linear nonhomogeneous neutral delay differential equations with constant coefficients and constant delays, and provide an interesting example.

Keywords: neutral delay differential equation; characteristic equation; periodic solution; asymptotic behavior
Mathematics Subject Classification: 34K06, 34K07, 34K13, 34K25, 34K40

1. Introduction and Preliminaries

Neutral delay differential equations (NDDEs) are a type of delay differential equations, characterized by the incorporation of temporal delays within the derivative terms of the state variables. NDDEs demonstrate broad applicability throughout multiple disciplines in scientific and engineering contexts [1, 2]. As an illustration, a previous study [2] introduced a feedback control mechanism designed for system output stabilization. A further implementation was demonstrated in another study [3], which employed a NDDE framework for modeling real-time dynamic substructuring experiments. In particular, these researchers [3] revealed excellent correspondence between theoretical predictions and empirical observations. Another piece of research [4] used linear NDDEs, which provide a more accurate representation of an *Escherichia coli* bacterial population's growth than conventional exponential growth models. Many results on NDDE theory are given in the books by Kolmanovski and Myshkis [5], and Hale and Verduyn Lunel [6].

Inspired by the pioneering work of Frasson and Tacuri [7], the aforementioned authors showed the application of Floquet Theory to NDDEs for the case where the delays are denoted as integer multiples of the common period of the periodic coefficients. Additionally, for the solutions of first-order NDDEs, they obtained asymptotic behaviors. The authors of a previous study [7] investigated the asymptotic behavior of solutions to linear homogeneous equations by calculating the resolvent of the monodromy operator. In this article, we obtain the asymptotic behavior and an estimation of the solutions of a linear nonhomogeneous equation using a suitable real root of the characteristic equation. Further recent outcomes were presented by Philos and Purnaras [8, 9]. The asymptotic behavior of solutions to first-order linear NDDEs with periodic coefficients and constant delays that are multiples of the common period was considered in a previous article [8]. It is worth noting that in this article, we extend the results obtained in the aforementioned article [8] to nonhomogeneous differential equations. The authors of [9] established both lower and upper estimates for the solutions through two admissible, distinct real roots of the pertinent characteristic equation. Presenting a study similar to the one in the article mentioned above [9] for periodic linear nonhomogeneous NDDEs will be the subject of a future study. These articles [7–9] inspired the form of the equation under consideration here.

This paper aims to obtain periodic solutions of a first-order nonhomogeneous linear NDDE, in which the coefficients are periodic and have constant delays, a common period exists for these coefficients, and the delays are multiples of the aforementioned period. Furthermore, utilizing a real root of the characteristic equation corresponding to this (with a suitable property), we obtain the asymptotic behavior for solutions of the given equation and construct an estimate of the solutions. The results given in a previous study [10] extended and improved some important results obtained by Farkas et al. [11] on the periodic solution and asymptotic behavior of a first-order linear (non-neutral) differential equation with a constant delay, as well as periodic coefficients. In addition to this, another important result given in the article in question [10] set an exponential estimate for the solutions. In this article, we extend the results obtained in these articles [10, 11] to nonhomogeneous linear NDDEs. It should be noted that, to the best of the authors' knowledge, periodic solutions of linear nonhomogeneous NDDEs have not been obtained analytically in the existing literature. In summary, recent results for periodic first-order linear (non-neutral) delay differential equations, given by Farkas et al. [11] and by Yeniçerioğlu and Yazıcı [10], can be derived (as a special case) from the results of this paper. The techniques we apply to obtain our results originate from a combination of the methods used in previous work [8–11]. In the article by Li, Jin and Zhang [12], interesting results were obtained regarding the existence of nonoscillatory solutions for a class of higher-order nonlinear differential equations.

In general, the theory of NDDEs presents some additional complexities not found in the corresponding theory of delay differential equations. Therefore, extending the results related to homogeneous delay differential equations to nonhomogeneous NDDEs is not easy.

This article consists of four sections: Section 1 provides an introduction to and preliminaries on first-order linear nonhomogeneous NDDEs containing periodic coefficients and constant delays, where the coefficients share a common period, and the delays are multiples of this period. Section

2 presents periodic solutions of the given equation using the variation of parameters method. These periodic solutions are expressed analytically, and two examples are given. Section 3 demonstrates the asymptotic behavior of the solutions of the given equation and then derives the estimates of the solutions. Three examples are shown at the end of the section. The final section presents a special case of linear nonhomogeneous NDDEs with constant coefficients and constant delays, and provides an example.

Consider the following neutral delay differential equation:

$$\left[u(x) + \sum_{i\in I} d_i u(x-\sigma_i)\right]' + \alpha(x)u(x) + \sum_{i\in I} \beta_i(x)u(x-\tau_i) = h(x), \tag{1.1}$$

where I is the initial segment of natural numbers; d_i for $i \in I$ are real numbers; h, α and β_i for $i \in I$ are continuous real-valued functions on the interval $[0,\infty)$; σ_i for $i \in I$ represents positive real numbers such that $\sigma_{i_1} \neq \sigma_{i_2}$ for $i_1, i_2 \in I$ with $i_1 \neq i_2$; and τ_i for $i \in I$ represents positive real numbers such that $\tau_{i_1} \neq \tau_{i_2}$ for $i_1, i_2 \in I$ with $i_1 \neq i_2$. Assume that at least one of the functions β_i for $i \in I$ is not identically zero on $[0,\infty)$. Moreover, suppose that the coefficients α and β_i for $i \in I$ are periodic functions with a common period $P > 0$ and positive integers n_i for $i \in I$ and m_i for $i \in I$ exist such that $\sigma_i = n_i P$ and $\tau_i = m_i P$ for $i \in I$.

We define

$$\tau = \max_{i\in I} \tau_i \quad \text{and} \quad \sigma = \max_{i\in I} \sigma_i,$$

and consider the positive real number

$$\gamma = \max\{\tau, \sigma\}.$$

As known, a continuous real-valued function u defined on the interval $[-\gamma, \infty)$ will be called a "solution" of the NDDE (1.1) if the function $u(x) + \sum_{i\in I} d_i u(x-\sigma_i)$ is continuously differentiable for $x \geq 0$ and u satisfies (1.1) for all $x \geq 0$.

Along with the NDDE (1.1), it is customary to specify an "initial condition" in the following form:

$$u(x) = \psi(x) \quad \text{for } -\gamma \leq x \leq 0, \tag{1.2}$$

where the initial function ψ is a given continuous real-valued function on the interval $[-\gamma, 0]$ satisfying the "consistency condition"

$$\psi'(0) + \sum_{i\in I} d_i \psi'(-\sigma_i) + \alpha(0)\psi(0) + \sum_{i\in I} \beta_i(0)\psi(-\tau_i) = h(0).$$

Equation (1.1) and initial function (1.2) constitute an "initial value problem" (IVP). It is well known (see, for example, Hale and Verduyn Lund [6]) that there is a unique solution u of the NDDE (1.1) which satisfies the initial condition (1.2); this unique solution u will be called the solution of the IVP (1.1)–(1.2).

In the case of the function h being identically zero on the interval $[0,\infty)$, the NDDE (1.1) is reduced to

$$\left[u(x) + \sum_{i\in I} d_i u(x-\sigma_i)\right]' + \alpha(x)u(x) + \sum_{i\in I} \beta_i(x)u(x-\tau_i) = 0. \tag{1.3}$$

The following notations are used in this article:

$$a = \frac{1}{p}\int_0^P \alpha(x)\,dx \quad \text{and} \quad b_i = \frac{1}{p}\int_0^P \beta_i(x)\,dx \quad \text{for} \quad i \in I.$$

We also associate the following equation with the differential equation (1.3):

$$\mu\left(1 + \sum_{i\in I} d_i e^{-\mu\sigma_i}\right) + a + \sum_{i\in I} b_i e^{-\mu\tau_i} = 0, \tag{1.4}$$

which will be called the "characteristic equation" of (1.3). Sufficient conditions for obtaining the real roots of the characteristic equation (1.4) are given by Philos and Purnaras [13, Chapter 3].

In the following parts, we use $\hat{\alpha}$ and $\hat{\beta}_i$ for $i \in I$ to denote the P-periodic extensions of the coefficients α and β_i for $i \in I$, respectively, on the interval $[-\gamma, \infty)$. Furthermore, for the real root μ_0 of (1.4), by ρ_{μ_0}, we use the continuous real-valued function defined on the interval $[-\gamma, \infty)$ to denote follows:

$$\rho_{\mu_0}(x) = \hat{\alpha}(x) + \sum_{i\in I} \hat{\beta}_i(x) e^{-\mu_0\tau_i} \quad \text{for } x \geq -\gamma. \tag{1.5}$$

Now, we set up some equalities needed below. For each index $i \in I$, we can use the assumption that the function $\hat{\beta}_i$ are P-periodic and that $\tau_i = m_i P$ and $\sigma_i = n_i P$ to solve for $x \geq 0$ and $i \in I$

$$\int_{x-\tau_i}^{x} \hat{\beta}_i(s)\,ds = \int_0^{\tau_i} \beta_i(s)\,ds = \left[\frac{1}{\tau_i}\int_0^{\tau_i} \beta_i(s)\,ds\right]\tau_i = \left[\frac{1}{P}\int_0^{P} \beta_i(s)\,ds\right]\tau_i = b_i\tau_i.$$

Similarly

$$\int_{x-\sigma_i}^{x} \hat{\beta}_i(s)\,ds = b_i\sigma_i.$$

We can verify that for every $x \geq 0$ and $i \in I$

$$\int_{x-\tau_i}^{x} \left|\hat{\beta}_i(s)\right|\,ds = \tilde{b}_i\tau_i, \tag{1.6}$$

where

$$\tilde{b}_i = \frac{1}{P}\int_0^{P} |\beta_i(s)|\,ds \quad \text{for } i \in I.$$

We clearly have

$$|b_i| \leq \tilde{b}_i \quad \text{for } i \in I.$$

Moreover, we have $|b_i| = \tilde{b}_i$ for $i \in I$ in the case where each one of the coefficients β_i for $i \in I$ is assumed to be of one sign on the interval $[0, \infty)$.

Our goal in this paper is to obtain periodic solutions of the NDDE (1.1) when the function h is P-periodic, as well as to obtain an asymptotic criterion and exponential estimates of the solutions of Equation (1.1).

The principal findings of this work are given together with the proofs of the first theorem in Section 2 and the two theorems in Section 3. In Section 4, the main results are applied to the special case of nonhomogeneous constant coefficient NDDEs.

2. Periodic solutions

In this section, we establish conditions under which Equation (1.1) has a periodic solution. We assume that h is always P-periodic, even if this is not explicitly stated. Consider the equation

$$u'(x) + \alpha(x)u(x) = 0. \tag{2.1}$$

As known, the general solution of Equation (2.1) is

$$u(x) = c \ \exp\left\{-\int_0^x \alpha(s)\,ds\right\},$$

where c is a constant. We apply the variation of constants formula to find the solution of (1.3). Suppose that

$$u(x) = c(x) \ \exp\left\{-\int_0^x \hat{\alpha}(s)\,ds\right\}, \tag{2.2}$$

where

$$\hat{\alpha}(x) = \begin{cases} \alpha(x)\,, & x \geq 0, \\ \alpha(x+\gamma)\,, & -\gamma \leq x \leq 0, \end{cases}$$

is a solution of (1.3). Replacing (2.2) into (1.3) yields the condition

$$\begin{aligned} c'(x) + \sum_{i\in I} d_i \left(c'(x-\sigma_i) - \alpha(x)\ c\ (x-\sigma_i)\right) \ \exp\left\{\int_{x-\sigma_i}^x \hat{\alpha}(s)\,ds\right\} \\ + \sum_{i\in I} \beta_i(x)\ c\ (x-\tau_i) \ \exp\left\{\int_{x-\tau_i}^x \hat{\alpha}(s)\,ds\right\} = 0 \end{aligned} \tag{2.3}$$

for all $x \geq 0$ on $c(x)$.

We define

$$b(x) = \sum_{i\in I} \hat{\beta}_i(x),$$

where

$$\hat{\beta}_i(x) = \begin{cases} \beta_i(x)\,, & x \geq 0, \\ \beta_i(x+\gamma)\,, & -\gamma \leq x \leq 0. \end{cases}$$

Suppose that (2.3) has a solution of the form

$$c(x) = \exp\left\{\int_0^x (\lambda_1\hat{\alpha}(s) + \lambda_2 b(s))\,ds\right\}. \tag{2.4}$$

Then, from (2.3), we obtain

$$\begin{aligned} (\lambda_1\hat{\alpha}(x) + \lambda_2 b(x)) + \sum_{i\in I} d_i \Bigg[(\lambda_1\hat{\alpha}(x-\sigma_i) + \lambda_2 b(x-\sigma_i)) \exp\left\{-\int_{x-\sigma_i}^x (\lambda_1\hat{\alpha}(s) + \lambda_2 b(s))\,ds\right\} \\ -\alpha(x) \exp\left\{-\int_{x-\sigma_i}^x (\lambda_1\hat{\alpha}(s) + \lambda_2 b(s))\,ds\right\}\Bigg] \exp\left\{\int_{x-\sigma_i}^x \hat{\alpha}(s)ds\right\} \\ + \sum_{i\in I} \beta_i(x) \exp\left\{-\int_{x-\tau_i}^x (\lambda_1\hat{\alpha}(s) + \lambda_2 b(s))\,ds\right\} \exp\left\{\int_{x-\tau_i}^x \hat{\alpha}(s)ds\right\} = 0 \end{aligned}$$

or

$$(\lambda_1\hat{\alpha}(x)+\lambda_2 b(x))+\sum_{i\in I}d_i\Big[\lambda_1\hat{\alpha}(x-\sigma_i)+\lambda_2 b(x-\sigma_i)-\alpha(x)\Big]\exp\left\{\int_{x-\sigma_i}^{x}((1-\lambda_1)\,\hat{\alpha}(s)-\lambda_2 b(s))\,ds\right\}$$
$$+\sum_{i\in I}\beta_i(x)\exp\left\{\int_{x-\tau_i}^{x}((1-\lambda_1)\,\hat{\alpha}(s)-\lambda_2 b(s))\,ds\right\}=0.$$

Since the functions α and b are P-periodic, from the last equation, we have

$$\begin{aligned}&(\lambda_1\alpha(x)+\lambda_2 b(x))+\sum_{i\in I}d_i\Big[(\lambda_1-1)\,\alpha(x)+\lambda_2 b(x)\Big]\exp\left\{\int_{0}^{\sigma_i}((1-\lambda_1)\,\alpha(s)-\lambda_2 b(s))\,ds\right\}\\&+\sum_{i\in I}\beta_i(x)\exp\left\{\int_{0}^{\tau_i}((1-\lambda_1)\,\alpha(s)-\lambda_2 b(s))\,ds\right\}=0.\end{aligned}\tag{2.5}$$

Next, for each index $i\in I$, we assume that α and β_i are P-periodic and that $\sigma_i=n_iP$ to solve for $x\geq 0$

$$\begin{aligned}\int_0^{\sigma_i}[(1-\lambda_1)\,\alpha(s)-\lambda_2 b(s)]\,ds&=\left\{\frac{1}{\sigma_i}\int_0^{\sigma_i}\left[(1-\lambda_1)\,\alpha(s)-\lambda_2\sum_{i\in I}\beta_i(s)\right]ds\right\}\sigma_i\\&=\left\{\frac{1}{P}\int_0^{P}\left[(1-\lambda_1)\,\alpha(s)-\lambda_2\sum_{i\in I}\beta_i(s)\right]ds\right\}\sigma_i\\&=\left\{(1-\lambda_1)\,\frac{1}{P}\int_0^{P}\alpha(s)ds-\lambda_2\sum_{i\in I}\left[\frac{1}{P}\int_0^{P}\beta_i(s)ds\right]\right\}\sigma_i\\&=\left[(1-\lambda_1)\,a-\lambda_2\sum_{i\in I}b_i\right]\sigma_i.\end{aligned}$$

Similarly, for each index $i\in I$, we assume that α and β_i are P-periodic and that $\tau_i=m_iP$ to solve for $x\geq 0$

$$\int_0^{\tau_i}[(1-\lambda_1)\,\alpha(s)-\lambda_2 b(s)]\,ds=\left[(1-\lambda_1)\;a-\lambda_2\sum_{i\in I}b_i\right]\tau_i.$$

Thus, from (2.5), we obtain

$$\begin{aligned}&(\lambda_1\alpha(x)+\lambda_2 b(x))+\sum_{i\in I}d_i\Big[(\lambda_1-1)\,\alpha(x)+\lambda_2 b(x)\Big]\exp\left\{\left[(1-\lambda_1)\;a-\lambda_2\sum_{i\in I}b_i\right]\sigma_i\right\}\\&+\sum_{i\in I}\beta_i(x)\exp\left\{\left[(1-\lambda_1)\;a-\lambda_2\sum_{i\in I}b_i\right]\tau_i\right\}=0\end{aligned}$$

or, by taking the definition of b into account, we get

$$\begin{aligned}&\alpha(x)\left(\lambda_1+(\lambda_1-1)\sum_{i\in I}d_i\exp\left\{\left[(1-\lambda_1)\;a-\lambda_2\sum_{i\in I}b_i\right]\sigma_i\right\}\right)\\&+\sum_{i\in I}\beta_i(x)\left(\lambda_2+\lambda_2\sum_{i\in I}d_i\exp\left\{\left[(1-\lambda_1)\;a-\lambda_2\sum_{i\in I}b_i\right]\sigma_i\right\}\right.\\&\left.+\exp\left\{\left[(1-\lambda_1)\;a-\lambda_2\sum_{i\in I}b_i\right]\tau_i\right\}\right)=0.\end{aligned}\tag{2.6}$$

If we assume that $\alpha(x) \neq 0$ and $\sum_{i\in I}\beta_i(x) \neq 0$ for $x \geq 0$ and

$$(1-\lambda_1)\ a - \lambda_2 \sum_{i\in I} b_i = 0 \tag{2.7}$$

hold. Then, from (2.6), we get

$$\lambda_1 + (\lambda_1 - 1)\sum_{i\in I} d_i = 0 \quad \text{and} \quad \lambda_2 + \lambda_2 \sum_{i\in I} d_i + 1 = 0.$$

From both equations, we obtain

$$\lambda_1 = \frac{\sum_{i\in I} d_i}{1+\sum_{i\in I} d_i} \quad \text{and} \quad \lambda_2 = -\frac{1}{1+\sum_{i\in I} d_i},$$

where $\sum_{i\in I} d_i \neq -1$. If we substitute these values (2.7), we get the following condition:

$$a + \sum_{i\in I} b_i = 0. \tag{2.8}$$

Moreover, by taking the definitions of λ_1 and λ_2 into account, from (2.4), it follows that

$$c(x) = \exp\left\{\frac{1}{1+\sum_{i\in I} d_i}\int_0^x \left[\hat{\alpha}(s)\sum_{i\in I} d_i - b(s)\right] ds\right\}$$

is a solution of (2.3). Hence, from (2.2), it follows that

$$u(x) = k\ \exp\left\{-\frac{1}{1+\sum_{i\in I} d_i}\int_o^x (\hat{\alpha}(s) + b(s))\, ds\right\}, \tag{2.9}$$

where k is a constant, is a solution of Equation (1.3). Furthermore, due to Condition (2.8), it can be easily seen that

$$\int_0^\omega \left[\alpha(s) + \sum_{i\in I} \beta_i(s)\right] ds = 0,$$

where

$$\omega = \min\left\{\min_{i\in I} \tau_i, \min_{i\in I} \sigma_i\right\}.$$

Then (2.9) is a ω-periodic solution of Equation (1.3).

We now turn our attention to the original nonhomogeneous equation (1.1). By implementing the variation of parameters method once more, we postulate that Equation (1.1) possesses a solution expressed in the following form:

$$u_P(x) = K(x)\ \exp\left\{-\frac{1}{1+\sum_{i\in I} d_i}\int_o^x [\hat{\alpha}(s) + b(s)]\, ds\right\}. \tag{2.10}$$

Using the condition (2.8), substituting (2.10) into (1.1) gives the equation

$$\begin{aligned}
&K'(x) - \frac{1}{1+\sum_{i\in I} d_i}\left[\alpha(x)+b(x)\right]\ K(x)\\
&+\sum_{i\in I} d_i\left\{K'\left(x-\sigma_i\right)-\frac{1}{1+\sum_{i\in I} d_i}\left[\alpha(x)+b(x)\right]\ K\left(x-\sigma_i\right)\right\}\\
&\times\exp\left\{\frac{1}{1+\sum_{i\in I} d_i}\int_{x-\sigma_i}^{x}\left[\hat{\alpha}(s)+b(s)\right]ds\right\}+\alpha(x)K(x)\\
&+\sum_{i\in I}\beta_i(x)K\left(x-\tau_i\right)\exp\left\{\frac{1}{1+\sum_{i\in I} d_i}\int_{x-\tau_i}^{x}\left[\hat{\alpha}(s)+b(s)\right]ds\right\}\\
&=h(x)\exp\left\{\frac{1}{1+\sum_{i\in I} d_i}\int_0^x\left[\alpha(s)+b(s)\right]ds\right\}
\end{aligned}$$

or

$$\begin{aligned}
&K'(x)+\frac{\sum_{i\in I} d_i}{1+\sum_{i\in I} d_i}\alpha(x)K(x)+\sum_{i\in I}\beta_i(x)\left[K\left(x-\tau_i\right)-\frac{1}{1+\sum_{i\in I} d_i}K(x)-\frac{\sum_{i\in I} d_i}{1+\sum_{i\in I} d_i}K\left(x-\sigma_i\right)\right]\\
&+\sum_{i\in I} d_i\left[K'\left(x-\sigma_i\right)-\frac{1}{1+\sum_{i\in I} d_i}\alpha(x)K\left(x-\sigma_i\right)\right]\\
&=h(x)\exp\left\{\frac{1}{1+\sum_{i\in I} d_i}\int_0^x\left[\alpha(s)+b(s)\right]ds\right\}.
\end{aligned}$$

The substituted (2.10) is a periodic solution of (1.1) if and only if $K(x)$ is periodic. However, this indicates that $K(x) = K(x-\tau_i)$ and $K(x) = K(x-\sigma_i)$ so the differential equation for $i \in I$, and thus, taking into account that it will be $K'(x) = K'(x-\sigma_i)$, the last differential equation for K is

$$K'(x)+\sum_{i\in I} d_i K'(x)=h(x)\exp\left\{\frac{1}{1+\sum_{i\in I} d_i}\int_0^x\left[\alpha(s)+b(s)\right]ds\right\}$$

or

$$K'(x)=\frac{1}{1+\sum_{i\in I} d_i}h(x)\exp\left\{\frac{1}{1+\sum_{i\in I} d_i}\int_0^x\left[\alpha(s)+b(s)\right]ds\right\}.$$

It follows that

$$K(x)=\frac{1}{1+\sum_{i\in I} d_i}\int_0^x h(v)\exp\left\{\frac{1}{1+\sum_{i\in I} d_i}\int_0^v\left[\alpha(s)+\sum_{i\in I}\beta_i(s)\right]ds\right\}dv. \tag{2.11}$$

Specifying that this function is the integral of a ω-periodic function, one can observe that it is a ω-periodic function if and only if

$$\int_0^\omega h(v)\exp\left\{\frac{1}{1+\sum_{i\in I} d_i}\int_0^v\left[\alpha(s)+\sum_{i\in I}\beta_i(s)\right]ds\right\}dv=0.$$

Substituting (2.11) into (2.10), the following result is obtained.

Theorem 1. *Suppose that $\alpha(x) \neq 0$, $\sum_{i\in I}\beta_i(x) \neq 0$ for $x \geq 0$, and $\sum_{i\in I} d_i \neq -1$. Assume that*

$$a + \sum_{i\in I} b_i = 0,$$

where

$$a = \frac{1}{P}\int_0^P \alpha(s)ds, \quad b_i = \frac{1}{P}\int_0^P b_i(s)ds, \quad i \in I.$$

Assume also that

$$\int_0^\omega h(v)\exp\left\{\frac{1}{1+\sum_{i\in I} d_i}\int_0^v \left[\alpha(s) + \sum_{i\in I}\beta_i(s)\right]ds\right\}dv = 0,$$

where

$$\omega = \min\left\{\min_{i\in I}\tau_i, \min_{i\in I}\sigma_i\right\}.$$

Then, for each $k \in \mathbb{R}$, we have

$$u(x) = k\ \exp\left\{-\frac{1}{1+\sum_{i\in I} d_i}\int_0^x \left[\alpha(s+\gamma) + \sum_{i\in I}\beta_i(s+\gamma)\right]ds\right\} + u_P(x), \quad x \geq -\gamma,$$

where

$$u_P(x) = \exp\left\{-\frac{1}{1+\sum_{i\in I} d_i}\int_0^x \left[\alpha(s+\gamma) + \sum_{i\in I}\beta_i(s+\gamma)\right]ds\right\}$$
$$\times\left\{\frac{1}{1+\sum_{i\in I} d_i}\int_0^x h(v)\exp\left\{\frac{1}{1+\sum_{i\in I} d_i}\int_0^v \left[\alpha(s+\gamma) + \sum_{i\in I}\beta_i(s+\gamma)\right]ds\right\}dv\right\}$$

is a ω-periodic solution of Eq (1.1).

Example 2.2. Consider

$$\begin{aligned} u'(x) + u'(x-1) - u'(x-2) - (2+\sin 2\pi x)u(x) + (1+sin2\pi x)u(x-1) \\ + (1+\cos 2\pi x)u(x-2) = \cos 2\pi x, \quad x \geq 0. \end{aligned} \tag{2.12}$$

In this equation, $\sum_{i=1}^2 d_i = 0 \neq -1$, $\alpha(x) = -(2+\sin 2\pi x) \neq 0$, and $\sum_{i=1}^2 \beta_i(x) = 2 + \sin 2\pi x + \cos 2\pi x \neq 0$ for $x \geq 0$. Since $a = \int_0^1(-2 - sin2\pi s)ds = -2$, $b_1 = \int_0^1(1 + sin2\pi s)ds = 1$, and $b_2 = \int_0^1(1 + sin2\pi s)ds = 1$, we have $a + b_1 + b_2 = 0$. Also, since $\omega = 1$, we get

$$\int_0^1 h(v)\exp\left\{\int_0^v (\cos 2\pi s)\,ds\right\}dv = \int_0^1 (\cos 2\pi v)\exp\left\{\frac{1}{2\pi}sin2\pi v\right\}dv = 0.$$

Thus, the conditions of Theorem 1 are satisfied. Then, for each $k \in \mathbb{R}$, we have

$$u(x) = k\ \exp\left\{-\frac{1}{2\pi}sin2\pi x\right\} + u_P(x) \quad \text{for} \quad x \geq -2,$$

where

$$u_P(x) = \exp\left\{-\frac{1}{2\pi}sin2\pi x\right\}\left(\exp\left\{\frac{1}{2\pi}sin2\pi x\right\} - 1\right).$$

Thus, for each $k \in \mathbb{R}$, we have

$$u(x) = (k-1)\exp\left\{-\frac{1}{2\pi}sin2\pi x\right\} + 1 \quad \text{for } x \geq -2$$

is a 1-periodic solution to Equation (2.12).

Example 2.3. Consider

$$u'(x) + u'(x-\pi) + \left(1 - \frac{\cos 2x}{2}\right)u(x) - \left(1 - \frac{\sin 2x}{2}\right)u(x-\pi) = \cos 2x - \sin 2x, \quad x \geq 0. \tag{2.13}$$

In this equation, $d_1 = 1 \neq -1$, $\alpha(x) = 1 - \frac{\cos 2x}{2} \neq 0$, and $\beta_1(x) = -1 + \frac{\sin 2x}{2} \neq 0$ for $x \geq 0$. Since $a = \frac{1}{\pi}\int_0^\pi \left(1 - \frac{\cos 2x}{2}\right) ds = 1$, and $b_1 = \frac{1}{\pi}\int_0^\pi \left(-1 + \frac{\sin 2x}{2}\right) ds = -1$, we have $a + b_1 = 0$. Since $\omega = \pi$, we get

$$\int_0^\pi (\cos 2v - \sin 2v)\exp\left\{\frac{1}{2}\int_0^v \left(\frac{\sin 2s - \cos 2s}{2}\right) ds\right\} dv = 0.$$

Thus, the conditions of Theorem 1 are fulfilled. Then, for each $k \in \mathbb{R}$, we have

$$u(x) = k\ \exp\left\{\frac{1}{8}(\cos 2x - \sin 2x - 1)\right\} + u_P(x) \quad \text{for} \quad x \geq -\pi,$$

where

$$u_P(x) = -2\left(1 - \exp\left\{\frac{1}{8}(\cos 2x - \sin 2x - 1)\right\}\right).$$

Thus, for each $k \in \mathbb{R}$, we have

$$u(x) = (k+2)\exp\left\{\frac{1}{8}(\cos 2x - \sin 2x - 1)\right\} - 2 \quad \text{for } x \geq -\pi$$

is a π-periodic solution to Equation (2.13).

3. An asymptotic result and estimation of the solutions

The main results of this section are presented in Theorems 2 and 4. Specifically, Theorem 2 establishes an asymptotic criterion for the solutions of the linear nonhomogeneous NDDE given by (1.1), while Theorem 4 provides an estimate for the solutions of the same equation. Examples are given at the end of this section.

Theorem 2. *Assume that α and β_i for $i \in I$ are periodic continuous real-valued functions having a common period $P > 0$, and that h is a continuous real-valued function on interval* $[0, \infty)$*. Suppose that μ_0 be a real root of the characteristic equation* (1.4) *and set*

$$\Gamma_{\mu_0} = 1 + \sum_{i \in I} d_i e^{-\mu_0 \sigma_i}. \tag{3.1}$$

Let the root μ_0 satisfy

$$\begin{aligned} &\sum_{i \in I} |d_i| \left(|\Gamma_{\mu_0}| + G_{\mu_0} \sigma_i \right) e^{-\mu_0 \sigma_i} + |\Gamma_{\mu_0}| \sum_{i \in I} \tilde{b}_i \tau_i e^{-\mu_0 \tau_i} \\ &+ |\Gamma_{\mu_0}| \int_0^\infty |h(s)| \exp\left[\frac{1}{\Gamma_{\mu_0}} \int_0^s \rho_{\mu_0}(v) dv \right] ds < |\Gamma_{\mu_0}|, \end{aligned} \tag{3.2}$$

where $G_{\mu_0} = \frac{1}{P} \int_0^P |\rho_{\mu_0}(s)| ds$, and also ρ_{μ_0} and $\tilde{b}_i$ are defined as in (1.5) *and* (1.6)*, respectively. Then, for any $\psi \in C([-\gamma, 0], \mathbb{R})$, the solution u of the IVP* (1.1)–(1.2) *satisfies*

$$\lim_{x \to \infty} \left\{ u(x) \exp\left[\frac{1}{\Gamma_{\mu_0}} \int_0^x \rho_{\mu_0}(s) ds \right] \right\} = \frac{L_{\mu_0}(\psi)}{1 + \xi_{\mu_0}}, \tag{3.3}$$

where

$$\begin{aligned} L_{\mu_0}(\psi) = \psi(0) &+ \sum_{i \in I} d_i \left\{ \psi(-\sigma_i) + \frac{e^{-\mu_0 \sigma_i}}{\Gamma_{\mu_0}} \int_{-\sigma_i}^0 \rho_{\mu_0}(s) \psi(s) \exp\left[\frac{1}{\Gamma_{\mu_0}} \int_0^s \rho_{\mu_0}(v) dv \right] ds \right\} \\ - \sum_{i \in I} e^{-\mu_0 \tau_i} &\int_{-\tau_i}^0 \hat{\beta}_i(s) \psi(s) \exp\left[\frac{1}{\Gamma_{\mu_0}} \int_0^s \rho_{\mu_0}(v) dv \right] ds + \int_0^\infty h(s) \exp\left[\frac{1}{\Gamma_{\mu_0}} \int_0^s \rho_{\mu_0}(v) dv \right] ds \end{aligned} \tag{3.4}$$

and

$$\xi_{\mu_0} = \sum_{i \in I} \left[d_i \left(1 - \mu_0 \sigma_i \right) e^{-\mu_0 \sigma_i} - b_i \tau_i e^{-\mu_0 \tau_i} \right]. \tag{3.5}$$

Proof. From (3.2), it follows immediately that

$$|\Gamma_{\mu_0}| \sum_{i \in I} |d_i| e^{-\mu_0 \sigma_i} < |\Gamma_{\mu_0}|, \quad \text{i.e.} \quad |\Gamma_{\mu_0}| \Big(1 - \sum_{i \in I} |d_i| e^{-\mu_0 \sigma_i} \Big) > 0.$$

Therefore, we always have

$$1 - \sum_{i \in I} |d_i| e^{-\mu_0 \sigma_i} > 0.$$

However, from (3.1), we get

$$\Gamma_{\mu_0} = 1 + \sum_{i \in I} d_i e^{-\mu_0 \sigma_i} \geq 1 - |d_i| e^{-\mu_0 \sigma_i}$$

and, consequently, Γ_{μ_0} is necessarily positive. Hence, (3.2) becomes

$$\varphi_{\mu_0} \equiv \sum_{i \in I} |d_i| \Big(1 + \frac{G_{\mu_0}}{\Gamma_{\mu_0}} \sigma_i \Big) e^{-\mu_0 \sigma_i} + \sum_{i \in I} \tilde{b}_i \tau_i e^{-\mu_0 \tau_i} + \int_0^\infty |h(s)| \exp\left[\frac{1}{\Gamma_{\mu_0}} \int_0^s \rho_{\mu_0}(v) dv \right] ds < 1. \tag{3.6}$$

In what follows, we proceed to establish several equalities that are essential for the subsequent developments. The P-periodicity of the functions $\hat{\alpha}$ and $\tilde{\beta}_i$ for $i \in I$ implies that the function ρ_{μ_0} is also P-periodic. Therefore, accounting for fact the that $\sigma_i = n_i P$ for $i \in I$, we solve for $i \in I$ and $x \geq 0$

$$\begin{aligned}\int_{x-\sigma_i}^{x} \rho_{\mu_0}(s)ds &= \int_0^{\sigma_i} \rho_{\mu_0}(s)ds = \left[\frac{1}{\sigma_i}\int_0^{\sigma_i} \rho_{\mu_0}(s)ds\right]\sigma_i = \left[\frac{1}{P}\int_0^{P} \rho_{\mu_0}(s)ds\right]\sigma_i \\ &= \left\{\left[\frac{1}{P}\int_0^{P} \alpha(s)ds\right] + \sum_{i\in I}\left[\frac{1}{P}\int_0^{P} \beta_i(s)ds\right]e^{-\mu_0\tau_i}\right\}\sigma_i \\ &= \left(a + \sum_{i\in I} b_i e^{-\mu_0\tau_i}\right)\sigma_i.\end{aligned}$$

Thus, since μ_0 is a root of (1.4), we have

$$\frac{1}{\Gamma_{\mu_0}}\int_{x-\sigma_i}^{x} \rho_{\mu_0}(s)ds = -\mu_0\sigma_i \quad \text{for every } x \geq 0 \quad \text{and all } i \in I. \tag{3.7}$$

In similar way, considering the fact that $\tau_i = m_i P$ for $i \in I$ and again using the hypothesis that μ_0 is a root of (1.4), we can obtain

$$\frac{1}{\Gamma_{\mu_0}}\int_{x-\tau_i}^{x} \rho_{\mu_0}(s)ds = -\mu_0\tau_i \quad \text{for every } x \geq 0 \quad \text{and all } i \in I. \tag{3.8}$$

Moreover, again, by taking the fact that $\sigma_i = n_i P$ for $i \in I$ into account, we get for $i \in I$ and $x \geq 0$, we get

$$\int_{x-\sigma_i}^{x} \left|\rho_{\mu_0}(s)\right| ds = \int_0^{\sigma_i} \left|\rho_{\mu_0}(s)\right| ds = \left[\frac{1}{\sigma_i}\int_0^{\sigma_i} \left|\rho_{\mu_0}(s)\right| ds\right]\sigma_i = \left[\frac{1}{P}\int_0^{P} \left|\rho_{\mu_0}(s)\right| ds\right]\sigma_i.$$

So, it holds that

$$\int_{x-\sigma_i}^{x} \left|\rho_{\mu_0}(s)\right| ds = G_{\mu_0}\sigma_i \quad \text{for every } x \geq 0 \quad \text{and all } i \in I. \tag{3.9}$$

By using (3.7) and (3.9) for a point $x = x_0 \geq 0$ and an index $i_0 \in I$, we obtain

$$|\mu_0| = \frac{1}{\Gamma_{\mu_0}\sigma_{i_0}}\left|\int_{x_0-\sigma_{i_0}}^{x_0} \rho_{\mu_0}(s)ds\right| \leq \frac{1}{\Gamma_{\mu_0}\sigma_{i_0}}\int_{x_0-\sigma_{i_0}}^{x_0} \left|\rho_{\mu_0}(s)\right| ds = \frac{G_{\mu_0}}{\Gamma_{\mu_0}},$$

i.e., $|\mu_0| \leq \frac{G_{\mu_0}}{\Gamma_{\mu_0}}$. Thus, using (1.6) and (3.5), we have

$$\begin{aligned}\left|\xi_{\mu_0}\right| &\leq \sum_{i\in I}\left[|d_i|\left(1 + |\mu_0|\sigma_i\right)e^{-\mu_0\sigma_i} + |b_i|\tau_i e^{-\mu_0\tau_i}\right] \\ &\leq \sum_{i\in I}\left[|d_i|\left(1 + \frac{G_{\mu_0}}{\Gamma_{\mu_0}}\sigma_i\right)e^{-\mu_0\sigma_i} + \tilde{b}_i\tau_i e^{-\mu_0\tau_i}\right] + \int_0^{\infty} |h(s)|\exp\left[\frac{1}{\Gamma_{\mu_0}}\int_0^{\infty} \rho_{\mu_0}(v)dv\right]ds \equiv \varphi_{\mu_0},\end{aligned}$$

where φ_{μ_0} is defined as in (3.6). We have thus proved that $\left|\xi_{\mu_0}\right| \leq \varphi_{\mu_0}$. Nevertheless, in view of (3.6), $\varphi_{\mu_0} < 1$ and thus we always have $1 + \xi_{\mu_0} > 0$.

Now, consider an arbitrary function $\psi(x) \in C([-\gamma, 0], \mathbb{R})$. Let u be the solution of the IVP (1.1)–(1.2) and define

$$y(x) = u(x) \exp\left\{\frac{1}{\Gamma_{\mu_0}} \int_0^x \rho_{\mu_0}(s)ds\right\} \quad \text{for} \quad x \geq -\gamma.$$

By using (3.7), for every $x \geq 0$, we then obtain

$$\begin{aligned} u(x) + \sum_{i \in I} d_i u(x - \sigma_i) &= \left\{y(x) + \sum_{i \in I} d_i y(x - \sigma_i) \times \exp\left[\frac{1}{\Gamma_{\mu_0}} \int_{x-\sigma_i}^x \rho_{\mu_0}(s)ds\right]\right\} \exp\left[-\frac{1}{\Gamma_{\mu_0}} \int_0^x \rho_{\mu_0}(s)ds\right] \\ &= \left[y(x) + \sum_{i \in I} d_i e^{-\mu_0 \sigma_i} y(x - \sigma_i)\right] \exp\left[-\frac{1}{\Gamma_{\mu_0}} \int_0^x \rho_{\mu_0}(s)ds\right], \end{aligned}$$

and by virtue of (3.8), for any $x \geq 0$, we get

$$\begin{aligned} \sum_{i \in I} \beta_i(x) u(x - \tau_i) &= \left\{\sum_{i \in I} \beta_i(x) y(x - \tau_i) \exp\left[\frac{1}{\Gamma_{\mu_0}} \int_{x-\tau_i}^x \rho_{\mu_0}(s)ds\right]\right\} \exp\left[-\frac{1}{\Gamma_{\mu_0}} \int_0^x \rho_{\mu_0}(s)ds\right] \\ &= \left[\sum_{i \in I} \beta_i(x) e^{-\mu_0 \tau_i} y(x - \tau_i)\right] \exp\left[-\frac{1}{\Gamma_{\mu_0}} \int_0^x \rho_{\mu_0}(s)ds\right]. \end{aligned}$$

Thus, using (1.5) and (3.1), from (1.1), we have

$$\begin{aligned} &\left\{\left[u(x) + \sum_{i \in I} d_i u(x - \sigma_i)\right]' + \alpha(x) u(x) + \sum_{i \in I} \beta_i(x) u(x - \tau_i)\right\} \exp\left[\frac{1}{\Gamma_{\mu_0}} \int_0^x \rho_{\mu_0}(s)ds\right] \\ &\qquad = h(x) \exp\left[\frac{1}{\Gamma_{\mu_0}} \int_0^x \rho_{\mu_0}(s)ds\right], \end{aligned}$$

$$\begin{aligned} &\left[y(x) + \sum_{i \in I} d_i y(x - \sigma_i) e^{-\mu_0 \sigma_i}\right]' - \frac{1}{\Gamma_{\mu_0}} \rho_{\mu_0}(x) \left[y(x) + \sum_{i \in I} d_i e^{-\mu_0 \sigma_i} y(x - \sigma_i)\right] + \alpha(x) y(x) \\ &\qquad + \sum_{i \in I} \beta_i(x) e^{-\mu_0 \tau_i} y(x - \tau_i) = h(x) \exp\left[\frac{1}{\Gamma_{\mu_0}} \int_0^x \rho_{\mu_0}(s)ds\right], \end{aligned}$$

$$\begin{aligned} &\left[y(x) + \sum_{i \in I} d_i y(x - \sigma_i) e^{-\mu_0 \sigma_i}\right]' - \frac{1}{\Gamma_{\mu_0}} \rho_{\mu_0}(x) \left[y(x) + \sum_{i \in I} d_i e^{-\mu_0 \sigma_i} y(x - \sigma_i)\right] \\ &\qquad + \left[g_{\mu_0}(x) - \sum_{i \in I} \beta_i(x) e^{-\mu_0 \tau_i}\right] y(x) + \sum_{i \in I} \beta_i(x) e^{-\mu_0 \tau_i} y(x - \tau_i) = h(x) \exp\left[\frac{1}{\Gamma_{\mu_0}} \int_0^x \rho_{\mu_0}(s)ds\right], \end{aligned}$$

$$\begin{aligned} &\left[y(x) + \sum_{i \in I} d_i y(x - \sigma_i) e^{-\mu_0 \sigma_i}\right]' + \left(1 - \frac{1}{\Gamma_{\mu_0}}\right) \rho_{\mu_0}(x) y(x) - \frac{1}{\Gamma_{\mu_0}} \rho_{\mu_0}(x) \sum_{i \in I} d_i e^{-\mu_0 \sigma_i} y(x - \sigma_i) \\ &\qquad - \sum_{i \in I} \beta_i(x) e^{-\mu_0 \tau_i} y(x) + \sum_{i \in I} \beta_i(x) e^{-\mu_0 \tau_i} y(x - \tau_i) = h(x) \exp\left[\frac{1}{\Gamma_{\mu_0}} \int_0^x \rho_{\mu_0}(s)ds\right], \end{aligned}$$

$$\left[y(x)+\sum_{i\in I}d_iy(x-\sigma_i)e^{-\mu_0\sigma_i}\right]'+\frac{1}{\Gamma_{\mu_0}}\left(\sum_{i\in I}d_ie^{-\mu_0\sigma_i}\right)\rho_{\mu_0}(x)y(x)-\frac{1}{\Gamma_{\mu_0}}\rho_{\mu_0}(x)\sum_{i\in I}d_ie^{-\mu_0\sigma_i}y(x-\sigma_i)$$
$$-\sum_{i\in I}\beta_i(x)e^{-\mu_0\tau_i}y(x)+\sum_{i\in I}\beta_i(x)e^{-\mu_0\tau_i}y(x-\tau_i)=h(x)\exp\left[\frac{1}{\Gamma_{\mu_0}}\int_0^x\rho_{\mu_0}(s)ds\right],$$

$$\left[y(x)+\sum_{i\in I}d_iy(x-\sigma_i)e^{-\mu_0\sigma_i}\right]'+\frac{1}{\Gamma_{\mu_0}}\rho_{\mu_0}(x)\sum_{i\in I}d_ie^{-\mu_0\sigma_i}\left[y(x)-y(x-\sigma_i)\right]$$
$$-\sum_{i\in I}\beta_i(x)e^{-\mu_0\tau_i}\left[y(x)-y(x-\tau_i)\right]=h(x)\exp\left[\frac{1}{\Gamma_{\mu_0}}\int_0^x\rho_{\mu_0}(s)ds\right]$$

for every $x\geq 0$. Therefore, the fact that u satisfies (1.1) for all $x\geq 0$ is equivalent to

$$\begin{aligned}\left[y(x)+\sum_{i\in I}d_ie^{-\mu_0\sigma_i}y(x-\sigma_i)\right]'&=-\frac{1}{\Gamma_{\mu_0}}\rho_{\mu_0}(x)\sum_{i\in I}d_ie^{-\mu_0\sigma_i}\left[y(x)-y(x-\sigma_i)\right]\\&+\sum_{i\in I}\beta_i(x)e^{-\mu_0\tau_i}\left[y(x)-y(x-\tau_i)\right]+h(x)\exp\left[\frac{1}{\Gamma_{\mu_0}}\int_0^x\rho_{\mu_0}(s)ds\right].\end{aligned}\tag{3.10}$$

Moreover, the initial condition (1.2) takes the following equivalent form:

$$y(x)=\psi(x)\exp\left[\frac{1}{\Gamma_{\mu_0}}\int_0^x\rho_{\mu_0}(s)ds\right]\quad\text{for}\quad x\in[-\gamma,0].\tag{3.11}$$

Furthermore, considering that the functions ρ_{μ_0} and $\hat{\beta}_i$ for $i\in I$ exhibit P-periodicity, and recognizing that the delays σ_i, $i\in I$ are integer multiples of P, it can be demonstrated that (3.10) is mathematically equivalent to

$$\begin{aligned}y(x)+\sum_{i\in I}d_ie^{-\mu_0\sigma_i}y(x-\sigma_i)&=L_{\mu_0}(\psi)-\frac{1}{\Gamma_{\mu_0}}\sum_{i\in I}d_ie^{-\mu_0\sigma_i}\int_{x-\sigma_i}^x\rho_{\mu_0}(s)y(s)ds\\&+\sum_{i\in I}e^{-\mu_0\tau_i}\int_{x-\tau_i}^x\hat{\beta}_i(s)y(s)ds-\int_x^\infty h(s)\exp\left[\frac{1}{\Gamma_{\mu_0}}\int_0^s\rho_{\mu_0}(v)dv\right]ds,\end{aligned}\tag{3.12}$$

where $L_{\mu_0}(\psi)$ is defined as in (3.4).

Next, we define

$$z(x)=y(x)-\frac{L_{\mu_0}(\psi)}{1+\xi_{\mu_0}}\quad\text{for}\quad x\geq 0.$$

By using (3.5) and (3.7), it is not difficult to verify that (3.12) is equivalent to the following equation:

$$\begin{aligned}z(x)+\sum_{i\in I}d_ie^{-\mu_0\sigma_i}z(x-\sigma_i)&=-\frac{1}{\Gamma_{\mu_0}}\sum_{i\in I}d_ie^{-\mu_0\sigma_i}\int_{x-\sigma_i}^x\rho_{\mu_0}(s)z(s)ds\\&+\sum_{i\in I}e^{-\mu_0\tau_i}\int_{x-\tau_i}^x\hat{\beta}_i(s)z(s)ds-\int_x^\infty h(s)\exp\left[\frac{1}{\Gamma_{\mu_0}}\int_0^s\rho_{\mu_0}(v)dv\right]ds\end{aligned}\tag{3.13}$$

for $x \geq 0$. Moreover, the initial condition (3.11) can be equivalently expressed as

$$z(x) = \psi(x) \exp\left[\frac{1}{\Gamma_{\mu_0}} \int_0^x \rho_{\mu_0}(s)ds\right] - \frac{L_{\mu_0}(\psi)}{1+\xi_{\mu_0}} \quad \text{for } x \in [-\gamma, 0]. \tag{3.14}$$

By definitions of y and z, we should prove the equality (3.3), i.e.,

$$\lim_{x\to\infty} z(x) = 0. \tag{3.15}$$

In the remaining part of the proof, we determine (3.15). Since $0 < \varphi_{\mu_0} < 1$, then

$$0 < \int_0^\infty |h(s)| \exp\left[\frac{1}{\Gamma_{\mu_0}} \int_0^s \rho_{\mu_0}(v)dv\right] ds < \varphi_{\mu_0} < 1.$$

Therefore, we can obtain an expression as follows:

$$\lim_{x\to\infty} \int_x^\infty |h(s)| \exp\left[\frac{1}{\Gamma_{\mu_0}} \int_0^s \rho_{\mu_0}(v)dv\right] ds = 0.$$

Hence, we can inductively define a sequence of points $(x_n)_{n\geq 1}$ in $[0, \infty)$ with

$$x_{n+1} - x_n \geq \gamma \qquad (n = 1, 2, \dots)$$

such that, for all $n = 1, 2, \dots,$

$$\int_{x_n}^\infty |h(s)| \exp\left[\frac{1}{\Gamma_{\mu_0}} \int_0^s \rho_{\mu_0}(v)dv\right] ds \leq \left(\varphi_{\mu_0}\right)^{n-1} \int_0^\infty |h(s)| \exp\left[\frac{1}{\Gamma_{\mu_0}} \int_0^s \rho_{\mu_0}(v)dv\right] ds. \tag{3.16}$$

Set $x_0 = -\gamma$, and we define

$$F_{\mu_0}(\psi) = \max\left\{1, \max_{x\in[x_0,x_1]} \left|\psi(x) \exp\left[\frac{1}{\Gamma_{\mu_0}} \int_0^x \rho_{\mu_0}(s)ds\right] - \frac{L_{\mu_0}(\psi)}{1+\xi_{\mu_0}}\right|\right\}. \tag{3.17}$$

Hence, $F_{\mu_0}(\psi) \geq 1$. In this case, from (3.14)

$$|z(x)| \leq F_{\mu_0}(\psi) \quad \text{for } x \in [x_0, x_1]. \tag{3.18}$$

We now prove the following inequality:

$$|z(x)| \leq F_{\mu_0}(\psi) \quad \text{for} \quad x \geq x_0. \tag{3.19}$$

Therefore, consider an arbitrary number $\varepsilon > 0$. We assume the following:

$$|z(x)| < F_{\mu_0}(\psi) + \varepsilon \quad \text{for} \quad x \geq x_0. \tag{3.20}$$

Let us assume that the inequality (3.20) is not satisfied. Due to (3.18) and by the continuity of z, a point $x^* > x_1$ exists such that

$$|z(x)| < F_{\mu_0}(\psi) + \varepsilon \quad \text{for} \quad x \in [x_0, x^*) \quad \text{and} \quad |z(x^*)| = F_{\mu_0}(\psi) + \varepsilon.$$

By using (1.6), (3.6), and (3.9), from (3.13), we obtain

$$\begin{aligned}
F_{\mu_0}(\psi)+\varepsilon &= |z(x^*)| \le \sum_{i\in I}|d_i|\left[|z(x^*-\sigma_i)|+\frac{1}{\Gamma_{\mu_0}}\int_{x^*-\sigma_i}^{x^*}\left|\rho_{\mu_0}(s)\right||z(s)|\,ds\right]e^{-\mu_0\sigma_i}\\
&\quad+\sum_{i\in I}e^{-\mu_0\tau_i}\int_{x^*-\tau_i}^{x^*}\left|\hat{\beta}_i(s)\right||z(s)|\,ds+\int_{x^*}^{\infty}|h(s)|\exp\left[\frac{1}{\Gamma_{\mu_0}}\int_0^s\rho_{\mu_0}(v)dv\right]ds\\
&\le\left(F_{\mu_0}(\psi)+\varepsilon\right)\left\{\sum_{i\in I}|d_i|\left[1+\frac{1}{\Gamma_{\mu_0}}\int_{x^*-\sigma_i}^{x^*}\left|\rho_{\mu_0}(s)\right|ds\right]e^{-\mu_0\sigma_i}\right.\\
&\quad\left.+\sum_{i\in I}e^{-\mu_0\tau_i}\int_{x^*-\tau_i}^{x^*}\left|\hat{\beta}_i(s)\right|ds+\int_0^{\infty}|h(s)|\exp\left[\frac{1}{\Gamma_{\mu_0}}\int_0^s\rho_{\mu_0}(v)dv\right]ds\right\}\\
&=\left(F_{\mu_0}(\psi)+\varepsilon\right)\left\{\sum_{i\in I}|d_i|\left[1+\frac{G_{\mu_0}}{\Gamma_{\mu_0}}\sigma_i\right]e^{-\mu_0\sigma_i}+\sum_{i\in I}\tilde{b}_i\tau_ie^{-\mu_0\tau_i}+\int_0^{\infty}|h(s)|\exp\left[\frac{1}{\Gamma_{\mu_0}}\int_0^s\rho_{\mu_0}(v)dv\right]ds\right\}\\
&<\left(F_{\mu_0}(\psi)+\varepsilon\right)\varphi_{\mu_0}<\left(F_{\mu_0}(\psi)+\varepsilon\right).
\end{aligned}$$

This is a contradiction; therefore, (3.20) holds true. Since (3.20) is provided for all $\varepsilon > 0$, (3.19) is always satisfied. Now, by virtue of (3.19), from (3.13), we get

$$\begin{aligned}
|z(x)| &\le \sum_{i\in I}|d_i|\left[|z(x-\sigma_i)|+\frac{1}{\Gamma_{\mu_0}}\int_{x-\sigma_i}^{x}\left|\rho_{\mu_0}(s)\right||z(s)|\,ds\right]e^{-\mu_0\sigma_i}\\
&\quad+\sum_{i\in I}e^{-\mu_0\tau_i}\int_{x-\tau_i}^{x}\left|\hat{\beta}_i(s)\right||z(s)|\,ds+\int_{x}^{\infty}|h(s)|\exp\left[\frac{1}{\Gamma_{\mu_0}}\int_0^s\rho_{\mu_0}(v)dv\right]ds\\
&\le F_{\mu_0}(\psi)\left\{\sum_{i\in I}|d_i|\left[1+\frac{1}{\Gamma_{\mu_0}}\int_{x}^{x-\sigma_i}\left|\rho_{\mu_0}(s)\right|ds\right]e^{-\mu_0\sigma_i}\right.\\
&\quad\left.+\sum_{i\in I}e^{-\mu_0\tau_i}\int_{x-\tau_i}^{x}\left|\hat{\beta}_i(s)\right|ds+\int_0^{\infty}|h(s)|\exp\left[\frac{1}{\Gamma_{\mu_0}}\int_0^s\rho_{\mu_0}(v)dv\right]ds\right\}\\
&\le F_{\mu_0}(\psi)\left\{\sum_{i\in I}|d_i|\left[1+\frac{G_{\mu_0}}{\Gamma_{\mu_0}}\sigma_i\right]e^{-\mu_0\sigma_i}+\sum_{i\in I}\tilde{b}_i\tau_ie^{-\mu_0\tau_i}+\int_0^{\infty}|h(s)|\exp\left[\frac{1}{\Gamma_{\mu_0}}\int_0^s\rho_{\mu_0}(v)dv\right]ds\right\}.
\end{aligned}$$

Therefore, in view of (3.6), we obtain

$$|z(x)| \le F_{\mu_0}(\psi)\varphi_{\mu_0} \quad \text{for all} \quad x \ge x_1. \tag{3.21}$$

Next, by using (3.6), (3.19), and (3.21), we show by induction that z satisfies the following inequality:

$$|z(x)| \le F_{\mu_0}(\psi)\left(\varphi_{\mu_0}\right)^n, \quad x \ge x_n \quad (n = 0, 1, 2, 3, \dots). \tag{3.22}$$

We observe that (3.22) with $n = 0$ coincides with (3.19), while (3.22) with $n = 1$ is the same as (3.21). Suppose that (3.22) is true for $n = k$, where k is a positive integer, i.e.,

$$|z(x)| \le F_{\mu_0}(\psi)\left(\varphi_{\mu_0}\right)^k, \quad x \ge x_k.$$

Using (3.16) and the fact that $F_{\mu_0}(\psi) \geq 1$, from (3.13), it follows that for $x \geq x_{k+1}$, we have

$$
\begin{aligned}
|z(x)| \leq & \sum_{i\in I} |d_i| \left[|z(x-\sigma_i)| + \frac{1}{\Gamma_{\mu_0}} \int_{x-\sigma_i}^{x} \left|\rho_{\mu_0}(s)\right| |z(s)|\, ds \right] e^{-\mu_0\sigma_i} \\
& + \sum_{i\in I} e^{-\mu_0\tau_i} \int_{x-\tau_i}^{x} \left|\hat{\beta}_i(s)\right| |z(s)|\, ds + \int_{x}^{\infty} |h(s)| \exp\left[\frac{1}{\Gamma_{\mu_0}} \int_0^s \rho_{\mu_0}(v)dv \right] ds \\
\leq & F_{\mu_0}(\psi) \left(\varphi_{\mu_0}\right)^k \left\{ \sum_{i\in I} |d_i| \left[1 + \frac{1}{\Gamma_{\mu_0}} \int_{x}^{x-\sigma_i} \left|\rho_{\mu_0}(s)\right| ds \right] e^{-\mu_0\sigma_i} \right. \\
& \left. + \sum_{i\in I} e^{-\mu_0\tau_i} \int_{x-\tau_i}^{x} \left|\hat{\beta}_i(s)\right| ds + \int_{x_{k+1}}^{\infty} |h(s)| \exp\left[\frac{1}{\Gamma_{\mu_0}} \int_0^s \rho_{\mu_0}(v)dv \right] ds \right\} \\
\leq & F_{\mu_0}(\psi) \left(\varphi_{\mu_0}\right)^k \left\{ \sum_{i\in I} |d_i| \left[1 + \frac{G_{\mu_0}}{\Gamma_{\mu_0}} \sigma_i \right] e^{-\mu_0\sigma_i} + \sum_{i\in I} \tilde{b}_i \tau_i e^{-\mu_0\tau_i} \right. \\
& \left. + \left(\varphi_{\mu_0}\right)^k \int_0^{\infty} |h(s)| \exp\left[\frac{1}{\Gamma_{\mu_0}} \int_0^s \rho_{\mu_0}(v)dv \right] ds \right\} \\
\leq & F_{\mu_0}(\psi) \left(\varphi_{\mu_0}\right)^k \left\{ \sum_{i\in I} |d_i| \left[1 + \frac{G_{\mu_0}}{\Gamma_{\mu_0}} \sigma_i \right] e^{-\mu_0\sigma_i} + \sum_{i\in I} \tilde{b}_i \tau_i e^{-\mu_0\tau_i} + \int_0^{\infty} |h(s)| \exp\left[\frac{1}{\Gamma_{\mu_0}} \int_0^s \rho_{\mu_0}(v)dv \right] ds \right\} \\
= & F_{\mu_0}(\psi) \left(\varphi_{\mu_0}\right)^{k+1}.
\end{aligned}
$$

We thus obtain

$$
|z(x)| \leq F_{\mu_0}(\psi) \left(\varphi_{\mu_0}\right)^{k+1}, \quad x \geq x_{k+1}.
$$

Therefore, by the induction principle, we conclude that (3.22) holds true for all non-negative integers n. Finally, because of (3.6), we have $\lim_{x\to\infty} \left(\varphi_{\mu_0}\right)^n = 0$. So, as (3.22) is true for all $n = 0, 1, 2, \ldots$, we can easily find (3.15), that is, $\lim_{x\to\infty} z(x) = 0$. Hence, this completes the proof of the theorem.

□

Corollary 3. *Suppose that*

$$
\alpha(x) + \sum_{i\in I} \beta_i(x) = 0 \quad \text{for} \quad x \in [0, \infty) \tag{3.23}
$$

and

$$
\sum_{i\in I} \left[|d_i| + \tilde{b}_i \tau_i \right] + \int_0^{\infty} |h(s)|\, ds < 1. \tag{3.24}
$$

For any $\psi \in C\left([-\gamma, 0], \mathbb{R}\right)$*, the solution* u *of the IVP* (1.1)–(1.2) *satisfies*

$$
\lim_{x\to\infty} u(x) = \frac{\psi(0) + \sum_{i\in I} \left[d_i \psi(-\sigma_i) - \int_{-\tau_i}^{0} \hat{\beta}_i(s)\psi(s)ds \right] + \int_0^{\infty} h(s)ds}{1 + \sum_{i\in I} \left[d_i - b_i \tau_i \right]}.
$$

Note: It is established from (3.24) that $1 + \sum_{i\in I} [d_i - b_i \tau_i] > 0$.

Proof. It immediately follows from (3.23) that $a + \sum_{i\in I} b_i = 0$ and hence $\mu_0 = 0$ is a real root of (1.4). By using (3.23) again, we see that, for $\mu_0 = 0$, we have $\rho_{\mu_0} = 0$ on the interval $[-\gamma, \infty)$, and $G_{\mu_0} = 0$. Moreover, according to (3.24), it is not difficult to verify that the root $\mu_0 = 0$ of (1.4) has the property (3.6). Therefore, applying Theorem 2 with $\mu_0 = 0$ leads to Corollary 3.

□

Theorem 4. *Let μ_0 be a real root of the characteristic equation* (1.4) *with the property* (3.6), *and let ξ_{μ_0} and φ_{μ_0} be defined by* (3.5) *and* (3.6), *respectively. Set*

$$\Omega(\mu_0) = \frac{\left(1 + \varphi_{\mu_0}\right)^2}{1 + \xi_{\mu_0}} + \varphi_{\mu_0}. \tag{3.25}$$

For any $\psi \in C([-\gamma, 0], \mathbb{R})$, the solution u of the IVP (1.1)–(1.2) *satisfies*

$$|u(x)| \leq \Omega(\mu_0)\, Y_{\mu_0}(\psi) \exp\left[-\frac{1}{\Gamma_{\mu_0}} \int_0^x \rho_{\mu_0}(s)ds\right] \quad \textit{for all} \quad x \geq 0, \tag{3.26}$$

where

$$Y_{\mu_0}(\psi) = \max\left\{1, \max_{-\gamma\leq x\leq 0} |\psi(x)|, \max_{-\gamma\leq x\leq 0}\left\{|\psi(x)| \exp\left[\frac{1}{\Gamma_{\mu_0}} \int_0^x \rho_{\mu_0}(s)ds\right]\right\}\right\}. \tag{3.27}$$

Proof. Assume that u is the solution of (1.1)–(1.2), and hat y and z are as in the proof of Theorem 2, i.e., for $x \geq -\gamma$

$$y(x) = u(x) \exp\left\{\frac{1}{\Gamma_{\mu_0}} \int_0^x \rho_{\mu_0}(s)ds\right\} \quad \text{and} \quad z(x) = y(x) - \frac{L_{\mu_0}(\psi)}{1 + \xi_{\mu_0}},$$

where ρ_{μ_0}, Γ_{μ_0}, and L_{μ_0} are defined as in (1.5), (3.1), and (3.4), respectively. Moreover, let $F_{\mu_0}(\psi)$ be defined by (3.17). As in the proof of Theorem 2, it can be demonstrated that z satisfies (3.21); in other words

$$|z(x)| \leq F_{\mu_0}(\psi)\varphi_{\mu_0} \quad \text{for all} \quad x \geq 0.$$

By the definition of z, it follows that

$$|y(x)| \leq F_{\mu_0}(\psi)\varphi_{\mu_0} + \frac{\left|L_{\mu_0}(\psi)\right|}{1 + \xi_{\mu_0}} \quad \text{for all} \quad x \geq 0. \tag{3.28}$$

On the other hand, using (3.6) and (3.27), from (3.4), we get

$$\begin{aligned}\left|L_{\mu_0}(\psi)\right| &\leq |\psi(0)| + \sum_{i\in I} |d_i| \left[|\psi(-\sigma_i)| + \frac{1}{\Gamma_{\mu_0}} \int_{-\sigma_i}^{0} \left|\rho_{\mu_0}(s)\right| |\psi(s)|\, ds\right] e^{-\mu_0\sigma_i} \\ &+ \sum_{i\in I} e^{-\mu_0\tau_i} \int_{-\tau_i}^{0} \left|\hat{\beta}_i(s)\right| |\psi(s)| \exp\left[\frac{1}{\Gamma_{\mu_0}} \int_0^s \rho_{\mu_0}(v)dv\right] ds + \int_0^\infty |h(s)| \exp\left[\frac{1}{\Gamma_{\mu_0}} \int_0^s \rho_{\mu_0}(v)dv\right] ds \\ &\leq \left\{1 + \sum_{i\in I} |d_i| \left[1 + \frac{1}{\Gamma_{\mu_0}} \int_{-\sigma_i}^{0} \left|\rho_{\mu_0}(s)\right| ds\right] e^{-\mu_0\sigma_i} + \sum_{i\in I} e^{-\mu_0\tau_i} \int_{-\tau_i}^{0} \left|\hat{\beta}_i(s)\right| ds\right. \\ &\left.+ \int_0^\infty |h(s)| \exp\left[\frac{1}{\Gamma_{\mu_0}} \int_0^s \rho_{\mu_0}(v)dv\right] ds\right\} Y_{\mu_0}(\psi) \\ &\leq \left\{1 + \sum_{i\in I} |d_i| \left[1 + \frac{G_{\mu_0}}{\Gamma_{\mu_0}}\sigma_i\right] e^{-\mu_0\sigma_i} + \sum_{i\in I} \tilde{b}_i\tau_i e^{-\mu_0\tau_i} + \int_0^\infty |h(s)| \exp\left[\frac{1}{\Gamma_{\mu_0}} \int_0^s \rho_{\mu_0}(v)dv\right] ds\right\} Y_{\mu_0}(\psi),\end{aligned}$$

i.e.,

$$\left|L_{\mu_0}(\psi)\right| \leq \left(1 + \varphi_{\mu_0}\right) Y_{\mu_0}(\psi). \tag{3.29}$$

Additionally, using (3.27) and (3.29), from (3.17), we obtain

$$\begin{aligned}F_{\mu_0}(\psi) &\leq \max\left\{1, Y_{\mu_0}(\psi) + \frac{\left|L_{\mu_0}(\psi)\right|}{1+\xi_{\mu_0}}\right\} = Y_{\mu_0}(\psi) + \frac{\left|L_{\mu_0}(\psi)\right|}{1+\xi_{\mu_0}} \leq Y_{\mu_0}(\psi) + \frac{\left(1+\varphi_{\mu_0}\right) Y_{\mu_0}(\psi)}{1+\xi_{\mu_0}} \\ &= \left(1 + \frac{1+\varphi_{\mu_0}}{1+\xi_{\mu_0}}\right) Y_{\mu_0}(\psi).\end{aligned}$$

Thus, by combining (3.28) and (3.29), we get

$$|y(x)| \leq \left(1 + \frac{1+\varphi_{\mu_0}}{1+\xi_{\mu_0}}\right) Y_{\mu_0}(\psi)\varphi_{\mu_0} + \frac{1+\varphi_{\mu_0}}{1+\xi_{\mu_0}} Y_{\mu_0}(\psi) = \left(\frac{\left(1+\varphi_{\mu_0}\right)^2}{1+\xi_{\mu_0}} + \varphi_{\mu_0}\right) Y_{\mu_0}(\psi) = \Omega\left(\mu_0\right) Y_{\mu_0}(\psi),$$

where $\Omega\left(\mu_0\right)$ is defined as in (3.25). Using the definition of y, we obtain

$$|u(x)| \leq \Omega\left(\mu_0\right) Y_{\mu_0}(\psi) \exp\left\{-\frac{1}{\Gamma_{\mu_0}} \int_0^x \rho_{\mu_0}(s)ds\right\} \quad \text{for all} \quad x \geq 0.$$

This completes the proof of Theorem 4.

□

In the examples that follow, we will apply Theorem 2 to the asymptotic behavior of the solutions and Theorem 4 to the exponential estimation of solutions. To do this, in each example, a suitable root of the characteristic equation (1.4) is first found. Later, it is checked whether Condition (3.2) (or (3.6)) holds for the suitable root μ_0. Finally, (3.3) from Theorem 2 and (3.26) from Theorem 4 are applied. Let us look at the following examples for easier understanding.

Example 3.4. We consider

$$[u(x) + u(x-2\pi)]' + \left(e^{-2\pi}\cos x - 1\right) u(x) - (\cos x + 1)\, u(x-2\pi) = \frac{x}{2} \tag{3.30}$$

for $x \geq 0$ and

$$u(x) = \psi(x), \quad -2\pi \leq x \leq 0, \tag{3.31}$$

where $\psi(x)$ is an arbitrary continuous function on the interval $[-2\pi, 0]$. Since

$$a = \frac{1}{2\pi}\int_0^{2\pi}\left(e^{-2\pi}\cos x - 1\right)dx = -1 \quad \text{and} \quad b_1 = \frac{1}{2\pi}\int_0^{2\pi}(-\cos x - 1)\,dx = -1,$$

the characteristic equation of the homogeneous equation of (3.30) is from (1.4)

$$\mu\left(1 + e^{-2\pi\mu}\right) - 1 - e^{-2\pi\mu} = 0. \tag{3.32}$$

We clearly observe that $\mu_0 = 1$ is a unique real root of the characteristic equation (3.32). In Figure 1, we show the location of the root of (3.32). We check the condition (3.2) or (3.6) in Theorem 2. Since $\Gamma_{\mu_0} = 1 + e^{-2\pi}$, $\rho_{\mu_0}(x) = -\left(1 + e^{-2\pi}\right)$, $\tilde{b}_1 = 1$, and $G_{\mu_0} = 1 + e^{-2\pi}$, from (3.6), we obtain

$$\begin{aligned}\varphi_{\mu_0} &= (1 + 2\pi)e^{-2\pi} + 2\pi e^{-2\pi} + \int_0^{\infty}\frac{s}{2}\exp\left[\frac{1}{1 + e^{-2\pi}}\int_0^{s} -(1 + e^{-2\pi})dv\right]ds \\ &\cong 0,014 + 0,012 + \int_0^{\infty}\frac{s}{2}e^{-s}ds \cong 0,026 + \frac{1}{2} < 1.\end{aligned}$$

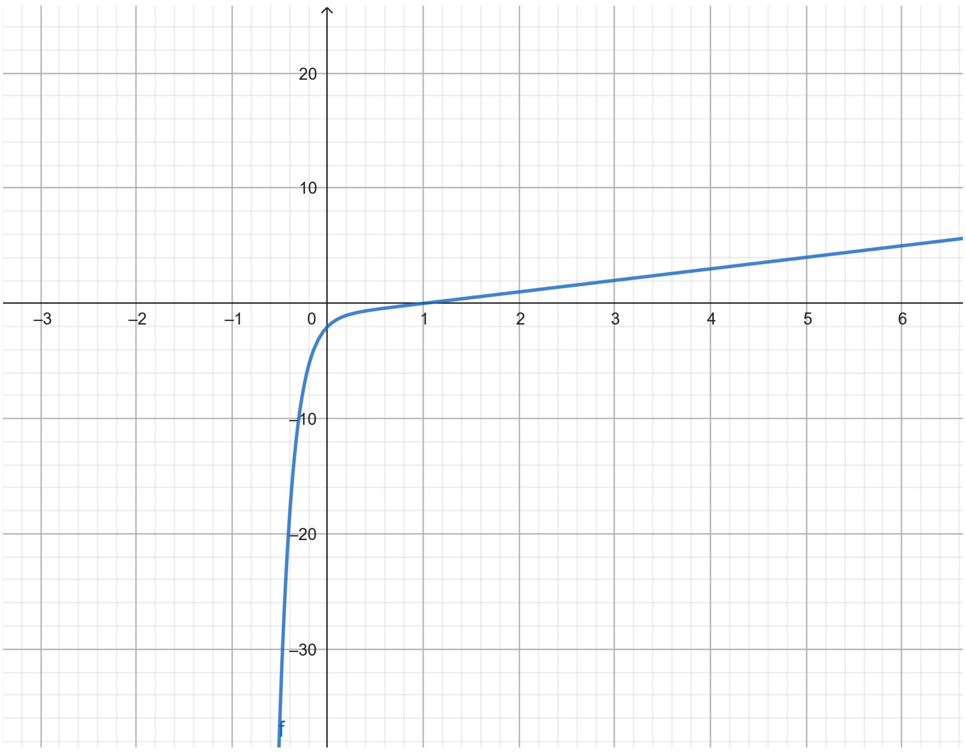

Figure 1. Location of the root of (3.32).

Therefore, (3.6) is satisfied. From (3.3) and (3.26), the solution u of (3.30) and (3.31) then satisfies

$$\lim_{x\to\infty}\{u(x)e^{-x}\} = \frac{L_{\mu_0}(\psi)}{1+e^{-2\pi}},$$

where

$$L_{\mu_0}(\psi) = \psi(0) + \left[\psi(-2\pi) - \int_{-2\pi}^{0}\psi(s)ds\right]e^{-2\pi} + e^{-2\pi}\int_{-2\pi}^{0}(\cos s+1)e^{-s}\psi(s)ds + \frac{1}{2},$$

and also

$$|u(x)| \leq \Omega(1)Y_1(\psi)e^{x} \quad \text{for all} \quad x \geq 0,$$

where

$$\Omega(1) = \frac{\left(\frac{3}{2}+0,026\right)^2}{1+e^{-2\pi}} + 0,026$$

and

$$Y_1(\psi) = \max\left\{1, \max_{-2\pi\leq x\leq 0}|\psi(x)|, \max_{-2\pi\leq x\leq 0}\left[\psi(x)e^{-2x}\right]\right\}.$$

Example 3.5. Consider

$$\left[u(x)+\frac{1}{2}u(x-1)\right]' - \left(\frac{1}{3}+\sin(2\pi x)\right)u(x) + \left(\frac{1}{3}+\sin(2\pi x)\right)u(x-1) = e^{-(x+2)} \tag{3.33}$$

for $x \geq 0$ and

$$u(x) = \psi(x), \quad -1 \leq x \leq 0, \tag{3.34}$$

where $\psi(x)$ is an arbitrary continuous function on the interval $[-1, 0]$. Since

$$a = \int_0^1 -\left(\frac{1}{3}+\sin(2\pi x)\right)dx = -\frac{1}{3} \quad \text{and} \quad b_1 = \int_0^1\left(\frac{1}{3}+\sin(2\pi x)\right)dx = \frac{1}{3},$$

the characteristic equation of the homogeneous equation of (3.33) is from (1.4) as follows:

$$\mu\left(1+\frac{1}{2}e^{-\mu}\right) - \frac{1}{3} + \frac{1}{3}e^{-\mu} = 0. \tag{3.35}$$

We clearly that $\mu_0 = 0$ is a unique real root of the characteristic equation (3.35). In Figure 2, we show the location of the root of (3.35). Since $\alpha(x)+\beta_1(x) = 0$, we look directly at Corollary 3. Since $\tilde{b}_1 = \frac{1}{3}$, from (3.24), we obtain

$$\left[\frac{1}{2}+\frac{1}{3}\right] + \int_0^{\infty} e^{-(s+2)}ds = \frac{5}{6} + \frac{1}{e^2} < 1.$$

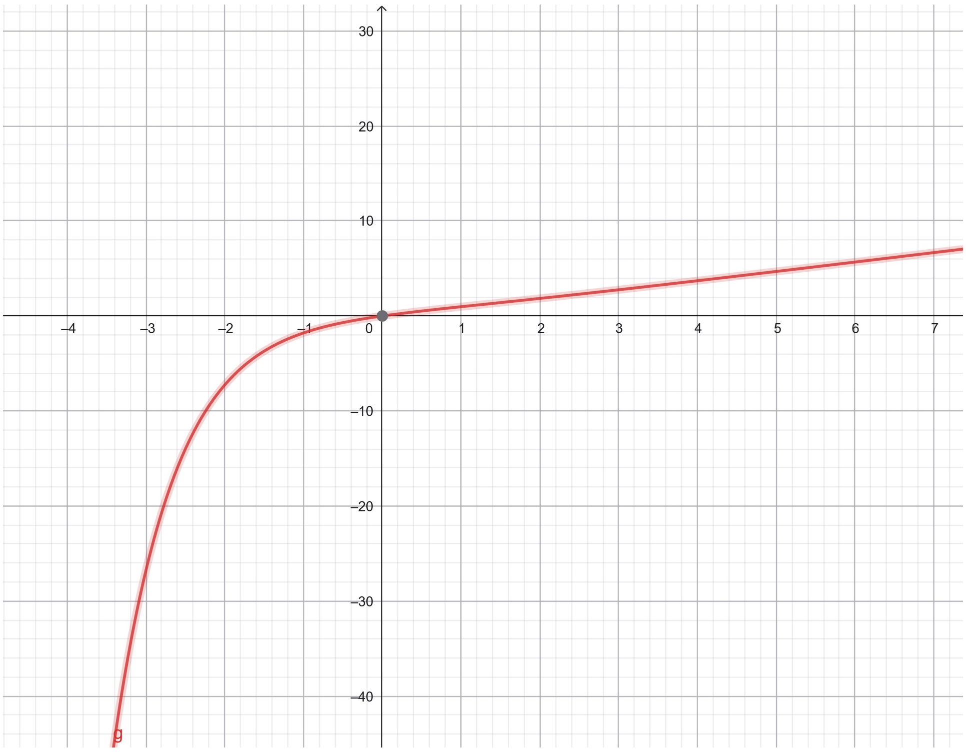

Figure 2. Location of the root of (3.35).

Therefore, (3.24) is satisfied. Then, the solution u of (3.33) and (3.34) satisfies

$$\lim_{x\to\infty} u(x) = \frac{\psi(0) + \left[\frac{1}{2}\psi(-1) - \int_{-1}^{0}\left(\frac{1}{3} + \sin(2\pi s)\right)\psi(s)ds\right] + \frac{1}{e^2}}{\frac{7}{6}}.$$

Likewise, from Theorem 4, we obtain

$$|u(x)| \leq \Omega(0)Y_0(\psi) \quad \text{for all} \quad x \geq 0,$$

where

$$\Omega(0) = \frac{\left(\frac{11}{6} + \frac{1}{e^2}\right)^2}{\frac{7}{6}} + \frac{5}{6} + \frac{1}{e^2} \quad \text{and} \quad Y_0(\psi) = \max\left\{1, \max_{-1\leq x\leq 0}|\psi(x)|\right\}.$$

Example 3.6. Consider

$$\left[u(x) - \frac{1}{e^2}u(x-1)\right]' - \cos(2\pi x)u(x) + \frac{1}{e^2}u(x-1) = 0 \quad \text{for} \quad x \geq 0, \tag{3.36}$$

and

$$u(x) = \psi(x), \quad -1 \leq x \leq 0, \tag{3.37}$$

where $\psi(x)$ is an arbitrary continuous function on the interval $[-1, 0]$. Since

$$a = \int_0^1 -cos(2\pi x)dx = 0 \quad \text{and} \quad b_1 = \int_0^1 \frac{1}{e^2}dx = \frac{1}{e^2},$$

the characteristic equation of the homogeneous equation of (3.36) is from (1.4) as follows:

$$\mu\left(1 - \frac{1}{e^2}e^{-\mu}\right) + \frac{1}{e^2}e^{-\mu} = e^{-\mu-2}(1-\mu) + \mu = 0. \tag{3.38}$$

We see that $\mu_1 \cong -1.485$ and $\mu_2 \cong -0.197$ are real roots of the characteristic equation (3.38). In Figure 3, we give the locations of the roots of (3.38). Let $\mu_0 = -1.485$. We check the condition (3.2) or (3.6) in Theorem 2. Since $\Gamma_{\mu_0} = 1 - e^{-0.515}$, $\tilde{b}_1 = \frac{1}{e^2}$, and $G_{\mu_0} = e^{-0.515}$, from (3.6), we obtain

$$\varphi_{\mu_0} = \frac{1}{e^2}\left(1 + \frac{e^{-0.515}}{1 - e^{-0.515}}\right)e^{1.485} + \frac{1}{e^2}e^{1.485} = e^{-0.515}\left(1 + \frac{1}{1 - e^{-0.515}}\right) \cong 2.082 > 1.$$

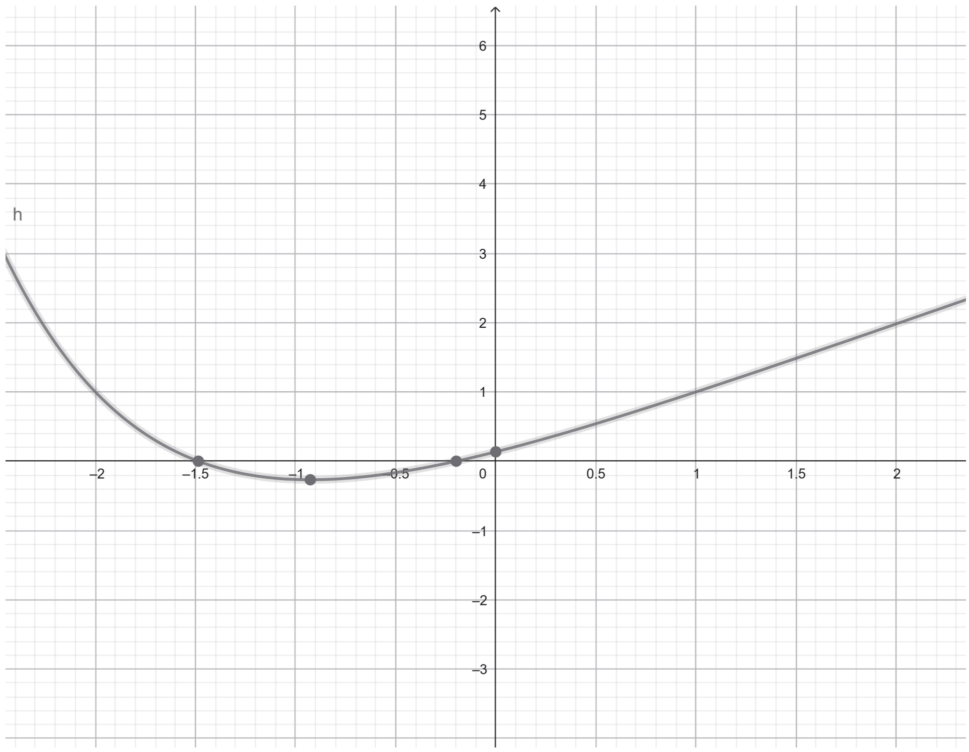

Figure 3. Locations of the roots of (3.38).

Therefore, Theorem 2 and Theorem 4 cannot be applied to Eq (3.36). On the other hand, for $\mu_0 = -0.197$, we get $\Gamma_{\mu_0} = 1 - e^{-1.803}$, $G_{\mu_0} = e^{-1.803}$, and

$$\varphi_{\mu_0} = \frac{1}{e^2}\left(1 + \frac{e^{-1.803}}{1 - ee^{-1.803}}\right)e^{0.197} + \frac{1}{e^2}e^{0.197} = e^{-1.803}\left(1 + \frac{1}{1 - e^{-1.803}}\right) \cong 0.362 < 1.$$

Therefore, (3.6) is satisfied. The solution u of (3.36) and (3.37) then satisfies

$$\lim_{x\to\infty}\left\{u(x)\exp\left[\frac{1}{1-e^{-1.803}}\int_0^x\left(\frac{1}{e^2}+\cos(2\pi s)\right)ds\right]\right\}=\frac{L_{\mu_0}(\psi)}{1-2.197e^{-1.803}},$$

and

$$|u(x)|\le\Omega(\mu_0)Y_{\mu_0}(\psi)\exp\left[-\frac{1}{1-e^{-1.803}}\int_0^x\left(\frac{1}{e^2}+\cos(2\pi s)\right)ds\right]\quad\text{for all}\quad x\ge 0,$$

where

$$\Omega(\mu_0)=\frac{(1+0.362)^2}{1-2.197e^{-1.803}}+0,362$$

and

$$Y_{\mu_0}(\psi)=\max\left\{1,\max_{-1\le x\le 0}|\psi(x)|,\max_{-1\le x\le 0}\left\{\psi(x)\exp\left[\frac{1}{1-e^{-1.803}}\int_0^x\left(\frac{1}{e^2}+\cos(2\pi s)\right)ds\right]\right\}\right\}.$$

4. The special case of linear nonhomogeneous delay differential equations with constant coefficients

In this section, we focus on the special case of first-order linear nonhomogeneous NDDEs with constant coefficients. The linear autonomous NDDE represents a specific variant of the NDDE (1.1).

$$\left[u(x)+\sum_{i\in I}d_iu(x-\sigma_i)\right]'+\alpha u(x)+\sum_{i\in I}\beta_iu(x-\tau_i)=h(x),\tag{4.1}$$

where α,β_i, and d_i for $i\in I$ are the real constants; τ_i for $i\in I$ are positive real numbers with $\tau_{i_1}\neq\tau_{i_2}$ for $i_1\neq i_2$; and σ_i for $i\in I$ are positive real numbers with $\sigma_{i_1}\neq\sigma_{i_2}$ for $i_1\neq i_2$. Moreover, $h(x)$ is a continuous real-valued function on the interval $[0,\infty)$.

The characteristic equation of the homogeneous equation of (4.1) is

$$\mu\left(1+\sum_{i\in I}d_ie^{-\mu\sigma_i}\right)+\alpha+\sum_{i\in I}\beta_ie^{-\mu\tau_i}=0.\tag{4.2}$$

The constant coefficients α and β_i in (4.1) can be considered to be P-periodic functions, for each real value $P>0$. Furthermore, as it concerns the autonomous NDDE (4.1), the hypothesis that positive integers n_i and m_i for $i\in I$ exist such that $\sigma_i=n_iP$ and $\tau_i=m_iT$ holds by itself. Given these considerations, the primary results of this study, namely, Theorem 2, Corollary 3, and Theorem 4, can be readily applied to the specific instance of the autonomous linear nonhomogeneous NDDE (4.1). Since Eq (4.1) features constant coefficients, the proofs for Theorem 5, Corollary 6, and Theorem 7 presented below become unnecessary.

Theorem 5. *Suppose that μ_0 is a real root of* (4.2) *with*

$$\tilde{\varphi}_{\mu_0}\equiv\sum_{i\in I}|d_i|\Big(1+|\mu_0|\sigma_i\Big)e^{-\mu_0\sigma_i}+\sum_{i\in I}|\beta_i|\tau_ie^{-\mu_0\tau_i}+\int_0^\infty|h(s)|e^{-\mu_0s}ds<1.\tag{4.3}$$

For any $\psi \in C\Big([-\gamma, 0], \mathbb{R}\Big)$*, the solution u of the IVP* (4.1) *and* (1.2) *then satisfies*

$$\lim_{x\to\infty}\{e^{\mu_0 x}u(x)\} = \frac{\tilde{L}_{\mu_0}(\psi)}{1+\sum_{i\in I}\Big[d_i(1-\mu_0\sigma_i)e^{-\mu_0\sigma_i}-\beta_i\tau_i e^{-\mu_0\tau_i}\Big]},$$

where

$$\tilde{L}_{\mu_0}(\psi) = \psi(0) + \sum_{i\in I} d_i\Big[\psi(-\sigma_i) - \mu_0 e^{-\mu_0\sigma_i}\int_{-\sigma_i}^{0} e^{-\mu_0 s}\psi(s)ds\Big] - \sum_{i\in I}\beta_i e^{-\mu_0\tau_i}\int_{-\tau_i}^{0} e^{-\mu_0 s}\psi(s)ds + \int_0^\infty h(s)e^{-\mu_0 s}ds.$$

Note: It is guaranteed by the property (4.3) that

$$1+\sum_{i\in I}\Big[d_i(1-\mu_0\sigma_i)e^{-\mu_0\sigma_i}-\beta_i\tau_i e^{-\mu_0\tau_i}\Big] > 0.$$

Application of Theorem 5 with $\mu_0 = 0$ leads to the following corollary.

Corollary 6. *Suppose that*

$$\alpha + \sum_{i\in I}\beta_i = 0 \quad \text{and} \quad \sum_{i\in I}\left[|d_i| + |\beta_i|\tau_i\right] + \int_0^\infty |h(s)|ds < 1. \tag{4.4}$$

For any $\psi \in C\Big([-\gamma, 0], \mathbb{R}\Big)$*, the solution u of the IVP* (4.1) *and* (1.2) *then satisfies*

$$\lim_{x\to\infty} u(x) = \frac{\psi(0) + \sum_{i\in I}\Big[d_i\psi(-\sigma_i) - \beta_i\int_{-\tau_i}^{0}\psi(s)ds\Big] + \int_0^\infty h(s)ds}{1+\sum_{i\in I}\left[d_i - \beta_i\tau_i\right]}.$$

Theorem 7. *Suppose that Theorem 5 is satisfied. Set*

$$\tilde{Y}_{\mu_0}(\psi) = \max\left\{1, \max_{-\gamma\le x\le 0}|\psi(x)|, \max_{-\gamma\le x\le 0}\Big[e^{-\mu_0 x}|\psi(x)|\Big]\right\}$$

For any $\psi \in C\Big([-\gamma, 0], \mathbb{R}\Big)$*, the solution u of the IVP* (4.1) *and* (1.2) *then satisfies*

$$|u(x)| \le \tilde{\Omega}(\mu_0)\tilde{Y}_{\mu_0}(\psi)e^{\mu_0 x} \quad \text{for all} \quad x \ge 0,$$

where

$$\tilde{\Omega}(\mu_0) = \frac{\Big(1+\tilde{\varphi}_{\mu_0}\Big)^2}{1+\sum_{i\in I}\Big[d_i(1-\mu_0\sigma_i)e^{-\mu_0\sigma_i}-\beta_i\tau_i e^{-\mu_0\tau_i}\Big]} + \tilde{\varphi}_{\mu_0}.$$

Example 4.4. Consider the following for $x \geq 0$:

$$\Big[u(x) + \frac{1}{4}u(x-\sigma_1) - \frac{1}{5}u(x-\sigma_2)\Big]' + \frac{1}{4}u(x) + \frac{1}{4}u(x-\frac{1}{2}) - \frac{1}{2}u(x-\frac{1}{4}) = \frac{e^{-x}}{4}, \tag{4.5}$$

$$u(x) = \psi(x), \quad -\gamma \leq x \leq 0, \tag{4.6}$$

where σ_1, σ_2 are any arbitrary constants, $\gamma = \max\left\{\sigma_1, \sigma_2, \frac{1}{2}, \frac{1}{4}\right\}$, and $\psi \in C([-\gamma, 0], \mathbb{R})$.

The characteristic equation of the homogeneous equation of (4.5) is

$$\mu\Big(1 + \frac{1}{4}e^{-\mu\sigma_1} - \frac{1}{5}e^{-\mu\sigma_2}\Big) + \frac{1}{4} + \frac{1}{4}e^{-\frac{\mu}{2}} - \frac{1}{2}e^{-\frac{\mu}{4}} = 0. \tag{4.7}$$

Here, we easily see that $\mu = 0$ is real root of (4.7). In Figure 4, we present the locations of the roots of (4.7) for different σ_1 and σ_2 values. From (4.3), for $\mu_0 = 0$, we obtain

$$\tilde{\varphi}_0 \equiv \frac{1}{4} + \frac{1}{5} + \frac{1}{4}.\frac{1}{2} + \frac{1}{2}.\frac{1}{4} + \int_0^\infty \frac{e^{-s}}{4} ds = \frac{19}{20} < 1.$$

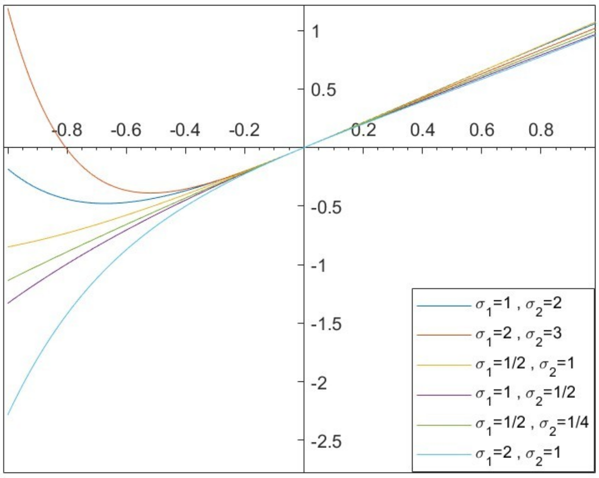

Figure 4. Locations of the roots of (4.7).

For any $\psi \in C([-\gamma, 0], \mathbb{R})$, the solution u of the IVP (4.5)–(4.6) then satisfies

$$\lim_{x\to\infty} u(x) = \frac{\psi(0) + \frac{1}{4}\psi(-\sigma_1) - \frac{1}{5}\psi(-\sigma_2) - \frac{1}{4}\int_{-\frac{1}{2}}^0 \psi(s)ds + \frac{1}{2}\int_{-\frac{1}{4}}^0 \psi(s)ds + \frac{1}{4}}{\frac{21}{20}}$$

and

$$|u(x)| \leq \tilde{\Omega}(0)\tilde{Y}_0(\psi) \quad \text{for all} \quad x \geq 0,$$

where

$$\tilde{\Omega}(0) = \frac{(1+\tilde{\varphi}_0)^2}{\frac{21}{20}} + \tilde{\varphi}_0 = \frac{\left(1+\frac{19}{20}\right)^2}{\frac{21}{20}} + \frac{19}{20} = \frac{32}{7} \quad \text{and} \quad \tilde{Y}_0(\psi) = \max\left\{1, \max_{-\gamma\le x\le 0} |\psi(x)|\right\}.$$

5. Conclusions

In this work, we first establish sufficient conditions ensuring the existence of periodic solutions for Equation (1.1). Subsequently, we prove that a fundamental asymptotic criterion exists for the solutions of the IVP (1.1)–(1.2). Ultimately, by using this asymptotic criterion, we reach a useful exponential estimate for the solutions of (1.1)–(1.2). These results are obtained by using an appropriate real root for the characteristic equation. This real root serves a significant function in determining the results of this study. We also present the application of the obtained results to the special case with constant coefficients. We provide six examples in this article.

Author contributions

Ali Fuat Yeniçerioğlu: Conceptualization, methodology, software development, validation, formal analysis, writing–original draft, writing–revised draft, investigation; Vildan Yazıcı: Conceptualization, formal analysis, validation, writing–original draft, writing–revised draft, investigation; Cüneyt Yazıcı: Conceptualization, formal analysis, validation, writing–original draft, writing–revised draft, investigation. All authors have read and approved the final version of the manuscript for publication.

Conflict of interest

The authors declare no conflicts of interest.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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