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Research article

Poisson quasi-Lindley and Poisson-new XLindley univariate and bivariate models: derivation techniques and automobile insurance applications

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Abstract: The Poisson quasi-Lindley and the Poisson-new XLindley distributions are revisited, emphasizing on alternative derivation techniques. These distributions can be derived as Poisson mixtures when (i) the probability density function of the mixing distribution is known, (ii) the moment generating function of the mixing distribution is known, or (iii) the regression function of the mixing continuous random variable on the mixed discrete random variable is of a known form. Furthermore, they can be derived by the addition of independent random variables. An indication that the Poisson-new XLindley distribution is a member of the class of Poisson quasi- Lindley models is also given. An Extended Poisson quasi-Lindley (EPQL) distribution is constructed following the above derivation procedures and, as a generalized binomial distribution, it is extensively studied, highlighting its role as a marginal distribution in bivariate settings. Two general and structurally different bivariate Poisson quasi-Lindley and Poisson-new XLindley distributions are then introduced utilizing various techniques, including mixing, generalization, addition of independent bivariate random variables, regression functions, and conditional distributions. These bivariate models exhibit positive correlation and over-dispersed marginals. Several of their characteristics are derived, including probability generating functions, probabilities and their recurrences, moments, conditional distributions, and regression functions. The special feature of these general models is that several of their members, including bivariate Poisson-new XLindley distributions, are fitted satisfactorily to different sets of automobile insurance data previously used in the literature. In particular, members of the first bivariate framework are applied to three sets of data involving the number of claims and claim amounts, while members of the second framework are fitted to data concerning material damage and bodily injury from portfolios of liability insurance policies. Finally, suggestions for future research are also provided.

Keywords: Poisson quasi-Lindley distribution; Poisson-new XLindley distribution; bivariate models; automobile insurance data; addition of independent bivariate RVs

Mathematics Subject Classification: 60E05, 62E15, 62P05

1. Introduction

Availability of count data, not only for one variable of interest but also for two (or more) dependent variables, is rapidly increasing in a variety of human activity sectors including medicine, ecology, genetics, economy, actuarial studies, and sports. One of the main problems faced by researchers in modeling count data is over-dispersion. Therefore, for modeling them, we should employ univariate distributions with the property of over-dispersion or bivariate (multivariate) models with over-dispersed marginals.

Since univariate distributions derived as Poisson mixtures possess this property, as pointed out by [1], there is an ongoing activity in recent years for introducing new univariate and subsequently bivariate (multivariate) Poisson-Lindley type models. In the univariate case various Poisson-Lindley type models were derived. We refer among others, to [2–7]. Bivariate (multivariate) Poisson-Lindley distributions were introduced by [8–13].

A Poisson-Lindley-type model that has attracted the attention of several authors is the Poisson quasi-Lindley distribution introduced by [14, 15]. Additional properties, extensions, and applications were studied among others by [16–19].

Recently, a simple one-parameter discrete model, the Poisson-new XLindley distribution, was introduced by [20, 21]. Further properties of this distribution were examined by [22–24], and applications were suggested in the areas of ecology, medicine, veterinary medicine, and actuarial science. A bivariate version of this model was introduced and studied by [25] with applications in soccer.

Poisson-Lindley-type models are usually derived as Poisson mixtures when the probability density function (PDF) of the mixing distribution is known. In this paper, the Poisson quasi-Lindley and the Poisson-new XLindley distributions are revisited, and alternative derivation procedures are suggested. We prove that these distributions can be obtained as Poisson mixtures when any one of: (i) the PDF, (ii) the moment generating function (MGF) or (iii) the conditional expectation of the continuous mixing random variable (RV) on the discrete mixed RV is known. Finally, they can also be derived by the addition of a geometric distribution with an independent inflated (or zero-modified or with added zeros) geometric distribution with the same parameter. Furthermore, we demonstrate that the one parameter Poisson-new XLindley distribution, as well as other models, can be regarded as members of the more general class of Poisson quasi-Lindley distributions.

We proceed by introducing and extensively studying an EPQL distribution. This distribution can be derived not only by the procedures suggested for the Poisson quasi-Lindley model but also by generalizing a binomial distribution when its exponent follows a Poisson quasi-Lindley model. This technique can be readily applied for introducing multivariate Poisson quasi-Lindley and multivariate Poisson-new XLindley distributions, see [12]. An additional reason for introducing this distribution is that it appears as marginal distribution in bivariate models.

However, the main contribution of this paper is the introduction of two general, structurally different Poisson quasi-Lindley distributions (X_1, X_2) with Poisson-new XLindley distributions as examples.

Each model has positive correlation and over-dispersed marginals and can be obtained by various techniques. The first one is derived by mixing a bivariate Poisson (Poisson-Bernoulli) model with a univariate Poisson quasi-Lindley distribution with known MGF. An alternative derivation is by assuming that the conditional distribution of the RV X_2, given the RV X_1, is binomial and requiring any one of the following three characteristics to be known: the probability function (PF) of the marginal RV X_1, the probability generating function (PGF) of the RV X_1, or the conditional expectation of the RV X_1 on the RV X_2. Finally, this bivariate Poisson quasi-Lindley distribution can be obtained by the addition of a bivariate geometric-Bernoulli distribution with an independent bivariate geometric-Bernoulli with added zeros in the $(0, 0)$ cell and the same parameters. The other bivariate model is preferably derived by generalizing a bivariate binomial distribution when its exponent follows a univariate Poisson quasi-Lindley distribution. Other methods involve mixing a well-known version of a bivariate Poisson distribution or adding a bivariate geometric distribution with an independent bivariate geometric distribution with added zeros in the $(0, 0)$ cell and the same parameters. Various properties of these distributions are obtained, including the PGF's recurrences for probabilities, moments, conditional distributions, and regression functions.

The special feature of these general models is that several of their members, including bivariate Poisson-new XLindley distributions, are fitted satisfactorily to various sets of data. A variety of examples related to two types of problems faced by automobile insurance companies are given. For demonstration purposes, we apply bivariate Poisson-new XLindley and three other members of each bivariate model to different data sets of automobile insurance portfolios. The first type of data refers to the number of claims and the claim size utilized in calculating bonus-malus premiums and we consider three different examples previously used by [13, 26–28]. The second type of data refers to material damage and bodily injury claims from a portfolio of liability policies, introduced in the literature by [29] and subsequently used by [8, 12, 13, 30, 31] to demonstrate the applicability of their bivariate models.

The rest of the paper is structured as follows. In Section 2, we briefly discuss the Poisson quasi-Lindley and the Poisson-new XLindley distributions focusing on procedures leading to their derivation. We also indicate interrelations between them. In Section 3, we introduce and extensively study an EPQL model. In Sections 4 and 5, bivariate general Poisson quasi-Lindley distributions are introduced and studied, emphasizing on derivation techniques. Sections 6 and 7 deal with applications of a variety of members of both bivariate models previously introduced, in various sets of real data from automobile insurance companies. Finally, Section 8 concludes.

2. The Poisson quasi-Lindley and the Poisson-new XLindley distributions

In this section various derivation techniques are suggested for the Poisson quasi-Lindley and Poisson-new XLindley models. Furthermore, interrelations between them are indicated, and relative properties are given.

2.1. The Poisson quasi-Lindley model

This distribution was introduced by [14] under the name quasi Poisson-Lindley and [15] as a Poisson mixture.

2.1.1. Definition and genesis

Definition 2.1. *An RV X is said to have a Poisson quasi-Lindley distribution with parameters $\alpha > 0$, $\theta > 0$ if its PF is*

$$P(X = x; \alpha, \theta) = \frac{\theta}{\alpha + 1} \frac{\alpha(\theta + 1) + \theta(x + 1)}{(\theta + 1)^{x+2}}, \quad x = 0, 1, \ldots, \tag{2.1}$$

or if its PGF is

$$G_X(s) = \frac{\theta}{\alpha + 1} \frac{\alpha(\theta - s + 1) + \theta}{(\theta - s + 1)^2}. \tag{2.2}$$

2.1.2. Derivation as a Poisson (λ) mixture

Consider a Poisson distribution with parameter λ, and assume that the parameter is not constant but a continuous RV with PDF $f(\lambda)$.

Definition 2.2. *A non-negative integer-valued RV X follows a mixed Poisson distribution if its PF $P(X = x)$ is given by*

$$P(X = x) = \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!} f(\lambda) d\lambda, \quad x = 0, 1, \ldots, \tag{2.3}$$

where $f(\lambda)$ is the PDF of the mixing distribution.

Consequently, the PGF of the RV X is

$$\begin{aligned} G_X(s) &= \int_0^\infty e^{\lambda(s-1)} f(\lambda) d\lambda \\ &= M_\Lambda(s - 1), \end{aligned} \tag{2.4}$$

where $M_\Lambda(\cdot)$ is the MGF of the mixing distribution evaluated at $s - 1$.

It is customary for Poisson mixtures to utilize the PDF $f(\lambda)$ of a known mixing distribution to derive the PF $P(X = x)$ of the mixed distribution from Eq (2.3). However, alternative approaches can also be employed. Assuming, for example, that the MGF $M_\Lambda(\cdot)$ is known, from Eq (2.4), we can derive the PGF $G_X(\cdot)$. In addition, from the knowledge of the regression function $E[\Lambda \mid X = x]$, the PF $P(X = x)$ can be obtained. These procedures are also of interest since often, they result in simplifications in the derivation of various characteristics of the mixed distribution, are more easily extended to higher dimensions (see for example [32]), and are useful in certain applications as suggested by [33].

i) Derivation when the PDF $f(\lambda)$ of the mixing distribution is known.

As assumed by [14, 15], when the mixing RV Λ follows a quasi-Lindley distribution with PDF

$$f(\lambda; \alpha, \theta) = \frac{\theta(\alpha + \theta\lambda)}{\alpha + 1} e^{-\theta\lambda} \quad \lambda > 0, \;\; \theta > 0, \;\; \alpha > 0 \tag{2.5}$$

introduced by [34] then, from Eq (2.3), they derived the PF of the Poisson quasi-Lindley distribution given by Eq (2.1).

ii) Derivation when the MGF $M_\Lambda(\cdot)$ of the mixing distribution is known.

Since the MGF of the quasi-Poisson distribution is

$$M_\Lambda(s) = \frac{\theta}{\alpha+1}\,\frac{\alpha(\theta-s)+\theta}{(\theta-s)^2}, \tag{2.6}$$

from Eq (2.4), the PGF of the Poisson quasi-Lindley distribution given by Eq (2.2) is immediately derived.

iii) Derivation when the regression function $m(x) = E[\Lambda \mid X = x]$ is known.

This technique is closely related to a characterization theorem proved by [11]. For relevant papers, see [33, 35].

Proposition 2.1. *Consider a Poisson mixture defined by Eq (2.3).*

Also, let

$$m(x) = \frac{x+1}{\theta+1}\,\frac{\alpha(\theta+1)+\theta(x+2)}{\alpha(\theta+1)+\theta(x+1)}.$$

Then $P(X = x)$ is given by Eq (2.1).

Proof. Since

$$m(x) = \int_0^\infty \lambda p(\lambda \mid x) d\lambda,$$

we obtain

$$E[\Lambda \mid X = x] = (x+1)\frac{P(X = x+1)}{P(X = x)},$$

or

$$\begin{aligned} P(X = x) &= P(X = 0)\prod_{k=0}^{x-1}\frac{m(k)}{k+1} \\ &= P(X = 0)\frac{1}{(\theta+1)^x}\,\frac{\alpha(\theta+1)+\theta(x+1)}{\alpha(\theta+1)+\theta} \\ &= \frac{\theta}{\alpha+1}\,\frac{\alpha(\theta+1)+\theta(x+1)}{(\theta+1)^{x+2}}, \quad x = 0, 1, 2, \dots . \end{aligned}$$

This expression corresponds to Eq (2.1) since

$$P(X = 0) = \frac{\theta}{\alpha+1}\,\frac{\alpha(\theta+1)+\theta}{(\theta+1)^2} \tag{2.7}$$

is determined from the initial condition $\sum_x P(X = x) = 1$.

2.1.3. Derivation by the addition of independent RVs

In this subsection, we prove that the Poisson quasi-Lindley model can be derived not only as a Poisson mixture but also by the addition of a geometric and an independent inflated (or zero-modified or with added zeros) geometric distribution with the same parameter.

Proposition 2.2. *Consider two independent RVs Z_1 and Z_2. Let Z_1 be a geometric distribution with parameter $\frac{\theta}{\theta+1}$ and PGF*

$$G_{Z_1}(s) = \frac{\theta}{\theta - s + 1}. \tag{2.8}$$

Also, assume that the RV Z_2 is distributed as an inflated geometric with inflation parameter ω and PGF given by [36]

$$G_{Z_2}(s) = \omega + (1-\omega)G_{Z_1}(s). \tag{2.9}$$

Then, if

$$\omega = \frac{\alpha}{\alpha+1}, \tag{2.10}$$

i)

$$G_{Z_2}(s) = \frac{\alpha(\theta - s + 1) + \theta}{(\alpha+1)(\theta - s + 1)}, \tag{2.11}$$

and
ii) the RV $Z_1 + Z_2$ follows a Poisson quasi-Lindley distribution with PGF given by Eq (2.2).

Proof. From Eqs (2.8)–(2.10),

$$\begin{aligned} G_{Z_2}(s) &= \frac{\alpha}{\alpha+1} + \left(1 - \frac{\alpha}{\alpha+1}\right)\frac{\theta}{(\theta - s + 1)} \\ &= \frac{\alpha(\theta - s + 1) + \theta}{(\alpha+1)(\theta - s + 1)}. \end{aligned}$$

In addition, from Eqs (2.8) and (2.11),

$$\begin{aligned} G_{Z_1+Z_2}(s) &= G_{Z_1}(s)G_{Z_2}(s) \\ &= \frac{\theta}{\theta - s + 1}\,\frac{\alpha(\theta - s + 1) + \theta}{(\alpha+1)(\theta - s + 1)} \\ &= \frac{\theta}{\alpha+1}\,\frac{\alpha(\theta - s + 1) + \theta}{(\theta - s + 1)^2}, \end{aligned}$$

which is the required PGF (2.2).

2.1.4. Properties

Considering the PGF given by Eq (2.2), we can easily obtain

$$\frac{\partial^x G(s)}{\partial s^x} = x!\frac{\theta}{\alpha+1}\,\frac{\alpha(\theta - s + 1) + \theta(x+1)}{(\theta - s + 1)^{x+2}}. \tag{2.12}$$

From Eq (2.12), expressions for probabilities and factorial moments can be derived. In particular,

$$P(X = x; \alpha, \theta) = \frac{\theta}{\alpha + 1} \frac{\alpha(\theta + 1) + \theta(x + 1)}{(\theta + 1)^{x+2}}, \quad \alpha, \theta > 0, \quad x = 0, 1, \ldots$$

and

$$\mu_{[\tau]:X} = \frac{\tau!}{\alpha + 1} \frac{\alpha + \tau + 1}{\theta^{\tau}}, \quad \tau = 1, 2, \ldots, \tag{2.13}$$

where

$$\mu_{[\tau]:X} = E(X^{(\tau)}) \quad \text{and} \quad X^{(\tau)} = X(X-1)\ldots(X-\tau+1).$$

From Eq (2.13),

$$E(X) = \frac{\alpha + 2}{(\alpha + 1)\theta}.$$

The probabilities can be calculated recursively from the relation

$$P(X = x + 1) = \frac{1}{\theta + 1} \frac{\alpha(\theta + 1) + \theta(x + 2)}{\alpha(\theta + 1) + \theta(x + 1)} P(X = x), \quad x = 0, 1, \ldots$$

with $P(X = 0)$ given by Eq (2.7).

2.2. *The Poisson-new XLindley model*

This one-parameter distribution was introduced by [20, 21] as a Poisson mixture.

2.2.1. Definition and genesis

An RV X is said to have a Poisson-new XLindley distribution with parameter $\theta > 0$ if its PF is

$$P(X = x; \theta) = \frac{\theta}{2} \frac{1 + \theta(x + 2)}{(\theta + 1)^{x+2}}, \quad x = 0, 1, \ldots \tag{2.14}$$

or if its PGF is

$$G_X(s) = \frac{\theta}{2} \frac{2\theta - s + 1}{(\theta - s + 1)^2}. \tag{2.15}$$

2.2.2. Derivation as a Poisson mixture

i) Derivation when the PDF $f(\lambda)$ of the mixing distribution is known.

This technique was used by [20, 21]. In particular, they assumed that the mixing RV Λ follows the new XLindley distribution, which is a special case of the one-parameter polynomial exponential distribution proposed by [37]. The new XLindley distribution has PDF

$$f(\lambda; \theta) = \frac{\theta(1 + \theta\lambda)}{2} e^{-\theta\lambda}, \quad \lambda > 0, \quad \theta > 0.$$

This distribution was introduced and extensively studied by [38]. Several alternative simple models related to the Lindley distribution were discussed by [39]. From the above equation and Eq (2.3), we obtain the PF of the Poisson-new XLindley distribution given by Eq (2.14).

ii) Derivation when the MGF of the mixing distribution is known.
The MGF of the new XLindley distribution is

$$M_{\Lambda}(s) = \frac{\theta}{2} \frac{2\theta - s}{(\theta - s)^2}. \tag{2.16}$$

From Eq (2.4), the PGF of the Poisson-new-XLindley distribution given by Eq (2.15) is immediately obtained.

iii) Derivation when the regression function $m(x) = E[\Lambda \mid X = x]$ is known.

Proposition 2.3. *Consider a Poisson mixture defined by Eq (2.3). Also, let*

$$m(x) = \frac{x+1}{\theta+1} \frac{1+\theta(x+3)}{1+\theta(x+2)}.$$

Then $P(X = x)$ is given by Eq (2.14).

Proof. Similar to the one for Proposition 2.1.

2.2.3. Derivation by the addition of independent RVs

Proposition 2.4. *Consider two independent RVs Z_1 and Z_2. Let Z_1 follow a geometric distribution with PGF given by Eq (2.8). Also, assume that the RV Z_2 is distributed as an inflated geometric with inflation parameter $\omega = \frac{1}{2}$ and PGF*

$$\begin{aligned} G_{Z_2}(s) &= \frac{1}{2} + \frac{1}{2} G_{Z_1}(s) \\ &= \frac{2\theta - s + 1}{2(\theta - s + 1)}. \end{aligned}$$

Then

$$G_{Z_1+Z_2}(s) = \frac{\theta}{2} \frac{2\theta - s + 1}{(\theta - s + 1)^2},$$

which is the PGF given by (2.15).

Proof. Similar to the one given for Proposition 2.2.

2.2.4. Properties

From the PGF given by Eq (2.15), we can easily obtain

$$\frac{\partial^x G(s)}{\partial s^x} = x! \frac{\theta}{2} \frac{(\theta - s + 1) + \theta(x+1)}{(\theta - s + 1)^{x+2}}. \tag{2.17}$$

Expressions for probabilities and factorial moments can be derived from (2.17). In particular,

$$P(X = x; \theta) = \frac{\theta}{2} \frac{1 + \theta(x+2)}{(\theta+1)^{x+2}}, \quad x = 0, 1, \ldots, \quad \theta > 0$$

and

$$\mu_{[\tau]:X} = \frac{\tau!}{2}\,\frac{\tau+2}{\theta^{\tau}}, \quad \tau = 1, 2, \dots \,. \tag{2.18}$$

From Eq (2.18),

$$E(X) = \frac{3}{2\theta}.$$

The probabilities can be calculated recursively from the relation

$$P(X = x+1) = \frac{1}{\theta+1}\,\frac{1+\theta(x+3)}{1+\theta(x+2)}P(X = x), \quad x = 0, 1, \dots,$$

with

$$P(X = 0) = \frac{\theta}{2}\,\frac{2\theta+1}{(\theta+1)^2}.$$

Remark 2.1. The Poisson-new XLindley distribution is a member of the class of Poisson quasi-Lindley distributions for $\alpha = 1$.

3. An EPQL distribution

In this section, we introduce and study in detail an EPQL model. This distribution is a member of three general classes of discrete distributions, inheriting important theoretical properties from each one of them. In particular, it can be derived (i) by generalizing a binomial distribution with respect to its exponent, (ii) as a Poisson mixture, and (iii) by addition of independent RVs. Moreover, it offers more flexibility for interpreting complex real-world data, e.g. for modeling counts with an exposure/weighting factor (p). Possible application areas are: accident theory (fatal or non-fatal accidents, also accidents involving material damage or bodily injury), biology (cell dynamics, genetics), physics (particle production), environmental science (weather patterns), and finance (option pricing). However, the main motivation for the introduction of the EPQL model is that it appears as marginal distribution in two structurally different bivariate Poisson quasi-Lindley distributions introduced and studied in the subsequent sections.

3.1. Derivation and genesis

Definition 3.1. *An RV Y is said to have an EPQL distribution with parameters, $\alpha > 0$, $\theta > 0$, $0 < p < 1$, if its PF is*

$$P(Y = y; \alpha, \theta, p) = p^y \frac{\theta}{\alpha+1}\,\frac{\alpha(\theta+p)+\theta(y+1)}{(\theta+p)^{y+2}}, \quad y = 0, 1, \dots, \tag{3.1}$$

or if its PGF is

$$G_Y(s) = \frac{\theta}{\alpha+1}\,\frac{\alpha(\theta-ps+p)+\theta}{(\theta-ps+p)^2}. \tag{3.2}$$

3.1.1. Derivation as a generalized binomial distribution

In addition to the derivation techniques described for the Poisson quasi-Lindley model, the EPQL distribution can also be obtained by assuming that the exponent of a binomial distribution is an RV distributed as a Poisson quasi-Lindley model. This derivation enables us to evaluate several characteristics of the EPQL distribution utilizing properties of the Poisson quasi-Lindley distribution. Furthermore, this technique can be used for the derivation of bivariate and multivariate Poisson quasi-Lindley distributions, see for example [11, 12].

Consider an RV Y with PGF

$$E(s^Y \mid N = n) = (q + ps)^n, \quad 0 < p < 1, \quad q = 1 - p,$$

where N is a non-negative integer-valued RV with PF $P(N = n)$ and PGF

$$E(s^N) = h_N(s).$$

Then the PGF of the RV Y is

$$G_Y(s) = h_N(q + ps). \tag{3.3}$$

If N is distributed as a Poisson quasi-Lindley distribution with PGF given by Eq (2.2), then

$$G_Y(s) = \frac{\theta}{\alpha + 1} \frac{\alpha(\theta - ps + p) + \theta}{(\theta - ps + p)^2},$$

which is the PGF of an EPQL model given by Eq (3.2).

3.1.2. Derivation as a Poisson (λp) mixture

Consider a class of mixed Poisson (λp) distributions, where $0 < p < 1$ is constant and the parameter λ is a continuous RV with PDF $f(\lambda)$, then

$$P(Y = y) = p^y \int_0^\infty e^{-\lambda p} \frac{\lambda^y}{y!} f(\lambda) d\lambda \tag{3.4}$$

and the MGF

$$G_Y(s) = M_\Lambda(p(s - 1)). \tag{3.5}$$

Utilizing Eqs (3.4) or (3.5), the following techniques can be used:

i) Derivation when the PDF of the mixing distribution is known.

If we assume that the RV Λ follows a quasi-Lindley distribution with PDF given by Eq (2.5), then from Eq (3.4) we can obtain the PF of the EPQL distribution given by Eq (3.1).

ii) Derivation when the MGF $M_\Lambda(\cdot)$ of the mixing distribution is known.

From Eqs (2.6) and (3.5), the PGF of the EPQL distribution given by Eq (3.2) is immediately derived.

iii) Derivation when the regression function $m(y) = E[\Lambda \mid Y = y]$ is known.

Proposition 3.1. *Consider a Poisson mixture defined by Eq (3.4). Also, let*

$$m(y) = \frac{y+1}{\theta+p}\frac{\alpha(\theta+p)+\theta(y+2)}{\alpha(\theta+p)+\theta(y+1)}.$$

Then, $P(Y=y)$ is given by Eq (3.1).

Proof. We have

$$\begin{aligned} m(y) &= \int_0^\infty \lambda p(\lambda \mid y) d\lambda \\ &= \frac{y+1}{pP(Y=y)}\int_0^\infty e^{-\lambda p}\frac{(\lambda p)^{y+1}}{(y+1)!}f(\lambda)d\lambda \\ &= \frac{y+1}{p}\frac{P(Y=y+1)}{P(Y=y)}. \end{aligned}$$

The remaining proof is similar to the one given for Proposition 2.1.

3.1.3. Derivation by the addition of independent RVs

Proposition 3.2. *Consider two independent RVs Z_3 and Z_4. Let Z_3 follow a geometric distribution with parameter $\frac{\theta}{\theta+p}$ and PGF*

$$G_{Z_3}(s) = \frac{\theta}{\theta-ps+p}.$$

Also, assume that the RV Z_4 is distributed as an inflated geometric with PGF given by

$$G_{Z_4}(s) = \frac{\alpha}{\alpha+1} + \left(1-\frac{\alpha}{\alpha+1}\right)G_{Z_3}(s).$$

Then

i)
$$G_{Z_4}(s) = \frac{\alpha(\theta-ps+p)+\theta}{(\alpha+1)(\theta-ps+p)},$$

and

ii) the RV Z_3+Z_4 follows an EPQL distribution with PGF given by

$$G_{Z_3+Z_4}(s) = \frac{\theta}{\alpha+1}\frac{\alpha(\theta-ps+p)+\theta}{(\theta-ps+p)^2}.$$

Proof. The proof is similar to the one given for Proposition 2.2.

3.2. Properties

From Eq (3.3), we can easily obtain

$$\frac{\partial^y G_Y(s)}{\partial s^y} = p^y\frac{\partial^y h_N(q+ps)}{\partial s^y}, \tag{3.6}$$

and from Eq (3.6),

$$P(Y=y)=\frac{p^y}{y!}h_N^{(y)}(q) \tag{3.7}$$

and

$$\mu_{[\tau]:Y}=p^\tau\mu_{[\tau]:N}. \tag{3.8}$$

If N is distributed as a Poisson quasi-Lindley distribution from Eqs (3.7) and (3.8), a number of characteristics of the EPQL model can be derived using corresponding properties of the Poisson quasi-Lindley distribution.

In particular,

$$P(Y=y;p,\alpha,\theta)=p^y\frac{\theta}{\alpha+1}\,\frac{\alpha(\theta+p)+\theta(y+1)}{(\theta+p)^{y+2}},\quad y=0,1,\dots\,.$$

The probabilities can be calculated recursively from the relation

$$P(Y=y+1)=\frac{p}{\theta+p}\,\frac{\alpha(\theta+p)+\theta(y+2)}{\alpha(\theta+p)+\theta(y+1)}P(Y=y),\quad y=0,1,\dots,$$

with

$$P(Y=0)=\frac{\theta}{\alpha+1}\,\frac{\alpha(\theta+p)+\theta}{(\theta+p)^2}.$$

In addition,

$$\mu_{[\tau]:Y}=p^\tau\frac{\tau!}{\theta^\tau}\,\frac{\alpha+\tau+1}{\alpha+1},\quad \tau=1,2,\dots,$$

$$E(Y)=p\frac{\alpha+2}{(\alpha+1)\theta},$$

and

$$Var(Y)=p\frac{p(\alpha^2+4\alpha+2)+\theta(\alpha+1)(\alpha+2)}{(\alpha+1)^2\theta^2}.$$

If we define the dispersion index of the RV Y as

$$DI_Y=\frac{Var(Y)}{E(Y)},$$

then

$$DI_Y=\frac{p(\alpha^2+4\alpha+2)}{\theta(\alpha+1)(\alpha+2)}+1.$$

The EPQL distribution, as the Poisson quasi-Lindley distribution, is unimodal and has the property of increasing failure or hazard (IFR) rate. This can be easily proved by using an approach suggested by [36]. Since the expression

$$\frac{P(Y=y+1)}{P(Y=y)}=\frac{p}{\theta+p}\left[1+\frac{\theta}{\alpha(\theta+p)+\theta(y+1)}\right]$$

is clearly a decreasing function of y, this implies unimodality of the corresponding distribution. Furthermore, because the ratio

$$\frac{P(Y=y+2)P(Y=y)}{[P(Y=y+1)]^2} = 1 - \left[\frac{\theta}{\alpha(\theta+p)+\theta(y+2)}\right]^2 < 1,$$

we can assume that $P(Y=y)$ is log-concave. An immediate consequence is that the EPQL distribution has the IFR property.

Remark 3.1. For $p = 1$, as expected, the EPQL distribution becomes a Poisson quasi-Lindley model.

Remark 3.2. For $\alpha = \theta$, the PGF (3.2) is written as

$$G_Y(s) = \frac{\theta^2}{\theta+1}\frac{\theta - ps + p + 1}{(\theta - ps + p)^2}, \tag{3.9}$$

which is the PGF of a marginal distribution in two bivariate Poisson-Lindley models examined by [11].

Remark 3.3. It is of interest to know that [40] also studied in detail a univariate distribution with PGF given by Eq (3.9), which they called a binomial-discrete Poisson-Lindley distribution.

Remark 3.4. For $\alpha = 1$, the characteristics of the EPQL distribution become properties of an extended Poisson-new XLindley distribution.

In particular,

$$P(Y=y;\alpha,\theta,p) = p\frac{\theta}{2}\frac{p+\theta(y+2)}{(\theta+p)^{y+2}}, \quad y = 0, 1, \ldots, \tag{3.10}$$

and

$$G_Y(s) = \frac{\theta}{2}\frac{2\theta - ps + p}{(\theta - ps + p)^2}. \tag{3.11}$$

4. Bivariate Poisson quasi-Lindley and Poisson-new XLindley distributions

In this section, motivated by the ongoing necessity of car insurance companies to continuously improve their bonus-malus policies we introduce and study bivariate versions of the distributions discussed in Section 2. Relative applications are given in Section 6.

4.1. A bivariate Poisson quasi-Lindley model

4.1.1. Definition and genesis

Definition 4.1. *A bivariate RV* (X_1, X_2) *is said to have a bivariate Poisson quasi-Lindley distribution with parameters* $\alpha > 0$, $\theta > 0$, $0 < p < 1$ *if its PF is*

$$P(X_1 = x_1, X_2 = x_2) = \binom{x_1}{x_2} p^{x_2} q^{x_1 - x_2} \frac{\theta}{\alpha+1}\frac{\alpha(\theta+1)+\theta(x_1+1)}{(\theta+1)^{x_1+2}}$$
$$x_1 = 0, 1, \ldots, \quad x_2 = 0, 1, \ldots, x_1, \quad q = 1 - p \tag{4.1}$$

or if its PGF is

$$G_{X_1,X_2}(s_1, s_2) = \frac{\theta}{\alpha+1}\,\frac{\alpha(\theta - qs_1 - ps_1s_2 + 1) + \theta}{(\theta - qs_1 - ps_1s_2 + 1)^2} \tag{4.2}$$

with marginals

$$G_{X_1}(s_1) = \frac{\theta}{\alpha+1}\,\frac{\alpha(\theta - s_1 + 1) + \theta}{(\theta - s_1 + 1)^2},$$

which corresponds to the PGF of a Poisson quasi-Lindley distribution discussed in Section 2 and

$$G_{X_2}(s_2) = \frac{\theta}{\alpha+1}\,\frac{\alpha(\theta - ps_2 + p) + \theta}{(\theta - ps_2 + p)^2},$$

which is the PGF of an EPQL distribution introduced and studied in Section 3.

4.1.2. Derivation as a bivariate Poisson mixture

A bivariate Poisson (Poisson-Bernoulli) distribution with PGF

$$G_{X_1,X_2}(s_1, s_2) = \exp\{\lambda[q(s_1 - 1) + p(s_1s_2 - 1)]\} \tag{4.3}$$

was derived by [41], see also [42], to express the joint distribution of the number of accidents and the number of fatal accidents. By assuming that λ is an RV with MGF $M_\Lambda(\cdot)$, then Eq (4.3) becomes

$$G_{X_1,X_2}(s_1, s_2) = M_\Lambda[q(s_1 - 1) + p(s_1s_2 - 1)]. \tag{4.4}$$

Furthermore, if Λ is a continuous RV with MGF given by Eq (2.6), from Eq (4.4) we obtain

$$G_{X_1,X_2}(s_1, s_2) = \frac{\theta}{\alpha+1}\,\frac{\alpha(\theta - qs_1 - ps_1s_2 + 1) + \theta}{(\theta - qs_1 - ps_1s_2 + 1)^2},$$

which is the PGF of a bivariate Poisson quasi-Lindley distribution given by Eq (4.2).

4.1.3. Derivation from the general structure $X_2 = Z_1 + Z_2 + \cdots + Z_{X_1}$

Assume that the RVs Z_i, $i = 1, 2, \ldots, X_1$ are independent, identically distributed (i.i.d.) Bernoulli (p) RVs also independent of the RV X_1. Then

$$P(X_2 \mid X_1 = x_1) = \binom{x_1}{x_2} p^{x_2} q^{x_1 - x_2}. \tag{4.5}$$

i) Derivation when the PF of the marginal RV X_1 is known.
Since

$$P(X_1 = x_1, X_2 = x_2) = \binom{x_1}{x_2} p^{x_2} q^{x_1 - x_2} P(X_1 = x_1),$$

if X_1 follows a Poisson quasi-Lindley distribution with PF given by Eq (2.1), then the PF of the bivariate Poisson quasi-Lindley distribution given by Eq (4.1) is immediately obtained.

ii) Derivation when the PGF of the marginal RV X_1 is known.

Since $G_{X_2|X_1}(s_1) = (q + ps_2)^{x_1}$, from [43, 44],

$$G_{X_1,X_2}(s_1, s_2) = G_{X_1}(s_1(q + ps_2)). \tag{4.6}$$

Consequently, if X_1 follows a Poisson quasi-Lindley distribution with PGF given by Eq (2.2), then from Eq (4.6),

$$G_{X_1,X_2}(s_1, s_2) = \frac{\theta}{\alpha + 1} \frac{\alpha(\theta - qs_1 - ps_1 s_2 + 1) + \theta}{\theta(qs_1 - ps_1 s_2 + 1)^2},$$

which is the PGF of a bivariate Poisson quasi-Lindley distribution.

iii) Derivation when the regression function $m(x_2) = E[X_1 \mid X_2 = x_2]$ is known.

This technique is closely related to a characterization theorem proved by [44].

Proposition 4.1. *Let the conditional distribution $P(X_2 \mid X_1 = x_1)$ be of the form (4.5). Also, let*

$$m(x_2) = x_2 + q\frac{x_2 + 1}{\theta + p} \frac{\alpha(\theta + p) + \theta(x_2 + 2)}{\alpha(\theta + p) + \theta(x_2 + 1)}.$$

Then $G_{X_1,X_2}(s_1, s_2)$ is given by Eq (4.2).

Proof. Using the combinatorial identity

$$x_1\binom{x_1}{x_2} = (x_2 + 1)\binom{x_1}{x_2 + 1} + x_2\binom{x_1}{x_2}$$

and Eq (4.5), we obtain

$$m(x_2) = x_2 + \frac{q}{p}(x_2 + 1)\frac{P(X_2 = x_2 + 1)}{P(X_2 = x_2)}. \tag{4.7}$$

Hence,

$$\begin{aligned} P(X_2 = x_2) &= P(X_2 = 0)\left(\frac{p}{q}\right)^{x_2} \prod_{k=0}^{x_2-1} \frac{1}{k + 1}(m(k) - k) \\ &= P(X_2 = 0)\left(\frac{p}{q}\right)^{x_2} \prod_{k=0}^{x_2-1} \frac{q}{\theta + p} \frac{\alpha(\theta + p) + \theta(k + 2)}{\alpha(\theta + p) + \theta(k + 1)} \\ &= P(X_2 = 0)p^{x_2} \frac{1}{(\theta + p)^{x_2}} \frac{\alpha(\theta + p) + \theta(x_2 + 1)}{\alpha(\theta + p) + \theta}. \end{aligned}$$

Since

$$P(X_2 = 0) = \frac{\theta}{\alpha + 1} \frac{\alpha(\theta + p) + \theta}{(\theta + p)^2},$$

from the initial condition $\sum_{x_2} P(X_2 = x_2) = 1$, we obtain

$$P(X_2 = x_2) = p^{x_2} \frac{\theta}{\alpha + 1} \frac{\alpha(\theta + p) + \theta(x_2 + 1)}{(\theta + p)^{x_2+2}}.$$

This is the PF of an EPQL distribution with parameters $\alpha > 0, \theta > 0, 0 < p < 1$ as denoted by Eq (3.1). Its corresponding PGF is given by Eq (3.2).

However, since

$$G_{X_2}(s) = G_{X_1}(q + ps),$$

we have

$$\begin{aligned} G_{X_1}(s) &= G_{X_2}\left(\frac{s-q}{p}\right) \\ &= \frac{\theta}{\alpha+1}\frac{\alpha(\theta - s + 1) + \theta}{(\theta - s + 1)^2}, \end{aligned}$$

and from Eq (4.6), the bivariate PGF given by Eq (4.2) is obtained.

4.1.4. Derivation by the addition of independent bivariate RVs

Bivariate Poisson quasi-Lindley distributions can be obtained using the general structure

$$\begin{aligned} R_1 &= U_1 + V_1, \\ R_2 &= U_2 + V_2, \end{aligned} \tag{4.8}$$

where (U_1, U_2) and (V_1, V_2) are independently distributed bivariate discrete RVs. Then the PGF of (R_1, R_2) is given by

$$G_{R_1,R_2}(s_1, s_2) = G_{U_1,U_2}(s_1, s_2)G_{V_1,V_2}(s_1, s_2). \tag{4.9}$$

This technique was initially suggested by [45] and also reported in the book by [46]. Bivariate Poisson generalized Lindley distributions were also derived by [13] using the above procedure.

Proposition 4.2. *Consider two independent bivariate RVs* (U_1, U_2) *and* (V_1, V_2). *Let* (U_1, U_2) *follow a bivariate geometric-Bernoulli distribution introduced by [47] with parameters* $\frac{q}{\theta+1}, \frac{p}{\theta+1}$ *and PGF*

$$G_{U_1,U_2}(s_1, s_2) = \frac{\theta}{\theta - qs_1 - ps_1s_2 + 1}. \tag{4.10}$$

In addition, assume that the RV (V_1, V_2) *follows a bivariate geometric-Bernoulli distribution with added zeros in the* $(0, 0)$ *cell and PGF*

$$G_{V_1,V_2}(s_1, s_2) = \omega + (1 - \omega)G_{U_1,U_2}(s_1, s_2).$$

For $\omega = \frac{\alpha}{\alpha+1}$,

i)
$$G_{V_1,V_2}(s_1, s_2) = \frac{\alpha(\theta - qs_1 - ps_1s_2 + 1) + \theta}{(\alpha + 1)(\theta - qs_1 - ps_1s_2 + 1)},$$

and

ii)
$$G_{R_1,R_2}(s_1, s_2) = \frac{\theta}{\alpha+1}\frac{\alpha(\theta - qs_1 - ps_1s_2 + 1) + \theta}{(\theta - qs_1 - ps_1s_2 + 1)^2},$$

which is the PGF of a bivariate Poisson quasi-Lindley distribution.

Proof. Similar to the one given for Proposition 2.2.

Remark 4.1. The above derivation supports empirical evidence (see Tables 1–3 in Section 6), that bivariate Poisson quasi-Lindley distributions defined by Eq (4.2) are appropriate for fitting data with a large number of observations in the $(0,0)$ cell.

4.1.5. Properties

Differentiating the PGF given by Eq (4.2), we obtain the marginal means (see also Sections 2 and 3)

$$E(X_1) = \frac{\alpha+2}{(\alpha+1)\theta}, \tag{4.11}$$

$$E(X_2) = p\frac{\alpha+2}{(\alpha+1)\theta}, \tag{4.12}$$

and

$$E(X_1X_2) = p\frac{2(\alpha+3)+(\alpha+2)\theta}{(\alpha+1)\theta^2}.$$

Consequently,

$$Cov(X_1,X_2) = p\frac{(\alpha^2+4\alpha+2)+(\alpha^2+3\alpha+2)\theta}{(\alpha+1)^2\theta^2},$$

which is positive.

To derive the conditional PGF of $G_{X_2|X_1=x_1}(s)$ of the RV X_2, given $X_1 = x_1$, we use the following result due to [48].

For a bivariate discrete RV (X_1, X_2) with PGF $G_{X_1,X_2}(s_1, s_2)$, the conditional PGF $G_{X_2|X_1=x_1}(s)$ of X_2 on X_1 is

$$G_{X_2|X_1=x_1}(s) = \frac{G^{(x_1,0)}(0,s)}{G^{(x_1,0)}(0,1)}, \tag{4.13}$$

where

$$G^{(x,y)}(u,v) = \frac{\partial^{x+y}G(s_1,s_2)}{\partial s_1^x \partial s_2^y}\bigg|\begin{matrix} s_1 = u. \\ s_2 = v. \end{matrix}$$

Consequently, from Eqs (4.2) and (4.13),

$$G_{X_2|X_1=x_1}(s) = (q+ps)^{x_1},$$

and as stated in Subsection 4.1.3 (i),

$$\begin{aligned} P(X_1 = x_1, X_2 = x_2) &= \binom{x_1}{x_2} p^{x_2} q^{x_1-x_2} P(X_1 = x_1) \\ &= \frac{x_1!p^{x_2}q^{x_1-x_2}}{x_2!(x_1-x_2)!}\,\frac{\theta}{\alpha+1}\,\frac{\alpha(\theta+1)+\theta(x_1+1)}{(\theta+1)^{x_1+2}}, \end{aligned} \tag{4.14}$$

$\theta > 0$, $x > 0$, $0 < p < 1$, $x_1 = 0, 1, \ldots$, and $x_2 = 0, 1, \ldots, x_1$.

Simple recurrences for the probabilities are:

$$P(X_1 = x_1 + 1, X_2 = x_2) = \frac{x_1 + 1}{x_1 + 1 - x_2} \frac{q}{\theta + 1} \frac{\alpha(\theta + 1) + \theta(x_1 + 2)}{\alpha(\theta + 1) + \theta(x_1 + 1)} P(X_1 = x_1, X_2 = x_2), \quad x_1 = 0, 1, \dots, \quad x_2 = 0, 1, \dots, x_1, \tag{4.15}$$

and

$$P(X_1 = x_1, X_2 = x_2 + 1) = \frac{x_1 - x_2}{x_2 + 1} \frac{p}{q} P(X_1 = x_1, X_2 = x_2), \quad x_1 = 0, 1, \dots, \quad x_2 = 0, 1, \dots, x_1, \tag{4.16}$$

with

$$P(X_1 = 0, X_2 = 0) = G(0, 0) = \frac{\theta}{\alpha + 1} \frac{\alpha(\theta + 1) + \theta}{(\theta + 1)^2}. \tag{4.17}$$

The conditional PGF $G_{X_1|X_2=x_2}(s)$ of X_1 on X_2, from Eqs (4.2) and (4.13), is

$$G_{X_1|X_2=x_2}(s) = s^{x_2} \frac{\alpha(\theta - qs + 1) + (x_2 + 1)\theta}{(\theta - qs + 1)^{x_2+2}} \frac{(\theta + p)^{x_2+2}}{\alpha(\theta + p) + \theta(x_2 + 1)}. \tag{4.18}$$

Differentiating Eq (4.18), we obtain

$$E[X_1 \mid X_2 = x_2] = x_2 + q \frac{x_2 + 1}{\theta + p} \frac{\alpha(\theta + p) + \theta(x_2 + 2)}{\alpha(\theta + p) + \theta(x_2 + 1)}.$$

The above regression can also be obtained from Eq (4.7).

Remark 4.2. For $\alpha = \theta$, the PGF (4.2) is written as

$$G_{X_1,X_2}(s_1, s_2) = \frac{\theta^2}{\theta + 1} \frac{\theta - qs_1 - ps_1 s_2 + 2}{(\theta - qs_1 - ps_1 s_2 + 1)^2},$$

which is the PGF of a bivariate Poisson-Lindley distribution introduced by [11].

4.2. Bivariate Poisson-new XLindley distribution

The results in this subsection will be stated briefly.

4.2.1. Definition and genesis

Definition 4.2. *A bivariate RV (X_1, X_2) is said to have a bivariate Poisson-new XLindley distribution with parameters $\theta > 0$, $0 < p < 1$, if its PF is*

$$P(X_1 = x_1, X_2 = x_2) = \binom{x_1}{x_2} p^{x_2} q^{x_1 - x_2} \frac{\theta}{2} \frac{1 + \theta(x_1 + 2)}{(\theta + 1)^{x_1+2}}, \quad x_1 = 0, 1, \dots, \quad x_2 = 0, 1, \dots, x_1, \tag{4.19}$$

or if its PGF is

$$G_{X_1,X_2}(s_1, s_2) = \frac{\theta}{2} \frac{2\theta - qs_1 - ps_1 s_2 + 1}{(\theta - qs_1 - ps_1 s_2 + 1)^2}, \tag{4.20}$$

with marginals

$$G_{X_1}(s_1) = \frac{\theta}{2}\frac{2\theta - s_1 + 1}{(\theta - s_1 + 1)^2},$$

and

$$G_{X_2}(s_2) = \frac{\theta}{2}\frac{2\theta - ps_2 + p}{(\theta - ps_2 + p)^2},$$

which are the PGFs of a Poisson-new XLindley and an extended Poisson-new XLindley distribution given by Eqs (2.15) and (3.11), respectively.

4.2.2. Derivation as a bivariate Poisson mixture

From Eq (4.4), if Λ is a continuous RV with MGF given by Eq (2.16), Eq (4.20) is derived.

4.2.3. Derivation from the general structure $X_2 = Z_1 + Z_2 + \cdots + Z_{X_1}$

Assume that the Z_i are independent and identically distributed (IID) Bernoulli (p) RVs

i) Derivation when the PF of the marginal RV X_1 is known.
From $P(X_2 \mid X_1 = x_1)$ given by Eq (4.5) and the PF of X_1 given by Eq (2.14), Eq (4.19) is obtained.

ii) Derivation when the PGF of the marginal RV X_1 is known.
When the PGF of X_1 is given by Eq (2.15), utilizing Eq (4.6), Eq (4.20) is derived.

iii) Derivation when the regression function $m(x_2) = E[X_1 \mid X_2 = x_2]$ is known.

Proposition 4.3. *Let the conditional distribution $P(X_2 \mid X_1 = x_1)$ be of the form (4.5). Also, let*

$$m(x_2) = x_2 + q\frac{x_2 + 1}{\theta + p}\frac{1 + \theta(x_2 + 3)}{p + \theta(x_2 + 2)}.$$

Then $G_{X_1,X_2}(s_1, s_2)$ is given by Eq (4.20).

Proof. Similar to the one given for Proposition 4.1.

4.2.4. Derivation by the addition of independent bivariate RVs

Proposition 4.4. *The bivariate Poisson-new XLindley distribution can be written as a convolution of a bivariate geometric-Bernoulli distribution with PGF given by Eq (4.10) and an independent bivariate inflated geometric-Bernoulli distribution with inflation parameter $\omega = \frac{1}{2}$. Then their sum follows a bivariate Poisson-new XLindley distribution with PGF given by Eq (4.20).*

Proof. Similar to the one given for Proposition 2.2.

4.2.5. Properties

Some characteristic properties are:

$$E(X_1) = \frac{3}{2\theta},$$
$$E(X_2) = p\frac{3}{2\theta},$$
$$E(X_1X_2) = p\frac{8+3\theta}{2\theta^2},$$
$$Cov(X_1, X_2) = p\frac{7+6\theta}{4\theta^2},$$

which is positive.

Recurrences for probabilities can be obtained from the relations

$$P(X_1 = x_1 + 1, X_2 = x_2) = \frac{x_1+1}{x_1+1-x_2}\frac{q}{\theta+1}\frac{1+\theta(x_1+3)}{1+\theta(x_1+2)}P(X_1 = x_1, X_2 = x_2),$$
$$x_1 = 0, 1, \ldots, \quad x_2 = 0, 1, \ldots, x_1,$$

and

$$P(X_1 = x_1, X_2 = x_2 + 1) = \frac{x_1 - x_2}{x_2+1}\frac{p}{q}P(X_1 = x_1, X_2 = x_2), \quad x_1 = 0, 1, \ldots, \quad x_2 = 0, 1, \ldots, x_1,$$

with

$$P(X_1 = 0, X_2 = 0) = \frac{\theta}{2}\frac{1+2\theta}{(\theta+1)^2}.$$

The PGF of the conditional distribution of X_1 on X_2 is

$$G_{X_1|X_2=x_2}(s) = s^{x_2}\frac{1-qs+\theta(x_2+2)}{(\theta-qs+1)^{x_2+2}}\frac{(\theta+p)^{x_2+2}}{p+\theta(x_2+2)}, \tag{4.21}$$

and from Eq (4.21),

$$E[X_1 \mid X_2 = x_2] = x_2 + q\frac{x_1+1}{\theta+p}\frac{p+\theta(x_2+3)}{p+\theta(x_2+2)}.$$

5. Extended bivariate Poisson quasi-Lindley and Poisson-new XLindley distributions

In this section, we introduce bivariate Poisson quasi-Lindley and Poisson-new XLindley distributions, which are structurally different from the bivariate models discussed in the previous section. However, these models can also be used in a variety of problems in accident data analysis, for example, in automobile portfolios to describe the interrelations between bodily injury and material damage. Relative applications are given in Section 7.

5.1. An extended bivariate Poisson quasi-Lindley distribution

5.1.1. Definition and genesis

Definition 5.1. *A bivariate RV (X_1, X_2) is said to have an extended bivariate Poisson quasi-Lindley distribution with parameters $\alpha > 0$, $\theta > 0$, $0 < p < 1$ if its PF is*

$$P(X_1 = x_1, X_2 = x_2) = \binom{x_1 + x_2}{x_2} p^{x_2} q^{x_1} \frac{\theta}{\alpha + 1} \frac{\alpha(\theta + 1) + \theta(x_1 + x_2 + 1)}{(\theta + 1)^{x_1 + x_2 + 2}},$$
$$x_i = 0, 1, \ldots, \quad i = 1, 2, \tag{5.1}$$

or if its PGF is

$$G_{X_1,X_2}(s_1, s_2) = \frac{\theta}{\alpha + 1} \frac{\alpha(\theta - qs_1 - ps_2 + 1) + \theta}{(\theta - qs_1 - ps_2 + 1)^2}. \tag{5.2}$$

The marginals have PGFs

$$G_{X_1}(s_1) = \frac{\theta}{\alpha + 1} \frac{\alpha(\theta - qs_1 + q) + \theta}{(\theta - qs_1 + q)^2},$$

and

$$G_{X_2}(s_2) = \frac{\theta}{\alpha + 1} \frac{\alpha(\theta - ps_2 + p) + \theta}{(\theta - ps_2 + p)^2}.$$

They are EPQL distributions with parameters, $\theta > 0$, $\alpha > 0$, $0 < q < 1$ and $\theta > 0$, $\alpha > 0$, $0 < p < 1$, respectively. It is of interest to note, that the distribution of the sum $X_1 + X_2$ has PGF

$$G_{X_1+X_2}(s) = \frac{\theta}{\alpha + 1} \frac{\alpha(\theta - s + 1) + \theta}{(\theta - s + 1)^2},$$

which is the PGF of a Poisson quasi-Lindley distribution.

5.1.2. Derivation by generalizing a bivariate binomial distribution

Consider a bivariate binomial distribution introduced by [49] with PGF

$$E(s_1^{X_1} s_2^{X_2} \mid N = n) = (qs_1 + ps_2)^n, \quad 0 < p < 1,$$

and $q = 1 - p$, where N is a non-negative integer-valued RV with PGF

$$E(s^N) = h_N(s).$$

Consequently,

$$G_{X_1,X_2}(s_1, s_2) = h_N(qs_1 + ps_2). \tag{5.3}$$

If N follows a Poisson quasi-Lindley distribution with PGF given by Eq (2.2), see also Subsection 3.1.1, then the PGF of the joint distribution of (X_1, X_2) is

$$G_{X_1,X_2}(s_1, s_2) = \frac{\theta}{\alpha + 1} \frac{\alpha(\theta - qs_1 - ps_2 + 1) + \theta}{(\theta - qs_1 - ps_2 + 1)^2},$$

which corresponds to Eq (5.2).

5.1.3. Derivation as a bivariate Poisson mixture

Consider a bivariate Poisson distribution with PGF

$$G_{X_1,X_2}(s_1, s_2) = \exp\{\lambda[q(s_1 - 1) + p(s_2 - 1)]\}.$$

Then, if λ is a continuous RV with MGF $M_\Lambda(\cdot)$, then

$$G_{X_1,X_2}(s_1, s_2) = M_\Lambda[q(s_1 - 1) + p(s_2 - 1)]. \tag{5.4}$$

In addition, if we assume that $M_\Lambda(\cdot)$ is given by Eq (2.6), then Eq (5.2) is obtained.

5.1.4. Derivation by the addition of independent bivariate RVs

As in the Subsection 4.1.4, we can construct extended bivariate Poisson quasi-Lindley distributions with PGF given by Eq (5.2) by the addition of independent RVs.

Proposition 5.1. *Consider two independent bivariate RVs* $(U_1, U_2), (V_1, V_2)$ *and a bivariate RV* (R_1, R_2) *defined by the structure (4.8) . Assume that* (U_1, U_2) *follows a bivariate geometric distribution with parameters* $\left(\frac{q}{\theta+1}, \frac{p}{\theta+1}\right)$ *and PGF*

$$G_{U_1,U_2}(s_1, s_2) = \frac{\theta}{\theta - qs_1 - ps_2 + 1}. \tag{5.5}$$

Furthermore, let the RV (V_1, V_2) *follow a bivariate geometric distribution with added zeros in the* $(0, 0)$ *cell and PGF*

$$G_{V_1,V_2}(s_1, s_2) = \omega + (1 - \omega)G_{U_1,U_2}(s_1, s_2).$$

For $\omega = \frac{\alpha}{\alpha+1}$,

i)
$$G_{V_1,V_2}(s_1, s_2) = \frac{\alpha(\theta - qs_1 - ps_2 + 1) + \theta}{(\alpha + 1)(\theta - qs_1 - ps_2 + 1)},$$

and

ii)
$$G_{R_1,R_2}(s_1, s_2) = \frac{\theta}{\alpha + 1} \frac{\alpha(\theta - qs_1 - ps_2 + 1) + \theta}{(\theta - qs_1 - ps_2 + 1)^2},$$

which is the PGF of an extended bivariate Poisson-quasi-Lindley distribution given by Eq (5.2).

Proof. Similar to the one given for Proposition 2.2.

Remark 5.1. The above derivation supports empirical evidence (see Table 4 in Section 7), that bivariate Poisson quasi-Lindley distributions defined by Eq (5.2) are appropriate for fitting data with a large number of observations in the $(0, 0)$ cell.

5.1.5. Properties

i) Probabilities and moments.

Differentiating Eq (5.3), we obtain

$$\frac{\partial^{x_1+x_2}G(s_1,s_2)}{\partial s_1^{x_1}\partial s_2^{x_2}} = q^{x_1}p^{x_2}h_N^{(x_1+x_2)}(qs_1+ps_2). \tag{5.6}$$

Consequently, from Eq (5.6),

$$\begin{aligned} P(X_1=x_1,X_2=x_2) &= \frac{q^{x_1}p^{x_2}}{x_1!x_2!}h_N^{(x_1+x_2)}(0) \\ &= \binom{x_1+x_2}{x_2}q^{x_1}p^{x_2}P(N=x_1+x_2), \end{aligned} \tag{5.7}$$

and

$$\mu_{[r,k]} = q^r p^k \mu_{[r+k]:N}, \tag{5.8}$$

where

$$\mu_{[r,k]} = E(X_1^{(r)}X_2^{(k)}).$$

Assume that the RV N follows a Poisson quasi-Lindley distribution. Then, from Eqs (5.7) and (2.1) Eq (5.1) is obtained, and from Eqs (5.8) and (2.13), we have

$$\mu_{[r,k]} = q^r p^k \frac{(r+k)!}{\alpha+1}\frac{\alpha+r+k+1}{\theta^{r+k}}. \tag{5.9}$$

From Eq (5.1), the probabilities can be calculated recursively from the relations

$$P(X_1=x_1+1,X_2=x_2) = \frac{x_1+1+x_2}{x_1+1}\frac{q}{\theta+1}\frac{\alpha(\theta+1)+\theta(x_1+x_2+2)}{\alpha(\theta+1)+\theta(x_1+x_2+1)}P(X_1=x_1,X_2=x_2), \quad x_i=0,1,\dots,\ \ i=1,2, \tag{5.10}$$

and

$$P(X_1=x_1,X_2=x_2+1) = \frac{x_1+x_2+1}{x_2+1}\frac{p}{\theta+1}\frac{\alpha(\theta+1)+\theta(x_1+x_2+2)}{\alpha(\theta+1)+\theta(x_1+x_2+1)}P(X_1=x_1,X_2=x_2), \quad x_i=0,1,\dots,\ \ i=1,2, \tag{5.11}$$

with

$$P(X_1=0,X_2=0) = \frac{\theta}{\alpha+1}\frac{\alpha(\theta+1)+\theta}{(\theta+1)^2}. \tag{5.12}$$

From Eq (5.9), we obtain the marginal means, as also expected from Section 3,

$$E(X_1) = q\frac{(\alpha+2)}{(\alpha+1)\theta}, \tag{5.13}$$

$$E(X_2) = p\frac{(\alpha+2)}{(\alpha+1)\theta}. \tag{5.14}$$

In adddition,

$$E(X_1X_2) = \frac{2pq}{\alpha+1}\,\frac{\alpha+3}{\theta^2}.$$

Hence,

$$Cov(X_1, X_2) = pq\frac{\alpha^2+4\alpha+2}{(\alpha+1)^2\theta^2},$$

which is positive.

ii) Conditional distributions.

From Eqs (4.13) and (5.3),

$$G_{X_2|X_1=x_1}(s) = \frac{h_N^{(x_1)}(ps)}{h_N^{(x_1)}(p)}. \tag{5.15}$$

Assume that the RV N follows a Poisson quasi-Lindley distribution.

Then, from Eq (2.12),

$$G_{X_2|X_1=x_1}(s) = \frac{\alpha(\theta-ps+1)+\theta(x_1+1)}{(\theta-ps+1)^{x_1+2}}\,\frac{(\theta+q)^{x_1+2}}{\alpha(\theta+q)+\theta(x_1+1)},$$

and we can easily prove, that

$$P(X_2 = x_2 \mid X_1 = x_1) = \binom{x_1+x_2}{x_1}p^{x_2}\frac{\alpha(\theta+1)+\theta(x_1+x_2+1)}{(\theta+1)^{x_1+x_2+2}}\,\frac{(\theta+q)^{x_1+2}}{\alpha(\theta+q)+\theta(x_1+1)}, \quad x_2 = 0, 1, \ldots .$$

The conditional probabilities can be easily computed by using the recurrence

$$P(X_2 = x_2+1 \mid X_1 = x_1) = \frac{x_1+x_2+1}{x_2+1}\,\frac{p}{\theta+1}\,\frac{\alpha(\theta+1)+\theta(x_1+x_2+2)}{\alpha(\theta+1)+\theta(x_1+x_2+1)}P(X_2 = x_2 \mid X_1 = x_1),$$

where

$$P(X_2 = 0 \mid X_1 = x_1) = \frac{\alpha(\theta+1)+\theta(x_1+1)}{(\theta+1)^{x_1+2}}\,\frac{(\theta+q)^{x_1+2}}{\alpha(\theta+q)+\theta(x_1+1)}.$$

From Eqs (5.15) and (2.12),

$$\begin{aligned}\mu_{[\tau|X_1=x_1]} &= p^\tau\frac{h_N^{(x_1+\tau)}(p)}{h_N^{(x_1)}(p)}\\ &= \frac{(x_1+\tau)!}{x_1!}\,\frac{p^\tau}{(\theta+q)^\tau}\,\frac{\alpha(\theta+q)+\theta(x_1+\tau+1)}{\alpha(\theta+q)+\theta(x_1+1)}.\end{aligned} \tag{5.16}$$

Hence,

$$E[X_2 = x_2 \mid X_1 = x_1] = \frac{p(x_1+1)}{\theta+q}\,\frac{\alpha(\theta+q)+\theta(x_1+2)}{\alpha(\theta+q)+\theta(x_1+1)}.$$

Corresponding relations for the conditional distribution of $X_1 \mid X_2 = x_2$ can also be derived.

Remark 5.2. For $\alpha = \theta$, the PGF (5.2) is written as

$$G_{X_1,X_2}(s_1, s_2) = \frac{\theta^2}{\theta+1}\,\frac{\theta-qs_1-ps_2+2}{(\theta-qs_1-ps_2+1)^2},$$

which is the PGF of another bivariate Poisson-Lindley distribution introduced by [11].

5.2. *Extended bivariate Poisson-new XLindley distribution*

The results in this subsection will be stated briefly.

5.2.1. Definition and genesis

Definition 5.2. *A bivariate RV* (X_1, X_2) *is said to have an extended bivariate Poisson-new XLindley distribution with parameters* $\theta > 0$*,* $0 < p < 1$*, if its PF is*

$$P(X_1 = x_1, X_2 = x_2) = \binom{x_1 + x_2}{x_2} p^{x_2} q^{x_1} \frac{\theta}{2} \frac{1 + \theta(x_1 + x_2 + 2)}{(\theta + 1)^{x_1+x_2+2}}, \quad x_i = 0, 1, \ldots, \quad i = 1, 2, \tag{5.17}$$

or if its PGF is

$$G_{X_1,X_2}(s_1, s_2) = \frac{\theta}{2} \frac{2\theta - qs_1 - ps_2 + 1}{(\theta - qs_1 - ps_2 + 1)^2}. \tag{5.18}$$

In addition,

$$G_{X_1}(s_1) = \frac{\theta}{2} \frac{2\theta - qs_1 + q}{(\theta - qs_1 + q)^2},$$

$$G_{X_2}(s_2) = \frac{\theta}{2} \frac{2\theta - ps_2 + p}{(\theta - ps_2 + p)^2},$$

$$G_{X_1+X_2}(s) = \frac{\theta}{2} \frac{2\theta - s + 1}{(\theta - s + 1)^2}.$$

The PGFs of the marginal X_1 *and* X_2 *are extended Poisson-new XLindley distributions, and the PGF of the sum* $X_1 + X_2$ *is a Poisson-new XLindley distribution.*

5.2.2. Derivation by generalizing a bivariate binomial distribution

From the general Eq (5.3), if N follows a Poisson-new XLindley distribution, then the joint PGF of (X_1, X_2) is given by Eq (5.18).

5.2.3. Derivation as a bivariate Poisson mixture

From Eq (5.4), if Λ is a continuous RV with MGF given by Eq (2.16), then Eq (5.18) is obtained.

5.2.4. Derivation by the addition of independent bivariate RVs

Proposition 5.2. *The bivariate extended Poisson-new XLindley distribution can be obtained by the addition of a bivariate geometric distribution with PGF given by Eq (5.5) and an independent bivariate inflated geometric distribution with inflation parameter* $\omega = \frac{1}{2}$*. Then the PGF of their sum is given by Eq (5.18).*

Proof. Similar to the one given for Proposition 2.2.

5.2.5. Properties

Some characteristics of the distribution are the following.

If N follows a Poisson-new XLindley distribution, from Eq (5.7), we obtain Eq (5.17), and from Eq (5.8), the expression

$$\mu_{[r,k]} = q^r p^k \frac{(r+k)!}{2} \frac{r+k+2}{\theta^{\tau+k}}, \tag{5.19}$$

respectively.

From Eq (5.17), the probabilities can be calculated recursively from the relations

$$P(X_1 = x_1+1, X_2 = x_2) = \frac{x_1+1+x_2}{x_1+1} \frac{q}{\theta+1} \frac{1+\theta(x_1+x_2+3)}{1+\theta(x_1+x_2+2)} P(X_1 = x_1, X_2 = x_2),$$
$$x_i = 0, 1, \ldots, \quad i = 1, 2,$$

and

$$P(X_1 = x_1, X_2 = x_2+1) = \frac{x_1+x_2+1}{x_2+1} \frac{p}{\theta+1} \frac{1+\theta(x_1+x_2+3)}{1+\theta(x_1+x_2+2)} P(X_1 = x_1, X_2 = x_2),$$
$$x_i = 0, 1, \ldots, \quad i = 1, 2,$$

with

$$P(X_1 = 0, X_2 = 0) = \frac{\theta}{2} \frac{2\theta+1}{(\theta+1)^2}.$$

Also, from Eq (5.19), we obtain

$$E(X_1) = q\frac{3}{2\theta},$$

$$E(X_2) = p\frac{3}{2\theta},$$

$$E(X_1X_2) = pq\frac{4}{\theta^2},$$

$$Cov(X_1, X_2) = pq\frac{7}{4\theta^2},$$

which is positive.

In addition, from the general Eq (5.15), if we assume that N is distributed according to a Poisson-new XLindley distribution and utilize Eq (2.17), we have

$$G_{X_2|X_1=x_1}(s) = \frac{1-ps+\theta(x_1+2)}{(\theta+1-ps)^{x_1+2}} \frac{(\theta+q)^{x_1+2}}{q+\theta(x_1+2)},$$

$$P(X_2 = x_2 \mid X_1 = x_1) = \binom{x_1+x_2}{x_1} p^{x_2} \frac{1+\theta(x_1+x_2+2)}{(\theta+1)^{x_1+x_2+2}} \frac{(\theta+q)^{x_1+2}}{q+\theta(x_1+2)}.$$

Furthermore, from Eq (5.16) and (2.17), we obtain

$$\mu_{[\tau|X_1=x_1]} = \frac{(x_1+\tau)!}{x_1!} \frac{p^\tau}{(\theta+q)^\tau} \frac{q+\theta(x_1+\tau+2)}{q+\theta(x_1+2)},$$

$$E[X_2 = x_2 \mid X_1 = x_1] = \frac{p(x_1+1)}{\theta+q} \frac{q+\theta(x_1+3)}{q+\theta(x_1+2)}.$$

6. Applications to data sets of automobile insurance claims and amounts of claims

The bonus-malus systems (BMSs) are pricing systems widely used in vehicle insurance. In the classical BMS, the premium assigned to each policyholder is based only on the number of claims without taking into account the claim size. Several authors, see relative references in [26], suggested that modifications of the BMS are appropriate, incorporating not only the number of claims but also the claim size. Furthermore, from empirical studies, it appears that these two variables are positively correlated. Consequently, bivariate discrete models with this property may be useful in the analysis of this type of data.

We demonstrate the applicability of the bivariate Poisson-new XLindley distribution and of three additional members of the bivariate Poisson quasi-Lindley class defined by Eq (4.1) by fitting them to three sets of insurance data previously used, among others, by [13, 26–28].

All sets of data originally came from a portfolio of 67856 one-year automobile insurance policies taken out in 2004 or 2005. The data set is available on the website of the Faculty of Business and Economics, Macquaire University, Sydney, Australia; see also [50]. Out of the 67856 policies in that portfolio, 4624 claims were made. There were 4333 policyholders who made claims once, 271 twice, 18 three times, and 2 four times. This set of data was tabulated by [26] using not only the number of claims (variable X_1) but also the total number of claims with the claim size larger than a threshold monetary value variable (X_2), $\Psi = 500$; see our Table 1. Relative tabulations when the threshold values are $\Psi = 1000$ and $\Psi = 3000$ are given by [27] and are also reported in our Tables 2 and 3 respectively.

From the bivariate Poisson quasi-Lindley model defined by Eq (4.1), a large number of distributions can be obtained for different values of the parameter α. For example, when $\alpha = 1$, the resulting distribution is, as expected, the bivariate Poisson-new XLindley model discussed in Subsection 4.2.

For demonstration purposes, we assume that the parameter α takes specific values, and the remaining two parameters θ and p are simply estimated by using the method of moments. However, more efficient estimation techniques exist such as the maximum likelihood method and need to be explored in future work. It is worth noting, that for the bivariate negative binomial-Bernoulli model introduced by [47] (with the bivariate geometric -Bernoulli defined by the PGF given by Eq (4.10) as a special case) and the bivariate negative binomial model (with the bivariate geometric with PGF defined by Eq (5.5) as a special case), several estimation techniques were proposed by [51] and [52], respectively. In particular, they examined a comparative study of the methods of maximum likelihood, moments, even points, double-zero proportion, and ratio of frequencies for large and small samples. We have elected to fit each one of the four models obtained when $a = 1, 2, 3,$ and 6 to the data sets of claims and amounts of claims recorded in Tables 1–3 using moment estimators.

Since the marginal means are given by expressions (4.11) and (4.12), the remaining parameters θ and p of the model are easily estimated from the equations

$$\bar{X}_1 = \frac{\alpha + 2}{(\alpha + 1)\theta}, \tag{6.1}$$

and

$$\bar{X}_2 = p\frac{\alpha + 2}{(\alpha + 1)\theta}. \tag{6.2}$$

Consequently, from Eqs (6.1) and (6.2),

$$\tilde{\theta} = \frac{\alpha + 2}{(\alpha + 1)\bar{X}_1}, \tag{6.3}$$

which does not depend on the value of p and

$$\tilde{p} = \frac{\bar{X}_2}{\bar{X}_1}, \tag{6.4}$$

which is independent of the value of the parameters α and θ.

Bivariate probabilities are easily calculated by using the recurrences

$$P(X_1 = x_1 + 1, X_2 = 0) = \frac{q}{\theta + 1} \frac{\alpha + (\alpha + x_1 + 2)\theta}{\alpha + (\alpha + x_1 + 1)\theta} P(X_1 = x_1, X_2 = 0), \quad x_1 = 0, 1, 2, 3$$

and

$$P(X_1 = x_1, X_2 = x_2 + 1) = \frac{x_1 - x_2}{x_2 + 1} \frac{p}{q} P(X_1 = x_1, X_2 = x_2) \;\; x_1 = 1, 2, 3, 4, \quad x_2 = 0, 1, 2, 3,$$

with

$$P(X_1 = 0, X_2 = 0) \quad \text{given by Eq (4.17)}$$

for the 67856 observations under consideration and for threshold values $\Psi = 500$, 1000, 3000, observed and expected frequencies are given in Tables 1, 2, and 3 respectively. In each table, the first line represents the observed frequencies, and the second, third, fourth, and fifth lines represent the expected frequencies for $\alpha = 1, 2, 3$, and 6 respectively. Parameter estimates and χ^2 values are also reported in these tables.

Denoting by $\bar{X}_i$, $i = 1, 2$, the sample marginal means; by $S^2_{X_i}$, $i = 1, 2$, the sample marginal variances; by DI_{X_i}, $i = 1, 2$, the sample index of dispersion (as we use the same notation with the population index of dispersion); by $S_{X_1X_2}$ the covariance; and by $\rho(X_1, X_2)$, the sample correlation coefficient, the corresponding calculated values of some characteristics of the data sets are given below.

In particular, since the values of the marginal variable X_1 remain the same for all Tables 1–3, we have $\bar{X}_1 = 0.072757$, $S^2_{X_1} = 0.077397$, and $DI_{X_1} = 1.063779$.

The remaining characteristics for Tables 1, 2 and 3 respectively, are as follows:

$$\bar{X}_2 = 0.042531, \quad S^2_{X_2} = 0.045144, \quad DI_{X_2} = 1.061435, \quad S_{X_1X_2} = 0.045937, \quad \rho(X_1, X_2) = 0.777131,$$

$$\bar{X}_2 = 0.029710, \quad S^2_{X_2} = 0.030596, \quad DI_{X_2} = 1.029829, \quad S_{X_1X_2} = 0.031395, \quad \rho(X_1, X_2) = 0.645159,$$

$$\bar{X}_2 = 0.012247, \quad S^2_{X_2} = 0.012480, \quad DI_{X_2} = 1.019056, \quad S_{X_1X_2} = 0.012564, \quad \rho(X_1, X_2) = 0.404263.$$

From the values of the correlation coefficients, it appears that strong positive relations exist between the number of claims and the amount of claims (especially for the data of Table 1).

To compute the corresponding chi-squared test statistics, we followed an approach suggested by [26]. We grouped classes to produce a theoretical class of 5 or larger. The degrees of freedom of the relative χ^2 statistic in accordance with [26], were $n - k - 2$, where n is the number of classes

considered and k the number of parameters. For Tables 1–3, seven categories were considered, and the calculated χ^2 values had 3 degrees of freedom. However, different groupings were selected for each table. In particular, for Table 1, the groupings were
$(0,0),(1,0),(1,1),(2,0),\{(2,1),(2,2)\},\{(3,2),(3,3)\},\{(3,0),(3,1),(4,0),(4,1),(4,2),(4,3),(4,4)\}$;
for Table 2,
$(0,0),(1,0),(1,1),(2,2),\{(2,0),(2,1)\},\{(3,0),(3,1)\},\{(3,2),(3,3),(4,0),(4,1),(4,2),(4,3),(4,4)\}$;
and for Table 3,
$(0,0),(1,0),(1,1),(2,2),(3,0),\{(2,0),(2,1)\},\{(3,1),(3,2),(3,3),(4,0),(4,1),(4,2),(4,3),(4,4)\}$.

The 5% critical value of the chi-squared statistic is $\chi^2_3(0.05) = 7.815$. Since the calculated χ^2 values of the groupings used in Tables 1–3 for the values of α under consideration are less than this critical value, we can assume that our models provide an adequate fit for these types of data[1].

From Remark 4.2, for $\alpha = \theta$, the bivariate Poisson quasi-Lindley distribution defined by Eq (4.1) becomes a bivariate Poisson-Lindley model introduced by [11]. Consequently, it is of interest to examine if this value of α provides an adequate fit for our models to the observed data sets given in Tables 1–3. Relative calculations of the expected frequencies of the corresponding bivariate Poisson-Lindley model were already performed by [13]. However, the evaluation of the chi-squared test statistic were computed by using different groupings from those used in this paper. Therefore, for comparison purposes, we calculated the χ^2 values for the groupings adopted for this study. From our calculations, the corresponding χ^2 values were: for $\Psi = 500$, $\chi^2(3) = 5.568$; for $\Psi = 1000$, $\chi^2(3) = 2.248$; and for $\Psi = 3000$, $\chi^2(3) = 3.007$. All of them were below the value of the $\chi^2_3(0.05) = 7.815$ statistic.

Overall, we can conclude that for the examples considered, for values of the parameter α at least in the range $[1,6]$, and for the groupings selected for each data set, a large number of members from the general class of bivariate Poisson quasi-Lindley distributions defined by Eq (4.1) fit the relative data sets satisfactorily. Of course, it is expected that for additional values of the parameter α, not necessarily the same for each data set, or alternative groupings of the data sets, several other members of the bivariate distribution will provide a satisfactory fit.

However, to select the best model among those that, according to Pearson's chi-squared test statistic, fit a specific set of data satisfactorily, we need additional criteria. Two model selection methods based on likelihood functions and frequently used in applications are the Akaike Information Criterion/Baysian Information Criterion (AIC/BIC). Using AIC/BIC, the best model is the one that minimizes information loss, balancing goodness of fit with parsimony.

[1]The goodness of fit was determined by standard Pearson's chi-squared test statistic given by

$$\chi^2 = \sum_{i=1}^{7} \frac{(\text{Observed}_i - \text{Expected}_i)^2}{\text{Expected}_i}.$$

In particular, for Table 1 with $\alpha = 1$, the corresponding observed (O_i) and expected (E_i) values are, respectively:

O_i :	63233	1840	2493	37	234	12	8
E_i :	63220.00	1808.09	2544.20	46.15	221.24	9.65	6.63

and the calculated χ^2 value was $\chi^2(3) = 4.999$ as reported in Table 1.

Table 1. Size of claims (X_1) and total number of claims with claim size larger than a threshold monetary value (X_2) $\Psi = 500$ from a portfolio of 67856 automobile insurance policies

Size of claims / Number of claims	0	1	2	3	4	Total
	63232					63232
	63220.00					63220.00
0	63235.02					63235.02
	63241.88					63241.88
	63249.22					63249.22
	1840	2493				4333
	1808.09	2544.20				4352.29
1	1796.52	2527.92				4324.44
	1791.28	2520.54				4311.82
	1785.71	2512.70				4298.41
	37	117	117			271
	46.15	129.87	91.37			267.39
2	48.02	135.13	95.07			278.22
	48.84	137.45	96.70			282.99
	49.70	139.86	98.40			287.96
	1	5	5	7		18
	1.11	4.67	6.57	3.08		15.43
3	1.23	5.21	7.33	3.44		17.21
	1.30	5.47	7.70	3.61		18.08
	1.37	5.77	8.12	3.81		19.07
	0	0	1	0	1	2
	0.03	0.14	0.30	0.28	0.10	0.85
4	0.03	0.17	0.37	0.34	0.12	1.03
	0.03	0.19	0.40	0.38	0.13	1.13
	0.04	0.21	0.44	0.42	0.15	1.26
	65110	2615	123	7	1	67856
	65075.38	2678.88	98.24	3.36	0.10	67855.96
	65080.82	2668.43	102.77	3.78	0.12	67855.92
Total	65083.33	2663.65	104.80	3.99	0.13	67855.90
	65086.04	2658.54	106.96	4.23	0.15	67855.92

For $\alpha = 1$ $\tilde{p} = 0.584566$ $\tilde{\theta} = 20.616569$ $\chi^2(3) = 4.999$

For $\alpha = 2$ $\tilde{p} = 0.584566$ $\tilde{\theta} = 18.325839$ $\chi^2(3) = 4.305$

For $\alpha = 3$ $\tilde{p} = 0.584566$ $\tilde{\theta} = 17.180474$ $\chi^2(3) = 4.494$

For $\alpha = 6$ $\tilde{p} = 0.584566$ $\tilde{\theta} = 15.707862$ $\chi^2(3) = 5.151$

Table 2. Size of claims (X_1) and total number of claims with claim size larger than a threshold monetary value (X_2) $\Psi = 1000$ from a portfolio of 67856 automobile insurance policies.

Size of claims / Number of claims	0	1	2	3	4	Total
0	63232					63232
	63220.00					63220.00
	63235.02					63235.02
	63241.88					63241.88
	63249.22					63249.22
1	2551	1782				4333
	2575.05	1777.24				4352.29
	2558.58	1765.87				4324.45
	2551.11	1760.71				4311.82
	2543.18	1755.24				4298.42
2	109	114	48			271
	93.60	129.20	44.59			267.39
	97.39	134.44	46.39			278.22
	99.07	136.75	47.19			283.01
	100.80	139.14	48.02			287.96
3	5	6	6	1		18
	3.20	6.62	4.57	0.11		14.50
	3.57	7.38	5.10	1.17		17.22
	3.75	7.76	5.35	1.23		18.09
	3.95	8.18	5.65	1.30		19.08
4	1	0	0	1	0	2
	0.11	0.29	0.30	0.14	0.02	0.86
	0.13	0.35	0.36	0.17	0.03	1.04
	0.14	0.38	0.40	0.18	0.03	1.13
	0.15	0.42	0.44	0.20	0.04	1.25
Total	65898	1902	54	2	0	67856
	65891.96	1913.35	49.46	0.25	0.02	67855.04
	65894.69	1908.04	51.85	1.34	0.03	67855.95
	65895.95	1905.60	52.94	1.41	0.03	67855.93
	65897.30	1902.98	54.11	1.50	0.04	67855.93

For $\alpha = 1$ $\tilde{p} = 0.408345$ $\tilde{\theta} = 20.616569$ $\chi^2(3) = 2.804$
For $\alpha = 2$ $\tilde{p} = 0.408345$ $\tilde{\theta} = 18.325839$ $\chi^2(3) = 0.957$
For $\alpha = 3$ $\tilde{p} = 0.408345$ $\tilde{\theta} = 17.180474$ $\chi^2(3) = 1.209$
For $\alpha = 6$ $\tilde{p} = 0.408345$ $\tilde{\theta} = 15.707862$ $\chi^2(3) = 1.861$

Table 3. Size of claims (X_1) and total number of claims with claim size larger than a threshold monetary value (X_2) $\Psi = 3000$ from a portfolio of 67856 automobile insurance policies.

Size of claims / Number of claims	0	1	2	3	4	Total
0	63232 63220.00 63235.02 63241.88 63249.22					63232 63220.00 63235.02 63241.88 63249.22
1	3576 3619.71 3596.51 3586.06 3574.90	757 732.58 727.90 725.77 723.51				4333 4352.29 4324.41 4311.83 4298.41
2	216 184.95 192.44 195.75 199.18	44 74.86 77.90 79.23 80.62	11 7.58 7.88 8.02 8.16			271 267.39 278.22 283.00 287.96
3	12 8.87 9.90 10.41 10.97	4 5.39 6.01 6.32 6.66	2 1.09 1.22 1.29 1.35	0 0.07 0.08 0.09 0.09		18 15.42 17.21 18.11 19.07
4	2 0.41 0.50 0.54 0.60	0 0.33 0.40 0.44 0.49	0 0.10 0.12 0.13 0.15	0 0.01 0.02 0.02 0.02	0 0.00 0.00 0.00 0.00	2 0.85 1.04 1.13 1.26
Total	67038 67033.94 67034.37 67034.64 67034.87	805 813.16 812.21 811.76 811.28	13 8.77 9.22 9.44 9.66	0 0.08 0.10 0.11 0.11	0 0.00 0.00 0.00 0.00	67856 67855.95 67855.90 67855.95 67855.92

For $\alpha = 1$ $\quad \tilde{p} = 0.168321$ $\quad \tilde{\theta} = 20.616569$ $\quad \chi^2(3) = 4.041$

For $\alpha = 2$ $\quad \tilde{p} = 0.168321$ $\quad \tilde{\theta} = 18.325839$ $\quad \chi^2(3) = 3.372$

For $\alpha = 3$ $\quad \tilde{p} = 0.168321$ $\quad \tilde{\theta} = 17.180474$ $\quad \chi^2(3) = 3.618$

For $\alpha = 6$ $\quad \tilde{p} = 0.168321$ $\quad \tilde{\theta} = 15.707862$ $\quad \chi^2(3) = 4.239$

7. Applications to insurance claims data referring to material damage and bodily injury

In this section, we demonstrate the applicability of the bivariate Poisson-new XLindley distribution and of three additional members of the bivariate Poisson quasi-Lindley class defined by Eq (5.1), by fitting them to a set of claims corresponding to a large automobile portfolio in France, which includes 181038 liability policies issued during the year 1989. The yearly claim frequencies have been divided into material damage (variable X_1) and bodily injury (variable X_2). This set of data was previously used by several authors, including [8, 12, 13, 29–31].

As in the previous section, we assume that the parameter α takes the values 1, 2, 3, and 6. Since the marginal means are given by expressions (5.13) and (5.14), estimators for the parameters θ and p are obtained from the equations

$$\bar{X}_1 = q\frac{\alpha+2}{(\alpha+1)\theta}, \tag{7.1}$$

and

$$\bar{X}_2 = p\frac{\alpha+2}{(\alpha+1)\theta}. \tag{7.2}$$

Hence, from Eqs (7.1) and (7.2),

$$\tilde{\theta} = \frac{\alpha+2}{(\alpha+1)(\bar{X}_1+\bar{X}_2)}, \tag{7.3}$$

which is independent of the parameter p, and

$$\tilde{p} = \frac{\bar{X}_2}{\bar{X}_1+\bar{X}_2}, \tag{7.4}$$

which does not depend on the value of the parameters α and θ. Since $\bar{X}_1 = 0.051006$ and $\bar{X}_2 = 0.005529$, from Eq (7.4), we obtain $\tilde{p} = 0.097802$.

Bivariate probabilities are easily calculated by using the recurrences

$$P(X_1 = x_1+1, X_2 = 0) = \frac{q}{\theta+1}\,\frac{\alpha+(\alpha+x_1+2)\theta}{\alpha+(\alpha+x_1+1)\theta}P(X_1 = x_1, X_2 = 0), \quad x_1 = 0,1,2,3$$

and

$$P(X_1 = x_1, X_2 = x_2+1) = \frac{x_1+x_2+1}{x_2+1}\,\frac{p}{\theta+1}\,\frac{\alpha+(\alpha+x_1+x_2+2)\theta}{\alpha+(\alpha+x_1+x_2+1)\theta}P(X_1 = x_1, X_2 = x_2),$$
$$x_1 = 0,1,2,3, \quad x_2 = 0,1,$$

with $P(X_1 = 0, X_2 = 0)$ given by Eq (5.12).

Table 4 gives the observed frequencies on the first line and the expected frequencies for $\alpha = 1,2,3$, and 6 on the second, third, fourth, and fifth lines, respectively, together with the parameter estimates and χ^2 values. We used the seven groupings $(0,0), (0,1), (0,2), (1,1), (2,1)$, $\{(1,0),(2,0),(3,0)\}$, and $\{(1,2),(2,2),(3,1),(3,2),(4,0),(4,1),(4,2)\}$. Since all computed χ^2 values with 3 degrees of freedom were less than $\chi^2_{0.05}(3) = 7.815$, we can assume that all four models fit the data satisfactory.

Table 4. Material damage (X_1) and bodily injury (X_2) claims from a portfolio of 181038 liability policies.

Bodily injuries / Material damage	0	1	≥ 2	Total
	171345	918	2	172265
	171294.16	907.10	4.28	172205.54
0	171319.17	902.49	4.47	172226.13
	171330.62	900.39	4.75	172235.56
	171342.90	898.15	4.64	172245.69
	8273	73	0	8346
	8367.76	79.02	0.53	8447.31
1	8325.23	82.46	0.59	8408.28
	8305.87	83.99	0.62	8390.48
	8285.20	85.60	0.66	8371.46
	380	5	0	394
	364.47	4.85	0.04	369.36
2	380.32	5.43	0.05	385.80
	387.39	5.72	0.06	393.17
	394.81	6.05	0.06	400.92
	31	1	0	32
	14.90	0.25	0.00	15.15
3	16.70	0.31	0.00	17.01
	17.59	0.34	0.00	17.93
	18.60	0.38	0.00	18.98
	1	0	0	1
	0.59	0.01	0.00	0.60
4	0.71	0.02	0.00	0.73
	0.78	0.02	0.00	0.80
	0.87	0.02	0.00	0.89
	180039	997	2	181038
	180041.88	991.23	4.85	181037.96
Total	180042.13	990.71	5.11	181037.95
	180042.25	990.46	5.23	181037.94
	180042.38	990.20	5.36	181037.94

For $\alpha = 1$ $\tilde{p} = 0.097802$ $\tilde{\theta}_1 = 26.532194$ $\chi^2(3) = 2.397$

For $\alpha = 2$ $\tilde{p} = 0.097802$ $\tilde{\theta}_2 = 23.584172$ $\chi^2(3) = 2.914$

For $\alpha = 3$ $\tilde{p} = 0.097802$ $\tilde{\theta}_3 = 22.110161$ $\chi^2(3) = 3.358$

For $\alpha = 6$ $\tilde{p} = 0.097802$ $\tilde{\theta}_6 = 20.215005$ $\chi^2(3) = 3.982$

From Remark 5.2, for $\alpha = \theta$, the bivariate Poisson quasi-Lindley distribution defined by Eq (5.1) becomes another form of the bivariate Poisson-Lindley distribution introduced by [11]. Consequently, as in the previous section, it is of interest to examine if this value of α provides an adequate fit for our model to the observed data in Table 4. Relative calculations of the expected frequencies of the corresponding bivariate Poisson-Lindley model were already performed by [12]. However, since different groupings were used in that paper for the evaluation of the chi-squared test statistic, for comparison purposes, we calculated the χ^2-values using the seven groupings mentioned above. The computed χ^2 value was $\chi^2(3) = 3.523$ which is less than $\chi^2_{(3)}(0.05) = 7.815$.

Overall, we can conclude that, like the models in the previous section, the class of models defined by Eq (5.1) appears to fit satisfactorily the data set of Table 4 for a wide range of values of the parameter α.

8. Conclusions and suggestions for further research

Derivation techniques for the Poisson quasi-Lindley and the Poisson–new XLindley distributions were discussed. In particular, they were derived as Poisson mixtures when any one of three characteristics of the mixing distribution is known or by the addition of independent RVs. An EPQL distribution was introduced and extensively studied. It was derived (i) by generalizing a binomial distribution with respect to its exponent, (ii) as a Poisson mixture, and (iii) by addition of independent RVs. It has a number of interesting characteristics, appears as marginal distribution in bivariate Poisson quasi-Lindley models, and offers more flexibility for interpreting complex real-world data. However, parameter estimation techniques should be discussed and compared (see for example [53]) for fitting the distribution to real and simulated data.

Two general and structurally different bivariate Poisson quasi-Lindley and Poisson–new XLindley distributions were introduced utilizing various techniques, including mixing, generalization, addition of independent RVs, regression functions, and conditional distributions. Several of their characteristics were derived, including PGFs, probabilities and their recurrences, moments, conditional distributions, and regression functions. Several members of each class of bivariate Poisson quasi-Lindley distributions, including bivariate Poisson–new XLindley distributions, were fitted satisfactorily according to the χ^2 criterion, by the method of moments, to a number of sets of automobile insurance data previously used in the literature. However, more efficient methods of estimation, such as maximum likelihood, should be considered. Furthermore, comparison between the different models should be examined using additional criteria, like the AIC/BIC information criteria.

Since one of the problems faced by insurance companies is the derivation of meaningful bonus–malus premiums, the class of bivariate models studied in Sections 4 and 6 can be used as the basis for further evaluation of the corresponding premiums for the data sets given in Tables 1–3; see for example [26–28]. Furthermore, both types of our bivariate models can be considered as possible alternatives for interpreting sets of data with a large number of observations in the $(0, 0)$ cell. Finally, since some authors have suggested various applications of the Poisson-new XLindley distribution, it may be of interest to examine relative applications of other members of the Poisson quasi-Lindley class.

Author contributions

Maria Vardaki: Conceptualization, Investigation, Validation, Writing–review & editing: Haralambos Papageorgiou: Conceptualization, Methodology, Formal analysis, Writing-original draft. Both authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors hereby declare that they have no conflicts of interest to disclose concerning the current study.

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