AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(2): 3750–3771.
DOI: 10.3934/math.2026153
Received: 04 November 2025
Revised: 26 January 2026
Accepted: 29 January 2026
Published: 09 February 2026

Research article

Investigation of remainder terms in the ergodic distribution and moments of a renewal-reward process with heavy-tailed demand

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Abstract: This study investigated the detailed asymptotic behavior of the remainder terms in the ergodic distribution and its moments for a semi-Markovian renewal-reward process modeling an (s, S)-type inventory system. We focused on systems in which the demand random variables were heavy-tailed, specifically regularly varying with index $-\alpha$, where $1 < \alpha < 2$. While the first two terms in the asymptotic expansion of such models are available in the literature, earlier works have not provided sharp quantitative descriptions of the remainder. Our aim was to derive rigorous expressions that capture the exact decay of the remainder in both the ergodic distribution function and in the corresponding moments. Building on Doney's refinement of the renewal theorem [1], which distinguishes three main settings: the non-critical case $\alpha \neq 3/2$, the critical case $\alpha = 3/2$ with square-integrable equilibrium distribution, and the case where such integrability fails, we established new asymptotic expansions that explicitly capture the decay structure of the remainder. Using this framework, we analyzed the remainder for both the ergodic distribution and its moments for each scenario.

Keywords: semi-Markovian process; renewal-reward models; heavy-tailed demand; regularly varying distribution; finite first moment; ergodic distribution; ergodic moments; asymptotic expansion; explicit remainder term
Mathematics Subject Classification: 60K05, 60K15

1. Introduction

Heavy-tailed distributions have become increasingly important in the modeling of stochastic systems that are subject to high variability and rare but significant events. In particular, distributions with regularly varying tails of index $-\alpha$ for $1 < \alpha < 2$ provide a realistic description of phenomena in which large deviations play a central role. Typical examples include insurance claims, financial returns, service failures, and inventory demand, where light-tailed models often underestimate both the frequency and magnitude of extreme outcomes. The mathematical implications of heavy-tailed behavior are substantial: classical limit theorems often require modification, variances may be infinite, and convergence to steady-state distributions can be slow or nonstandard. In renewal-reward and semi-Markovian processes, heavy-tailed interarrival or demand distributions have a direct impact on the long-term behavior of performance measures such as renewal functions, cost accumulations, and ergodic distributions. These features motivate the use of refined asymptotic tools for accurate performance approximation and risk-aware decision-making; the present study addresses this need by analyzing the impact of regularly varying demand on a semi-Markovian renewal-reward model.

Renewal processes, renewal-reward processes, and their variations have long been fundamental tools in the mathematical modeling of real-world systems across diverse fields. In particular, they have been used in warranty cost analysis [2], stochastic inventory and outbound shipment decisions [3], reliability engineering [4], warranty analysis and renewal function estimation [5], computational warranty-cost models based on renewal processes [6], stationary inventory models via generalized renewal functions [7], expected warranty-cost modeling for two-attribute warranties [8], nonlinear renewal theory with applications to sequential analysis [9], and renewal-process approaches in quality control and sampling plans [10]. These processes provide a structured way to model random events by assuming that interarrival times and demand sizes are independent and identically distributed (i.i.d.) random variables.

In many practical settings, renewal processes are not isolated but interact with reward or cost structures. Depending on the system, rewards (or costs) may either accumulate continuously over time or occur at event epochs. Such renewal-reward functionals and their asymptotics have been studied, for example, in the context of randomly stopped averages [11], queueing systems [12], inventory performance and sensitivity analysis [13], and renewal-reward processes with retrospective reward structures [14]. Foundational background on renewal theory and stochastic-process methodology can be found in [15, 16], while related analytical tools and applied models are discussed in [17, 18]. Further results on cumulant/moment methods for renewal processes appear in [19], and an algorithmic perspective on stochastic models is provided in [20]. These extensive applications have led to a broad and growing body of literature.

The renewal function $U(t)$ represents the expected number of renewals in the interval $[0, t]$ and plays a central role in renewal-reward process analysis. This function is central to both theoretical investigations and practical implementations. In semi-Markovian stochastic models, frequently employed in inventory control theory, renewal events do not occur at fixed intervals but are governed by random time distributions. As a result, understanding the renewal function and its long-run behavior is crucial for characterizing system performance and making informed control decisions.

In this study, we focus on a semi-Markovian renewal–reward process that models an (s, S)-type inventory system, where the demand random variables are heavy-tailed. More precisely, the demand random variables have finite mean and belong to the class of regularly varying distributions with tail index $-\alpha$, where $1 < \alpha < 2$. In an (s, S)-type model with uniformly distributed interference of chance and a regularly varying tail assumption $1 < \alpha < 2$, Kamışlık et al. [21] established two-term asymptotic expansions for the ergodic distribution, but without quantifying how the remaining terms behave; moreover, the critical case $\alpha = \frac{3}{2}$ was not analyzed via the three separate regimes considered in the present work. However, the rate at which the approximation error decays directly affects the accuracy of long-run performance evaluations, making its explicit determination essential.

The dynamics of the system can be formulated through the following model.

The model. The inventory level at the initial time $t = 0$ is assumed to be at its maximum capacity, denoted by $X(0) = S$, where S represents the upper bound of the stock level. Over time, the inventory decreases due to successive demands occurring at random time points $T_1, T_2, \ldots, T_n, \ldots$. These demands, represented by a sequence of independent random variables $\eta_1, \eta_2, \ldots, \eta_n, \ldots$, reduce the stock level incrementally until it reaches or falls below a predefined threshold s, where $0 < s < S < \infty$. The evolution of the stock level in the depot can thus be expressed as follows:

$$X_1 = S - \eta_1, \quad X_2 = S - (\eta_1 + \eta_2), \quad \ldots, \quad X_n = S - \sum_{i=1}^{n} \eta_i.$$

Here, the random variable η_n denotes the quantity of the n-th demand. The time points at which the stock transitions from one level to another are given by

$$T_n = \sum_{i=1}^{n} \xi_i, \quad n = 1, 2, \ldots.$$

Here the random variable ξ_i represents the interarrival time between consecutive demands. As each demand occurs, the inventory level decreases discretely by an amount corresponding to the demand size. This depletion process continues until a random time τ_1, defined as the first passage time at which the stock level reaches or drops below the control level s. At this point, an immediate replenishment occurs, restoring the stock level to S, marking the completion of one replenishment cycle. Subsequently, a new cycle starts with the stock level reset to S, and the depletion process repeats in an analogous manner across successive cycles.

Semi-Markovian renewal–reward processes and related modifications have been investigated in the literature, including models with normally distributed interference of chance [22], $\Gamma(g)$-distributed demand [23], and moment-based approximations for stochastic control models of type (s, S) [24]. Heavy-tailed demand with infinite variance in an (s, S) setting has been studied in [21]. Weak convergence results for the ergodic distribution in (s, S)-type inventory models are discussed in [25], while asymptotic approaches for semi-Markovian (s, S) models and renewal–reward processes with general interference of chance can be found in [26,27]. In addition to these studies, there are significant works examining the process $X(t)$ under heavy-tailed demand. For example, the classical (s, S)-type inventory model with subexponential demand and finite variance was examined in [28]. By expressing the final term of the ergodic distribution and the moments of the ergodic distribution using big-O notation, two-term asymptotic expansions were obtained. Moreover, [21] analyzed a related model under regularly varying demand with infinite variance and uniformly distributed interference of chance, and derived a two-term asymptotic expansion for the ergodic distribution of the process.

Differently from the litreature, in this study, much more precise results have been obtained for both the ergodic distribution of the process $X(t)$ and the moments of the ergodic distribution. While our earlier work [21] utilized Geluk's classical approach [29] to estimate the remainder in the renewal function, the current study adopts Doney's sharper asymptotic framework [1], which enables a more precise characterization of the decay behavior of the error term. Geluk's method [29] provides upper bounds that are insightful but do not capture the leading asymptotic profile. In contrast, Doney's result [1] yields an explicit expression that reveals the exact contribution of the remainder term, especially near the critical case $\alpha = 3/2$, where standard approximations are less effective. Doney [1] treated three principal regimes separately: the non-critical case $\alpha \neq \frac{3}{2}$, the critical case $\alpha = \frac{3}{2}$ with square-integrable equilibrium distribution, and the case where such integrability fails.

Using Doney's refinement of the renewal theorem [1], we analyze the remainder for both the ergodic distribution of the considered process and its moments. In each case, we derive detailed asymptotic expansions accompanied by explicitly quantified remainders across the entire range $1 < \alpha < 2$. Importantly, we consider both the general case $\alpha \neq 3/2$ and the critical case $\alpha = 3/2$ within a unified asymptotic framework.

Our approach not only refines existing methodologies but also offers a more precise characterization of the tail behavior in these stochastic models. These findings contribute to a deeper understanding of ergodic properties in systems governed by heavy-tailed demand distributions and provide a solid theoretical foundation for applications in inventory modeling, risk assessment, and stochastic process analysis.

To our knowledge, this is the first detailed, case-by-case treatment of an (s, S)-type semi-Markovian inventory model with regularly varying demand covering the full range $1 < \alpha < 2$.

The remainder of this paper is organized as follows. In Section 2, we introduce the necessary preliminaries, including the theoretical framework, key concepts from the theory of regular variation, and the mathematical construction of the semi-Markovian renewal-reward process. We also present precise formulas for the ergodic distribution and its moments that will serve as the foundation for our asymptotic analysis. Section 3 is devoted to the main results, including deriving improved asymptotic approximations for both the ergodic distribution and the higher-order ergodic moments in the presence of heavy-tailed demand distributions. Special attention is given to the critical case $\alpha = 3/2$, for which refined two-term expansions are established. Finally, Section 4 concludes the paper with a summary of our contributions and a discussion of possible directions for future research.

2. Theoretical background and preliminaries

In this section, we introduce the essential notations and provide a mathematical formulation of the model prior to addressing the main problem.

2.1. Theory of regular variation and important results

This section presents the key definitions and foundational results that will be used throughout the study. The well-established material is primarily drawn from [30–32].

Definition 2.1. A positive measurable function $f : [0, \infty) \to [0, \infty)$ is said to be *regularly varying at infinity* with index α, written $f \in RV_\alpha$, if for every $\lambda > 0$,

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda^\alpha. \tag{2.1}$$

In the special case $\alpha = 0$, that is, when for all $\lambda > 0$,

$$\lim_{x\to\infty} \frac{L(\lambda x)}{L(x)} = 1,$$

the function L is called *slowly varying* and we write $L \in SV$.

Remark 2.2. Every regularly varying function f of index α can be expressed in the form

$$f(x) = x^{\alpha} L(x),$$

where L is a slowly varying function.

Theorem 2.3 (Karamata's theorem [32]). *Let L be slowly varying and locally bounded in* $[x_0, \infty)$ *for some* $x_0 \geq 0$. *Then*

(a) for $\alpha > -1$,

$$\int_{x_0}^{x} t^{\alpha} L(t)\, dt \sim (\alpha + 1)^{-1} x^{\alpha+1} L(x), \quad x \to \infty;$$

(b) for $\alpha < -1$,

$$\int_{x}^{\infty} t^{\alpha} L(t)\, dt \sim -(\alpha + 1)^{-1} x^{\alpha+1} L(x), \quad x \to \infty.$$

The following propositions focus on integral transforms of regularly varying functions and their asymptotic behaviors when $\alpha = -1$.

Proposition 2.4. *Let* $L \in SV$ *be locally bounded on* $[a, \infty)$. *Then*

$$\tilde{L}(x) = \int_{a}^{x} \frac{L(t)}{t} dt$$

is slowly varying, and $\tilde{L}(x)/L(x) \to \infty$.

Proposition 2.5. *If* $\int_{x}^{\infty} \frac{L(t)}{t} dt < \infty$, *then*

$$\tilde{L}(x) = \int_{x}^{\infty} \frac{L(t)}{t} dt$$

is slowly varying and $\tilde{L}(x)/L(x) \to \infty$.

Karamata's result is frequently useful in various applications. It essentially states that the integrals of regularly varying functions also exhibit regular variation. More specifically, the slowly varying function can be factored out of the integral. This theorem is particularly valuable when analyzing integrals involving regularly varying functions, as it simplifies their asymptotic behavior. One of the theorems that is particularly useful when dealing with the integrals of regularly varying functions is given below (see [32, Theorem 2.7]).

Theorem 2.6 ([32]). *Let L be slowly varying on* $(0, \infty)$ *and bounded on each finite subinterval of* $(0, \infty)$. *Suppose the integral*

$$\int_{0}^{\gamma} t^{-\eta} f(t)\, dt$$

is well-defined for some given real function f and a given number $\eta > 0$. *Then as* $x \to \infty$,

$$\int_{0}^{\gamma} f(t) L(xt)\, dt \sim L(x) \int_{0}^{\gamma} f(t)\, dt,$$

for $\eta > 0$*; and for* $\eta = 0$ *providing L is non-decreasing on* $(0, \infty)$.

Definition 2.7. A non-negative random variable X and its distribution are said to be regularly varying with index $\alpha \geq 0$ if the right distribution tail $\overline{F}(x)$ is regularly varying with index $-\alpha$.

Remark 2.8. According to Remark 2.2 and Definition 2.7, it is straightforward to observe that for any regularly varying random variable with index α, the tail distribution can be expressed as

$$\overline{F}(x) = x^{-\alpha} L(x),$$

where L denotes a slowly varying function.

In this section, we introduced only the theoretical background required for our analysis. The extensive theory of regularly varying functions, including their properties, historical development, and applications, is systematically treated in [30–32]. We now proceed to provide the mathematical formulation of the stochastic process considered in this study.

2.2. Model assumptions and construction of the process $X(t)$

Let $\{\xi_n\}$ and $\{\eta_n\}$, for $n \geq 1$, be two independent sequences of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, where the variables in each sequence are independent and identically distributed. Assume that ξ_n and η_n take only positive values, with their corresponding distribution functions given by

$$\Phi(t) = P(\xi_1 \leq t),\ t > 0, \quad \text{and} \quad F(x) = P(\eta_1 \leq x),\ x > 0.$$

We define renewal sequences $\{T_n\}$ and $\{Y_n\}$ using the initial sequences $\{\xi_n\}$ and $\{\eta_n\}$ as follows:

$$T_n = \sum_{i=1}^{n} \xi_i, \quad Y_n = \sum_{i=1}^{n} \eta_i, \quad n = 1, 2, \ldots; \quad T_0 = Y_0 = 0.$$

Additionally, let us introduce a sequence of integer-valued random variables:

$$N_0 = 0, \quad N_{n+1} = \min\left\{k \geq N_n + 1 : S - (Y_k - Y_{N_n}) < s\right\}, \quad n = 0, 1, 2, \ldots.$$

Following random variables, τ_n denotes the times at which the stock level falls below the control threshold s for the n^{th} time:

$$\tau_n = T_{N_n}, \quad n = 1, 2, \ldots; \quad \tau_0 = 0.$$

Furthermore, we define the counting process $\nu(t)$ as:

$$\nu(t) = \max\{n \geq 0 : T_n \leq t\}.$$

Using these notations, we construct the following stochastic process:

$$X(t) = S - Y_{\nu(t)} + Y_{N_k}, \quad \text{for} \quad \tau_k \leq t < \tau_{k+1}, \quad k = 0, 1, 2, \ldots.$$

2.3. Precise formulas for the ergodic distribution of the process $X(t)$ and its moments

The primary objective of this paper is to examine the asymptotic behavior of the system as the parameter $\beta = S - s$ in the ergodic distribution becomes sufficiently large. Additionally, we study the n^{th}-order moments ($n = 1, 2, \dots$) of ergodic distribution of the stochastic process $X(t)$, particularly in cases where the sequence of demands $\{\eta_n\}$, $n \geq 1$, follows a regularly varying distribution with tail index $1 < \alpha < 2$. In this specific case, regularly varying random variables are known to have infinite variance.

Let us define the ergodic distribution of the process $X(t)$ as $Q_X(x) = \lim_{t\to\infty} P\{X(t) \leq x\}$. The following proposition presents a result on the ergodicity of the process $X(t)$, based on [33]:

Proposition 2.9 ([33]). *Suppose the initial sequences $\{\xi_n\}$ and $\{\eta_n\}$, for $n \geq 1$, satisfy the following conditions:*
(1) $E\xi_1 < \infty$;
(2) The random variable η_1 does not follow an arithmetic distribution.
Then, the process $X(t)$ is ergodic, and its ergodic distribution function has the explicit form:

$$Q_X(x) = 1 - \frac{U(S - x)}{U(S - s)}, \quad s \leq x \leq S, \tag{2.2}$$

where $U(x) = \sum_{m=0}^{\infty} F_{\eta}^{*m}(x)$ denotes the renewal function associated with the i.i.d. sequence $\{\eta_n\}_{n\geq 1}$ (with the convention $F_{\eta}^{*0}(x) = \mathbf{1}\{x \geq 0\}$).

If the standardized form of the process $X(t)$ is defined as $W_\beta(t) \equiv \frac{1}{\beta}(X(t) - s)$, then the exact formula for the ergodic distribution of $W_\beta(t)$ is given as follows:

$$Q_{W_\beta}(x) = 1 - \frac{U(\beta(1 - x))}{U(\beta)}, \quad \beta \equiv S - s. \tag{2.3}$$

By applying Proposition 2.9, n^{th}-order moments ($n = 1, 2, \dots$) of the ergodic distribution of the process $X(t)$ is derived as follows (see [28]):

Proposition 2.10. *If the conditions in Proposition 2.9 hold and n^{th}-order moments of the ergodic distribution of the process $\tilde{X}(t) = X(t) - s$ are finite, then they can be expressed using the renewal function $U(x)$ as follows:*

$$E(\tilde{X}^n) = \frac{nU_n(\beta)}{U(\beta)}, \tag{2.4}$$

where $U_n(\beta)$ is defined as:

$$U_n(\beta) = \beta^{n-1} * U(\beta) \equiv \int_0^{\beta} (\beta - t)^{n-1} U(t)dt, \quad n = 1, 2, \dots. \tag{2.5}$$

As evident from the exact formulas of both the ergodic distribution and ergodic moments, the renewal function generated by demand random variables is essential for this study. Altough for very few distributions (e.g., exponential, Erlang) exact formulas for renewal function $U(x)$ can be obtained, in general cases, this is not easy. Therefore, when analyzing real-life problems that involve the renewal function, utilizing the asymptotic expansion or approximations of the renewal function is a preferred approach.

3. Asymptotic analysis of the ergodic distribution and n^{th}-order moments of the ergodic distribution of the process $X(t)$

In this part of the study, our goal is to obtain improved approximate results for the ergodic distribution of the process $X(t)$ in the case where the demand random variables have a regularly varying distribution with infinite variance, i.e., $\overline{F}(x) = P(\eta_1 > x) = x^{-\alpha}L(x)$, for $1 < \alpha < 2$. To state our results, let us first introduce the equilibrium distribution and excess distribution of a random variable η_1 as follows:

$$F_I(x) = 1/m \int_0^x \overline{F}(u)du, \quad \overline{F_I}(x) = 1/m \int_x^\infty \overline{F}(u)du. \tag{3.1}$$

Note that

$$\overline{F}(x) = P(\eta_1 > x) = x^{-\alpha}L(x), \quad 1 < \alpha < 2, \quad \text{and} \quad m = E(\eta_1). \tag{3.2}$$

As previously noted, this study requires approximate expressions for the renewal function associated with the demand random variables. The corresponding renewal process, denoted by $N(z)$, is defined as $N(z) = inf\{n \geq 1 : \sum_{i=1}^n \eta_i > z\}$, where η_i represents the sequence of i.i.d. demand random variables [15, p. 184]. The renewal function $U(z)$ is the expected number of demands in the interval of time $[0, t]$, and is given by

$$U(t) = E[N(t)] = \sum_{n=0}^\infty F^{*n}(t). \tag{3.3}$$

There exist many asymptotic expansions and approximations in the literature for renewal functions generated by random variables whose tail distributions are given as in (3.2) (see, for example, [29, 34–36]).

A notable refinement relevant to this study is presented in [1], where significantly sharper approximations are obtained compared to earlier results for this class of problems. First of all, let us define the real-valued functions $\overline{G}$ and g as follows:

$$g(y) = 2f_I(y) - (f_I(y) * f_I(y)) \quad \text{and} \quad \overline{G}(x) = \int_x^\infty g(z)dz. \tag{3.4}$$

Here $f_I(x)$ denotes the density function of the equilibrium distribution $F_I(x)$, which is given by $f_I(x) = m^{-1}\overline{F}(x)$. The expression $f_I(y) * f_I(y)$ represents the two-fold (second-order) convolution of $f_I(y)$ with itself. Moreover we define the constant

$$c_\alpha = \frac{(3 - 2\alpha)\Gamma(2 - \alpha)^2}{\Gamma(4 - 2\alpha)}, \quad \text{where } \Gamma(\cdot) \text{ is a well-known gamma function.}$$

By setting

$$U(x) - m^{-1}x - m^{-1} \int_0^x \overline{F_I}(y)\, dy = m^{-1}V(x). \tag{3.5}$$

Doney [1] obtained the following approximations as $x \to \infty$:

$$V(x) \sim \frac{|c_\alpha| x \overline{F_I}(x)^2}{|2\alpha - 3|}, \qquad \text{if } \alpha \neq 3/2, \tag{3.6}$$

$$V(x) \to \int_0^\infty \overline{G}(y)\, dy, \qquad \text{if } \alpha = 3/2 \text{ and } \int_0^\infty \overline{F_I}(y)^2\, dy < \infty, \tag{3.7}$$

$$V(x) = o\left(\int_0^x \overline{F_I}(y)^2\, dy\right), \qquad \text{if } \alpha = 3/2 \text{ and } \int_0^\infty \overline{F_I}(y)^2\, dy = \infty. \tag{3.8}$$

Now we will present improved approximations for the ergodic distribution of the process $X(t)$ while the tail distribution of the demand random variables satisfies condition (3.2) by using Doney's approximations. To obtain these results, we will use the following lemmas:

Lemma 3.1. *Let $\overline{F}(x) = x^{-3/2}L(x)$ with $L \in SV$ and $m = E(\eta_1) \in (0, \infty)$, and set*

$$L_s(x) := \int_1^x \frac{L(y)^2}{y}\, dy.$$

Moreover assume that

$$\int_0^\infty \overline{F}_I(y)^2\, dy = \infty,$$

for $\overline{F}_I$ is defined by (3.1).

Then, as $x \to \infty$,

$$\int_0^x \overline{F}_I(y)^2\, dy \;\sim\; \frac{4}{m^2}\, L_s(x).$$

Proof. Write

$$\overline{F}_I(y) = \frac{1}{m}\int_y^\infty u^{-3/2}L(u)\, du = \frac{1}{m}\, y^{-1/2}L(y)\int_1^\infty t^{-3/2}\frac{L(yt)}{L(y)}\, dt \qquad (u = yt). \tag{3.9}$$

We recall Potter's theorem in the form of [30, Thm. 1.5.6(i)]: If $L \in SV$, then for every $C > 1$ and every $\varepsilon > 0$, there exists $x_0 = x_0(C, \varepsilon)$ such that

$$\frac{L(y)}{L(x)} \le C \max\left\{\left(\frac{y}{x}\right)^{\varepsilon}, \left(\frac{y}{x}\right)^{-\varepsilon}\right\}, \qquad x \ge x_0,\ y \ge x_0. \tag{3.10}$$

In our setting, we apply this statement with $x = y$ and $y = yt$ (so that $y/x = t$). Due to our change of variables $u = yt$ in (3.9), then $t \in [1, \infty)$, and we restrict to $t \ge 1$, in which case $\max\{t^{\varepsilon}, t^{-\varepsilon}\} = t^{\varepsilon}$. Moreover, interchanging x and y in (3.10) yields the corresponding lower estimate. Consequently, for any fixed $C > 1$ and any fixed $\varepsilon > 0$, there exists $y_0 = y_0(C, \varepsilon)$ such that for all $y \ge y_0$ and all $t \ge 1$,

$$C^{-1}t^{-\varepsilon} \le \frac{L(yt)}{L(y)} \le Ct^{\varepsilon}. \tag{3.11}$$

In (3.11), we choose $C = 1 + \delta$ with an arbitrary $\delta > 0$ and restrict to $\varepsilon \in (0, \frac{1}{2})$ to ensure that the integrals $\int_1^\infty t^{-3/2\pm\varepsilon}\, dt$ are finite. So, we fix $\varepsilon \in (0, \frac{1}{2})$ and $\delta > 0$, and then there exists $y_0 = y_0(\varepsilon, \delta)$ such that for all $y \ge y_0$ and all $t \ge 1$,

$$(1+\delta)^{-1}t^{-\varepsilon} \;\le\; \frac{L(yt)}{L(y)} \;\le\; (1+\delta)\, t^{\varepsilon}.$$

Note that here y_0 may depend on (ε, δ); since we first fix (ε, δ) and then let $y \to \infty$, the inequalities hold for all sufficiently large y. Hence, for all $y \ge y_0$,

$$\frac{1}{m}\, y^{-1/2}L(y)\,(1+\delta)^{-1}\int_1^\infty t^{-3/2-\varepsilon}dt \;\le\; \overline{F}_I(y) \;\le\; \frac{1}{m}\, y^{-1/2}L(y)\,(1+\delta)\int_1^\infty t^{-3/2+\varepsilon}dt. \tag{3.12}$$

Since $\varepsilon < 1/2$, the integrals appearing on the left- and right-hand sides of inequality (3.12) can be calculated as follows:

$$\int_1^\infty t^{-3/2-\varepsilon}\,dt = \frac{2}{1+2\varepsilon}, \qquad \int_1^\infty t^{-3/2+\varepsilon}\,dt = \frac{2}{1-2\varepsilon}.$$

Define

$$a_{\varepsilon,\delta} := \frac{2}{(1+\delta)(1+2\varepsilon)}, \qquad b_{\varepsilon,\delta} := \frac{2(1+\delta)}{1-2\varepsilon}.$$

Then, for $y \geq y_0$,

$$\frac{a_{\varepsilon,\delta}}{m}\, y^{-1/2}L(y) \;\leq\; \overline{F}_I(y) \;\leq\; \frac{b_{\varepsilon,\delta}}{m}\, y^{-1/2}L(y). \tag{3.13}$$

Inequality (3.14) follows directly from (3.13) upon squaring both sides:

$$\frac{a_{\varepsilon,\delta}^2}{m^2}\,\frac{L(y)^2}{y} \;\leq\; \overline{F}_I(y)^2 \;\leq\; \frac{b_{\varepsilon,\delta}^2}{m^2}\,\frac{L(y)^2}{y}, \qquad y \geq y_0. \tag{3.14}$$

Integrating (3.14) over $[y_0, x]$ gives

$$\frac{a_{\varepsilon,\delta}^2}{m^2}\int_{y_0}^x \frac{L(y)^2}{y}\,dy \;\leq\; \int_0^x \overline{F}_I(y)^2\,dy + \int_0^{y_0} F_I(y)^2\,dy \;\leq\; \frac{b_{\varepsilon,\delta}^2}{m^2}\int_{y_0}^x \frac{L(y)^2}{y}\,dy.$$

Noting that $\int_0^{y_0} \overline{F}_I(y)^2 dy$ is a finite constant (independent of x), we obtain (3.15):

$$\frac{a_{\varepsilon,\delta}^2}{m^2}\int_{y_0}^x \frac{L(y)^2}{y}\,dy \;\leq\; \int_0^x \overline{F}_I(y)^2\,dy + O(1) \;\leq\; \frac{b_{\varepsilon,\delta}^2}{m^2}\int_{y_0}^x \frac{L(y)^2}{y}\,dy. \tag{3.15}$$

By the assumption $\int_0^\infty \overline{F}_I(y)^2\,dy = \infty$, we have $\int_0^x \overline{F}_I(y)^2\,dy \to \infty$, hence the $O(1)$ term is negligible. In particular, the upper bound implies $\int_{y_0}^x \frac{L(y)^2}{y}\,dy \to \infty$, so

$$\int_{y_0}^x \frac{L(y)^2}{y}\,dy = L_s(x) + O(1).$$

Dividing the previous inequality by $L_s(x)$ and letting $x \to \infty$ gives

$$\frac{a_{\varepsilon,\delta}^2}{m^2} \;\leq\; \liminf_{x\to\infty} \frac{\int_0^x \overline{F}_I(y)^2\,dy}{L_s(x)} \;\leq\; \limsup_{x\to\infty} \frac{\int_0^x \overline{F}_I(y)^2\,dy}{L_s(x)} \;\leq\; \frac{b_{\varepsilon,\delta}^2}{m^2}.$$

Finally, since $a_{\varepsilon,\delta} \to 2$ and $b_{\varepsilon,\delta} \to 2$ as $\varepsilon \downarrow 0$ and $\delta \downarrow 0$, we obtain

$$\int_0^x \overline{F}_I(y)^2\,dy \;\sim\; \frac{4}{m^2}\,L_s(x), \qquad x \to \infty.$$

Finally, by applying Proposition 2.4, we have $L_s \in SV$ and $L_s(x)/L(x)^2 \to \infty$. □

Lemma 3.2. *Let* $\{\eta_i\},\ i \geq 1$, *be a sequence of regularly varying random variables with exponent* $-\alpha$, $1 < \alpha < 2$, *i.e.,*

$$\overline{F}(t) = P\{\eta_1 > t\} = t^{-\alpha} L(t).$$

Let $m = E(\eta_1)$. *Then, as* $x \to \infty$, *the following approximations are obtained for the renewal function generated by the random variables* $\{\eta_i\},\ i \geq 1$.

Case 1. $\alpha \neq 3/2$:

$$U(x) \sim \frac{x}{m} + \frac{1}{m^2} \frac{1}{(\alpha-1)(2-\alpha)} x^{2-\alpha} L(x) + \frac{1}{m} \frac{|c_\alpha| x^{3-2\alpha} L^2(x)}{(\alpha-1)^2 |2\alpha-3|}. \tag{3.16}$$

Case 2. $\alpha = 3/2$ *and* $\int_0^\infty \overline{F_I}(y)^2\, dy < \infty$:

$$U(x) = \frac{x}{m} + \frac{4}{m^2} x^{1/2} L(x) + o\big(x^{1/2} L(x)\big), \qquad x \to \infty. \tag{3.17}$$

Case 3. $\alpha = 3/2$ *and* $\int_0^\infty \overline{F_I}(y)^2\, dy = \infty$:

$$U(x) = \frac{x}{m} + \frac{4}{m^2} x^{1/2} L(x) + o(L_s(x)). \tag{3.18}$$

Note that in (3.18), $L_s(x)$ *is a slowly varying function such that*

$$L_s(x) := \int_1^x \frac{L(y)^2}{y}\, dy \text{ satisfies } \lim_{x\to\infty} \frac{L_s(x)}{L^2(x)} = \infty. \tag{3.19}$$

Proof. **Case 1.** $\alpha \neq 3/2$: It is well known that (see, for example, [32])

$$\overline{F}_I(x) \sim \frac{1}{m} \frac{x^{1-\alpha}}{(\alpha-1)} L(x) \in RV(1-\alpha) \quad \text{for} \quad 1 < \alpha < 2 \quad \text{and} \quad \alpha \neq 3/2. \tag{3.20}$$

From here we obtain

$$V(x) \sim \frac{|c_\alpha| x^{3-2\alpha} L^2(x)}{(\alpha-1)^2 |2\alpha-3|}. \tag{3.21}$$

In this case, from Theorem 2.3(a), it is clear that

$$\frac{1}{m} \int_0^x \overline{F_I}(y) dy \sim \frac{1}{m^2} \frac{1}{(\alpha-1)(2-\alpha)} x^{2-\alpha} L(x). \tag{3.22}$$

Substituting (3.21) and (3.22) into (3.5) yields the desired result.

Case 2. $\alpha = 3/2$ and $\int_0^\infty \overline{F_I}(y)^2\, dy < \infty$: From Karamata's theorem (since $\alpha = 3/2 > 1$),

$$\overline{F_I}(x) \sim \frac{2}{m} x^{-1/2} L(x), \qquad \int_0^x \overline{F_I}(y)\, dy \sim \frac{4}{m} x^{1/2} L(x).$$

Following (3.5), we define

$$V(x) := m\Big(U(x) - \frac{x}{m} - \frac{1}{m} \int_0^x \overline{F_I}(y)\, dy\Big).$$

By the definitions of g and $\overline{G}$ in (3.4), the remainder term $V(x)$ can be represented as the renewal-type convolution

$$V(x) = \int_0^x \overline{G}(x-y)\,\overline{F_I}(y)\,dy. \tag{3.23}$$

Under the assumptions $\alpha = \frac{3}{2}$ and

$$\int_0^\infty \overline{F_I}(y)^2\,dy < \infty,$$

the equilibrium density f_I is square-integrable, and therefore the convolution $f_I * f_I$ is well-defined and integrable on $(0,\infty)$. In this case, Doney [1] showed that $\overline{G}(x)$ is integrable, i.e.,

$$\int_0^\infty \overline{G}(y)\,dy < \infty.$$

Consequently, from (3.23) and dominated convergence,

$$V(x) \;=\; \int_0^x \overline{G}(x-y)\,\overline{F_I}(y)\,dy \;\xrightarrow[x\to\infty]{}\; \int_0^\infty \overline{G}(y)\,dy \;=\; C.$$

Combining (3.5) and the following asymptotic equivalences:

$$\overline{F_I}(x) \sim \frac{2}{m}\,x^{-1/2}L(x), \qquad \int_0^x \overline{F_I}(y)\,dy \sim \frac{4}{m}\,x^{1/2}L(x),$$

we obtain

$$U(x) = \frac{x}{m} + \frac{1}{m}\int_0^x \overline{F_I}(y)\,dy + \frac{1}{m}V(x) = \frac{x}{m} + \frac{4}{m^2}\,x^{1/2}L(x) + \frac{C}{m} + o(1), \qquad x \to \infty.$$

For any slowly varying function L, we have $t^\alpha L(t) \to \infty$ for every $\alpha > 0$ (and $t^\alpha L(t) \to 0$ for every $\alpha < 0$); see [37, Lemma 2.1 (SV5)]. Hence $x^{1/2}L(x) \to \infty$. Then the constant term $\frac{C}{m}$ is asymptotically negligible, which gives:

$$U(x) = \frac{x}{m} + \frac{4}{m^2}\,x^{1/2}L(x) + o(x^{1/2}L(x)).$$

This completes the derivation.

Case 3. $\alpha = 3/2$ and $\int_0^\infty \overline{F_I}(y)^2 dy = \infty$: Let $\overline{F}(x) = x^{-3/2}L(x)$ with $L \in SV$ and $E(\eta_1) = m \in (0,\infty)$.

By the definition of $\overline{F}_I$ and Karamata's theorem (see Theorem 2.3(b)),

$$\overline{F}_I(x) \sim \frac{2}{m}x^{-1/2}L(x),$$

so that $\overline{F}_I \in RV_{-1/2}$. Therefore, by Karamata's theorem (see Theorem 2.3(a)),

$$\frac{1}{m}\int_0^x \overline{F}_I(y)\,dy \sim \frac{2}{m}\,x\,\overline{F}_I(x) \sim \frac{4}{m^2}x^{1/2}L(x). \tag{3.24}$$

For the critical index $\alpha = 3/2$ and under $\int_0^\infty \overline{F}_I(y)^2 dy = \infty$, Doney's second-order estimate (see Eq (3.8)) gives

$$V(x) = o\left(\int_0^x \overline{F}_I(y)^2 \, dy\right).$$

Moreover, by Lemma 3.1,

$$\int_0^x \overline{F}_I(y)^2 \, dy \sim \frac{4}{m^2} L_s(x),$$

and therefore

$$V(x) = o(L_s(x)), \tag{3.25}$$

where L_s is the slowly varying function defined in Lemma 3.1. Inserting (3.24) and (3.25) into (3.5) yields

$$U(x) = \frac{x}{m} + \frac{4}{m^2} x^{1/2} L(x) + o(L_s(x)), \qquad x \to \infty,$$

which is the desired result. □

Building upon Lemma 3.2, Theorem 3.3 provides new approximations for the ergodic distribution of the process $X(t)$. For the noncritical regime $\alpha \neq \frac{3}{2}$, we obtain an explicit two-term approximation together with a rigorously controlled remainder term. This refines the available asymptotic descriptions by making the correction term and the error order explicit. Moreover, in the critical case $\alpha = \frac{3}{2}$, we derive a two-term expansion

$$Q_{W_\beta}(x) = x + (\text{correction term}) + o(\cdot),$$

whose leading term x is consistent with the standard scaling limit in classical (s, S)-type models (see, for example, [24]), while the order of the $o(\cdot)$ remainder is inherited from Doney's [1] second–order renewal estimate in the critical regime.

Theorem 3.3. *Let the conditions of Proposition 2.9 be satisfied. Assume further that the distribution of η_1 is regularly varying as specified in Lemma 3.2. Under these conditions, for each $x \in (0, 1)$, approximate expressions for the ergodic distribution of the process $Q_{W_\beta}(x)$ are derived below as $\beta = S - s \to \infty$.*

Case 1. $1 < \alpha < 2$; $\alpha \neq 3/2$:

$$\begin{aligned} Q_{W_\beta}(x) &\sim x + c_{21}\left\{(1-x) - (1-x)^{2-\alpha}\right\}\beta^{1-\alpha}L(\beta) \\ &\quad + \left\{c_{21}^2\left[(x-1) - (1-x)^{2-2\alpha}\right] - c_3 x + c_{31}\left[1 - (1-x)^{3-2\alpha}\right]\right\}\beta^{2-2\alpha}L^2(\beta). \end{aligned} \tag{3.26}$$

For

$$\begin{aligned} &c_1 = \frac{1}{m}, \quad c_2 = \frac{1}{m^2(\alpha-1)(2-\alpha)}, \\ &c_{21} = \frac{c_2}{c_1} = \frac{1}{m(\alpha-1)(2-\alpha)}, \quad c_\alpha = \frac{(3-2\alpha)\Gamma(2-\alpha)^2}{\Gamma(4-2\alpha)}, \quad \Gamma(\cdot) \quad \textit{denotes the gamma function}, \\ &c_3 = \frac{|c_\alpha|}{m|2\alpha-3|(\alpha-1)^2} = \frac{\Gamma(2-\alpha)^2}{m\Gamma(4-2\alpha)(\alpha-1)^2}, \quad c_{31} = \frac{c_3}{c_1} = \frac{\Gamma(2-\alpha)^2}{\Gamma(4-2\alpha)(\alpha-1)^2}, \quad m = E(\eta_1). \end{aligned} \tag{3.27}$$

Case 2. *$\alpha = 3/2$ and $\int_0^\infty \overline{F_I}(y)^2\,dy < \infty$:*

$$Q_{W_\beta}(x) = x + \frac{4}{m}\left((1-x) - (1-x)^{1/2}\right)\beta^{-1/2}L(\beta) + o\left(\beta^{-1/2}L(\beta)\right). \tag{3.28}$$

Case 3. *$\alpha = 3/2$ and $\int_0^\infty \overline{F_I}(y)^2\,dy = \infty$:*

$$Q_{W_\beta}(x) = x + \left[\frac{4}{m}\left((1-x) - (1-x)^{1/2}\right)\right]\beta^{-1/2}L(\beta) + o\left(\beta^{-1}L_s(\beta)\right). \tag{3.29}$$

Note that in (3.29), $L_s(x)$ is a slowly varying function defined by (3.19).

Proof. **Case 1.** First of all, since $L(x)$ is slowly varying, $L(cx) \sim L(x)$ for any constant c. Hence, $L(\beta(1-x)) \sim L(\beta)$ for each $x \in (0,1)$ and $\beta \to \infty$. On the other hand, $L^2(x)$ is slowly varying for any slowly varying $L(x)$. Hence $L^2(\beta(1-x)) \sim L^2(\beta)$ for each $x \in (0,1)$ as $\beta \to \infty$.

From (3.16), we have

$$(U(\beta))^{-1} \sim \frac{m}{\beta}\left\{1 - c_{21}\beta^{1-\alpha}L(\beta) + (c_{21}^2 - c_3)\beta^{2-2\alpha}L^2(\beta)\right\}, \tag{3.30}$$

$$U(\beta(1-x)) \sim \{c_1(1-x)\}\beta + \{c_2(1-x)^{2-\alpha}\}\beta^{2-\alpha}L(\beta) + \{c_3(1-x)^{3-2\alpha}\}\beta^{3-2\alpha}L^2(\beta). \tag{3.31}$$

Substituting (3.30) and (3.31) on the right-hand side of (2.3), we obtain the following approximation for $Q_{W_\beta}(x)$ by using a Taylor series expansion:

$$\begin{aligned}
Q_{W_\beta}(x) &= 1 - \frac{U(\beta(1-x))}{U(\beta)} \\
&\sim \left\{x + \left[\frac{c_2}{c_1}[1-(1-x)^{2-\alpha}]\right]\beta^{1-\alpha}L(\beta) + \left[\frac{c_3}{c_1}[1-(1-x)^{3-2\alpha}]\right]\beta^{2-2\alpha}L^2(\beta)\right\} \\
&\times \left\{1 - c_{21}\beta^{1-\alpha}L(\beta) + (c_{21}^2 - c_3)\beta^{2-2\alpha}L^2(\beta)\right\} \\
&= x + c_{21}\left\{(1-x) - (1-x)^{2-\alpha}\right\}\beta^{1-\alpha}L(\beta) \\
&+ \left\{c_{21}^2\left[(x-1) - (1-x)^{2-2\alpha}\right] - c_3 x + c_{31}\left[1 - (1-x)^{3-2\alpha}\right]\right\}\beta^{2-2\alpha}L^2(\beta).
\end{aligned}$$

Case 2. From (3.17), we have

$$(U(\beta))^{-1} \sim \frac{m}{\beta}\left\{1 - \left(\frac{4}{m}\right)\beta^{-1/2}L(\beta) + o\left(\beta^{-1/2}L(\beta)\right)\right\}, \tag{3.32}$$

$$U(\beta(1-x)) \sim \frac{(1-x)}{m}\beta + \left\{\frac{4}{m^2}(1-x)^{1/2}\right\}\beta^{1/2}L(\beta) + o(\beta^{1/2}L(\beta)). \tag{3.33}$$

By substituting of (3.32) and (3.33) on the right-hand side of (2.3), the desired result holds using similar techniques as in Case 1.

Case 3. From (3.18), we have

$$(U(\beta))^{-1} \sim \frac{m}{\beta}\left\{1 - \frac{4}{m}\beta^{-1/2}L(\beta) + o\left(\beta^{-1}L_s(\beta)\right)\right\}, \tag{3.34}$$

such that $L_s(x)$ is a slowly varying function satisfying the conditions of Lemma 3.1 and

$$lim_{x\to\infty}\frac{L_s(x)}{L^2(x)} = \infty.$$

Moreover,

$$U(\beta(1-x)) \sim \frac{(1-x)}{m}\beta + \left\{\frac{4}{m^2}(1-x)^{1/2}\right\}\beta^{1/2}L(\beta) + o(L_s(\beta)). \tag{3.35}$$

Substitution of (3.34) and (3.35) on the right-hand side of (2.3) allows the desired result to hold. □

With Theorem 3.3, we derived an approximation for the ergodic distribution of the process $W_\beta(t)$ under certain specified conditions. A second key objective of this work is to establish approximate results for the moments of the ergodic distribution of the process $X(t)$, denoted by $E(\tilde{X}^n)$ for $n \geq 1$. The exact expressions for these moments are presented with (2.4) in Section 2.3. Prior to introducing the approximations for the ergodic moments of order n, we first present the following lemmas. In Lemmas 3.4–3.6, we study the asymptotic behavior of $U_n(\beta)$ case by case.

Lemma 3.4. *Assume that the conditions of Proposition 2.9 hold. Suppose further that the random variables* (η_i) *generating the renewal function U satisfy the assumptions of Lemma 3.2 (Case 1). Let* $U_n(\beta)$ *be defined by* (2.5). *Then, as* $\beta \to \infty$,

$$U_n(\beta) \sim \frac{\beta^{n+1}}{m}B(2,n) + c_2 B(n, 3-\alpha)\beta^{n+2-\alpha}L(\beta) + c_3 B(n, 4-2\alpha)\beta^{n+3-2\alpha}L_1(\beta), \tag{3.36}$$

where

$$c_2 = \frac{1}{m^2(\alpha-1)(2-\alpha)}, \qquad c_3 = \frac{|c_\alpha|}{m|2\alpha-3|(\alpha-1)^2} = \frac{\Gamma(2-\alpha)^2}{m\Gamma(4-2\alpha)(\alpha-1)^2},$$

$B(\cdot,\cdot)$ *is the beta function,* $\Gamma(\cdot)$ *denotes the gamma function, and* $L_1(\beta) = L(\beta)^2$.

Proof. Taking (3.16) into account, we have

$$\begin{aligned} U_n(\beta) &= \int_0^\beta (\beta-t)^{n-1}U(t)\,dt \\ &\sim \int_0^\beta (\beta-t)^{n-1}\left\{\frac{t}{m} + c_2\,t^{2-\alpha}L(t) + c_3\,t^{3-2\alpha}L_1(t)\right\}dt =: I_{11}(\beta) + I_{12}(\beta) + I_{13}(\beta). \end{aligned}$$

For $I_{11}(\beta)$,

$$I_{11}(\beta) = \frac{1}{m}\int_0^\beta (\beta-t)^{n-1}t\,dt = \frac{1}{m}\beta^{n+1}\int_0^1 (1-u)^{n-1}u\,du = \frac{1}{m}\beta^{n+1}B(2,n). \tag{3.37}$$

For $I_{12}(\beta)$, by Theorem 2.6,

$$\begin{aligned} I_{12}(\beta) &= c_2\int_0^\beta (\beta-t)^{n-1}t^{2-\alpha}L(t)\,dt \\ &= c_2\,\beta^{n+2-\alpha}\int_0^1 (1-u)^{n-1}u^{2-\alpha}L(\beta u)\,du \\ &\sim c_2\,\beta^{n+2-\alpha}L(\beta)\,B(n, 3-\alpha). \end{aligned} \tag{3.38}$$

Since $L_1 = L^2 \in SV$, Theorem 2.6 also yields

$$\begin{aligned} I_{13}(\beta) &= c_3 \int_0^\beta (\beta - t)^{n-1} t^{3-2\alpha} L_1(t)\, dt \\ &= c_3\, \beta^{n+3-2\alpha} \int_0^1 (1-u)^{n-1} u^{3-2\alpha} L_1(\beta u)\, du \\ &\sim c_3\, \beta^{n+3-2\alpha} L_1(\beta)\, B(n, 4-2\alpha). \end{aligned} \tag{3.39}$$

Combining (3.37)–(3.39) gives (3.36). □

Lemma 3.5. *Assume that the conditions of Proposition 2.9 hold. Suppose further that* (η_i) *satisfies the assumptions of Lemma 3.2 (Case 2). Let* $U_n(\beta)$ *be defined by* (2.5). *Then, as* $\beta \to \infty$,

$$U_n(\beta) = \frac{1}{m} B(2, n)\, \beta^{n+1} + \frac{4}{m^2} B\left(n, \frac{3}{2}\right) \beta^{n+\frac{1}{2}} L(\beta) + o\left(\beta^{n+\frac{1}{2}} L(\beta)\right). \tag{3.40}$$

Proof. Taking (3.17) into account, we derive

$$\begin{aligned} U_n(\beta) &= \int_0^\beta (\beta - t)^{n-1} U(t)\, dt \\ &= \int_0^\beta (\beta - t)^{n-1} \left\{ \frac{t}{m} + \frac{4}{m^2} t^{1/2} L(t) + H(t) \right\} dt \\ &= I_{21}(\beta) + I_{22}(\beta) + I_{23}(\beta), \end{aligned}$$

where $H(t) = o\left(t^{1/2} L(t)\right)$. The first two terms are computed exactly as in (3.37) and (3.38), yielding

$$I_{21}(\beta) = \int_0^\beta (\beta - t)^{n-1} \frac{t}{m} dt = \frac{1}{m} \beta^{n+1} B(2, n)$$

and

$$I_{22}(\beta) = \frac{4}{m^2} \int_0^\beta (\beta - t)^{n-1} t^{1/2} L(t) dt = \frac{4}{m^2} B\left(n, \frac{3}{2}\right) \beta^{n+\frac{1}{2}} L(\beta).$$

It remains to show that $I_{23}(\beta) = o\left(\beta^{n+\frac{1}{2}} L(\beta)\right)$, where

$$I_{23}(\beta) = \int_0^\beta (\beta - t)^{n-1} H(t)\, dt$$

and $H(t) = o\left(t^{1/2} L(t)\right)$ as $t \to \infty$. Fix $\varepsilon > 0$. Then there exists $T > 0$ such that

$$|H(t)| \le \varepsilon\, t^{1/2} L(t), \qquad t \ge T.$$

Split $I_{23}(\beta) = J_1(\beta) + J_2(\beta)$ with

$$J_1(\beta) := \int_0^T (\beta - t)^{n-1} H(t)\, dt, \qquad J_2(\beta) := \int_T^\beta (\beta - t)^{n-1} H(t)\, dt.$$

Since T is fixed and H is locally integrable, we have $J_1(\beta) = O(\beta^{n-1})$, hence $J_1(\beta) = o\Big(\beta^{n+\frac{1}{2}}L(\beta)\Big)$. Moreover, for $\beta > T$,

$$\begin{aligned}|J_2(\beta)| &\leq \varepsilon \int_T^{\beta} (\beta - t)^{n-1} t^{1/2} L(t)\, dt \\ &= \varepsilon \beta^{n+\frac{1}{2}} \int_{T/\beta}^{1} (1-u)^{n-1} u^{1/2} L(\beta u)\, du \\ &\sim \varepsilon \beta^{n+\frac{1}{2}} L(\beta) \int_0^1 (1-u)^{n-1} u^{1/2}\, du = \varepsilon\, B\left(n, \frac{3}{2}\right) \beta^{n+\frac{1}{2}} L(\beta),\end{aligned}$$

by Theorem 2.6. Since ε is arbitrary, it follows that $I_{23}(\beta) = o\Big(\beta^{n+\frac{1}{2}}L(\beta)\Big)$. Combining the asymptotics of I_{21}–I_{23} yields (3.40). □

Lemma 3.6. *Assume that the conditions of Proposition 2.9 hold. Suppose further that* (η_i) *satisfies the assumptions of Lemma 3.2 (Case 3). Let* $U_n(\beta)$ *be defined by* (2.5). *Then, as* $\beta \to \infty$*,*

$$U_n(\beta) = \frac{1}{m} B(2, n) \beta^{n+1} + \frac{4}{m^2} B\left(n, \frac{3}{2}\right) \beta^{n+\frac{1}{2}} L(\beta) + o\left(\beta^n L_s(\beta)\right), \tag{3.41}$$

where $L_s \in SV$ *is as in Lemma 3.1 and satisfies*

$$\lim_{\beta \to \infty} \frac{L_s(\beta)}{L(\beta)^2} = \infty.$$

Proof. Taking (3.18) into account, write

$$\begin{aligned}U_n(\beta) &= \int_0^{\beta} (\beta - t)^{n-1} U(t)\, dt \\ &= \int_0^{\beta} (\beta - t)^{n-1} \left\{ \frac{t}{m} + \frac{4}{m^2} t^{1/2} L(t) + G(t) \right\} dt \\ &= I_{31}(\beta) + I_{32}(\beta) + I_{33}(\beta),\end{aligned}$$

where $G(t) = o(L_s(t))$ as $t \to \infty$.

The evaluations of $I_{31}(\beta)$ and $I_{32}(\beta)$ are identical to those of $I_{21}(\beta)$ and $I_{22}(\beta)$ in the proof of Lemma 3.5 (with the same change of variables), hence

$$I_{31}(\beta) = \frac{1}{m} \beta^{n+1} B(2, n), \qquad I_{32}(\beta) \sim \frac{4}{m^2} B\left(n, \frac{3}{2}\right) \beta^{n+\frac{1}{2}} L(\beta).$$

It remains to show that

$$I_{33}(\beta) = \int_0^{\beta} (\beta - t)^{n-1} G(t)\, dt = o(\beta^n L_s(\beta))\,.$$

Fix $\varepsilon > 0$. Since $G(t) = o(L_s(t))$, there exists $T > 0$ such that

$$|G(t)| \leq \varepsilon\, L_s(t), \qquad t \geq T.$$

Split $I_{33}(\beta) = K_1(\beta) + K_2(\beta)$ with

$$K_1(\beta) := \int_0^T (\beta - t)^{n-1} G(t)\, dt, \qquad K_2(\beta) := \int_T^\beta (\beta - t)^{n-1} G(t)\, dt.$$

Since T is fixed, $K_1(\beta) = O(\beta^{n-1}) = o(\beta^n L_s(\beta))$. Moreover, for $\beta > T$,

$$\begin{aligned} |K_2(\beta)| &\le \varepsilon \int_T^\beta (\beta - t)^{n-1} L_s(t)\, dt \\ &= \varepsilon \beta^n \int_{T/\beta}^1 (1-u)^{n-1} L_s(\beta u)\, du \sim \varepsilon \beta^n L_s(\beta) \int_0^1 (1-u)^{n-1}\, du \\ &= \varepsilon\, B(1,n) \beta^n L_s(\beta), \end{aligned}$$

where we used Theorem 2.6 (with $L_s \in SV$). Since ε is arbitrary, $I_{33}(\beta) = o(\beta^n L_s(\beta))$. Combining I_{31}–I_{33} yields (3.41). □

Theorem 3.7. *Let the conditions of Proposition 2.9 be satisfied. Assume further that the distribution of the demand random variable η_1 is as specified in Lemma 3.2. Under these conditions, approximate expressions for the moments of ergodic distribution of the process $Q_{W_\beta}(x)$ are derived below as $\beta = S - s \to \infty$.*

Case 1. $1 < \alpha < 2$, $\alpha \neq 3/2$,

$$E\left(\tilde{X}^n\right) \sim a_1\beta^n + (a_2 - a_1a_4)\beta^{n+1-\alpha}L(\beta) + (a_3 - a_2a_4 - a_5a_1 + a_4^2a_1)\beta^{n+2-2\alpha}L^2(\beta), \tag{3.42}$$

for

$$\begin{aligned} &a_1 = \frac{1}{n+1}, \quad a_2 = \frac{n!(2-\alpha)!}{(\alpha-1)(2-\alpha)(n+2-\alpha)!}, \quad a_3 = \frac{nB(n, 4-2\alpha)\Gamma(2-\alpha)^2}{\Gamma(4-2\alpha)(\alpha-1)^2}, \\ &a_4 = \frac{1}{(\alpha-1)(2-\alpha)}, \quad a_5 = \frac{\Gamma(2-\alpha)^2}{\Gamma(4-\alpha)}; \quad \Gamma(\cdot) \quad \textit{is the gamma function}, \quad n = 1,2,3,.... \end{aligned} \tag{3.43}$$

Case 2. $\alpha = 3/2$ *and* $\int_0^\infty \overline{F_I}(y)^2\, dy < \infty$,

$$E\left(\tilde{X}^n\right) \sim a_1\beta^n + b_1\beta^{n-\frac{1}{2}}L(\beta) + o\left(\beta^{n-\frac{1}{2}}L(\beta)\right), \tag{3.44}$$

for

$$a_1 = \frac{1}{n+1}, \quad b_1 = \left(\frac{4n}{m}\right)\left[B\left(n, \frac{3}{2}\right) - B\left(n, 2\right)\right]; \quad B(\cdot,\cdot) \quad \textit{is the beta function}, \quad m_1 = E(\eta_1).$$

Case 3. $\alpha = 3/2$ *and* $\int_0^\infty \overline{F_I}(y)^2\, dy = \infty$,

$$E\left(\tilde{X}^n\right) \sim a_1\beta^n + b_1\beta^{n-\frac{1}{2}}L(\beta) + o\left(\beta^{n-1}L_s(\beta)\right), \tag{3.45}$$

for

$$a_1 = \frac{1}{n+1}, \quad b_1 = \left(\frac{4n}{m}\right)\left[B\left(n, \frac{3}{2}\right) - B\left(n, 2\right)\right]; \quad B(\cdot,\cdot) \quad \textit{is the beta function}, \quad m_1 = E(\eta_1).$$

Note that in (3.45), $L_s(x)$ is a slowly varying function that satisfies the conditions of Lemma 3.1 such that

$$lim_{x\to\infty}\frac{L_s(x)}{L^2(x)} = \infty.$$

Proof. By (2.4),

$$E(\tilde{X}^n) = \frac{n\,U_n(\beta)}{U(\beta)}.$$

Case 1 follows from Lemma 3.4 and (3.30). Case 2 follows from Lemma 3.5 and (3.32). Case 3 follows from Lemma 3.6 and (3.34). □

4. Conclusions

In this study, we have carried out a detailed asymptotic analysis of the ergodic distribution and its moments for a semi-Markovian renewal–reward process arising from an (s, S)-type inventory system with heavy-tailed, regularly varying demand. The emphasis has been on identifying the leading terms and the correct error orders uniformly over the range $1 < \alpha < 2$, with particular attention to the transition at the critical index $\alpha = \frac{3}{2}$.

Unlike our earlier work, which relied on Geluk's classical renewal theorem [29], the present paper is based on Doney's refined renewal expansion [1]. Geluk's approach is well suited for obtaining leading-order asymptotics and bounds, but it typically leaves the remainder implicit. By contrast, Doney's framework yields a sharper second-order description in the heavy-tailed regime and, crucially for our purposes, provides the appropriate error scale in the critical setting. This allows us to write the approximations in a form that is both explicit and directly usable in the inventory model.

A main contribution of the paper is a uniform treatment of the non-critical and critical regimes within a single framework. When $\alpha \neq \frac{3}{2}$, we obtain two-term expansions with explicit constants, making transparent how the tail index controls the rate at which the normalized ergodic quantities approach their limits. When $\alpha = \frac{3}{2}$, we recover the correct second term and distinguish the two sub-regimes determined by $\int_0^\infty \overline{F}_I(y)^2\,dy$, which governs the order of the remainder. In this sense, the critical case is handled at the level of precision required for performance evaluation, rather than only at the level of first-order limits.

Overall, the results provide a more explicit description of the asymptotic structure of the ergodic distribution and its moments, with remainder terms stated on the correct scale. From a modeling viewpoint, this is most relevant precisely in settings where heavy tails drive the long-run behavior, such as inventory systems with sporadic large demands and related renewal–reward models with slowly decaying fluctuations.

Possible extensions include treating more general interference-of-chance mechanisms, allowing dependence or long-range effects in the demand sequence, and developing analogous expansions for multi-item or networked inventory systems where several renewal–reward components interact.

Use of Generative-AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author gratefully acknowledges Tahir Khaniyev (TOBB University of Economics and Technology) and Tülay Yazır (Karadeniz Technical University) for their valuable comments and support.

Conflict of interest

The author declares no conflict of interest.

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