Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(2): 3636–3646.
DOI:10.3934/math.2026148
Received: 17 November 2025
Revised: 22 January 2026
Accepted: 29 January 2026
Published: 06 February 2026

Research article

Recurrence and transience for infinite dimensional hypercube

Nan Fan, Caishi Wang* and Jijun Zhao

School of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China

* **Correspondence:** Email: cswangnwnu@163.com.

Abstract: The infinite-dimensional hypercube (IDH) is a novel example of a locally infinite graph and can play an important role in understanding Anderson localization and other related physical phenomena. In this paper, we investigate the IDH from the perspective of transience and recurrence. We examine the Green function associated with the heat semigroup on the IDH and establish several of its fundamental properties. By using the unilateral Green function, which we introduce, we provide conditions for the heat semigroup to be transient. Finally, we prove that the transience (or recurrence) of the heat semigroup is equivalent to that of a discrete-time Markov chain defined on the IDH.

Keywords: infinite dimensional hypercube; heat semigroup; Green function; transience; discrete-time Markov chain
Mathematics Subject Classification: 60J10

1. Introduction

As a special class of finite graphs, hypercubes are widely used to build mathematical models across diverse research fields. In statistical physics, hypercubes serve as the configuration spaces of various mean-field spin glass models [8]. In mathematical biology, the REM House-of-Cards model adopts a hypercube as its space of gene types [1]. Moreover, in quantum information theory, hypercubes are often used to represent the testing grounds of quantum annealing algorithms designed for searches in unstructured energy landscapes [2].

From a topological perspective, however, hypercubes are essentially of finite dimension. Recently, an infinite dimensional analogue of hypercubes, called the infinite dimensional hypercube (IDH), has been introduced in Reference [9]. It has been shown that the IDH plays an active role in building an alternative model for understanding Anderson localization in disordered quantum systems, as well as in addressing other problems in mathematical physics (see [9, 10]). The IDH is a connected infinite graph, and its vertex set has a group structure, which is very similar to that of the d-dimensional integer lattice $\mathbb{Z}^d$. However, the IDH is locally infinite; namely, each of its vertices has an infinite number of neighbors, which is in sharp contrast to the local finiteness of $\mathbb{Z}^d$. Graphs are naturally classified into two categories: locally finite ones and locally infinite ones, although these two categories are not complementary sets of each other. The IDH actually provides a novel example of locally infinite graphs and seems to deserve attention from various aspects. We note that the vertex set of the IDH has a group structure, which distinguishes the IDH from other locally infinite graphs studied in the literature [3].

The study of the long-time behavior of discrete-time Markov chains is a classical research field in probability theory, wherein the most fundamental problem might lie in determining whether a given Markov chain exhibits recurrence or transience. Schmidt [7] investigated, within the framework of Dirichlet forms, recurrence and transience of discrete-time Markov chains on a class of weighted graphs. Among others, he found the close connection between the notion of recurrence/transience for discrete-time Markov chains and that for regular Dirichlet forms. Kostenko and Nicolussi [6] considered the transience of the operator semigroup generated by a Laplacian on the Cayley graph of a finitely generated group. Recently, Keller and Muranova [4] characterized recurrence and transience for non-Archimedean and directed graphs. There is also interesting work on recurrence and transience for growing trees and hypercubes [5].

In this paper, motivated by the preceding discussion, we investigate the IDH from the perspectives of recurrence and transience. More precisely, analyze the transience of the heat semigroup on the IDH and examine its relationship with a discrete-time Markov chain. Our main contributions are summarized as follows. We first examine the heat semigroup on the IDH together with its Green function and establish several fundamental properties of the function. We then introduce a function, called the unilateral Green function, and use it to derive sufficient conditions for the heat semigroup to be transient. Finally, we construct a discrete-time Markov chain on the IDH and prove that this Markov chain is transient if and only if the heat semigroup is transient.

2. Preliminaries

In this section, we describe our setup by introducing the infinite dimensional hypercube and the related Laplacian, heat semigroup, and Dirichlet form.

Let Γ stand for the family of finite subsets of $\mathbb{N} := \{m \in \mathbb{Z} \mid m \geq 0\}$, namely

$$\Gamma = \{\sigma \mid \sigma \subset \mathbb{N},\ \#(\sigma) < \infty\}, \tag{1}$$

where $\#(\sigma)$ means the cardinality of σ as a set. Note that the empty set $\emptyset$ is an element of Γ with $\#(\emptyset) = 0$. Two elements $\sigma, \tau \in \Gamma$ are said to be adjacent, written $\sigma \sim \tau$, if $\#(\sigma \triangle \tau) = 1$. Here $\sigma \triangle \tau$ denotes the symmetric difference of σ and τ. With the adjacency relation "$\sim$", Γ forms an infinite graph, which we denote by $(\Gamma, \sim)$.

Remark 2.1. *The graph* $(\Gamma, \sim)$ *is called the infinite dimensional hypercube (IDH), while* Γ *is called the vertex set of the IDH.*

For a vertex $\sigma \in \Gamma$, we write $\mathcal{N}(\sigma) = \{\tau \mid \tau \in \Gamma,\ \tau \sim \sigma\}$, which is known as the nearest neighbor set of σ. It can be shown that $\mathcal{N}(\sigma)$ has a representation of the form

$$\mathcal{N}(\sigma) = \{\sigma \triangle k \mid k \in \mathbb{N}\}, \tag{2}$$

where $\sigma \triangle k := \sigma \triangle \{k\}$. Thus, as a graph, the IDH is locally infinite, i.e., each of its vertices has infinitely many nearest neighbors. The IDH has many amazing properties. For example, it is connected and has a topological group structure. Moreover, it can be viewed as the inductive limit of hypercubes.

Let $\mathcal{H}$ be the space of square summable real-valued functions defined on the vertex set Γ, i.e.,

$$\mathcal{H} = l^2(\Gamma) = \Big\{ f\colon \Gamma \to \mathbb{R} \,\Big|\, \sum_{\sigma\in\Gamma} |f(\sigma)|^2 < \infty \Big\} \tag{3}$$

with the usual linear operation and inner product, that is, $\mathcal{H} = l^2(\Gamma, m)$ with a counting measure m defined by $m(\{\gamma\}) = 1$ for all $\gamma \in \Gamma$. We denote by $\langle \cdot, \cdot \rangle$ the inner product in $\mathcal{H}$ and write $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$. It is known that $\mathcal{H}$ is a separable Hilbert space of infinite dimension. We denote by $\{\psi_\gamma \mid \gamma \in \Gamma\}$ the canonical orthonormal basis (ONB) for $\mathcal{H}$, where ψ_γ is the function on Γ given by

$$\psi_\gamma(\sigma) = \begin{cases} 1, & \sigma = \gamma, \\ 0, & \sigma \in \Gamma, \sigma \neq \gamma. \end{cases} \tag{4}$$

We use $l^\infty(\Gamma)$ to mean the Banach space of all bounded functions on Γ with the usual linear operation and the norm $\|\cdot\|_\infty$ defined by

$$\|f\|_\infty = \sup_{\sigma\in\Gamma} |f(\sigma)|, \quad f \in l^\infty(\Gamma). \tag{5}$$

It is easy to see that $\mathcal{H}$ is a proper linear subspace of $l^\infty(\Gamma)$ and $\|f\|_\infty \leq \|f\|$ for $f \in \mathcal{H}$.

To overcome the local infiniteness of the IDH, a function needs to be introduced. A function $w\colon \mathbb{N} \to (0, \infty)$ is called a *weight* on $\mathbb{N}$ if it satisfies the requirement that $|w| := \sum_{k=0}^{\infty} w(k) < \infty$. The summability condition for the weight function ensures the boundedness of the Laplacian.

Lemma 2.1. *[11] Let w be a weight on $\mathbb{N}$. Then there exists a positive self-adjoint bounded linear operator Δ_w on $\mathcal{H}$ such that*

$$\Delta_w f(\sigma) = \sum_{k=0}^{\infty} w(k)[f(\sigma) - f(\sigma \triangle k)], \quad \sigma \in \Gamma, \tag{6}$$

where $f \in \mathcal{H}$ and $\Delta_w f(\sigma)$ means the value of the function $\Delta_w f$ at σ.

The operator Δ_w indicated above is called the w-Laplacian on the IDH, while the operator semigroup $\{e^{-t\Delta_w} \mid t \in \mathbb{R}_+\}$ on $\mathcal{H}$ generated by $-\Delta_w$ is known as the heat semigroup on the IDH.

Lemma 2.2. *[11] Let w be a weight on $\mathbb{N}$ and $\mathcal{E}_w$ be the form on $\mathcal{H}$ defined by*

$$\mathcal{E}_w(f, g) = \langle f, \Delta_w g \rangle, \quad f, g \in \mathcal{H}. \tag{7}$$

Then $\mathcal{E}_w$ is a Dirichlet form on $\mathcal{H}$. And moreover, it admits a representation of the form

$$\mathcal{E}_w(f, g) = \frac{1}{2} \sum_{\sigma\in\Gamma} \sum_{k=0}^{\infty} w(k)(f(\sigma) - f(\sigma \triangle k))(g(\sigma) - g(\sigma \triangle k)), \quad f, g \in \mathcal{H}. \tag{8}$$

This lemma, together with the general theory of Dirichlet forms, shows that the heat semigroup $\{e^{-t\Delta_w} \mid t \in \mathbb{R}_+\}$ is actually a Markov semigroup. It can be shown that the heat semigroup has a representation of the form

$$e^{-t\Delta_w} f(\sigma) = \sum_{\tau\in\Gamma} p_t(\sigma, \tau) f(\tau), \quad f \in \mathcal{H}, \; t \in \mathbb{R}_+, \tag{9}$$

where $p_t(\sigma,\tau) = \langle e^{-t\Delta_w}\psi_\sigma, \psi_\tau\rangle$. Conventionally, the family $\{p_t \mid t \in \mathbb{R}_+\}$ of functions on $\Gamma \times \Gamma$ is called the heat kernel on the IDH.

Lemma 2.3. *[11] Let w be a weight on $\mathbb{N}$. Then, the heat kernel $\{p_t \mid t \in \mathbb{R}_+\}$ admits the following properties:*

(1) $p_t(\sigma,\tau) = p_t(\tau,\sigma) \geq 0$ *for all σ, $\tau \in \Gamma$ and $t \in \mathbb{R}_+$;*
(2) $p_{s+t}(\sigma,\tau) = \sum_{\gamma\in\Gamma} p_s(\sigma,\gamma)p_t(\gamma,\tau)$ *for all σ, $\tau \in \Gamma$ and s, $t \in \mathbb{R}_+$;*
(3) $\sum_{\tau\in\Gamma} p_t(\sigma,\tau) = 1$ *for all σ, $\tau \in \Gamma$ and $t \in \mathbb{R}_+$.*

3. Green function and transience of heat semigroup

In this section, we first examine the Green function of the heat semigroup and then use it to investigate the transience of the heat semigroup. Throughout this section, w is assumed to be a fixed weight on $\mathbb{N}$.

Recall that the heat semigroup $\{e^{-t\Delta_w} \mid t \in \mathbb{R}_+\}$ is Markovian, hence the operator $e^{-t\Delta_w}$ is positivity preserving for each $t \in \mathbb{R}_+$. This, together with the uniform continuity of the heat semigroup $\{e^{-t\Delta_w} \mid t \in \mathbb{R}_+\}$, implies that the function $t \mapsto e^{-t\Delta_w}\psi_\sigma(\tau)$ is nonnegative and continuous on $[0,\infty)$ for any σ, $\tau \in \Gamma$. In view of these facts, we come to the next definition.

Definition 3.1. *The Green function of the heat semigroup $\{e^{-t\Delta_w} \mid t \in \mathbb{R}_+\}$ is the function $G_w\colon \Gamma\times\Gamma \to [0,\infty]$ given by*

$$G_w(\sigma,\tau) = \int_0^\infty e^{-t\Delta_w}\psi_\sigma(\tau)dt, \quad \sigma,\ \tau \in \Gamma, \tag{10}$$

where ψ_σ is the basis vector of the canonical ONB for $\mathcal{H}$.

Clearly, the Green function is positive in the sense that $G_w(\sigma,\tau) \geq 0$ for all σ, $\tau \in \Gamma$. The next proposition shows that it also admits symmetry and some kind of invariance.

Proposition 3.1. *G_w is symmetric, and moreover for all σ, τ, $\gamma \in \Gamma$, it holds true that*

$$G_w(\sigma\triangle\gamma, \tau\triangle\gamma) = G_w(\sigma,\tau), \tag{11}$$

where $\sigma\triangle\gamma$ signifies the symmetric difference of σ and γ.

Proof. Since $e^{-t\Delta_w}\psi_\sigma(\tau) = \langle e^{-t\Delta_w}\psi_\sigma, \psi_\tau\rangle = p_t(\sigma,\tau)$, we have

$$G_w(\sigma,\tau) = \int_0^\infty p_t(\sigma,\tau)\,dt,$$

for all σ, $\tau \in \Gamma$. This, together with the symmetry of the heat kernel, implies the symmetry of G_w. Now let σ, τ, $\gamma \in \Gamma$ be given. Then, by Proposition 3.1 of [11], we have

$$p_t(\sigma\triangle\gamma, \tau\triangle\gamma) = p_t(\sigma,\tau),$$

which implies (11). □

Referring to the discussion of w-Laplacian Δ_w in [11], the operator norm of Δ_w satisfies $\|\Delta_w\| \leq 2|w|$. This, together with $\Delta_w \geq 0$, implies that the spectrum of Δ_w has an estimate of the form $\mathrm{spec}(\Delta_w) \subset [0, 2|w|]$. Thus, for any $\alpha > 0$, the operator $\alpha I + \Delta_w$ is invertible and its inverse $(\alpha I + \Delta_w)^{-1}$ is also bounded, where I is the identity operator on $\mathcal{H}$. Usually, $(\alpha I + \Delta_w)^{-1}$ is known as the resolvent of $-\Delta_w$. By convention, we write simply $(\alpha + \Delta_w)^{-1} = (\alpha I + \Delta_w)^{-1}$. The next proposition gives the links between the Green function and the resolvent.

Proposition 3.2. *For all vertices σ, $\tau \in \Gamma$, it holds true that*

$$G_w(\sigma, \tau) = \lim_{\alpha \to 0^+} (\alpha + \Delta_w)^{-1} \psi_\sigma(\tau), \tag{12}$$

where $\lim_{\alpha\to 0^+}$ *means the right limit at* 0.

Proof. Let σ, $\tau \in \Gamma$ be given. According to the spectral theory of self-adjoint operators, there exists a unique projection-valued measure π on the Borel space $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ such that $\mathrm{supp}(\pi) \subset [0, 2|w|]$ and

$$\Delta_w = \int_0^{2|w|} \lambda \pi(d\lambda). \tag{13}$$

Write $\mu_{\sigma,\tau}(\cdot) = \langle \pi(\cdot)\psi_\sigma, \psi_\tau \rangle$. Then $\mu_{\sigma,\tau}$ is a signed measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ with the property that $\mathrm{supp}(\mu_{\sigma,\tau}) \subset [0, 2|w|]$. By the functional calculus of self-adjoint operators, we have

$$e^{-t\Delta_w}\psi_\sigma(\tau) = \langle e^{-t\Delta_w}\psi_\sigma, \psi_\tau \rangle = \int_0^{2|w|} e^{-t\lambda} \mu_{\sigma,\tau}(d\lambda).$$

Note that the function $t \mapsto e^{-t(\alpha+\lambda)}$ is integrable on $[0, \infty)$ for all $\alpha > 0$ and all $\lambda \geq 0$. Thus, by the Fubini theorem, we have

$$\int_0^\infty e^{-t\alpha} e^{-t\Delta_w}\psi_\sigma(\tau)dt = \int_0^\infty e^{-t\alpha}\left(\int_0^{2|w|} e^{-t\lambda}\mu_{\sigma,\tau}(d\lambda)\right)dt = \int_0^{2|w|}\left(\int_0^\infty e^{-t(\alpha+\lambda)}dt\right)\mu_{\sigma,\tau}(d\lambda) = \int_0^{2|w|} \frac{1}{\alpha+\lambda}\mu_{\sigma,\tau}(d\lambda),$$

which, together with $(\alpha + \Delta_w)^{-1} = \int_0^{2|w|} \frac{1}{\alpha+\lambda}\pi(d\lambda)$, implies that

$$\int_0^\infty e^{-t\alpha} e^{-t\Delta_w}\psi_\sigma(\tau)dt = \langle (\alpha + \Delta_w)^{-1}\psi_\sigma, \psi_\tau \rangle = (\alpha + \Delta_w)^{-1}\psi_\sigma(\tau).$$

It then follows from the monotone convergence theorem that

$$G_w(\sigma, \tau) = \lim_{\alpha\to 0^+} \int_0^\infty e^{-t\alpha} e^{-t\Delta_w}\psi_\sigma(\tau)\mathrm{d}t = \lim_{\alpha\to 0^+} (\alpha + \Delta_w)^{-1}\psi_\sigma(\tau).$$

This completes the proof. □

In the following, we define $F_w(\sigma) := G_w(\emptyset, \sigma)$, $\sigma \in \Gamma$, where $\emptyset \in \Gamma$ is the empty subset of $\mathbb{N}$. This identity allows us to reduce the study of the two-point Green function to that of a single-point function centered at the origin $\emptyset$ of Γ. We call F_w the *unilateral Green function* of the heat semigroup. The next proposition actually provides a useful sufficient condition for the heat semigroup to be transient.

Proposition 3.3. *Suppose there exists some $\sigma_0 \in \Gamma$ such that $F_w(\sigma_0) < \infty$. Then the heat semigroup $\{e^{-t\Delta_w} \mid t \geq 0\}$ is transient, i.e., $G_w(\sigma, \tau) < \infty$, $\forall\, \sigma, \tau \in \Gamma$.*

Proof. By definition, we know that $G_w(\sigma_0, \emptyset) = F_w(\sigma_0) < \infty$. Thus, due to the connectivity of the IDH $(\Gamma, \sim)$, it suffices to prove that $G_w(\gamma, \tau) < \infty$ for all $\gamma \in \Gamma$ with $\gamma \sim \sigma$ whenever $\sigma, \tau \in \Gamma$ satisfy $G_w(\sigma, \tau) < \infty$.

Let $\sigma, \tau \in \Gamma$ satisfy that $G_w(\sigma, \tau) < \infty$ and $\gamma \in \Gamma$ satisfy that $\gamma \sim \sigma$. Since $\gamma \sim \sigma$, there exists a unique $k_0 \in \mathbb{N}$ such that $\gamma = \sigma \triangle k_0$. Recall that the heat kernel is defined by $p_t(\sigma, \gamma) = \langle e^{-t\Delta_w}\psi_\sigma, \psi_\gamma\rangle$, where $\{\psi_\sigma\}_{\sigma\in\Gamma}$ is the canonical orthonormal basis of $\mathcal{H}$. The operator Δ_w is bounded and self-adjoint with $\|\Delta_w\| \leq 2|w|$. Hence, the heat semigroup $\{e^{-t\Delta_w} \mid t \geq 0\}$ admits a norm-convergent power series expansion,

$$e^{-t\Delta_w} = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!}\Delta_w^n, \qquad t \geq 0.$$

Consequently,

$$p_t(\sigma, \gamma) = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!}\langle \Delta_w^n \psi_\sigma, \psi_\gamma\rangle. \tag{14}$$

For $n = 0$, we have

$$\langle \Delta_w^0 \psi_\sigma, \psi_\gamma\rangle = \langle \psi_\sigma, \psi_\gamma\rangle = 0.$$

For $n = 1$, we compute $\langle -\Delta_w\psi_\sigma, \psi_\gamma\rangle = -\Delta_w\psi_\sigma(\gamma)$. Using the definition of w-Laplacian Δ_w, $\psi_\sigma(\gamma) = 0$ and $\gamma\triangle k_0 = \sigma$, we obtain

$$\Delta_w\psi_\sigma(\gamma) = -w(k_0).$$

Thus, $\langle -\Delta_w\psi_\sigma, \psi_\gamma\rangle = w(k_0) > 0$. Hence, the linear term in (14) is $tw(k_0)$.
For $n \geq 2$, we bound the absolute value of each term

$$\left|\frac{(-t)^n}{n!}\langle \Delta_w^n\psi_\sigma, \psi_\gamma\rangle\right| \leq \frac{t^n}{n!}\|\Delta_w^n\| \leq \frac{t^n}{n!}\|\Delta_w\|^n \leq \frac{t^n}{n!}(2|w|)^n.$$

Now choose $t_0 > 0$ sufficiently small so that

$$t_0 w(k_0) > \sum_{n=2}^{\infty} \frac{t_0^n}{n!}(2|w|)^n.$$

Such a t_0 exists because the right-hand side is $O(t_0^2)$ as $t_0 \to 0^+$. Then from (14) we obtain

$$p_{t_0}(\sigma, \gamma) \geq t_0 w(k_0) - \sum_{n=2}^{\infty} \frac{t_0^n}{n!}(2|w|)^n > 0.$$

Thus, there exists some $t_0 > 0$ such that $p_{t_0}(\sigma, \gamma) > 0$.

Expanding $e^{-t_0\Delta_w}\psi_\sigma$ in the ONB $\{\psi_\kappa \mid \kappa \in \Gamma\}$, we have

$$e^{-t_0\Delta_w}\psi_\sigma = \sum_{\kappa\in\Gamma}\langle e^{-t_0\Delta_w}\psi_\sigma, \psi_\kappa\rangle\psi_\kappa = \sum_{\kappa\in\Gamma} p_{t_0}(\sigma, \kappa)\psi_\kappa.$$

Thus, in the norm convergence, we further have

$$e^{-(t+t_0)\Delta_w}\psi_\sigma = e^{-t\Delta_w}e^{-t_0\Delta_w}\psi_\sigma = \sum_{\kappa\in\Gamma} p_{t_0}(\sigma, \kappa)e^{-t\Delta_w}\psi_\kappa,$$

which implies that

$$e^{-(t+t_0)\Delta_w}\psi_\sigma(\tau) = \sum_{\kappa\in\Gamma} p_{t_0}(\sigma,\kappa)e^{-t\Delta_w}\psi_\kappa(\tau) \geq p_{t_0}(\sigma,\gamma)e^{-t\Delta_w}\psi_\gamma(\tau).$$

Integrating both sides gives

$$\int_0^\infty e^{-(t+t_0)\Delta_w}\psi_\sigma(\tau)dt \geq p_{t_0}(\sigma,\gamma)\int_0^\infty e^{-t\Delta_w}\psi_\gamma(\tau)dt = p_{t_0}(\sigma,\gamma)G_w(\gamma,\tau),$$

which together with

$$G_w(\sigma,\tau) = \int_0^\infty e^{-t\Delta_w}\psi_\sigma(\tau)dt \geq \int_0^\infty e^{-(t+t_0)\Delta_w}\psi_\sigma(\tau)dt$$

yields $G_w(\sigma,\tau) \geq p_{t_0}(\sigma,\gamma)G_w(\gamma,\tau)$. Thus $G_w(\gamma,\tau) < \infty$. □

Remark 3.1. *Suppose there exists no $\sigma_0 \in \Gamma$ such that $F_w(\sigma_0) < \infty$. Then, for all σ, $\tau \in \Gamma$, it holds that $G_w(\sigma,\tau) = F_w(\sigma\triangle\tau) = \infty$. In this case the heat semigroup $\{e^{-t\Delta_w} \mid t \geq 0\}$ is said to be recurrent. Thus, recurrence and transience of the heat semigroup are mutually exclusive.*

Recall that the form $\mathcal{E}_w$ defined (7) is a Dirichlet form. A function $\xi\colon \Gamma \to \mathbb{R}_+$ is called a *reference function* for $\mathcal{E}_w$ if it holds that

$$\sum_{\gamma\in\Gamma} |f(\gamma)|\xi(\gamma) \leq \sqrt{\mathcal{E}_w(f)}, \quad \forall f \in \mathcal{H}, \tag{15}$$

where $\mathcal{E}_w(f) := \mathcal{E}_w(f,f)$. Clearly, the constant function $\xi(\gamma) \equiv 0$ is a reference function for $\mathcal{E}_w$, which is known as the trivial reference for $\mathcal{E}_w$. We denote by $\mathcal{H}_+$ the cone consisting of nonnegative functions in $\mathcal{H}$, namely

$$\mathcal{H}_+ = \{f \in \mathcal{H} \mid f \geq 0\}. \tag{16}$$

Given nonnegative functions u, v defined on Γ, we abuse the symbol $\langle u,v\rangle$, which is defined as

$$\langle u,v\rangle = \sum_{\gamma\in\Gamma} u(\gamma)v(\gamma).$$

Clearly, the symbol $\langle u,v\rangle$ means the same as the inner product of u and v whenever u, $v \in \mathcal{H}_+$. For $f \in \mathcal{H}_+$, we denote by $\mathfrak{G}f$ the nonnegative function given by

$$(\mathfrak{G}f)(\sigma) = \int_0^\infty e^{-t\Delta_w}f(\sigma)dt, \quad \sigma \in \Gamma. \tag{17}$$

Note that $(\mathfrak{G}f)(\sigma)$ may take the value ∞. In the following, we denote by $l^1(\Gamma)$ the space of absolutely summable functions on Γ with the usual linear operation and norm.

Proposition 3.4. *Suppose that $\mathcal{E}_w$ has a reference function $\xi \in l^1(\Gamma)\cap l^\infty(\Gamma)$ satisfying that $\xi(\gamma) > 0$ for all $\gamma \in \Gamma$. Then $F_w(\sigma) < \infty$ for all $\sigma \in \Gamma$. In particular, the heat semigroup $\{e^{-t\Delta_w} \mid t \geq 0\}$ is transient.*

Proof. It is easy to see that $\xi \in \mathcal{H}$. On the other hand, it follows from (15) and the strict positivity of ξ that $\mathcal{E}_w(f) = 0$ implies $f = 0$. By Lemma 2.7 of Reference [7], we have

$$\sqrt{\langle \xi, \mathfrak{G}\xi \rangle} = \sup_{f \in \mathcal{H}, f \neq 0} \frac{\langle |f|, \xi \rangle}{\sqrt{\mathcal{E}_w(f)}},$$

which together with (15) gives $\langle \xi, \mathfrak{G}\xi \rangle \leq 1$. Thus $(\mathfrak{G}\xi)(\gamma) \leq \frac{1}{\xi(\gamma)} < \infty$ for all $\gamma \in \Gamma$. Now let $\sigma \in \Gamma$ be given. Then, by straightforward calculations, we find

$$F_w(\sigma) = \int_0^\infty e^{-t\Delta_w} \psi_\sigma(\emptyset) dt = \frac{1}{\xi(\emptyset)} \int_0^\infty \langle e^{-t\Delta_w} \psi_\sigma, \xi(\emptyset)\psi_\emptyset \rangle dt,$$

which, together with $\xi(\emptyset)\psi_\emptyset \leq \xi$ and $\langle e^{-t\Delta_w}\psi_\sigma, \xi \rangle = e^{-t\Delta_w}\xi(\sigma)$, implies that

$$F_w(\sigma) \leq \frac{1}{\xi(\emptyset)} \int_0^\infty e^{-t\Delta_w} \xi(\sigma) dt = \frac{(\mathfrak{G}\xi)(\sigma)}{\xi(\emptyset)} < \infty.$$

This completes the proof. □

4. Transience of Markov chain

In this section, we investigate the transience (recurrence) of a discrete-time Markov chain on the IDH and its connection to that of the heat semigroup.

Let us consider the *adjacency operator* A_w on $(\Gamma, \sim)$, which is defined as

$$A_w f(\sigma) = \sum_{k=0}^{\infty} w(k) f(\sigma \triangle k), \quad \sigma \in \Gamma,\ f \in \mathcal{H}, \tag{18}$$

where w is a weight on $\mathbb{N}$. Clearly, A_w is a self-adjoint bounded linear operator on $\mathcal{H}$. Define a function Q_w on $\Gamma \times \Gamma$ as

$$Q_w(\sigma, \tau) := \frac{1}{|w|} \langle A_w \psi_\sigma, \psi_\tau \rangle, \quad \sigma,\ \tau \in \Gamma. \tag{19}$$

Due to the self-adjointness of A_w, we have $Q_w(\sigma, \tau) = Q_w(\tau, \sigma) \geq 0$, $\forall\, \sigma,\ \tau \in \Gamma$. For $\sigma \in \Gamma$, we have

$$\sum_{\tau \in \Gamma} Q_w(\sigma, \tau) = 1.$$

In other words, Q_w is a transition probability matrix on Γ. Thus, there exists a discrete-times Markov chain $X = \{X_n \mid n \geq 0\}$ defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and valued in Γ such that $\mathbb{P}\{X_0 = \gamma\} = \mu(\gamma)$, $\gamma \in \Gamma$, and

$$\mathbb{P}\{X_{n+1} = \tau | X_n = \sigma\} = Q_w(\sigma, \tau), \quad \sigma,\ \tau \in \Gamma,\ n \geq 0, \tag{20}$$

where μ is a probability distribution on Γ, which is known as the initial distribution of X.

As is well known, transience and recurrence of a discrete-time Markov chain are mutually exclusive. So, we need only to consider the transience of X and its links with that of the heat semigroup.

In the following, we use $Q_w^{(n)}$ to mean the usual nth power of the transition probability matrix Q_w, where $n \geq 0$. It can be shown that

$$\frac{1}{|w|^n} A_w^n f(\sigma) = \sum_{\tau\in\Gamma} Q_w^{(n)}(\sigma,\tau) f(\tau), \quad \sigma \in \Gamma,\ f \in \mathcal{H}, \tag{21}$$

where A_w^n is the nth power of the adjacency operator A_w.

The next lemma is an immediate consequence of a general result in the theory of Markov chains [7].

Lemma 4.1. *Let w be a weight on $\mathbb{N}$. Then X is transient if and only if*

$$\sum_{n=0}^{\infty} Q_w^{(n)}(\sigma,\tau) < \infty, \quad \forall\, \sigma,\ \tau \in \Gamma. \tag{22}$$

It follows easily from the definitions of Δ_w and A_w that $\Delta_w = |w|I - A_w$. Hence

$$e^{-t\Delta_w} = e^{-t|w|I + tA_w} = e^{-t|w|} e^{tA_w} = e^{-t|w|} \sum_{n=0}^{\infty} \frac{t^n}{n!} A_w^n, \quad t \geq 0,$$

in view of $e^{-t|w|I} = e^{-t|w|} \cdot I$, where the operator series converges in the operator norm. Given $\sigma,\ \tau \in \Gamma$, one can find

$$\langle e^{-t\Delta_w} \psi_\sigma, \psi_\tau \rangle = e^{-t|w|} \sum_{n=0}^{\infty} \frac{t^n}{n!} \langle A_w^n \psi_\sigma, \psi_\tau \rangle, \quad t \geq 0,$$

which together with (21) implies that

$$\langle e^{-t\Delta_w} \psi_\sigma, \psi_\tau \rangle = e^{-t|w|} \sum_{n=0}^{\infty} \frac{|w|^n t^n}{n!} Q_w^{(n)}(\sigma,\tau), \quad t \geq 0.$$

Thus, by the formula $\int_0^\infty t^n e^{-t|w|} dt = \frac{n!}{|w|^{n+1}}$ and the definition of the Green function G_w, one comes to the next useful formula

$$G_w(\sigma,\tau) = \frac{1}{|w|} \sum_{n=0}^{\infty} Q_w^{(n)}(\sigma,\tau). \tag{23}$$

Formula (23) reveals that the Green function $G_w(\sigma,\tau)$ is finite if and only if the series $\sum_{n=0}^{\infty} Q_w^{(n)}(\sigma,\tau)$ converges. Consequently, the transience of the heat semigroup (which requires $G_w(\sigma,\tau) < \infty$ for all $\sigma,\tau \in \Gamma$) is equivalent to the transience of the Markov chain (which, by Lemma 4.1, is characterized by the convergence of the same series for all σ,τ). This, together with Lemma 4.1, proves the next theorem.

Theorem 4.1. *The Markov chain X is transient if and only if the heat semigroup $\{e^{-t\Delta_w} \mid t \in \mathbb{R}_+\}$ is transient.*

Transience of the heat semigroup, characterized by the finiteness of the Green function, naturally extends to the probabilistic setting. The equivalence between the analytic notion of transience (via the Green function) and the probabilistic notion (via expected occupation time) is established in the Theorem above. Through this equivalence, we come to the next corollary, which provides a useful sufficient condition for a Markov chain to be transient.

Corollary 4.1. *Suppose there exists some $\sigma_0 \in \Gamma$ such that $F_w(\sigma_0) < \infty$. Then the Markov chain X is transient.*

Remark 4.1. *From the perspective of the associated continuous-time Markov chain $\{X_t\}_{t\geq 0}$ generated by $-\Delta_w$,*

$$F_w(\sigma) = \mathbb{E}_\emptyset\left[\int_0^\infty \mathbf{1}_{\{X_t=\sigma\}}\,\mathrm{d}t\right]$$

represents the expected total time spent at vertex σ when starting from the origin $\emptyset$. Thus, finiteness of $F_w(\sigma)$ for some σ indicates transience. This function, therefore, serves as a natural bridge between the analytic properties of the heat semigroup and the probabilistic behavior of the underlying stochastic process.

Author contributions

Nan Fan, Caishi Wang, and Jijun Zhao: Writing-Review & Editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is supported by the National Natural Science Foundation of China under Grant 12261080.

Conflict of interest

This work does not have any conflicts of interest.

References

1. E. Baake, H. Wagner, Mutation-selection models solved exactly with methods of statistical mechanics, *Genetical Research*, **78** (2001), 93–117. https://doi.org/10.1017/S0016672301005110
2. T. Jörg, F. Krzakala, J. Kurchan, A. C. Maggs, Simple glass models and their quantum annealing, *Phys. Rev. Lett.*, **101** (2008), 147204. https://doi.org/10.1103/PhysRevLett.101.147204
3. M. Keller, C. Rose, Anchored heat kernel upper bounds on graphs with unbounded geometry and anti-trees, *Calc. Var.*, **63** (2024), 20. https://doi.org/10.1007/s00526-023-02622-3
4. M. Keller, A. Muranova, Recurrence and transience for non-Archimedean and directed graphs, *Math. Nachr.*, **298** (2025), 3307–3330. https://doi.org/10.1002/mana.70040
5. S. Kumamoto, S. Kijima, T. Shirai, An analysis of the recurrence/transience of random walks on growing trees and hypercubes, *Proceedings of 3rd Symposium on Algorithmic Foundations of Dynamic Networks*, 2024, 1–17. https://doi.org/10.4230/LIPIcs.SAND.2024.17

6. A. Kostenko, N. Nicolussi, Laplacians on infinite graphs: discrete vs. continuous, In: *European congress of mathematics*, Berlin: EMS Press, 2023, 295–323. https://doi.org/10.4171/8ECM/02
7. M. Schmidt, Global properties of Dirichlet forms on discrete spaces, *Diss. Math.*, **522** (2017), 1–43. https://doi.org/10.4064/dm738-7-2016
8. M. Talagrand, *Mean field models for spin glasses, volume I: basic examples*, Berlin: Springer-Verlag, 2011. https://doi.org/10.1007/978-3-642-15202-3
9. L. X. Zhang, C. S. Wang, Random Schrödinger operator on infinite-dimensional hypercube (I): ergodicity and density of states, *J. Stat. Phys.*, **190** (2023), 138. https://doi.org/10.1007/s10955-023-03153-z
10. L. X. Zhang, C. S. Wang, Hamiltonian of free field on infinite-dimensional hypercube, *Infin. Dimens. Anal. Qu.*, **27** (2024), 2350027. https://doi.org/10.1142/s0219025723500273
11. J. J. Zhao, C. S. Wang, N. Fan, Continuous-time Markov processes on infinite dimensional hypercube, *Infin. Dimens. Anal. Qu.*, in press. https://doi.org/10.1142/S0219025725500146

AIMS Press