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Research article

The Steiner antipodal number of zero-divisor graphs of finite commutative rings

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Abstract: Let R be a finite commutative ring with unity. The zero-divisor graph $\Gamma(R)$ is defined such that its vertex set comprises all nonzero zero-divisors of R, with two distinct vertices being adjacent if and only if their product is zero. This study provides a closed-form expression for the n-eccentricity of each vertex and computes the Steiner antipodal number of $\Gamma(R)$ under the following conditions: (i) $R = \mathbb{Z}_m$, (ii) R is a reduced ring, and (iii) R is a finite direct product of rings of the form $\mathbb{Z}_m$. Moreover, we establish the existence of a zero-divisor graph with a Steiner antipodal number equal to some positive integer m.

Keywords: zero-divisor graph; n-eccentricity; n-diameter; Steiner n-antipodal graph; Steiner antipodal number
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1. Introduction

Associating graphs with algebraic structures has long been recognized as a powerful tool for analyzing and revealing their intrinsic properties. One notable example is the construction of a graph from a commutative ring with identity, where the set of all nonzero zero-divisors forms the vertices.

Two vertices are connected by an edge if and only if their product is zero [1]. This construction yields the *zero-divisor graph* of the ring, denoted by $\Gamma(R)$, which has been extensively studied in recent decades. Various structural and distance-based parameters of zero-divisor graphs over finite commutative rings have been studied. These include investigations on Wiener indices [2,5], Laplacian eigenvalues [3], and radio numbers [4], all of which contribute to understanding their algebraic and metric properties. This paper focuses on the *Steiner antipodal number*, a significant graph invariant that has received considerable attention in graph theory.

For a subset S of vertices in a graph G, the *Steiner distance*, denoted $d_G(S)$, is defined as the minimum number of edges in any connected subgraph that contains all vertices of S. Such a subgraph is referred to as a *Steiner tree* for S in G. The *Steiner n-eccentricity* of a vertex u, denoted $e_n(u)$, is the maximum Steiner distance among all subsets $S \subset V(G)$ of size n that include u, i.e., $e_n(u) = \max\{d_G(S) \mid u \in S \subseteq V(G),\ |S| = n\}$. The *Steiner n-radius*, $rad_n(G)$, is the minimum n-eccentricity over all vertices in G, and the *Steiner n-diameter*, $diam_n(G)$, is the maximum.

Singleton [6] introduced the concept of an *antipodal graph* in 1968. Given a graph G, its antipodal graph shares the same vertex set, with two vertices adjacent if their distance in G equals the diameter of G. Building on this, Arockiaraj et al. introduced the concepts of the Steiner n-antipodal graph and the Steiner antipodal number in [7,8]. A subset of n vertices in G is said to be n-antipodal if the Steiner distance among them equals $diam_n(G)$. The *Steiner n-antipodal graph*, denoted $SA_n(G)$, has the same vertex set as G, where any n vertices form a clique in $SA_n(G)$ if and only if they are n-antipodal in G. The Steiner antipodal number, $a_S(G)$, is the smallest positive integer n such that $SA_n(G) \cong K_p$, where $p = |V(G)|$.

The Steiner antipodal number of zero-divisor graphs is not only of theoretical interest but also has practical relevance. In modular arithmetic–based systems such as cryptography and error-detecting codes, it measures the minimal number of control nodes required to cover all maximal distance interactions, ensuring resilience against weak links caused by zero-divisors. In parallel computing and distributed database systems, where reduced rings and finite direct products of rings naturally arise, it quantifies the minimal redundancy or monitoring nodes needed to preserve connectivity and robustness across subsystems. These applications highlight its usefulness in optimizing secure communication networks, fault-tolerant computation, and distributed architectures.

In our investigation, we consider R to be a finite commutative ring with unity. Following the approach in [9], an equivalence relation is defined on the vertex set $V(\Gamma(R))$ based on open neighborhoods. The open neighborhood of a vertex v is given by $\mathcal{N}(v) = \{u \in V(\Gamma(R)) \mid uv \in E(\Gamma(R))\}$. Define a relation $\sim$ on $V(\Gamma(R))$ such that $u \sim v$ if and only if $\mathcal{N}(u) \setminus \{v\} = \mathcal{N}(v) \setminus \{u\}$. This defines an equivalence relation, and the resulting equivalence classes are called *equi-neighbor classes*. Let $\mathscr{C}_{x_1}, \mathscr{C}_{x_2}, \ldots, \mathscr{C}_{x_k}$ be such classes, with $x_1, x_2, \ldots, x_k$ representing elements from each class. These foundational results will be instrumental in proving our main theorems.

Lemma 1.1 ([7]). *If G is a graph with $a_S(G) = n$, then $rad_n(G) = diam_n(G)$.*

Lemma 1.2 ([7]). *For any complete bipartite graph K_{q_1,q_2} with $q_1 \leq q_2$ and $q_1 \neq 1$, it holds that $a_S(K_{q_1,q_2}) = q_2 + 1$.*

Lemma 1.3 ([7]). *For a graph G, $a_S(G) = 2$ if and only if G is either a complete graph or totally disconnected.*

Lemma 1.4 ([1]). *Let R be a commutative ring. Then $\Gamma(R)$ is connected and $diam(\Gamma(R)) = 3$.*

Lemma 1.5 ([10]). *Let d be a divisor of n. Then $|A_d| = \phi\left(\frac{n}{d}\right)$, where $1 \leq d \leq n$.*

The structure of this paper is organized as follows. In Section 2, we introduce a methodology to analyze the zero-divisor graph $\Gamma(\mathbb{Z}_m)$, focusing on computing the n-eccentricity for each vertex and evaluating the Steiner antipodal number. Section 3 addresses the Steiner antipodal number for zero-divisor graphs of reduced finite commutative rings. In Section 4, we extend the results to the case where the ring is a finite direct product of rings of the form $\mathbb{Z}_n$. Lastly, Section 5 explores the inverse problem—identifying graph structures corresponding to a given Steiner antipodal number.

2. The Steiner antipodal number of the zero-divisor graph of $\mathbb{Z}_m$

In this section, we consider the ring $\mathbb{Z}_m$, the set of integers modulo m. Building on the equivalence relation $\sim$ introduced in Section 1, we apply it to the vertex set of the zero-divisor graph $\Gamma(\mathbb{Z}_m)$.

The following lemmas describe the structural properties and adjacency conditions among equi-neighbor classes in $\Gamma(\mathbb{Z}_m)$.

Lemma 2.1. *In the graph $\Gamma(\mathbb{Z}_m)$, each equi-neighbor class corresponds uniquely to a divisor d of m. Furthermore, every element within such a class can be written in the form αd, where $\gcd(\alpha, m/d) = 1$.*

Proof. Let $m = q_1^{a_1} q_2^{a_2} \dots q_h^{a_h}$, where q_i are distinct primes.

Existence: Assume, for contradiction, that an equi-neighbor class $\mathscr{C}_x$ does not contain any divisor of m. Let $n_1 \in \mathscr{C}_x$ be of the form $n_1 = \alpha q_{i_1}^{a_{i_1}} \dots q_{i_k}^{a_{i_k}}$, for some $1 \leq i_j \leq h$. There exists a divisor d_1 of m such that $m \mid n_1 d_1$, implying that $m = d_1 d_2$ and $n_1 = d_2 n_2$ for some $n_2 \nmid m$. Suppose k is a vertex adjacent to n_1. Then $m \mid k n_1 = k d_2 n_2$, and since $n_2 \nmid m$, it follows that $m \mid k d_2$. Thus, k is also adjacent to d_2, implying $\mathscr{N}(d_2) \setminus \{n_1\} = \mathscr{N}(n_1) \setminus \{d_2\}$, so $d_2 \in \mathscr{C}_x$. Hence, $\mathscr{C}_x$ contains a divisor of m.

Uniqueness: Suppose two distinct divisors, $d_1 < d_2$, both belong to the same class $\mathscr{C}_x$. Let d be such that $m = d_2 d$ and $m \nmid d d_1$. Then $d \in \mathscr{N}(d_2)$ but $d \notin \mathscr{N}(d_1)$, which contradicts $\mathscr{N}(d_1) = \mathscr{N}(d_2)$. Therefore, each class contains a unique divisor d of m, and its elements are precisely those of the form αd with $\gcd(\alpha, m/d) = 1$. □

Lemma 2.2. *Let $\Gamma(Z_m)$ be the zero-divisor graph of Z_m. A vertex in class $\mathscr{C}_{d_i}$ is adjacent to a vertex in the class $\mathscr{C}_{d_j}$ if and only if $d_i d_j \equiv 0 \pmod{m}$.*

Proof. Let $u \in \mathscr{C}_{d_i}$ and $v \in \mathscr{C}_{d_j}$. By Lemma 2.1, $u = \alpha d_i$, $v = \beta d_j$ with $\gcd(\alpha, m/d_i) = \gcd(\beta, m/d_j) = 1$. Then u and v are adjacent iff $uv \equiv 0 \pmod{m}$, which is equivalent to $\alpha\beta d_i d_j \equiv 0 \pmod{m}$, and since α and β are units modulo respective divisors, this is equivalent to $d_i d_j \equiv 0 \pmod{m}$. □

Corollary 2.3 ([9]). *(i) The induced subgraph $G(\mathscr{C}_d)$ is either a complete graph or an empty graph. (ii) For $d_1 \neq d_2$, each vertex in $\mathscr{C}_{d_1}$ is adjacent to either all or none of the vertices in $\mathscr{C}_{d_2}$.*

Remark 2.4. *The number of equi-neighbor classes is given by $d(m)-2$, where $d(m)$ denotes the number of positive divisors of m. Hence, these classes can be labeled $\mathscr{C}_{x_1}, \mathscr{C}_{x_2}, \dots, \mathscr{C}_{x_k}$ corresponding to the divisors $x_1, x_2, \dots, x_k$ of m.*

The following lemma establishes that the Steiner k-eccentricity is constant within each equi-neighbor class, which in turn allows the eccentricity of the entire class to be determined by computing it for a single representative element.

Lemma 2.5. *For any $u, v \in \mathscr{C}_{x_i}$ in $\Gamma(\mathbb{Z}_m)$, it holds that $e_k(u) = e_k(v)$.*

Proof. Assume, for contradiction, that $e_k(u) = l_1$ and $e_k(v) = l_2$ with $l_2 > l_1$, and $u, v \in \mathscr{C}_{x_i}$. Let ST_1 and ST_2 be Steiner trees of k vertices attaining l_1 and l_2, respectively. Removing v from ST_2 yields a subgraph ST_2' with $k-1$ vertices and at most $l_2 - 1$ edges. Since u and v have the same neighborhood (excluding each other), adding u in place of v results in a Steiner tree of size l_2 for u, contradicting $e_k(u) = l_1 < l_2$. □

To compute the Steiner k-eccentricity in $\Gamma(\mathbb{Z}_{q^a})$, we define the following partition of the vertex set:

$$\mathscr{B}_0 = \mathscr{C}_{q^{a-1}}, \quad \mathscr{B}_1 = \bigcup_{k=1}^{a-2} \mathscr{C}_{q^k}.$$

Thus, $V(\Gamma(\mathbb{Z}_m)) = \mathscr{B}_0 \cup \mathscr{B}_1$, where every vertex in $\mathscr{B}_0$ is adjacent to every vertex in $\mathscr{B}_1$, and $\mathscr{C}_{q^k}$ is complete for $k \geq \lceil a/2 \rceil$, while other classes are independent sets.

Lemma 2.6. *Let $m = q^a$ with $a > 2$ and q prime. Then:*

(i) For $3 \leq k \leq l := |\mathscr{B}_1|$,

$$e_k(u) = \begin{cases} k, & \text{if } u \in \mathscr{B}_1, \\ k-1, & \text{if } u \in \mathscr{B}_0. \end{cases}$$

(ii) For $k > l$, we have $e_k(u) = k - 1$ for all $u \in V(\Gamma(\mathbb{Z}_m))$.

Proof. **Case 1:** $3 \leq k \leq l$.

Subcase 1: $u \in \mathscr{B}_1$. Then $u = \alpha q^b$ for some $b \leq a-2$ and $\gcd(\alpha, m/q^b) = 1$. Since u is adjacent to all vertices in $\mathscr{B}_0$, choosing $k-1$ neighbors from $\mathscr{B}_1$ yields $e_k(u) = k$.

Subcase 2: $u \in \mathscr{B}_0$. Then $u = \alpha q^{a-1}$, and $\mathscr{N}(u) = V(\Gamma(\mathbb{Z}_m)) \setminus \{u\}$. Hence, $e_k(u) = k-1$.

Case 2: $k > l$.

Any Steiner tree with k vertices must include a vertex from $\mathscr{B}_0$, so $e_k(u) = k-1$ for all u. □

In Lemma 2.6, we determined the k-eccentricity of all vertices; using this result, we now compute the Steiner antipodal number of $\Gamma(\mathbb{Z}_{q^a})$.

Theorem 2.7. *For a prime q and $a > 2$, the Steiner antipodal number of $\Gamma(\mathbb{Z}_m)$, where $m = q^a$, is $a_S(\Gamma(\mathbb{Z}_m)) = l + 1$, where $l = |\mathscr{B}_1|$.*

Proof. From Lemma 2.6, we know that $rad_k = k-1$ and $diam_k = k$ for $k \leq l$. Then by Lemma 1.1, it follows that $a_S(\Gamma(\mathbb{Z}_m)) > l$. For $k = l+1$, one can choose l vertices from $\mathscr{B}_1$ and one vertex $u \in \mathscr{B}_0$ to achieve Steiner distance l. These vertices form a clique in $SA_{l+1}(\Gamma(\mathbb{Z}_m))$, and since the selection is arbitrary, $SA_{l+1}(\Gamma(\mathbb{Z}_m)) \cong K_{q^{a-1}-1}$. □

Remark 2.8. *For $m = q^2$, the graph $\Gamma(\mathbb{Z}_m) \cong K_{q-1}$ is complete. Therefore, $a_S(\Gamma(\mathbb{Z}_m)) = 2$ by Lemma 1.2.*

Example 2.9. *The Steiner n-eccentricity of all vertices in the zero-divisor graph $\Gamma(\mathbb{Z}_m)$, where $m = 5^6$, is illustrated below.*

Let $\mathscr{B}_0 = \mathscr{C}_{5^5}$ and $\mathscr{B}_1 = \mathscr{C}_5 \cup \mathscr{C}_{5^2} \cup \mathscr{C}_{5^3} \cup \mathscr{C}_{5^4}$. As shown in Figure 1, every vertex in $\mathscr{B}_0$ is adjacent to each vertex in $\mathscr{B}_1$. The equi-neighbor classes $\mathscr{C}_5$ and $\mathscr{C}_{5^2}$ are independent sets, while $\mathscr{C}_{5^3}$, $\mathscr{C}_{5^4}$, and

$\mathscr{C}_{5^5}$ form complete subgraphs. The set $\mathscr{B}_1$ contains 3120 elements, so $|\mathscr{B}_1| = l = 3120$. For each k with $3 \leq k \leq l$, the Steiner k-eccentricity $e_k(u)$ is given by:

- *$e_k(u) = k$ if $u \in \mathscr{B}_1$;*
- *$e_k(u) = k - 1$ if $u \in \mathscr{B}_0$.*

For $k > l$, we have $e_k(u) = k - 1$ for all $u \in V(\Gamma(\mathbb{Z}_m))$.

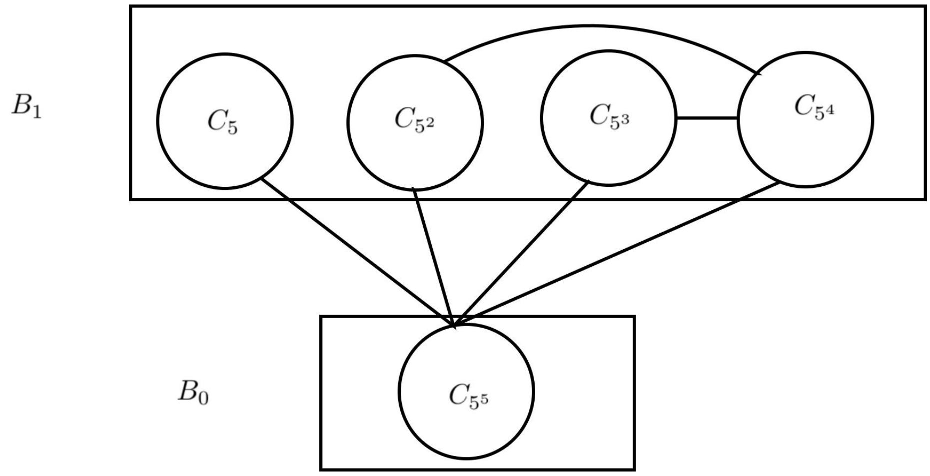

Figure 1. Zero-divisor graph of $\mathbb{Z}_{5^6}$.

To determine the n-eccentricity of all vertices in $\Gamma(Z_m)$, where $m = q_1^{a_1} q_2^{a_2} \ldots q_h^{a_h}$ and the q_i's are distinct primes, we partition the vertex set $V(\Gamma(Z_m))$ as follows:

$$\mathscr{L}_1 = \{u | u = \alpha q_\beta^{b_\beta}, 1 \leq \beta \leq h, 1 \leq b_\beta \leq a_\beta\},$$

$$\mathscr{L}_2 = \{u | u = \alpha q_{\beta_1}^{b_{\beta_1}} q_{\beta_2}^{b_{\beta_2}}, 1 \leq \beta_i \leq h, 1 \leq b_{\beta_i} \leq a_{\beta_i}\} \ldots,$$

$$\mathscr{L}_{h-2} = \{u | u = \alpha q_{\beta_1}^{b_{\beta_1}} q_{\beta_2}^{b_{\beta_2}}, \ldots q_{\beta_{h-2}}^{b_{\beta_{h-2}}}, 1 \leq \beta_i \leq h, 1 \leq b_{\beta_i} \leq a_{\beta_i}\},$$

$$\mathscr{L}_{h-1} = \{u | u = \alpha q_{\beta_1}^{b_{\beta_1}} q_{\beta_2}^{b_{\beta_2}}, \ldots q_{\beta_{h-1}}^{b_{\beta_{h-1}}}, 1 \leq \beta_i \leq h, 1 \leq b_{\beta_i} \leq a_{\beta_i}\} \setminus \mathscr{L},$$

$$\mathscr{L}_h = \{u | u = \alpha q_1^{b_1} q_2^{b_2}, \ldots q_h^{b_h}, 1 \leq b_i \leq a_i\} \setminus \mathscr{L}$$

and $\mathscr{L} = \bigcup_{\beta=1}^{h} \mathscr{C}_{x_\beta}$, where $x_\beta = q_1^{a_1} q_2^{a_2} \ldots q_\beta^{a_\beta - 1} \ldots q_h^{a_h}$, and by Lemma 1.5, we have $|\mathscr{C}_{x_\beta}| = q_\beta - 1$. In addition, each vertex in $\mathscr{L}_i$ for $1 \leq i \leq h$ is adjacent to at least one equi-neighbor class in $\mathscr{L}$.

In Lemmas 2.10–2.12, we determine the k-eccentricity of all vertices in the sets $\mathscr{L}_1$, $\mathscr{L}_k (2 \leq k \leq h)$ and $\mathscr{L}$, respectively. Using these lemmas, we then establish the Steiner antipodal number of $\Gamma(Z_m)$ in Theorem 2.13.

Lemma 2.10. *Let $u \in \mathscr{L}_1$, and suppose $m = q_1^{a_1} q_2^{a_2} \ldots q_h^{a_h}$, where $q_1 < q_2 < \ldots < q_h$ are distinct primes. Then the Steiner n-eccentricity of u is given by:*

$$e_n(u) = \begin{cases} 2n - 1, & \text{if } 3 \leq n \leq h, \\ h + n - 1, & \text{if } h + 1 \leq n \leq l_0, \end{cases}$$

where $l_0 = |V(\Gamma(\mathbb{Z}_m))| - |\mathscr{L}|$. *Additionally, for* $0 \le j \le h-1$ *and* $1 \le n_j \le q_{h-j} - 1$, *if* $e_{l_j}(u) = d_j$, *then* $e_{l_j+n_j}(u) = d_j + n_j - 1$, *where*

$$l_j = |V(\Gamma(\mathbb{Z}_m))| - \left|\bigcup_{\beta=1}^{h-j} \mathscr{C}_{x_\beta}\right|$$

with $x_\beta = q_1^{a_1} q_2^{a_2} \dots q_\beta^{a_\beta - 1} \dots q_h^{a_h}$.

Proof. Let $u = \alpha q_\beta^{b_\beta} \in \mathscr{L}_1$, with $1 \le b_\beta \le a_\beta$ and $1 \le \beta \le h$.

Case 1: $3 \le n \le h$**.**

By Lemma 2.2, the neighborhood $\mathscr{N}(u) = \mathscr{C}_{x_\beta}$ when $b_\beta = 1$, or $\mathscr{C}_{x_\beta} \subseteq \mathscr{N}(u)$ otherwise. For $\gamma \ne \beta$, we have $\mathscr{C}_{x_\gamma} \nsubseteq \mathscr{N}(u)$. Thus, each $u \in \mathscr{L}_1$ is adjacent to exactly one equi-neighbor class in $\mathscr{L}$. Let $v \in \mathscr{C}_{x_\gamma}$. Since v is connected to all vertices in $\mathscr{L} \setminus \mathscr{C}_{x_\gamma}$, a path $u - x_\beta - x_\gamma - q_\gamma$ gives the longest distance between u and q_γ. Choosing $n-2$ vertices from distinct $\mathscr{C}_{q_\delta}$ classes along with u and q_γ yields a Steiner tree of length $3 + 2(n-2) = 2n - 1$.

Case 2: $h + 1 \le n \le l_0$**.**

Let ST_1 be a Steiner tree with h vertices from Case 1, having length $2h - 1$. Additional vertices chosen from outside $\mathscr{L}$, each increases the length by one (as every such vertex is adjacent to some vertex in $\mathscr{L}$). Thus, $e_n(u) = 2h - 1 + (n - h) = h + n - 1$.

Case 3: $l_0 < n \le l_1$**.**

Extend the Steiner tree ST_1 to ST_2 using l_0 vertices, with $e_{l_0}(u) = d_0$. Adding a vertex $v_1 \in \mathscr{C}_{x_i}$ does not change the distance. Adding $v_2 \in \mathscr{C}_{x_i}$ (with $v_2 \ne v_1$) increases the distance by one. Since $|\mathscr{C}_{x_h}|$ is the largest, the extra vertices are best chosen from $\mathscr{C}_{x_h}$ to maximize distance: $e_n(u) = d_0 + (n - l_0) - 1 = d_0 + n_0 - 1$.

Case 4: $l_j < n \le l_{j+1}$**,** $1 \le j \le h-1$**.**

Use the Steiner tree ST_3 formed with l_j vertices (as in Case 3) and let $e_{l_j}(u) = d_j$. Adding $n - l_j$ vertices from $\mathscr{C}_{x_{h-j}}$ increases the distance by $n - l_j - 1$. Hence, $e_n(u) = d_j + n - l_j - 1 = d_j + n_j - 1$. □

Lemma 2.11. *Let* $u \in \mathscr{L}_k$ *for some* $2 \le k \le h$, *and let* $m = q_1^{a_1} q_2^{a_2} \dots q_h^{a_h}$ *with* $q_1 < q_2 < \dots < q_h$ *distinct primes. Then:*

(i) $e_n(u) = \begin{cases} 2n-2, & \text{if } u \in \bigcup_{i=0}^{n-2} \mathscr{L}_{h-i}, \\ 2n-1, & \text{otherwise}, \end{cases}$ *for* $3 \le n \le h$;

(ii) $e_n(u) = h + n - 1$, *if* $h + 1 \le n \le l_0$, *where* $l_0 = |V(\Gamma(\mathbb{Z}_m))| - |\mathscr{L}|$;

(iii) If $e_{l_j}(u) = d_j$, *then* $e_{l_j+n_j}(u) = d_j + n_j - 1$, *for* $0 \le j \le h-1$, $1 \le n_j \le q_{h-j} - 1$, *and*

$$l_j = |V(\Gamma(\mathbb{Z}_m))| - \left|\bigcup_{\beta=1}^{h-j} \mathscr{C}_{x_\beta}\right|,$$

where $x_\beta = q_1^{a_1} q_2^{a_2} \dots q_\beta^{a_{\beta-1}} \dots q_h^{a_h}$.

Proof. Let $u \in \mathscr{L}_k$ be represented as $u = \alpha q_{i_1}^{\beta_{i_1}} \dots q_{i_k}^{\beta_{i_k}}$ with $\beta_{i_j} \le a_{i_j}$ and $1 \le j \le k$.

Case 1: $3 \le n \le h$**.**

Let $S = \{i_1, i_2, \ldots, i_k\}$ be the index set of the primes forming u, and define $S' = \{1, 2, \ldots, h\} \setminus S$.

Subcase 1a: If $S' \neq \emptyset$, then the equi-neighbor classes $\mathscr{C}_{x_{i_1}}, \ldots, \mathscr{C}_{x_{i_k}}$ lie in $\mathscr{N}(u)$, while vertices in these classes are not adjacent to any vertex in $\mathscr{C}_{q_j}$ for $j \in S'$. By Lemma 1.4, the distance from any $v \in \mathscr{C}_{q_j}$ to u is 3. Adding another vertex $w \in \mathscr{C}_{q_i}$, $i \in S'$, yields a Steiner tree of length 5.

Selecting $n - 3 \leq |S'|$ vertices from distinct classes $\mathscr{C}_{q_j}$ (not containing v or w), we obtain a Steiner tree of length $5 + 2(n - 3) = 2n - 1$.

If $n - 3 > |S'|$, first construct a Steiner tree using all $|S'|$ vertices from distinct $\mathscr{C}_{q_j}$, with $e_{|S'|+1}(u) = 2|S'| + 1$. Adding more vertices from $\mathscr{C}_{q_j}$, $j \in S$, increases the length incrementally, leading to $e_n(u) = 2n - 2$.

Subcase 1b: If $S' = \emptyset$, select $n - 1$ vertices from different $\mathscr{C}_{q_i}$ to obtain $e_n(u) = 2(n - 1) = 2n - 2$.

Cases 2-3. Cases 2 and 3 follow by direct application of the structure and results from Lemma 2.10. □

Lemma 2.12. *Let $u \in \mathscr{L}$, and suppose $m = q_1^{a_1} q_2^{a_2} \ldots q_h^{a_h}$ with $q_1 < q_2 < \ldots < q_h$ distinct primes. Then:*

(i) $e_n(u) = \begin{cases} 2n - 2, & \text{if } 3 \leq n \leq h, \\ h + n - 2, & \text{if } h + 1 \leq n \leq l_0 + 1, \end{cases}$ *where* $l_0 = |V(\Gamma(\mathbb{Z}_m))| - |\mathscr{L}|$;

(ii) If $u \in \mathscr{C}_{x_h}$, then $e_{l_j+n_j}(u) = d_j + n_j - 1$, where $e_{l_j}(u) = d_j$, $0 \leq j \leq h - 1$, $1 \leq n_j \leq q_{h-j} - 1$, and

$$l_j = |V(\Gamma(\mathbb{Z}_m))| - \left| \bigcup_{\beta=1}^{h-j} \mathscr{C}_{x_\beta} \right|,$$

with $x_\beta = q_1^{a_1} q_2^{a_2} \ldots q_\beta^{a_\beta - 1} \ldots q_h^{a_h}$;

(iii) If $u \in \mathscr{C}_{x_\beta}$ for $\beta \neq h$, then $e_{t_j+r_j}(u) = c_j + r_j - 1$, where $e_{t_j}(u) = c_j$ and

$$t_i = \begin{cases} t_{i-1} + |\mathscr{C}_{x_{h-i+2}}|, & \text{if } i \neq 1 \text{ and } h - i + 2 > \beta, \\ t_{i-1} + |\mathscr{C}_{x_{h-i+1}}|, & \text{if } i \neq 1 \text{ and } h - i + 2 \leq \beta, \\ l_0 + |\mathscr{C}_{x_\beta}|, & \text{if } i = 1, \end{cases}$$

for $1 \leq i \leq h - 1$, $0 \leq j \leq h - 1$, and the range of r_j is given by:

$$r_j \in \begin{cases} [1, q_\beta - 1], & \text{if } j = 0, \\ [1, q_{h-j+1} - 1], & \text{if } 1 \leq j \leq h - \beta, \\ [1, q_{h-j} - 1], & \text{if } h - \beta + 1 \leq j \leq h - 1. \end{cases}$$

Proof. Let $u \in \mathscr{C}_{x_\beta} \subseteq \mathscr{L}$, so $u = \alpha x_\beta$ for some β.

Case 1. *Case 1a:* $3 \leq n \leq h$.

Since each $v \in V(\Gamma(\mathbb{Z}_m))$ is adjacent to some $\mathscr{C}_{x_\gamma}$, and u is adjacent to all $\mathscr{C}_{x_\gamma}$ with $\gamma \neq \beta$, the distance from u to such v is at most 2. Choosing $n - 1$ vertices from distinct $\mathscr{C}_{q_\gamma}$ in $\mathscr{L}_1$ (with $\gamma \neq \beta$), and including u, yields a Steiner tree of length $2(n - 1)$.

Case 1b: $h + 1 \leq n \leq l_0 + 1$.

Adding each vertex from $V(\Gamma(\mathbb{Z}_m)) \setminus \mathscr{L}$ increases the Steiner distance by 1. Thus, $e_n(u) = 2h - 2 + (n - h) = h + n - 2$.

Case 2. If $u \in \mathscr{C}_{x_h}$, results follow directly from Lemma 2.10 (Cases 3 and 4).

Case 3. If $u \in \mathscr{C}_{x_\beta}$ with $\beta \neq h$:

Case 3a: $l_0 + 2 \leq n \leq l_1$.

Each added vertex $w \in \mathscr{C}_{x_\beta}$ increases the tree length by 1, so $e_{l_0+r_0}(u) = c_0 + r_0 - 1$ for $1 \leq r_0 \leq q_\beta - 1$.

Case 3b: $t_1 + 1 \leq n \leq t_{h-\beta+1}$.

Adding $v \in \mathscr{C}_{x_i}$ $(i \neq \beta)$ does not increase length. Additional vertices from the same class increase length by 1. Choosing r_1 vertices from $\mathscr{C}_{x_h}$ gives $e_{t_1+r_1}(u) = c_1 + r_1 - 1$.

Similarly, for $\mathscr{C}_{x_{h-1}}, \ldots, \mathscr{C}_{x_{\beta+1}}$, we get $e_{t_i+r_i}(u) = c_i + r_i - 1$ for respective r_i.

Case 3c: $t_{h-\beta+1} + 1 \leq n \leq t_{h-1}$.

Using the same logic for $\mathscr{C}_{x_{\beta-1}}, \ldots, \mathscr{C}_{x_1}$ yields $e_{t_i+r_i}(u) = c_i + r_i - 1$. □

Theorem 2.13. *Let $m = q_1^{a_1} q_2^{a_2} \ldots q_h^{a_h}$, with $q_1 < q_2 < \ldots < q_h$ being distinct primes and $h \geq 3$. Suppose:*

(i) If $q_1 + q_2 \leq q_i + 1$ for all $i \geq 3$, then

$$a_S(\Gamma(\mathbb{Z}_m)) = \begin{cases} |V(\Gamma(\mathbb{Z}_m))|, & \text{if } q_1 = 2, \\ |V(\Gamma(\mathbb{Z}_m))| - q_1 + 2, & \text{if } q_1 \neq 2. \end{cases}$$

(ii) Otherwise, if $q_1 + q_2 > q_i + 1$ for some $i \geq 3$, let $j = \max\{i : q_1 + q_2 > q_i + 1,\ i \geq 3\}$, and define

$$a_S(\Gamma(\mathbb{Z}_m)) = l_0 + (q_h - 1) + (q_{h-1} - 1) + \ldots + (q_j - 1) + 1,$$

where $l_0 = |V(\Gamma(\mathbb{Z}_m))| - |\mathscr{L}|$.

Proof. **Case (i):** Assume $q_1 + q_2 \leq q_i + 1$ for all $i \geq 3$. Let $t = |V(\Gamma(\mathbb{Z}_m))| - (q_1 - 1)$. Suppose $a_S(\Gamma(\mathbb{Z}_m)) = t$. Consider $u_1 \in \mathscr{C}_{x_1}$ and $u_2 \in \mathscr{C}_{x_2}$. Construct a Steiner tree ST_1 with t vertices, including u_1, u_2. Its length is $l_0 + (q_2 + q_3 + \ldots + q_h) - 2(h-1) + (h-1) - 1$. However, from Lemma 2.12, the diameter $\text{diam}_t(\Gamma(\mathbb{Z}_m))$ equals $l_0 + (q_2 + q_3 + \ldots + q_h) - 2(h-1) + (h-1)$, leading to a contradiction. So $a_S(\Gamma(\mathbb{Z}_m)) \geq t + 1$.

Subcase (a): If $q_1 = 2$, then $t = |V(\Gamma(\mathbb{Z}_m))| - 1$, implying $a_S(\Gamma(\mathbb{Z}_m)) = |V(\Gamma(\mathbb{Z}_m))|$.

Subcase (b): If $q_1 \neq 2$, choose l_0 vertices from $\mathscr{L}_k$ $(1 \leq k \leq h)$, and $(q_h - 1), (q_{h-1} - 1), \ldots, (q_2 - 1)$ vertices from respective $\mathscr{C}_{x_i}$, plus one from $\mathscr{C}_{x_1}$. Let u be any vertex not in $\mathscr{C}_{x_1}$ and form a Steiner tree ST_2 with those vertices. Then:

$$\text{diam}_{t+1}(\Gamma(\mathbb{Z}_m)) = l_0 + (q_2 + q_3 + \ldots + q_h) - 2(h-1) + (h-1),$$

which ensures $SA_{t+1}(\Gamma(\mathbb{Z}_m))$ is a complete graph.

Case (ii): If $q_1 + q_2 > q_i + 1$ for some $i \geq 3$, set $t_1 = l_0 + (q_h - 1) + (q_{h-1} - 1) + \ldots + (q_j - 1)$. Assume $a_S(\Gamma(\mathbb{Z}_m)) = t_1$ and build ST_3 with $u_1 \in \mathscr{C}_{x_1}$, $u_2 \in \mathscr{C}_{x_2}$, l_0 vertices from $\mathscr{L}_k$, all from $\mathscr{C}_{x_i}$ with $i \geq j$, and $(q_j - 1)$ from $\mathscr{C}_{x_1}, \mathscr{C}_{x_2}$. The Steiner distance is

$$l_0 + (h-1) + (q_h - 2) + (q_{h-1} - 2) + \ldots + (q_{j+1} - 2) + (q_j - 3).$$

But Lemma 2.12 gives

$$\mathrm{diam}_{t_1}(\Gamma(\mathbb{Z}_m)) = l_0 + (h-1) + (q_h - 2) + \ldots + (q_j - 2),$$

which is greater than the Steiner tree length, so $a_S(\Gamma(\mathbb{Z}_m)) \geq t_1 + 1$.

Now, let $u_3 \notin \{\mathscr{C}_{x_1}, \mathscr{C}_{x_2}, \ldots, \mathscr{C}_{x_{j-1}}\}$. Selecting l_0 vertices from $\mathscr{L}_k$ and $(q_h - 1), \ldots, (q_j - 1)$ vertices from respective $\mathscr{C}_{x_i}$ plus one vertex $v_3 \in \mathscr{C}_{x_i}$ $(1 \leq i \leq j-1)$, all such configurations form Steiner trees of maximum length. Hence $SA_{t_1+1}(\Gamma(\mathbb{Z}_m))$ is a complete graph. □

Example 2.14. *Consider the zero-divisor graph $\Gamma(\mathbb{Z}_m)$ with $m = 3 \cdot 5 \cdot 7 \cdot 11 = 1155$.*

Define the following sets:

$$\mathscr{L}_1 = \{\mathscr{C}_3, \mathscr{C}_5, \mathscr{C}_7, \mathscr{C}_{11}\}, \quad \mathscr{L}_2 = \{\mathscr{C}_{3\cdot5}, \mathscr{C}_{3\cdot7}, \mathscr{C}_{3\cdot11}, \mathscr{C}_{5\cdot7}, \mathscr{C}_{5\cdot11}, \mathscr{C}_{7\cdot11}\},$$

$$\mathscr{L}_3 = \emptyset, \quad \mathscr{L}_4 = \emptyset, \mathscr{L} = \{\mathscr{C}_{x_1}, \mathscr{C}_{x_2}, \mathscr{C}_{x_3}, \mathscr{C}_{x_4}\},$$

where $x_1 = 5 \cdot 7 \cdot 11,\ x_2 = 3 \cdot 7 \cdot 11,\ x_3 = 3 \cdot 5 \cdot 11,\ x_4 = 3 \cdot 5 \cdot 7$.

Let $l_0 = |\mathscr{L}_1| + |\mathscr{L}_2| = 488 + 164 = 652$.

- *If $u \in \mathscr{L}_1$, then $e_3(u) = 5$, $e_4(u) = 7$.*
- *If $u \in \mathscr{L}_2$, then $e_3(u) = 5$, $e_4(u) = 6$.*
- *If $u \in \mathscr{L}$, then $e_3(u) = 4$, $e_4(u) = 6$.*
- *For $u \notin \mathscr{L}$, $e_{4+i}(u) = 7 + i$, $1 \leq i \leq 648$.*
- *For $u \in \mathscr{L}$, $e_{4+i}(u) = 6 + i$, $1 \leq i \leq 648$.*

Hence,

$$e_{l_0}(u) = \begin{cases} 654, & \text{if } u \in \mathscr{L}, \\ 655, & \text{otherwise}, \end{cases} \quad e_{l_0+1}(u) = 655, \quad e_{654}(u) = 656.$$

Case 1: $u \in \mathscr{C}_{x_1}$.
$t_1 = 654, \quad e_{t_1+r_1}(u) = 656 + r_1 - 1, \quad 1 \leq r_1 \leq 10,$
$t_2 = 664, \quad e_{t_2+r_2}(u) = 665 + r_2 - 1, \quad 1 \leq r_2 \leq 6,$
$t_3 = 670, \quad e_{t_3+r_3}(u) = 670 + r_3 - 1, \quad 1 \leq r_3 \leq 4.$

Case 2: $u \in \mathscr{C}_{x_2}$.
$e_{l_0+r_0}(u) = 655 + r_0 - 1, \quad 1 \leq r_0 \leq 4,$
$t_1 = 656, \quad e_{t_1+r_1}(u) = 658 + r_1 - 1, \quad 1 \leq r_1 \leq 10,$
$t_2 = 666, \quad e_{t_2+r_2}(u) = 667 + r_2 - 1, \quad 1 \leq r_2 \leq 6,$
$t_3 = 677, \quad e_{t_3+r_3}(u) = 672 + r_3 - 1, \quad 1 \leq r_3 \leq 2.$

Case 3: $u \in \mathscr{C}_{x_3}$.
$e_{l_0+r_0}(u) = 655 + r_0 - 1, \quad 1 \leq r_0 \leq 6,$
$t_1 = 658, \quad e_{t_1+r_1}(u) = 660 + r_1 - 1, \quad 1 \leq r_1 \leq 10,$
$t_2 = 668, \quad e_{t_2+r_2}(u) = 669 + r_2 - 1, \quad 1 \leq r_2 \leq 4,$
$t_3 = 672, \quad e_{t_3+r_3}(u) = 672 + r_3 - 1, \quad 1 \leq r_3 \leq 2.$

Case 4: $u \in \mathscr{C}_{x_4} \cup \mathscr{L}_1 \cup \mathscr{L}_2$.
$l_1 = 662, \quad e_{l_0+n_0}(u) = 655 + n_0 - 1, \quad 1 \leq n_0 \leq 10,$

$l_2 = 668, \quad e_{l_1+n_1}(u) = 664 + n_1 - 1, \quad 1 \leq n_1 \leq 6,$
$l_3 = 672, \quad e_{l_2+n_2}(u) = 669 + n_2 - 1, \quad 1 \leq n_2 \leq 4,$
$e_{l_3+n_3}(u) = 672 + n_3 - 1, \quad 1 \leq n_3 \leq 2.$

The graph corresponding to this example is illustrated in Figure 2.

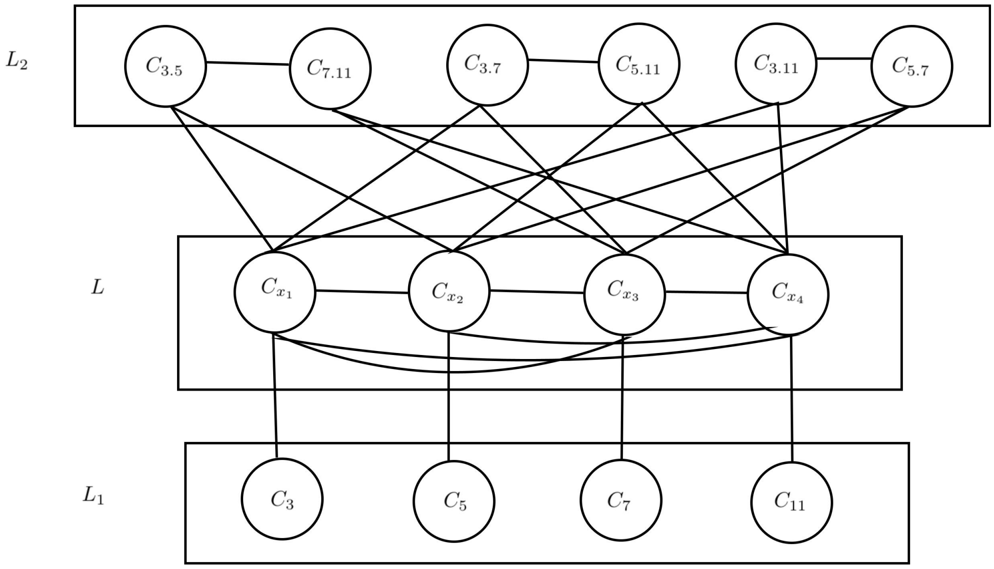

Figure 2. Zero-divisor graph $\Gamma(\mathbb{Z}_{3\cdot5\cdot7\cdot11})$.

Example 2.15. *Consider the zero-divisor graph* $\Gamma(\mathbb{Z}_m)$ *where* $m = 2^2 \cdot 3 \cdot 5^2 = 300$. *Define the following sets:*

$$\begin{aligned}
\mathscr{L}_1 &= \{\mathscr{C}_2, \mathscr{C}_{2^2}, \mathscr{C}_3, \mathscr{C}_5, \mathscr{C}_{5^2}\},\\
\mathscr{L}_2 &= \{\mathscr{C}_{2\cdot3}, \mathscr{C}_{2\cdot5}, \mathscr{C}_{2\cdot5^2}, \mathscr{C}_{2^2\cdot3}, \mathscr{C}_{2^2\cdot5}, \mathscr{C}_{3\cdot5}, \mathscr{C}_{3\cdot5^2}\},\\
\mathscr{L}_3 &= \{\mathscr{C}_{2\cdot3\cdot5}\},\\
\mathscr{L} &= \{\mathscr{C}_{x_1}, \mathscr{C}_{x_2}, \mathscr{C}_{x_3}\}, \text{where} \quad x_1 = 2 \cdot 3 \cdot 5^2, \quad x_2 = 2^2 \cdot 5^2, \quad x_3 = 2^2 \cdot 3 \cdot 5.
\end{aligned}$$

Now compute $l_0 = |\mathscr{L}_1| + |\mathscr{L}_2| + |\mathscr{L}_3| = 140 + 68 + 4 = 212, l_1 = 216$. *The Steiner n-eccentricities are given by:*

$$e_3(u) = \begin{cases} 5, & \text{if } u \in \mathscr{L}_1, \\ 4, & \text{if } u \notin \mathscr{L}_1, \end{cases} \qquad e_{3+i}(u) = \begin{cases} 5+i, & \text{if } u \notin \mathscr{L}, \\ 4+i, & \text{if } u \in \mathscr{L}, \end{cases} \qquad 1 \leq i \leq 119.$$

In particular, for all u, we have $e_{l_0+1}(u) = e_{213}(u) = 214$.

The remaining computations follow analogously to those in Example 2.14. The corresponding graph is depicted in Figure 3.

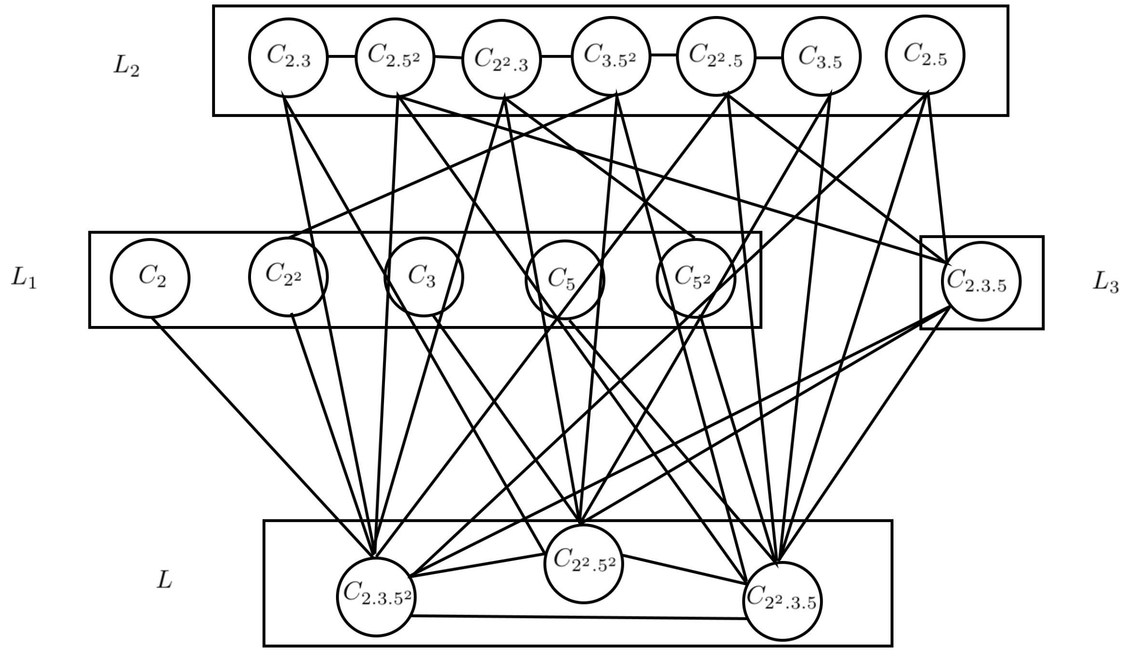

Figure 3. Zero-divisor graph $\Gamma(\mathbb{Z}_{2^2.3.5^2})$.

3. The Steiner antipodal number of the zero-divisor graph of a finite commutative reduced ring with unity

Let R be a finite commutative reduced ring with unity. It is well known that such a ring is isomorphic to a direct product of finite fields, i.e., $R \cong \mathbb{F}_{p_1} \times \mathbb{F}_{p_2} \times \cdots \times \mathbb{F}_{p_h}$, where each $p_i = q_i^{a_i}$ for some prime q_i and integer $a_i \geq 1$, and $\mathbb{F}_p$ denotes the finite field with p elements.

Since each $\mathbb{F}_{p_i}$ has only one zero-divisor (namely 0), the set of zero-divisors in R consists of all elements having at least one coordinate equal to zero.

We apply the equivalence relation $\sim$ (as introduced in Section 1) to the vertex set $V(\Gamma(R))$. Define the set:

$$E = \{a = (a_1, a_2, \ldots, a_h) \in Z(R) : a_i = 0 \text{ or } a_i = 1\}.$$

The next lemma describes the structure of equi-neighbor classes in this context.

Lemma 3.1. *Each equi-neighbor class contains exactly one element from E.*

Proof. **Existence:** Let $\mathscr{C}_u$ be an equi-neighbor class. Suppose $\mathscr{C}_u$ does not contain any element from E. Take $a = (a_1, a_2, \ldots, a_h) \in \mathscr{C}_u$ with $a_i \in F_{p_i}$, and assume that at least one coordinate a_i is neither 0 nor 1. Let $A = \{i \in \{1, 2, \ldots, h\} : a_i \neq 0\}$. Clearly, $A \neq \emptyset$. Define $b = (b_1, b_2, \ldots, b_h)$ where $b_i = 1$ if $i \in A$ and $b_i = 0$ otherwise. Then $b \in E$.

Now, for any $c = (c_1, \ldots, c_h)$ adjacent to a, it must be that $c_i = 0$ for all $i \in A$, or $a_i = 1$. Hence, c is also adjacent to b. Thus, $\mathscr{N}(a) = \mathscr{N}(b)$ and $b \in \mathscr{C}_u$.

Uniqueness: Suppose $a \neq b \in \mathscr{C}_u$ and both $a, b \in E$. Then there exists some i such that $a_i \neq b_i$, implying one is 0 and the other is 1. Without loss of generality, assume $a_i = 1$ and $b_i = 0$. Let

$c = (c_1, \ldots, c_h)$ where $c_i = 1$ and $c_j = 0$ for all $j \neq i$. Then c is adjacent to b but not to a, contradicting $\mathcal{N}(a) = \mathcal{N}(b)$. Hence, such a, b cannot both exist in the same class. □

Lemma 3.2. *Two vertices in $\mathcal{C}_u$ and $\mathcal{C}_v$ are adjacent if and only if $uv = 0$.*

Corollary 3.3. *(1) The induced subgraph $G(\mathcal{C}_u)$ is totally disconnected.*
(2) For $u \neq v$, a vertex of $\mathcal{C}_u$ is either adjacent to all or none of the vertices in $\mathcal{C}_v$.

Remark 3.4. *The number of equi-neighbor classes equals the number of nonzero elements in E; hence, $|E| = 2^h - 2$.*

To compute the Steiner n-eccentricity of each vertex in $\Gamma(R)$, partition the vertex set $V(\Gamma(R))$ as follows: Let $\mathcal{L}_i$ denote the set of vertices $u = (u_1, \ldots, u_h)$ such that exactly i coordinates are zero, for $1 \leq i \leq h-1$. Define $\mathcal{L} = \mathcal{L}_{h-1}$. Note that $\mathcal{L}_{h-1}$ includes the equi-neighbor class $\mathcal{C}_{e_i}$, where e_i has 1 in the i-th coordinate and 0 elsewhere. For any $u \in \mathcal{L}_1$ with $u_i = 0$, we have $\mathcal{N}(u) = \mathcal{C}_{e_i}$ for $1 \leq i \leq h$.

Lemma 3.5. *Let $u \in \mathcal{L}_1$ and $R = F_{p_1} \times \cdots \times F_{p_h}$ with $p_1 \leq p_2 \leq \cdots \leq p_h$. Then*

$$e_n(u) = \begin{cases} 2n-1, & \text{if } 3 \leq n \leq h, \\ h+n-1, & \text{if } h+1 \leq n \leq l_0, \end{cases}$$

where $l_0 = |V(\Gamma(R))| - |\mathcal{L}|$. Furthermore, $e_{l_j+n_j}(u) = d_j + n_j - 1$, where $0 \leq j \leq h-1$, $1 \leq n_j \leq q_{h-j}^{a_{h-j}} - 1$, $e_{l_j}(u) = d_j$, and $l_j = |V(\Gamma(R))| - \left|\bigcup_{\beta=1}^{h-j} \mathcal{C}_{e_\beta}\right|$.

Proof. Follows directly from Lemma 2.10. □

Lemma 3.6. *Let $u \in \mathcal{L}_k$ with $2 \leq k \leq h-2$. Then:*

(1) For $3 \leq n \leq h$, $e_n(u) = \begin{cases} 2n-2, & \text{if } u \in \bigcup_{i=2}^{n-2} \mathcal{L}_{h-i}, \\ 2n-1, & \text{otherwise}. \end{cases}$
(2) For $h+1 \leq n \leq l_0$, we have $e_n(u) = h+n-1$, where $l_0 = |V(\Gamma(R))| - |\mathcal{L}|$.
(3) If $e_{l_j}(u) = d_j$, then $e_{l_j+n_j}(u) = d_j + n_j - 1$ for $0 \leq j \leq h-1$, $1 \leq n_j \leq q_{h-j}^{a_{h-j}} - 1$, with $l_j = |V(\Gamma(R))| - \left|\bigcup_{\beta=1}^{h-j} \mathcal{C}_{e_\beta}\right|$.

Proof. The result is a direct consequence of Lemma 2.11. □

Lemma 3.7. *Let $u \in \mathcal{L}$. Then:*

(1) For $3 \leq n \leq h$, $e_n(u) = 2n-2$, and for $h+1 \leq n \leq l_0+1$, $e_n(u) = h+n-2$, where $l_0 = |V(\Gamma(R))| - |\mathcal{L}|$.
(2) If $u \in \mathcal{C}_{e_h}$, then $e_{l_j+n_j}(u) = d_j + n_j - 1$, where $e_{l_j}(u) = d_j$, $1 \leq j \leq h-1$, $1 \leq n_j \leq q_{h-j}^{a_{h-j}} - 1$, and

$$l_j = |V(\Gamma(R))| - \left|\bigcup_{\beta=1}^{h-j} \mathcal{C}_{e_\beta}\right|.$$

(3) If $u \in \mathcal{C}_{e_\beta}$ with $\beta \neq h$, then $e_{t_j+r_j}(u) = c_j + r_j - 1$, where $e_{t_j}(u) = c_j$, and

$$t_i = \begin{cases} t_{i-1} + |\mathcal{C}_{e_{h-i+2}}|, & \text{if } i \neq 1 \text{ and } h-i+2 > \beta, \\ t_{i-1} + |\mathcal{C}_{e_{h-i+1}}|, & \text{if } i \neq 1 \text{ and } h-i+2 \leq \beta, \\ l_0 + |\mathcal{C}_{e_\beta}|, & \text{if } i = 1, \end{cases}$$

for $1 \leq i \leq h-1$, $0 \leq j \leq h-1$, *and the range of* r_j *is*

$$r_j \in \begin{cases} [1, q_{\beta}^{a_{\beta}} - 1], & \text{if } j = 0, \\ [1, q_{h-j+1}^{a_{h-j+1}} - 1], & \text{if } 1 \leq j \leq h-\beta, \\ [1, q_{h-j}^{a_{h-j}} - 1], & \text{if } h-\beta+1 \leq j \leq h-1. \end{cases}$$

Proof. The proof can be derived from Lemma 2.12. □

Theorem 3.8. *Let* $R = F_{p_1} \times F_{p_2} \times \ldots \times F_{p_h}$, *where* $p_i = q_i^{a_i}$ *for some primes* q_i *and integers* a_i, *such that* $p_1 \leq p_2 \leq \ldots \leq p_h$ *and* $h \geq 3$. *If* $p_1 + p_2 \leq p_i + 1$ *for all* $i \geq 3$, *then*

$$a_S(\Gamma(R)) = \begin{cases} |V(\Gamma(R))|, & \text{if } p_1 = 2, \\ |V(\Gamma(R))| - p_1 + 2, & \text{if } p_1 \neq 2. \end{cases}$$

Additionally, $a_S(\Gamma(R)) = l_0 + \sum_{i=j}^{h}(p_i - 1) + 1$, *where* $l_0 = |V(\Gamma(R))| - |\mathscr{L}|$ *and* $j = \max\{i : p_1 + p_2 > p_i + 1,\ i \geq 3\}$.

Proof. The result follows directly from Theorem 2.13, which characterizes the Steiner antipodal number based on the structure of the zero-divisor graph. □

Example 3.9. *In this example, we compute the Steiner n-eccentricities of all vertices in the zero-divisor graph of the reduced ring* $R = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$.

Let

$$\mathscr{L}_1 = \{\mathscr{C}_{(0,1,1,1)}, \mathscr{C}_{(1,0,1,1)}, \mathscr{C}_{(1,1,0,1)}, \mathscr{C}_{(1,1,1,0)}\},$$

$$\mathscr{L}_2 = \{\mathscr{C}_{(0,0,1,1)}, \mathscr{C}_{(0,1,0,1)}, \mathscr{C}_{(0,1,1,0)}, \mathscr{C}_{(1,0,0,1)}, \mathscr{C}_{(1,0,1,0)}, \mathscr{C}_{(1,1,0,0)}\},$$

$$\mathscr{L} = \{\mathscr{C}_{(1,0,0,0)}, \mathscr{C}_{(0,1,0,0)}, \mathscr{C}_{(0,0,1,0)}, \mathscr{C}_{(0,0,0,1)}\}.$$

We have $l_0 = |\mathscr{L}_1| + |\mathscr{L}_2| = 36 + 28 = 64$, $\quad l_1 = 68$, $\quad l_2 = 70$, $\quad l_3 = 72$.

Steiner eccentricities:

- *If* $u \in \mathscr{L}_1$: $e_3(u) = 5$, $e_4(u) = 7$;
- *If* $u \in \mathscr{L}_2$: $e_3(u) = 5$, $e_4(u) = 6$;
- *If* $u \in \mathscr{L}$: $e_3(u) = 4$, $e_4(u) = 6$.

For $1 \leq i \leq 60$,

$$e_{4+i}(u) = \begin{cases} 6+i, & \text{if } u \in \mathscr{L}, \\ 7+i, & \text{if } u \notin \mathscr{L}. \end{cases}$$

Therefore,

$$e_{l_0}(u) = e_{64}(u) = \begin{cases} 66, & \text{if } u \in \mathscr{L}, \\ 67, & \text{otherwise}; \end{cases} \quad e_{65}(u) = 67, \quad e_{66}(u) = 68 \text{ for all } u.$$

The remaining cases are similar to those discussed in Example 2.14, and the associated graph is illustrated in Figure 4.

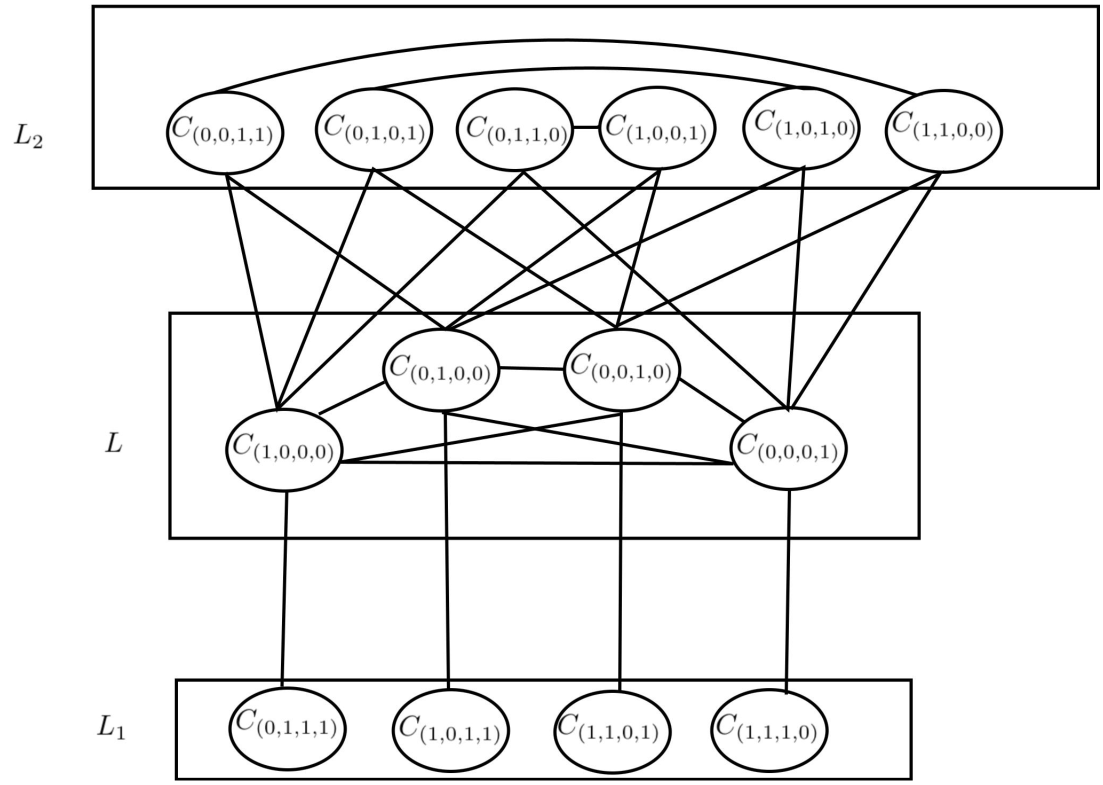

Figure 4. Zero-divisor graph $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5)$.

4. The Steiner antipodal number of the zero-divisor graph of a finite product of integer modulo rings

In this section, we examine the Steiner antipodal number of the zero-divisor graph $\Gamma(R)$ where R is a finite product of integer modulo rings. Let

$$R = \mathbb{Z}_{q_1^{a_1}} \times \mathbb{Z}_{q_2^{a_2}} \times \cdots \times \mathbb{Z}_{q_h^{a_h}},$$

where each q_i is a prime number and $a_i \geq 2$. We partition the vertex set of $\Gamma(R)$ using the equivalence relation defined in Section 1.

Define the sets $\mathscr{L}_i$ as follows:

- For $1 \leq i \leq h-2$, let $\mathscr{L}_i$ be the set of all h-tuples with exactly i coordinates equal to zero or exactly i coordinates being non-zero divisors.
- Let $\mathscr{L} = \bigcup_{\beta=1}^{h} \mathscr{C}_{e_\beta}$, where each $e_\beta = (0, \ldots, q_\beta^{a_\beta - 1}, \ldots, 0)$ is a tuple with the non-zero entry in the β-th position.
- Define $\mathscr{L}_{h-1}$ as the set of h-tuples with exactly $h-1$ zeros or $h-1$ non-zero divisors, excluding the elements in $\mathscr{L}$.
- Let $\mathscr{L}_h$ consist of tuples with all entries as non-zero divisors, or one zero entry and the remaining $h-1$ as non-zero divisors, again excluding elements in $\mathscr{L}$.

We now present results concerning the Steiner n-eccentricity of vertices in $\Gamma(R)$.

Lemma 4.1. *Let $u \in \mathscr{L}_1$ and R be as defined above with $q_1 \leq q_2 \leq \cdots \leq q_h$. Then*

$$e_n(u) = \begin{cases} 2n-1, & \text{for } 3 \leq n \leq h, \\ h+n-1, & \text{for } h+1 \leq n \leq l_0, \end{cases}$$

where $l_0 = |V(\Gamma(R))| - |\mathscr{L}|$. Furthermore, if $e_{l_j}(u) = d_j$, then $e_{l_j+n_j}(u) = d_j + n_j - 1$, for $0 \leq j \leq h-1$, $1 \leq n_j \leq q_{h-j} - 1$, and

$$l_j = |V(\Gamma(R))| - \left| \bigcup_{\beta=1}^{h-j} \mathscr{C}_{e_\beta} \right|.$$

Proof. Follows from Lemma 2.10. □

Lemma 4.2. *Let $u \in \mathscr{L}_k$ for $2 \leq k \leq h-2$. Then*

(1) For $3 \leq n \leq h$, $e_n(u) = \begin{cases} 2n-2, & \text{if } u \in \bigcup_{i=0}^{n-2} \mathscr{L}_{h-i}, \\ 2n-1, & \text{otherwise}. \end{cases}$
(2) For $h+1 \leq n \leq l_0$, $e_n(u) = h+n-1$.
(3) If $e_{l_j}(u) = d_j$, then $e_{l_j+n_j}(u) = d_j + n_j - 1$ for $0 \leq j \leq h-1$, $1 \leq n_j \leq q_{h-j} - 1$.

Proof. A direct consequence of Lemma 2.11. □

Lemma 4.3. *Let $u \in \mathscr{L}$. Then:*

(1) For $3 \leq n \leq h$, $e_n(u) = \begin{cases} 2n-2, & \\ h+n-2, & \text{for } h+1 \leq n \leq l_0+1, \end{cases}$ where $l_0 = |V(\Gamma(R))| - |\mathscr{L}|$.
(2) If $u \in \mathscr{C}_{e_h}$, then $e_{l_j+n_j}(u) = d_j + n_j - 1$, for $1 \leq j \leq h-1$, $1 \leq n_j \leq q_{h-j} - 1$.
(3) If $u \in \mathscr{C}_{e_\beta}$ with $\beta \neq h$, define t_i recursively by

$$t_i = \begin{cases} l_0 + |\mathscr{C}_{e_\beta}|, & \text{if } i = 1, \\ t_{i-1} + |\mathscr{C}_{e_{h-i+2}}|, & \text{if } h-i+2 > \beta, \\ t_{i-1} + |\mathscr{C}_{e_{h-i+1}}|, & \text{otherwise}. \end{cases}$$

Then for $0 \leq j \leq h-1$, let $e_{t_j+r_j}(u) = c_j + r_j - 1$, where:

$$r_j \in \begin{cases} [1, q_\beta - 1], & \text{if } j = 0, \\ [1, q_{h-j+1} - 1], & \text{if } 1 \leq j \leq h-\beta, \\ [1, q_{h-j} - 1], & \text{if } h-\beta+1 \leq j \leq h-1. \end{cases}$$

Proof. Follows from Lemma 2.12. □

Theorem 4.4. *Let $R = \mathbb{Z}_{q_1^{a_1}} \times \mathbb{Z}_{q_2^{a_2}} \times \cdots \times \mathbb{Z}_{q_h^{a_h}}$, with primes q_i and $a_i \geq 2$, such that $q_1 \leq q_2 \leq \cdots \leq q_h$ and $h \geq 3$. If $q_1 + q_2 \leq q_i + 1$ for all $i \geq 3$, then*

$$a_S(\Gamma(R)) = \begin{cases} |V(\Gamma(R))|, & \text{if } q_1 = 2, \\ |V(\Gamma(R))| - q_1 + 2, & \text{if } q_1 \neq 2. \end{cases}$$

Moreover, $a_S(\Gamma(R)) = l_0 + (q_h - 1) + (q_{h-1} - 1) + \cdots + (q_j - 1) + 1$, where $l_0 = |V(\Gamma(R))| - |\mathscr{L}|$, and $j = \max\{i : q_1 + q_2 > q_i + 1,\ i \geq 3\}$.

Proof. This follows directly from Theorem 2.13, which relates the Steiner antipodal number to the graph structure. □

Example 4.5. *Let* $R = \mathbb{Z}_{2^2}\times\mathbb{Z}_{3^2}\times\mathbb{Z}_{3^3}$. *Define the sets* $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$, *and* $\mathscr{L}$ *as in the original manuscript. Let* $l_0 = 750$, $l_1 = 752$, *and* $l_2 = 754$. *Then for u:*

$$e_3(u) = \begin{cases} 5, & \text{if } u \in \mathscr{L}_1, \\ 4, & \text{otherwise}, \end{cases} \qquad e_{3+i}(u) = \begin{cases} 5+i, & \text{if } u \notin \mathscr{L}, \\ 4+i, & \text{if } u \in \mathscr{L}, \end{cases} \qquad \text{for } 1 \leq i \leq 747.$$

Also, $e_{751}(u) = 752$ *for all u. The rest of the behavior is similar to Example 2.14 and visualized in Figure 5.*

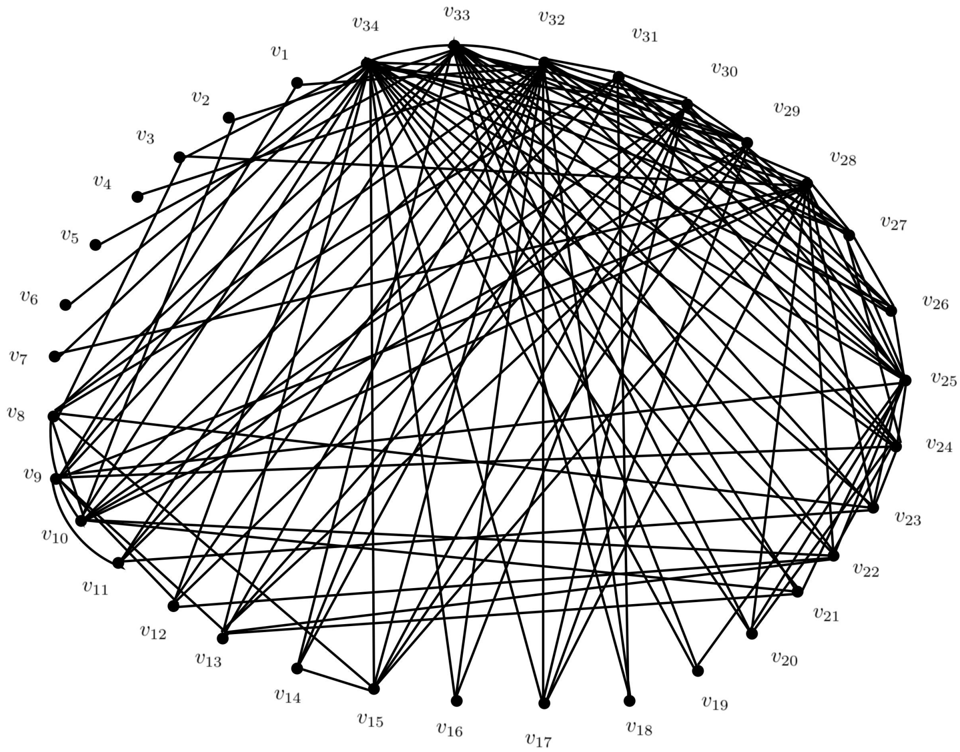

Figure 5. Zero-divisor graph $\Gamma(\mathbb{Z}_{2^2}\times\mathbb{Z}_{3^2}\times\mathbb{Z}_{3^3})$.

5. Inverse problem for the Steiner antipodal number

The inverse problem for the Steiner antipodal number asks whether, for a given non-negative integer k, there exists a zero-divisor graph $\Gamma(Z_m)$ such that its Steiner antipodal number is equal to k. In this work, we address this question by computational means, using the characterizations developed in Section 2.

Tables 1 and 2 list all values of m for which $\Gamma(Z_m)$ has Steiner antipodal numbers in the range 2 to 100. These results are obtained using a Python algorithm specifically designed to compute the Steiner antipodal number for $\Gamma(Z_m)$, for all integers $1 \leq m \leq 1000$.

Table 1. Zero-divisor graphs $\Gamma(\mathbb{Z}_m)$ and their Steiner antipodal numbers (Part 1).

S.No	Possible zero-divisor graphs of $\mathbb{Z}_m$	Steiner antipodal number
1	$m = q^2$ for any prime $q > 2$	2
2	$m = 6, 8$	3
3	$m = 10, 15$	5
4	$m = 12, 14, 16, 21, 27, 35$	7
5	$m = 18, 20, 22, 33, 55, 77$	11
6	$m = 26, 39, 65, 91, 143$	13
7	$m = 24, 28, 32$	15
8	$m = 17q$, prime $q < 17$	17
9	$m = 19q, 45, 133$, prime $q < 19$	19
10	$m = 30, 125$	21
11	$m = 23q, 40, 44, 46$, prime $q < 23$	23
12	$m = 63, 81$	25
13	$m = 52$	27
14	$m = 29q, 42, 50$, prime $q < 29$	29
15	$m = 31q, 48, 56, 64$, prime $q < 31$	31
16	$m = 75$	33
17	$m = 54, 68$	35
18	$m = 37q, 99$, prime $q < 37$	37
19	$m = 76$	39
20	$m = 41q$, prime $q < 41$	41
21	$m = 451$	42
22	$m = 43q, 60, 117, 343$, prime $q < 43$	43
23	$m = 66, 70$	45
24	$m = 47q, 72, 80, 88$, prime $q < 47$	47
25	$m = 175$	51
26	$m = 53q, 78$, prime $q < 53$	53
27	$m = 98, 104, 105, 153$	55
28	$m = 84$	57
29	$m = 59q, 100, 116$, prime $q < 59$	59
30	$m = 61q, 135, 147, 171$, prime $q < 61$	61
31	$m = 96, 112, 124, 128$	63

Table 2. Zero-divisor graphs $\Gamma(\mathbb{Z}_m)$ and their Steiner antipodal numbers (Part 2).

S.No	Possible zero-divisor graphs of $\mathbb{Z}_m$	Steiner antipodal number
32	$m = 90$	65
33	$m = 67q$, prime $q < 67$	67
34	$m = 102, 110$	69
35	$m = 71q, 108, 136, 275$, prime $q < 71$	71
36	$m = 73q, 207, 245$, prime $q < 73$	73
37	$m = 148$	75
38	$m = 114$	77
39	$m = 79q, 152, 189, 243$, prime $q < 79$	79
40	$m = 130, 325$	81
41	$m = 83q, 164, 165$, prime $q < 83$	83
42	$m = 120$	85
43	$m = 120, 172$	87
44	$m = 89q, 126$, prime $q < 89$	89
45	$m = 132, 140, 261$	91
46	$m = 138, 154$	93
47	$m = 144, 160, 176, 184, 188$	95
48	$m = 97q, 185, 279$, prime $q < 97$	97

5.1. Python code for computing Steiner antipodal number

The following Python function estimates the Steiner antipodal number for a given composite number m based on its prime structure:

```
def prime_divisors(n):
    v = []
    i = 1
    while i < n:
        k = 0
        if n % i == 0:
            j = 1
            while j <= i:
                if i % j == 0:
                    k += 1
                j += 1
        if k == 2:
            v.append(i)
        i += 1
    return v
def euler_totient(n):
    a = prime_divisors(n)
    i = 1
    for p in a:
```

```
        i *= (1 - (1 / p))
    return int(i * n)
def antipodal(n):
    v = prime_divisors(n)
    a = len(v)
    c = 0
    b = 0
    if a == 0:
        print("Given number is prime")
    elif a == 1 and n == v[0] * v[0]:
        print("2 is the antipodal number")
    elif a == 1 or a == 2:
        b = n - euler_totient(n) + 1 - v[0]
        print(b, "is the antipodal number")
    else:
        j = 0
        for i in range(1, a - 1):
            if v[0] + v[1] - 2 <= v[a - i] - 1:
                j += 1
        if j == a - 2 and v[0] == 2:
            b = n - euler_totient(n) - 1
            print(b, "is the antipodal number of Z_n")
        elif j == a - 2 and v[0] != 2:
            b = n - euler_totient(n) - v[0] + 1
            print(b, "is the antipodal number of Z_n")
        else:
            for i in range(a - j - 1):
                c = b + (v[i] - 1)
                b = c
            r = n - euler_totient(n) - b
            print(r, "is the antipodal number of Z_n")
n = int(input("Enter an integer: "))
antipodal(n)
```

5.2. Python code for solving the inverse problem (values from 2 to 100)

The following Python function searches for values of n such that the Steiner antipodal number of $\Gamma(Z_n)$ matches each integer m in the range $[3, 100]$:

```
def antipodal():
    for m in range(3, 101):
        for n in range(6, 1000):
            v = prime_divisors(n)
            a = len(v)
```

```
            c = 0
            b = 0
            r = 0
            if a == 0:
                r = 0
            elif a == 1 and n == v[0] * v[0]:
                r = 2
            elif a == 1 or a == 2:
                b = n - euler_totient(n) + 1 - v[0]
                r = b
            else:
                j = 0
                for i in range(1, a - 1):
                    if v[0] + v[1] - 2 <= v[a - i] - 1:
                        j += 1
                if j == a - 2 and v[0] == 2:
                    b = n - euler_totient(n) - 1
                    r = b
                elif j == a - 2 and v[0] != 2:
                    b = n - euler_totient(n) - v[0] + 1
                    r = b
                else:
                    for i in range(a - j - 1):
                        c = b + (v[i] - 1)
                        b = c
                    r = n - euler_totient(n) - b
            if m == r:
                print(m, "is the antipodal number for n =", n)
                break
            elif n == 999:
                print(m, "is not the antipodal number of any n")
antipodal()
```

This algorithm enables us to forecast which integers n yield a specified Steiner antipodal number m. The results are validated and consistent with the theoretical bounds established in previous sections.

6. Conclusions

In this study, we examined the n-eccentricity of vertices and determined the Steiner antipodal number for zero-divisor graphs of Z_m, reduced rings, and finite direct products of rings over Z_m. We also presented an algorithmic procedure for computing the Steiner antipodal number of $\Gamma(Z_m)$, illustrated with examples. Our findings not only enrich the study of structural invariants of zero-divisor graphs with applications to network design and fault-tolerant architectures, but also motivate the extension of Steiner invariants to more general algebraic frameworks.

In addition, our analysis can be extended to compute the Steiner antipodal number for related algebraic graphs such as non-commuting graphs, total graphs, and extended zero-divisor graphs, thereby broadening the applicability of these results to a wider class of algebraic structures.

Author contributions

Gurusamy Rajendran and Sankari alias Deepa Ramamoorthy: Conceptualization, methodology, validation, formal analysis, investigation, writing original draft preparation, writing review and editing, visualization, and supervision; Arockiaraj Sonasalam: Methodology, validation, investigation, review and editing, and visualization; Grienggrai Rajchakit: Writing original draft preparation, writing review and editing, project administration, and funding acquisition. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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