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Research article

Regions of variability for generalized Janowski functions

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Abstract: Let $r \in \mathbb{C}$, $s \in [-1, 0)$, $0 \le \alpha < 1$. Then, $Q[r, s, \alpha]$ stands for the set of analytic functions q that is within the open unit disk E, with $q(0) = 1$, and satisfies the explicit representation

$$q(\zeta) = \frac{1 + ((1-\alpha) r + \alpha s) \chi(\zeta)}{1 + s\chi(\zeta)},$$

where $\chi(0) = 0$ and $|\chi(\zeta)| < 1$. In this article, we find the regions of variability $W_\lambda(\zeta_0, r, s, \alpha)$ for $\int_0^{z_0} q(\rho)\, d\rho$ when q ranges over the class $Q_\lambda[r, s, \alpha]$ defined as

$$Q_\lambda[r, s, \alpha] = \{q \in Q[r, s, \alpha] : q'(0) = ((1-\alpha)(r-s))\lambda\}$$

for any fixed $\zeta_0 \in E$ and $\lambda \in \overline{E}$. As a corollary, the region of variability appears for the alternate sets of parameters as well.

Keywords: region of variability; Janowski functions; generalized Janowski functions; Schwarz function
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1. Introduction

Let $\mathcal{A}$ be the set of analytic functions (AFs) in $\mathcal{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ that are expressed as

$$\phi(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n \zeta^n. \tag{1.1}$$

Consider Ω to be a topological vector space equipped with the topology defined by uniform convergence over compact subsets of the set $\mathcal{D}$. Let us define the class $\mathcal{B}$ to include all functions χ that are analytic on $\mathcal{D}$, satisfy the condition $|\chi(\zeta)| < 1$ for every $\zeta \in \mathcal{D}$, and are normalized such that $\chi(0) = 0$.

It can be said that an AFs ϕ is subordinate to another function g, represented as $\phi \prec g$, if there exists $\chi \in \mathcal{B}$, for which the identity

$$\phi(\zeta) = g(\chi(\zeta))$$

holds for all $\zeta \in \mathcal{D}$. In the special case where g is univalent (i.e., one-to-one) on $\mathcal{D}$, this subordination implies that $\phi(0) = g(0)$, and that the image of ϕ is contained within that of g, meaning $\phi(\mathcal{D}) \subset g(\mathcal{D})$.

Now, let us introduce the family $\mathcal{Q}[r, s]$, containing functions q that are analytic in $\mathcal{D}$, fulfill $q(0) = 1$, and can be expressed in the form

$$q(\zeta) = \frac{1 + r\chi(\zeta)}{1 + s\chi(\zeta)}, \ \chi \in \mathcal{B}, \zeta \in \mathcal{D}, \tag{1.2}$$

where $r \in \mathbb{C}$, $s \in [-1, 0)$ with $r \neq s$. Note that $\mathcal{Q}[1, -1] = \mathcal{Q}$.

For arbitrarily fixed numbers $r \in \mathbb{C}$, $s \in [-1, 0)$ with $r \neq s$ and $0 \leq \alpha < 1$, let $\mathcal{Q}[r, s, \alpha]$ be the class of AFs q such that $q(0) = 1$, and $q \in \mathcal{Q}[r, s, \alpha]$ if and only if

$$q(\zeta) = \frac{1 + ((1-\alpha) r + \alpha s) \chi(\zeta)}{1 + s\chi(\zeta)}, \ \zeta \in \mathcal{D}. \tag{1.3}$$

Note that $\mathcal{Q}[1, -1, 0] = \mathcal{Q}$. Let $q \in \mathcal{Q}[r, s, \alpha]$, then, applying the Herglotz representation, there exists a unique positive unit measure $\mu \in (-\pi, \pi]$, where

$$q(\zeta) = \int_{-\pi}^{\pi} \frac{1 + ((1-\alpha) r + \alpha s) e^{it}\zeta}{1 + se^{it}\zeta} d\mu(t), \ \zeta \in \mathcal{D}. \tag{1.4}$$

As evident from (1.3) that

$$\chi_q(\zeta) = \frac{q(\zeta) - 1}{((1-\alpha) r + \alpha s) - sq(\zeta)}, \ \zeta \in \mathcal{D}, \tag{1.5}$$

conversely, we have

$$q'(0) = (((1-\alpha) r + \alpha s) - s) \chi_q'(0). \tag{1.6}$$

Let $q \in \mathcal{Q}[r, s, \alpha]$. Then, call on the Schwarz lemma, that is, $\left|\chi_q'(0)\right| \leq 1$ (see [1]). This yields

$$q'(0) = (1-\alpha)(r-s)\lambda$$

for some $\lambda \in \overline{\mathcal{D}}$. By using (1.5), one can compute

$$\frac{\chi_q''(0)}{2} = \frac{q''(0)}{2(1-\alpha)(r-s)} + s\lambda^2.$$

Now, if we let

$$g(\zeta) = \begin{cases} \frac{\frac{\chi_q(\zeta)}{\zeta} - \lambda}{1 - \overline{\lambda}\frac{\chi_q(\zeta)}{\zeta}}, & 1 > |\lambda|, \\ 0 & |\lambda| = 1, \end{cases}$$

consequently

$$g^{'}(0) = \begin{cases} \frac{1}{1-|\lambda|^2}\left(\frac{\chi_q(\zeta)}{\zeta}\right)^{'} |_{\zeta=0} = \frac{1}{1-|\lambda|^2}\frac{\chi_q''(\zeta)}{2}, & |\lambda| < 1, \\ 0, & |\lambda| = 1. \end{cases}$$

The classical Schwarz lemma tells us that for any analytic map g that dominates the unit disk and fixes the origin, we must have $|g(\zeta)| \leq |\zeta|$ and $|g'(0)| \leq 1$. The boundary cases are realized when g takes the form $g(\zeta) = e^{i\alpha}\zeta$ for some $\alpha \in \mathbb{R}$. The inequality $|g'(0)| \leq 1$ guarantees the existence of a point a in the open unit disk such that $g'(0) = a$. Thus, we have

$$q''(0) = 2(1-\alpha)(r-s)\left[\left(1 - |\lambda|^2\right)a - s\lambda^2\right].$$

Hence, for any λ in $\overline{\mathcal{D}}$ and any ζ_0 in $\mathcal{D}$, we obtain $W_\lambda(\zeta_0, r, s, \alpha)$, expressed as $\int_0^{\zeta_0} q(\delta)\, d\delta$, when q varies within the family $Q_\lambda[r, s, \alpha]$, characterized by

$$Q_\lambda[r, s, \alpha] = \{q \in Q[r, s, \alpha] : q'(0) = (((1-\alpha)r + \alpha s) - s)\lambda\},$$

and

$$W_\lambda(\zeta_0, r, s, \alpha) = \left\{\int_0^{\zeta_0} q(\delta)\, d\delta, \qquad q \in Q_\lambda[r, s, \alpha]\right\}.$$

The area of complex analysis, and specifically that of univalent functions, has been a popular research area in the literature. A principal target of this area is the fundamentally remarkable result, the Schwarz lemma ([1], 1989), and from that point, many contributions have been made to study the properties of some classes of univalent functions. Most relevant to our work, Yanagihara [2, 3] studied regions of variation for functions that have bounded variation in 2005 and for convex functions in 2006. Then, in many avenues, Ponnusamy and Vasudevarao [4, 5] shared results on the region of variation of subclasses of univalent functions, for functions with a positive real part, and other classes of univalent functions [6]. In the year 2008, Ponnusamy, along with other collaborators [7], also studied the region of variation for close-to-convex functions. Still in the year 2008, Ponnusamy et al. [8] investigated region of variation of univalent functions $\varphi(\zeta)$, where $\zeta\varphi^{'}(\zeta)$ is spirallike, while Ul-Haq [9] studied the regions of variability for Janowski convex functions in 2014. In 2020, Raza et al. [10] examined the region of variability for a subclass of analytic functions, and Bukhari et al. examined the region of variability of Bazilevic functions in [11]. In 2025, Khan et al. [12] published a comprehensive study of geometric perspectives on the variability of spirallike functions in relation to Janowski functions with respect to a boundary point. Collectively, their works contribute to an understanding of the properties and behavior of various classes of univalent functions, and their connections between the theory of functions and geometric properties.

While the aforementioned studies have significantly advanced our understanding of variability regions for various function classes, most investigations have focused on specific subfamilies with fixed geometric domains. Of particular relevance to our work, Ul-Haq [9] studied the regions of variability for Janowski convex functions in 2014, and in 2015, Raza et al. [13] investigated variability regions $V_\lambda(\zeta_0)$ for the integral functional $\int_0^{\zeta_0} p(\rho)\, d\rho$ when p ranges over the class $\mathcal{P}_\lambda[A, B]$ of Janowski functions. However, their study considered only the case where boundary parameters are fixed, without exploring the effect of convex combinations of these parameters on the resulting variability regions. The present paper addresses this gap by introducing and investigating the class $\mathcal{Q}_\lambda[r, s, \alpha]$ of generalized Janowski functions, where the additional parameter $\alpha \in [0, 1)$ provides a convex combination structure that interpolates between different geometric configurations. Our main contribution lies in determining the complete regions of variability $W_\lambda(\zeta_0, r, s, \alpha)$ for the integral functional $\int_0^{z_0} q(\rho)\, d\rho$ when q ranges over $\mathcal{Q}_\lambda[r, s, \alpha]$. This work directly extends and generalizes the results of Raza et al. [13] in the following significant ways: **(i)** Setting $\alpha = 0$ in our main theorems recovers their variability regions as a special case, thereby validating our approach while demonstrating that their results are embedded within our broader framework, and **(ii)** the introduction of the parameter α enables a continuous transition between boundary behaviors, yielding new geometric insights into how variability regions evolve as functions move between different Janowski-type domains. Moreover, our explicit characterization implies how the geometry of these regions depends smoothly on α, a case not studied in previous work. The explicit characterization of these regions not only enriches the geometric function theory but also provides a flexible tool for applications where intermediate geometric properties are of interest. The remainder of this paper is organized as follows: Section 2 presents preliminary results and basic properties of $W_\lambda(\zeta_0, r, s, \alpha)$. Section 3 contains our main theorems on variability regions with detailed proofs, and in Section 4, we conclude our work.

2. Basic properties of $W_\lambda(\zeta_0, r, s, \alpha)$

Proposition 2.1. *(i) The set $W_\lambda(\zeta_0, r, s, \alpha)$ forms a compact subset of the complex plane $\mathbb{C}$.*

(ii) The set $W_\lambda(\zeta_0, r, s, \alpha)$ forms a convex set of the complex plane $\mathbb{C}$.

(iii) If $|\lambda| = 1$ or $\zeta_0 = 0$, then,

$$W_\lambda(\zeta_0, r, s, \alpha) = \left\{\zeta_0 + \frac{(1-\alpha)(r-s)}{s}\left(\zeta_0 - \frac{1}{s\lambda}\log(1 + s\lambda\zeta_0)\right)\right\},$$

and if $|\lambda| < 1$ and $\zeta_0 \neq 0$, then,

$$\left\{\zeta_0 + \frac{(1-\alpha)(r-s)}{s}\left(\zeta_0 - \frac{1}{s\lambda}\log(1 + s\lambda\zeta_0)\right)\right\}$$

is the interior point of $W_\lambda(\zeta_0, r, s, \alpha)$.

Proof. (i) As $\mathcal{Q}_\lambda[r, s, \alpha]$ is a compact subset of $\mathbb{C}$, thus, $W_\lambda(\zeta_0, r, s, \alpha)$ is also compact.

(ii) Let $q_1, q_2 \in \mathcal{Q}_\lambda[r, s, \alpha]$. Then,

$$q(\zeta) = (1-t)\, q_1(\zeta) + t q_2(\zeta),\ t \in [0, 1]$$

is also in $\mathcal{Q}_\lambda[r, s, \alpha]$. Therefore, $W_\lambda(\zeta_0, r, s, \alpha)$ is convex.

(iii) Since if $|\lambda| = |\chi'_\phi(0)| = 1$, then, by use of The Shwarz lemma, we obtain $\chi_\phi(\zeta) = \lambda\zeta$, which yields

$$q(\zeta) = \frac{1 + \lambda\left((1-\alpha)\, r + \alpha s\right)\zeta}{1 + \lambda s\zeta}.$$

This implies that

$$q\left(\zeta\right) = 1 + \left(\frac{(1-\alpha)\,(r-s)}{s}\right)\left\{1 - \frac{1}{1 + s\lambda\zeta}\right\}.$$

Therefore,

$$W_\lambda\left(\zeta_0, r, s, \alpha\right) = \int_0^{\zeta_0} q\left(\delta\right) d\delta = \left\{\zeta_0 + \frac{(1-\alpha)\,(r-s)}{s}\left(\zeta_0 - \frac{1}{s\lambda}\log(1 + s\lambda\zeta_0)\right)\right\}.$$

This also trivially holds true when $\zeta_0 = 0$. For $\lambda \in \mathcal{D}$ and $\alpha \in \overline{\mathcal{D}}$, set

$$\delta(\zeta, \lambda) = \frac{\zeta + \lambda}{1 + \overline{\lambda}\zeta},$$

$$\int_0^{\zeta} G_{a,\lambda}\left(\sigma\right) d\sigma = \int_0^{\zeta} \frac{1 + \left((1-\alpha)\, r + \alpha s\right)\sigma\delta(a\sigma, \lambda)}{1 + s\sigma\delta(a\sigma, \lambda)} d\sigma,\ \zeta \in \mathcal{D}. \tag{2.1}$$

Then,

$$\int_0^{\zeta} G_{a,\lambda}\left(\sigma\right) d\sigma \text{ is in } Q_\lambda[r, s, \alpha],$$

and

$$\chi_\phi(\zeta) = \zeta\delta(a\zeta, \lambda).$$

For fixed $\lambda \in \mathcal{D}$ and $\zeta_0 \in \mathcal{D}\backslash\{0\}$, the function

$$\mathcal{D} \ni a \mapsto \int_0^{\zeta_0} G_{a,\lambda}\left(\sigma\right) d\sigma = \int_0^{\zeta_0} \frac{1 + (\overline{\lambda}a + ((1-\alpha)\, r + \alpha s)\,\lambda)\sigma + ((1-\alpha)\, r + \alpha s)\, a\sigma^2}{1 + (\overline{\lambda}a + s\lambda)\sigma + sa\sigma^2} d\sigma$$

is analytic and non-constant on $a \in \mathcal{D}$, implying it's an open mapping. Hence,

$$\int_0^{\zeta_0} G_{0,\lambda}\left(\sigma\right) d\sigma = \left\{\zeta_0 + \frac{(1-\alpha)\,(r-s)}{s}\left(\zeta_0 - \frac{1}{s\lambda}\log(1 + s\lambda\zeta_0)\right)\right\}$$

is an element of the interior of

$$\left\{\int_0^{\zeta_0} G_{a,\lambda}\left(\sigma\right) d\sigma : a \in \mathcal{D}\right\} \subset W_\lambda[\zeta_0, r, s, \alpha].$$

□

Remark 2.2. Proposition 2.1 is sufficient to find $W_\lambda[\zeta_0, r, s, \alpha]$ for $0 \leq \lambda < 1$ and $\zeta_0 \in \mathcal{D}\backslash\{0\}$.

3. Main results

In this part, we present and demonstrate several known and new results.

Lemma 3.1. *For $q \in \mathcal{Q}_\lambda[r, s, \alpha]$, we have*

$$|q(\zeta) - x(\zeta, \lambda, \alpha)| \leq y(\zeta, \lambda, \alpha), \ \zeta \in \mathcal{D}, \lambda \in \overline{\mathcal{D}}, \tag{3.1}$$

where

$$\begin{aligned} x(\zeta, \lambda, \alpha) &= \frac{(1 + \lambda((1-\alpha) r + \alpha s)\zeta)\left(1 + s\overline{\lambda\zeta}\right)}{1 - s^2|\zeta|^4 + 2s(1 - |\zeta|^2)\Re(\lambda\zeta) + |\lambda|^2|\zeta|^2(s^2 - 1)} \\ &- \frac{|\zeta|^2(\lambda + s\overline{\zeta})\left(\overline{\lambda} + ((1-\alpha) r + \alpha s)\zeta\right)}{1 - s^2|\zeta|^4 + 2s(1 - |\zeta|^2)\Re(\lambda\zeta) + |\lambda|^2|\zeta|^2(s^2 - 1)} \end{aligned} \tag{3.2}$$

and

$$y(\zeta, \lambda, \alpha) = \frac{|((1-\alpha) r + \alpha s) - s|(1 - |\lambda|^2)|\zeta|^2}{1 - s^2|\zeta|^4 + 2s(1 - |\zeta|^2)\Re(\lambda\zeta) + |\lambda|^2|\zeta|^2(s^2 - 1)}. \tag{3.3}$$

The inequality is attained for $\zeta_0 \in \mathcal{D}\backslash\{0\}$ precisely when $\phi(\zeta) = G_{e^{i\theta},\lambda}(\zeta)$, so there exists a $\theta \in \mathbb{R}$.

Proof. Let $q \in \mathcal{Q}_\lambda[r, s, \alpha]$. Then, for some $\chi_q \in \mathcal{B}$, we have

$$\left|\frac{\frac{\chi_q(\zeta)}{\zeta} - \lambda}{1 - \overline{\lambda}\frac{\chi_q(\zeta)}{\zeta}}\right| \leq |\zeta|. \tag{3.4}$$

From (1.5), this can be expressed in the same way as

$$\left|\frac{q(\zeta) - b(\zeta, \lambda, \alpha)}{q(\zeta) + c(\zeta, \lambda, \alpha)}\right| \leq |\tau(\zeta, \lambda)|\,|\zeta|, \tag{3.5}$$

where

$$b(\zeta, \lambda, \alpha) = \frac{1 + \lambda((1-\alpha) r + \alpha s)\zeta}{1 + \lambda s\zeta}, \tag{3.6}$$

$$c(\zeta, \lambda, \alpha) = -\frac{\overline{\lambda} + ((1-\alpha) r + \alpha s)\zeta}{s\zeta + \overline{\lambda}} \tag{3.7}$$

and

$$\tau(\zeta, \lambda) = \frac{\overline{\lambda} + s\zeta}{1 + \lambda s\zeta}. \tag{3.8}$$

Elementary calculations yield that inequality (3.5) is equivalent to

$$\left|q(\zeta) - \frac{|\zeta|^2\,|\tau(\zeta, \lambda)|^2\,c(\zeta, \lambda, \alpha) + b(\zeta, \lambda, \alpha)}{1 - |\zeta|^2\,|\tau(\zeta, \lambda)|^2}\right| \leq \frac{|\zeta|\,|\tau(\zeta, \lambda)|\,|c(\zeta, \lambda, \alpha) + b(\zeta, \lambda, \alpha)|}{1 - |\zeta|^2\,|\tau(\zeta, \lambda, \alpha)|^2}. \tag{3.9}$$

Now, we have

$$1 - |\zeta|^2\,|\tau(\zeta, \lambda)|^2 = \frac{1 - s^2|\zeta|^4 + 2s(1 - |\zeta|^2)\Re(\lambda\zeta) + |\lambda|^2|\zeta|^2(s^2 - 1)}{|1 + \lambda\zeta s|^2},$$

$$c(\zeta,\lambda,\alpha)+b(\zeta,\lambda,\alpha) = \frac{(s-((1-\alpha)r+\alpha s))(1-|\lambda|^2)\zeta}{(1+\lambda s\zeta)(s\zeta+\overline{\lambda})}$$

and

$$b(\zeta,\lambda,\alpha)+|\zeta|^2|\tau(\zeta,\lambda)|^2c(\zeta,\lambda,\alpha) = \frac{(1+\lambda((1-\alpha)r+\alpha s)\zeta)\left(1+s\overline{\lambda\zeta}\right)}{|1+\lambda\zeta s|^2} - \frac{|\zeta|^2\left(\lambda+s\overline{\zeta}\right)\left(\overline{\lambda}+((1-\alpha)r+\alpha s)\zeta\right)}{|1+\lambda\zeta s|^2}.$$

Direct computation yields

$$x(\zeta,\lambda,\alpha) = \frac{|\zeta|^2|\tau(\zeta,\lambda)|^2c(\zeta,\lambda,\alpha)+b(\zeta,\lambda,\alpha)}{1-|\zeta|^2|\tau(\zeta,\lambda)|^2}$$

and

$$y(\zeta,\lambda,\alpha) = \frac{|\zeta||\tau(\zeta,\lambda)||c(\zeta,\lambda,\alpha)+b(\zeta,\lambda,\alpha)|}{1-|\zeta|^2|\tau(\zeta,\lambda)|^2}.$$

These relations and (3.9) yield (3.1). The equality case in (3.1) is attained when $q = G_{i\theta,\lambda}(\zeta)$, for some $\zeta \in \mathcal{D}$. Conversely, equality in (3.1) for some $\zeta \in \mathcal{D}\backslash\{0\}$, implies equality in (3.4). Hence, Schwarz's lemma implies the existence of $\theta \in \mathbb{R}$, such that

$$\chi_q(\zeta) = \zeta\delta(e^{i\theta}\zeta,\lambda), \forall\zeta \in \mathcal{D}.$$

It follows that

$$Q = G_{i\theta,\lambda}.$$

□

Remark 3.2. The geometric interpretation of Lemma 3.1 is that q is contained within the closed disk of radius $y(\zeta,\lambda,\alpha)$ and centered at $x(\zeta,\lambda,\alpha)$.

In the case $\lambda = 0$, we obtain the following.

Corollary 3.3. *For $q \in Q_0[r,s,\alpha]$, we have*

$$\left|q(\zeta) - \frac{1-((1-\alpha)r+\alpha s)s|\zeta|^4}{1-s^2|\zeta|^4}\right| \le \frac{(((1-\alpha)r+\alpha s)-s)|\zeta|^2}{1-s^2|\zeta|^4},\ \zeta \in \mathcal{D}\backslash\{0\}.$$

The equality case is attained if and only if $q = G_{i\theta,0}$.

Theorem 3.4. *Let C^1- be a curve in $\mathcal{D}$ with $\zeta(0) = 0$ and $\zeta(1) = \zeta_0$ and $\gamma : \zeta(t), 0 \le t \le 1$. Then,*

$$W_\lambda(\zeta_0,r,s,\alpha) \subset \{\chi \in \mathbb{C} : |\chi - Q(\lambda,\gamma,\alpha)| \le R(\lambda,\gamma,\alpha)\},$$

where

$$Q(\lambda,\gamma,\alpha) = \int_0^1 x(\zeta(t),\lambda,\alpha)D(t), \quad R(\lambda,\gamma,\alpha) = \int_0^1 y(\zeta(t),\lambda,\alpha)D(t),$$

and $D(t) = \zeta'(t)dt$.

Proof. As $q \in \mathcal{Q}_\lambda[r, s, \alpha]$, we can apply Lemma 3.1, to obtain

$$\begin{aligned} \left| \int_0^1 q(\zeta(t)) D(t) - Q(\lambda, \gamma, \alpha) \right| &= \left| \int_0^1 q(\zeta(t)) D(t) - \int_0^1 x(\zeta(t), \lambda, \alpha) D(t) \right| \\ &= \left| \int_0^1 (q(\zeta(t)) - x(\zeta(t), \lambda, \alpha)) D(t) \right|. \end{aligned} \tag{3.10}$$

Using $D(t) = \zeta'(t)dt$ in (3.10), we have

$$\left| \int_0^1 q(\zeta(t)) \zeta'(t)dt - Q(\lambda, \gamma, \alpha) \right| \leq \int_0^1 y(\zeta(t), \lambda, \alpha)|\zeta'(t)|dt = R(\lambda, \gamma, \alpha).$$

This establishes the result. □

The next result relies on the following lemma.

Lemma 3.5. *For* $|\lambda| < 1$ *and* $\theta \in \mathbb{R}$, *the function*

$$Y(\zeta) = \int_0^\zeta \frac{e^{i\theta}\xi^2}{\left(1 + (\bar{\lambda}e^{i\theta} + s\lambda)\xi + se^{i\theta}\xi^2\right)^2} d\xi, \quad \zeta \in \mathcal{D},$$

has zeros of multiplicity 3 at the origin and no zero in $\mathcal{D}$. Furthermore, there exists a normalized starlike univalent function s in $\mathcal{D}$, such that $Y(\zeta) = 3^{-1}e^{i\theta}s^3(\zeta)$.

This lemma is a result of work by Ponnusamy et al., as reported in [6].

Theorem 3.6. *Let* $\theta \in (-\pi, \pi]$, $\zeta_0 \in \mathcal{D}\backslash\{0\}$. *Then,* $\int_0^{\zeta_0} G_{e^{i\theta},\lambda}(\sigma)\, d\sigma \in \partial W_\lambda(\zeta_0, r, s, \alpha)$. *Moreover ,*

$$\int_0^{\zeta_0} q(\delta)\, d\delta = \int_0^{\zeta_0} G_{e^{i\theta},\lambda}(\sigma)\, d\sigma$$

implies $q = G_{e^{i\theta},\lambda}$ for some $q \in \mathcal{Q}_\lambda[r, s, \alpha]$ and $\theta \in (-\pi, \pi]$.

Proof. It follows from (2.1) that

$$\begin{aligned} G_{a,\lambda}(\zeta) &= \frac{1 + ((1-\alpha)r + \alpha s)\zeta\delta(a\zeta, \lambda)}{1 + s\zeta\delta(a\zeta, \lambda)} \\ &= \frac{1 + (\bar{\lambda}a + ((1-\alpha)r + \alpha s)\lambda)\zeta + ((1-\alpha)r + \alpha s)a\zeta^2}{1 + (\bar{\lambda}a + s\lambda)\zeta + sa\zeta^2}. \end{aligned}$$

Thus, from (3.6), (3.7), and (3.8), it follows that

$$G_{a,\lambda}(\zeta) - b(\zeta, \lambda, \alpha) = \frac{(((1-\alpha)r + \alpha s) - s)(1 - |\lambda|^2)a\zeta^2}{\left(1 + (\bar{\lambda}a + s\lambda)\zeta + sa\zeta^2\right)(1 + \lambda s\zeta)},$$

and

$$G_{a,\lambda}(\zeta) + c(\zeta,\lambda,\alpha) = \frac{(s-((1-\alpha)r+\alpha s))(1-|\lambda|^2)\zeta}{\left(1+(\overline{\lambda}a+s\lambda)\zeta+sa\zeta^2\right)(s\zeta+\overline{\lambda})}.$$

Therefore,

$$\begin{aligned}
& G_{a,\lambda}(\zeta) - x(\zeta,\lambda,\alpha) \\
= \; & G_{a,\lambda}(\zeta) - \frac{b(\zeta,\lambda,\alpha)+c(\zeta,\lambda,\alpha)|\zeta|^2\,|\tau(\zeta,\lambda)|^2}{1-|\zeta|^2\,|\tau(\zeta,\lambda)|^2} \\
= \; & \frac{1}{1-|\zeta|^2\,|\tau(\zeta,\lambda)|^2}\left[G_{a,\lambda}(\zeta) - b(\zeta,\lambda,\alpha) - |\zeta|^2\,|\tau(\zeta,\lambda)|^2\left(G_{a,\lambda}(\zeta)+c(\zeta,\lambda,\alpha)\right)\right] \\
= \; & \frac{(((1-\alpha)r+\alpha s)-s)(1-|\lambda|^2)\left[a\zeta(1+s\overline{\lambda\zeta})+|\zeta|^2(s\overline{\zeta}+\lambda)\right]}{(1-s^2|\zeta|^4+2s(1-|\zeta|^2)\Re(\lambda\zeta)+|\lambda|^2|\zeta|^2(s^2-1))\left[1+(\overline{\lambda}a+s\lambda)\zeta+sa\zeta^2\right]}.
\end{aligned}$$

Putting $a = e^{i\theta}$, we obtain

$$\begin{aligned}
& G_{e^{i\theta},\lambda}(\zeta) - x(\zeta,\lambda,\alpha) \\
= \; & \frac{r(\zeta,\lambda,\alpha)e^{i\theta}\zeta^2}{|\zeta|^2}\frac{\left(1+(\overline{\lambda}e^{i\theta}+s\lambda)\zeta+se^{i\theta}\zeta^2\right)\overline{\left(1+(\overline{\lambda}e^{i\theta}+s\lambda)\zeta+se^{i\theta}\zeta^2\right)}}{\left(1+(\overline{\lambda}e^{i\theta}+s\lambda)\zeta+se^{i\theta}\zeta^2\right)^2} \\
= \; & \frac{r(\zeta,\lambda,\alpha)e^{i\theta}\zeta^2}{|\zeta|^2}\frac{\left|1+(\overline{\lambda}e^{i\theta}+s\lambda)\zeta+se^{i\theta}\zeta^2\right|^2}{\left(1+(\overline{\lambda}e^{i\theta}+s\lambda)\zeta+se^{i\theta}\zeta^2\right)^2}.
\end{aligned}$$

Now, using $Y(\zeta)$ defined in Lemma 3.5, it follows that

$$G_{e^{i\theta},\lambda}(\zeta) - x(\zeta,\lambda,\alpha) = y(\zeta,\lambda,\alpha)\frac{Y'(\zeta)}{|Y'(\zeta)|}. \tag{3.11}$$

As in Lemma 3.5, we can write $Y = 3^{-1}e^{i\theta}s^3$, where s is starlike in $\mathcal{D}$ with $s(0) = s'(0) - 1 = 0$, and for $\forall\ \zeta_0 \in \mathcal{D}\backslash\{0\}$, the linear segment joining 0 and $s(\zeta_0)$ is entirely in $s(\mathcal{D})$. Consider the curve γ_0 given by

$$\gamma_0 : \zeta(t) = s^{-1}\left(ts(\zeta_0)\right), t \in [0,1].$$

Since

$$Y(\zeta(t)) = 2^{-1}e^{i\theta}\left(s(\zeta(t))\right)^2 = 3^{-1}e^{i\theta}\left(ts(\zeta_0)\right)^3 = t^3Y(\zeta_0)$$

and differentiation w.r.t t, we get

$$Y^{'}(\zeta(t))\zeta^{'}(t) = 3t^2Y(\zeta_0), \quad t \in [0,1]. \tag{3.12}$$

This, along with (3.11), yields

$$\int_0^{\zeta_0} G_{e^{i\theta},\lambda}(\sigma)\,d\sigma - Q(\lambda,\gamma_0,\alpha) = \int_0^1 \left(G_{e^{i\theta},\lambda}(\zeta(t)) - x(\zeta(t),\lambda,\alpha)\right)\zeta'(t)dt$$

$$= \int_0^1 y(\zeta(t),\lambda,\alpha)\frac{Y'(\zeta(t))\zeta'(t)}{|Y'(\zeta(t))\zeta'(t)|}|\zeta'(t)|dt$$
$$= \frac{Y(\zeta_0)}{|Y(\zeta_0)|}\int_0^1 y(\zeta(t),\lambda,\alpha)|\zeta'(t)|dt$$
$$= \frac{Y(\zeta_0)}{|Y(\zeta_0)|}R(\lambda,\gamma_0,\alpha). \quad (3.13)$$

This implies that

$$\int_0^{\zeta_0} G_{e^{i\theta},\lambda}(\sigma)\,d\sigma \in \partial\overline{\mathcal{D}}\left(Q(\lambda,\gamma_0,\alpha),R(\lambda,\gamma_0,\alpha)\right),$$

where $Q(\lambda,\gamma_0,\alpha)$ and $R(\lambda,\gamma_0,\alpha)$ are specified as in Corollary 3.4. By Corollary 3.4, it follows that

$$\int_0^{\zeta_0} G_{e^{i\theta},\lambda}(\sigma)\,d\sigma \in \partial W_\lambda(\zeta_0,r,s,\alpha).$$

To establish uniqueness, let us assume that

$$\int_0^{\zeta_0} q(\sigma)\,d\sigma = \int_0^{\zeta_0} G_{e^{i\theta},\lambda}(\sigma)\,d\sigma,$$

$\exists\,\theta \in (-\pi,\pi]$, and $q \in \mathcal{Q}_\lambda[r,s,\alpha]$. Suppose

$$h(t) = \frac{\overline{Y(\zeta_0)}}{|Y(\zeta_0)|}\left(q(\zeta(t)) - x(\zeta(t),\lambda,\alpha)\right)\zeta'(t),$$

where γ_0 is parameterized by $\gamma_0 : \zeta(t),\ 0 \le t \le 1$. Then, the continuity of h follows, and

$$|h(t)| = \frac{\left|\overline{Y(\zeta_0)}\right|}{|Y(\zeta_0)|}|\zeta'(t)|\left|(q(\zeta(t)) - x(\zeta(t),\lambda,\alpha))\right|.$$

Applying Theorem 3.1, we obtain

$$|h(t)| \le y(\zeta(t),\lambda,\alpha)|\zeta'(t)|.$$

Moreover, (3.13), implies

$$\int_0^1 \Re h(t)dt = \int_0^1 \Re\left[\frac{\overline{Y(\zeta_0)}}{|Y(\zeta_0)|}\left(q(\zeta(t)) - x(\zeta(t),\lambda,\alpha)\right)\zeta'(t)\right]dt$$
$$= \Re\left[\frac{\overline{Y(\zeta_0)}}{|Y(\zeta_0)|}\int_0^{\zeta_0}\{G_{e^{i\theta},\lambda}(\sigma)\,d\sigma - Q(\zeta(t),\lambda,\alpha)\}\right]$$

$$= \int_0^1 \Re y(\zeta(t), \lambda, \alpha)|\zeta'(t)|dt.$$

Thus,

$$h(t) = y(\zeta(t), \lambda, \alpha)|\zeta'(t)|, \ \forall\, t \in [0, 1].$$

From (3.11) and (3.12), this implies that $\int_0^{\zeta_0} q(\sigma)\, d\sigma = \int_0^{\zeta_0} G_{e^{i\theta},\lambda}(\sigma)\, d\sigma$ on γ_0. For analytic functions, the identity theorem gives us $q = G_{e^{i\theta},\lambda}$, $\zeta \in \mathcal{D}$. □

Remark 3.7. Theorem 3.6, is essential to prove the Theorem 3.8.

Theorem 3.8. *Assume $\lambda \in \mathcal{D}$ and $\zeta_0 \in \mathcal{D}\backslash\{0\}$. Thus, $\partial W_\lambda(\zeta_0, r, s, \alpha)$ is the Jordan curve described as:*

$$(-\pi, \pi] \ni \theta \mapsto \int_0^{\zeta_0} G_{e^{i\theta},\lambda}(\sigma)\, d\sigma = \int_0^{\zeta_0} \frac{1 + ((1-\alpha)\, r + \alpha s)\, \sigma\delta(e^{i\theta}\sigma, \lambda)}{1 + s\sigma\delta(e^{i\theta}\sigma, \lambda)} d\sigma.$$

If $\int_0^{\zeta_0} q(\sigma)\, d\sigma = \int_0^{\zeta_0} G_{e^{i\theta},\lambda}(\sigma)\, d\sigma$ for some $q \in \mathcal{Q}_\lambda[r, s, \alpha]$ and $\theta \in (-\pi, \pi]$, then $q(\zeta) = G_{e^{i\theta},\lambda}(\zeta)$.

Proof. First, we have to show that the curve

$$(-\pi, \pi] \ni \theta \mapsto \int_0^{\zeta_0} G_{e^{i\theta},\lambda}(\sigma)\, d\sigma = \int_0^{\zeta_0} \frac{1 + ((1-\alpha)\, r + \alpha s)\, \sigma\delta(e^{i\theta}\sigma, \lambda)}{1 + s\sigma\delta(e^{i\theta}\sigma, \lambda)} d\sigma$$

is simple. Let us assume that

$$\int_0^{\zeta_0} G_{e^{i\theta_1},\lambda}(\sigma)\, d\sigma = \int_0^{\zeta_0} G_{e^{i\theta_2},\lambda}(\sigma)\, d\sigma,$$

$$\int_0^{\zeta_0} \frac{1 + ((1-\alpha)\, r + \alpha s)\, \sigma\delta(e^{i\theta_1}\sigma, \lambda)}{1 + s\sigma\delta(e^{i\theta_1}\sigma, \lambda)} d\sigma = \int_0^{\zeta_0} \frac{1 + ((1-\alpha)\, r + \alpha s)\, \sigma\delta(e^{i\theta_2}\sigma, \lambda)}{1 + s\sigma\delta(e^{i\theta_2}\sigma, \lambda)} d\sigma,$$

$\exists\, \theta_1, \theta_2 \in (-\pi, \pi]$, and $\theta_1 \neq \theta_2$. Then, Theorem 3.6 yields

$$G_{e^{i\theta_1},\lambda}(\zeta_0) = G_{e^{i\theta_2},\lambda}(\zeta_0),$$

implying the relation

$$\tau\left(\frac{\chi_{G_{e^{i\theta_1},\lambda}}(\zeta)}{\zeta}, \lambda\right) = \tau\left(\frac{\chi_{G_{e^{i\theta_2},\lambda}}(\zeta)}{\zeta}, \lambda\right).$$

Consequently,

$$\frac{s(\zeta e^{i\theta_1} + \lambda) + \overline{\lambda}(1 + \overline{\lambda} e^{i\theta_1}\zeta)}{1 + \overline{\lambda} e^{i\theta_1}\zeta + \lambda s(\zeta e^{i\theta_1} + \lambda)} = \frac{s(\zeta e^{i\theta_2} + \lambda) + \overline{\lambda}(1 + \overline{\lambda} e^{i\theta_2}\zeta)}{1 + \overline{\lambda} e^{i\theta_2}\zeta + \lambda s(\zeta e^{i\theta_2} + \lambda)}.$$

After some simplification, we get $\zeta e^{i\theta_1} = \zeta e^{i\theta_2}$, giving a contradiction. It follows that the curve is simple. $W_\lambda(\zeta_0, r, s, \alpha)$ is compact convex subset of $\mathbb{C}$ with a non-empty interior, therefore, $\partial W_\lambda(\zeta_0, r, s, \alpha)$ forms a simple closed curve. From Theorem 3.6, the boundary $\partial W_\lambda(\zeta_0, r, s, \alpha)$ consists of the curve $(-\pi, \pi] \ni \theta \mapsto \int_0^{\zeta_0} G_{e^{i\theta},\lambda}(\sigma)\, d\sigma$. The simple closed curve cannot contain any simple closed curve other than itself. Thus, $\partial W_\lambda(\zeta_0, r, s, \alpha)$ is given by

$$(-\pi, \pi] \ni \theta \mapsto \int_0^{\zeta_0} G_{e^{i\theta},\lambda}(\sigma)\, d\sigma = \int_0^{\zeta_0} \frac{1 + ((1-\alpha)\, r + \alpha s)\, \sigma\delta(e^{i\theta}\sigma, \lambda)}{1 + s\sigma\delta(e^{i\theta}\sigma, \lambda)} d\sigma.$$

□

4. Conclusions

In this work, we have successfully determined the regions of variability $W_\lambda(\zeta_0, r, s, \alpha)$ for the integral functional $\int_0^{z_0} q(\rho) d\rho$ when q belongs to the class $Q_\lambda[r, s, \alpha]$. Our analysis provides explicit characterizations of these variability regions for functions in the broader class $Q[r, s, \alpha]$ of analytic functions with prescribed boundary behavior in the unit disk.

The main results extend classical variability theory to the families of analytic functions with specific geometric properties. The derived regions offer new understanding of the geometric properties of integral functionals over function classes and provide tools for optimization problems in complex analysis.

The corollary results for alternate parameter sets demonstrate the applicability of our approach and suggest potential applications in univalent function theory, geometric function theory, and related areas of complex analysis. These findings contribute to the broader understanding of extremal problems for analytic functions subject to geometric conditions.

Author contributions

All authors of this article have been contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no competing interests.

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