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Research article

Existence of positive solutions for a class of nonlinear elliptic equations with Hardy potential

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Abstract: This paper is concerned with a class of nonlinear elliptic equations with a generalized Hardy potential

$$-\Delta u = \frac{\lambda}{|x|^{\alpha}}u - b(x)g(u),\ \ x \in \Omega\backslash\{0\},$$

where $\alpha > 0$, $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded smooth domain containing the origin. We establish the existence, nonexistence, and asymptotic behavior of positive solutions. When the potential function $|x|^{-\alpha}$ has strong singularity at the origin, we obtain some qualitative properties of positive solutions.

Keywords: semilinear elliptic differential equation; upper and lower solutions; blow-up problem; uniqueness; asymptotic behavior
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1. Introduction

Partial differential equations are extremely important as a crucial branch of modern mathematics, such as elliptic-type equations and their direct application to nonlinear fluctuation equations [1], the theory of electromagnetism [2, 3], dynamics [4–7], the fields of finance and quantum mechanics [8], and gravitational lensing effects [9], as well as computational science and engineering.

In recent years, elliptic equations with Hardy potential have attracted the attention of many scholars, and this kind of research has important practical and theoretical significance in the fields of heat conduction theory [10], fluid dynamics, and so on. Many excellent conclusions have been obtained on the existence, multiplicity, regularity and asymptotic behavior of their solutions. The study of single equations could be found in [11–15], among them, the authors deals with the Laplacian case in [11, 12]; In [13], the authors considered the case of the P-Laplacian equation with the weighted function; and in [14, 15], the authors deals with a more general quasilinear elliptic problems. The case of elliptic differential systems was investigated in [16–18] and so on.

In [19], F.C. Cîrstea et al. studied the following nonlinear elliptic equations with Hardy potential

$$-\Delta u = \frac{\lambda}{|x|^2}u - |x|^\sigma u^q,\ x \in \Omega\backslash\{0\} \tag{1.1}$$

and obtained a complete classification of the singularities of all their positive solutions when $\lambda \le H := \left(\frac{N-2}{2}\right)^2$. It is well known that H is the classical Hardy constant [20]. In [21], Wei studied the exact singularity of all positive solutions when $\lambda > H$ and obtained the following theorem.

Theorem 1.1. *Suppose that $\lambda > H$ and $u(x)$ is an arbitrary positive solution of (1.1). Then*

$$\lim_{x\to 0} |x|^{2+\sigma} u^{p-1}(x) = l,$$

where

$$l = \lambda + \frac{2+\sigma}{p-1}\left(\frac{2+\sigma}{p-1} + 2 - N\right).$$

Remark. For comparison purposes, we have made appropriate equivalence modifications to the conclusions of Theorem 1.1 in the original paper [21]. A similar approach has been applied to the next theorem.

In [22], Cheng et al. studied the following nonlinear elliptic equations:

$$-\Delta u = \frac{\lambda}{|x|^\alpha}u - |x|^\sigma u^p,\ x \in \Omega\backslash\{0\}, \tag{1.2}$$

and obtained the existence, nonexistence, and asymptotic behavior of the positive solutions. One of their main results can be stated here:

Theorem 1.2. *Suppose that $\alpha > 2$, $\theta + \alpha > 0$, and $p > 1$. Then the following two statements are true:*

(*i*) *If $\lambda < 0$, then equation* (1.2) *with the zero Dirichlet boundary condition has no positive solutions.*

(*ii*) *If $\lambda > 0$, then equation* (1.2) *with the zero Dirichlet boundary condition has a unique positive solution $u(x)$ satisfying*

$$\lim_{|x|\to 0^+} |x|^{\alpha+\theta} u^{p-1}(x) = \lambda. \tag{1.3}$$

We should point out that the conclusion (*ii*) in Theorem 1.2 requires a slight modification: $\lambda > 0$ should be changed to $\lambda > 0$ being sufficiently large, because in the proof of Proposition 2.2 in their article (see [22] for details), the condition $\lambda > 0$ being sufficiently large is necessary.

A natural question is if the conclusion (*ii*) in Theorem 1.2 holds, does there exist a more precise lower bound $\lambda_* \ge 0$ for λ such that when $\lambda > \lambda_*$, (1.3) holds? This inspires us for further in-depth research on this problem.

It is also worth noting that although equations (1.1) and (1.2) differ only in the use of the parameter α and the number 2, from Theorem 1.1 and Theorem 1.2, we can find that their conclusions are fundamentally distinct: The asymptotic limit of one depends not only on the parameter λ but also on parameters p and σ , and the space dimension N, while the other depends solely on λ. In a sense, $\alpha = 2$ constitutes a critical value.

In this paper, we consider the elliptic equation with a generalized Hardy potential

$$-\Delta u = \frac{\lambda}{|x|^\alpha}u - b(x)g(u),\ x \in \Omega\backslash\{0\}, \tag{1.4}$$

where $\lambda \in \mathbb{R}$, $\alpha > 0$, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $0 \in \Omega$.

For convenience, we briefly record the following conditions, respectively. For the restrictions on $b(x)$ and $g(u)$, the readers can refer to the literature [23].

(b_1) $b(x)$ is a positive continuous function in $\Omega\backslash\{0\}$.

(b_2) There exist constants θ and $m > 0$ such that

$$\lim_{|x|\to 0} \frac{b(x)}{|x|^\theta} = m.$$

(g_1) $g(0) = 0$ and $\frac{g(u)}{u}$ is an increasing function.

(g_2) There exist constants q and k, with $q > 1$ and $k > 0$, such that

$$\lim_{u\to\infty} \frac{g(u)}{u^q} = k.$$

(g_3) There exist constants p and s, with $p > 1$ and $s > 0$, such that

$$\lim_{u\to 0} \frac{g(u)}{u^p} = s.$$

Our main results are stated as follows.

Theorem 1.3. *Suppose that $b(x)$ satisfies the conditions (b_1) and (b_2), $g(u)$ satisfies the conditions (g_1), (g_2) and (g_3), then we have:*

(*i*) *If $\alpha > 2$, $\theta+\alpha > 0$, $q > \frac{2(\alpha+\theta)}{N-2}+1$, and $\lambda \le 0$, then equation* (1.4) *with the zero Dirichlet boundary condition has no positive solution.*

(*ii*) *If $\alpha > 2$, $\theta + \alpha > 0$, and $\lambda > 0$, then equation* (1.4) *with the zero Dirichlet boundary condition has at least one positive solution. And if $\lambda > \frac{\alpha^\alpha}{4(\alpha-2)^{\alpha-2}}\lambda_1 [B_1(0)]$, then for any solutions $u(x)$ of* (1.4)*, there exist two positive constants C_3 and C_4 such that*

$$C_4|x|^{-\frac{\alpha+\theta}{q-1}} \le u(x) \le C_3|x|^{-\frac{\alpha+\theta}{q-1}}, \ \text{as } x \to 0.$$

That is, the equation (1.4) has Keller–Osserman solutions. Moreover, if $g(u)$ is convex in u for $u > 0$, then equation (1.4) with the zero Dirichlet boundary condition has a unique positive solution $u(x)$ and satisfies

$$\lim_{|x|\to 0^+} |x|^{\frac{\alpha+\theta}{q-1}} u(x) = \left(\frac{\lambda}{mk}\right)^{\frac{1}{q-1}}. \tag{1.5}$$

Remark. (1) Recall that if there exist constants $C_1, C_2 > 0$ such that

$$C_1|x|^{-\frac{\alpha+\theta}{q-1}} \le u(x) \le C_2|x|^{-\frac{\alpha+\theta}{q-1}},$$

where $x \in B_1(0)\backslash\{0\}$, then the positive solution $u(x)$ of equation (1.4) is called the Keller–Osserman solution.

(2) Under different parameter conditions on α and θ, we present a result on the formal lower bound λ_* of λ, which provides a relatively comprehensive answer to the question we previously raised. In addition, considering the limiting cases of the conditions (via a formal comparison) and comparing them with $\alpha = 2$, we find that the original condition $\lambda > H$ is replaced by $\lambda > \lambda_1 [B_1(0)]$. It is easy to see that $\lambda_1 [B_1(0)] > H$.

(3) As we all know that in quantum mechanics, the first eigenvalue of the $-\triangle$ operator represents the ground state energy of the system. Since equation (1.4) is a standing wave equation derived from the Schröinger equation with a Hardy potential, the condition $\lambda > \lambda_1\,[B_1(0)]$ in this theorem indicates that if the equation admits a unique ground state solution with positive energy (or, equivalently, if its ground state wave function u is positive), then the ground state energy λ_1 must have a positive lower bound, and the value of $\lambda_1\,[B_1(0)]$ can be interpreted as an estimate of this ground state energy.

Corollary 1.4. *Suppose that $b(x)$ satisfies the conditions (b_1) and (b_2) and $g(u)$ satisfies the conditions (g_1), (g_2), and (g_3). Moreover, assume that $g(u)$ is convex in u for $u > 0$. When $m = 1$, $k = 1$, $\alpha > 2$, and $\theta + \alpha > 0$, the following two statements are correct:*

(i) If $q > \frac{2(\alpha+\theta)}{N-2} + 1$ and $\lambda \le 0$, then equation (1.4) with the zero Dirichlet boundary condition has no positive solution.

(ii) If $\lambda > 0$ is sufficiently large, then equation (1.4) with the zero Dirichlet boundary condition has a unique positive solution $u(x)$ such that

$$\lim_{|x|\to 0^+} |x|^{\frac{\alpha+\theta}{q-1}} u(x) = \lambda^{\frac{1}{q-1}}. \tag{1.6}$$

Remark. The above corollary can be seen as a natural generalization of [22], as well as a useful complement to the findings of [21] (see [21] and [22] for details).

2. Preliminaries

To facilitate the study, a brief description of the notations and citations used in this paper is provided below:

$$\Omega^{\delta} = \{x \in \Omega,\ |x| > \delta\},\ \Omega^{\delta,\rho} = \{x \in \Omega,\ \delta < |x| < \rho\},$$
$$B_\delta(0) = \{x \in \mathbb{R}^N : |x| < \delta\}.$$

Lemma 2.1. *[24] Suppose that Ω is a bounded domain in $\mathbb{R}^N$, $\alpha(x)$ and $\beta(x)$ are continuous functions in Ω with $\|\alpha\|_{L^\infty(\Omega)} < \infty$, and $\beta(x)$ is non–negative and not identically zero. Let $u_1,\ u_2 \in C^1(\Omega)$ be positive in Ω and satisfy in the weak sense*

$$\Delta u_1 + \alpha(x)u_1 - \beta(x)g(u_1) \le 0 \le \Delta u_2 + \alpha(x)u_2 - \beta(x)g(u_2), \quad x \in \Omega, \tag{2.1}$$

and

$$\limsup_{x\to\partial\Omega}[u_2(x) - u_1(x)] \le 0,$$

where $g(u)$ is continuous and $g(u)/u$ is strictly increasing and nonnegative for u in the range $\min_\Omega\{u_1, u_2\} < u < \max_\Omega\{u_1, u_2\}$. Then $u_2 \le u_1$ in Ω.

Let $\lambda_1[c, b, \Omega]$ denote the first eigenvalue of

$$\begin{cases} -\Delta u + c(x)u = \lambda b(x)u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{2.2}$$

where $b(x), c(x) \in C(\overline{\Omega})$, and $b(x)$ is a positive function. In short, set

$$\lambda_1[b, \Omega] = \lambda_1[0, b, \Omega],\ \lambda_1(\Omega) = \lambda_1[0, 1, \Omega].$$

For the first eigenvalue $\lambda_1[c, b, \Omega]$, the following three lemmas hold true.

Lemma 2.2. *[25] Let $\lambda_1[c, b, \Omega]$ be defined as above.*

(i) If $b_1(x) \le b_2(x)$ in Ω, then $\lambda_1[c, b_2, \Omega] \le \lambda_1[c, b_1, \Omega]$ and the equality holds if and only if $b_1(x) = b_2(x)$.

(ii) If $c_1(x) \le c_2(x)$ in Ω, then $\lambda_1[c_1, b, \Omega] \le \lambda_1[c_2, b, \Omega]$ and the equality holds if and only if $c_1(x) = c_2(x)$.

(iii) If $0 < \delta_1 < \delta_2$ in Ω, then $\lambda_1[c, b, \Omega^{\delta_1}] \le \lambda_1[c, b, \Omega^{\delta_2}]$ and $\lambda_1[c, b, \Omega^{\delta}] \to \lambda_1[c, b, \Omega]$ as $\delta \to 0^+$.

Lemma 2.3. *[26] The equation*

$$\begin{cases} -\Delta u = \lambda D(x)u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{2.3}$$

has a positive strict supersolution if and only if

$$\lambda_1[D, \Omega] > \lambda.$$

Lemma 2.4. *[25] For ϵ, $\delta > 0$, we have*

(i) $\lim\limits_{\varepsilon\to 0}\lambda_1\left[\frac{1}{|x|^2+\epsilon}, \Omega\right] = H$,

(ii) $\lim\limits_{\varepsilon\to 0}\lambda_1\left[\frac{1}{|x|^2}, \Omega^{\delta}\right] = H$,

where $H = \frac{(N-2)^2}{4}$.

Now we give some lemmas that will be useful for further study.

Lemma 2.5. *Assume that $b(x)$ satisfies the conditions (b_1) and (b_2) and $g(u)$ satisfies the conditions (g_1), (g_2), and (g_3). Then for any $\lambda > 0$, $\alpha > 2$, and $\theta + \alpha > 0$, the following Dirichlet problem*

$$\begin{cases} -\Delta u - \lambda |x|^{-\alpha} u + b(x)g(u) = 0, & x \in \Omega\backslash\{0\}, \\ u > 0, & x \in \Omega\backslash\{0\}, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{2.4}$$

has a minimal positive solution and a maximal positive solution.

Proof. First, we prove that (2.4) has a maximal positive solution $U(x)$. For any $\delta > 0$, we consider the following equation:

$$\begin{cases} -\Delta u - \lambda |x|^{-\alpha} u + b(x)g(u) = 0, & x \in \Omega^{\delta}, \\ u = +\infty, & |x| = \delta, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{2.5}$$

By (b_1) and (b_2), when $x \in \Omega^{\delta}$, $b(x)$ is a positive continuous function, and it has a strictly positive infimum. Notice that $g(u)$ satisfies (g_1) and (g_2). From Theorem 6.18 in Du [26], we know that the equation (2.5) has a unique positive solution, and we denote it by $U_\delta(x)$, obviously, for any $0 \le \delta_1 \le \delta_2$, $U_{\delta_1}(x) = 0 = U_{\delta_2}(x)$ on $\partial\Omega$, and $U_{\delta_1}(x) < +\infty = U_{\delta_2}(x)$ on $|x| = \delta_2$. Then by Lemma 2.1, we have $U_{\delta_1}(x) \le U_{\delta_2}(x)$ for any $x \in \Omega^{\delta_2}$. So, $U_\delta(x)$ is increasing in δ. When $\delta \to 0^+$, $U_\delta(x)$ is monotonically decreasing and $U_\delta(x) \ge 0$, thus

$$U(x) = \lim_{\delta\to 0^+} U_\delta(x), \ x \in \Omega\backslash\{0\}.$$

By the regularity argument of the elliptic equation, $U(x)$ is a positive solution of (2.4). Suppose that w is an arbitrary positive solution of (2.4). When $x \in \partial\Omega$, we have $w(x) = U_\delta(x) = 0$. And for any $\delta > 0$, when $|x| = \delta$, $w(x) < U_\delta(x) = +\infty$. Thus, Lemma 2.1 implies that

$$w(x) \le U_\delta(x),\ x \in \Omega^\delta.$$

Let $\delta \to 0^+$, and we have

$$w(x) \le U(x),\ x \in \Omega \backslash \{0\}.$$

This shows that $U(x)$ is the maximal positive solution of (2.4).

Now we show the existence of the minimal positive solution of the problem (2.4). For any $\lambda > 0$, there exists $\rho_1 > 0$ small enough such that $\left[\frac{(N-2)^2}{4} + 1\right]\rho_1^{\alpha-2} < \lambda$. According to the condition (b_2), there exists $\rho_2 > 0$ such that $\frac{m}{2}|x|^\theta \le b(x) \le 2m|x|^\theta$ for $|x| < \rho_2$. According to Lemma 2.4, if we let $\rho = \min\{\rho_1, \rho_2\}$, we can find that $\delta_0 \in (0, \rho)$, such that for every $\delta \in (0, \delta_0]$,

$$\lambda_1\left[\frac{1}{|x|^2}, \Omega^{\delta,\rho}\right] \le \lambda_1\left[\frac{1}{|x|^2}, \Omega^{\delta_0,\rho}\right] \le \frac{(N-2)^2}{4} + 1. \tag{2.6}$$

We can get, for such δ and ρ,

$$\lambda_1\left[\frac{1}{|x|^2}, \Omega^{\delta,\rho}\right]\frac{1}{|x|^2} \le \left[\frac{(N-2)^2}{4} + 1\right]\frac{\rho^{\alpha-2}}{|x|^\alpha},\ \text{for } x \in \Omega^{\delta,\rho}. \tag{2.7}$$

Let $\phi_{\delta,\rho} > 0$ (in $\Omega^{\delta,\rho}$) be the eigenfunction corresponding to $\lambda_1\left[\frac{1}{|x|^2}, \Omega^{\delta,\rho}\right]$. Notice that $\alpha > 2$, then

$$-\Delta(\phi_{\delta,\rho}) = \lambda_1\left[\frac{1}{|x|^2}, \Omega^{\delta,\rho}\right]\frac{1}{|x|^2}\phi_{\delta,\rho} \le \left[\frac{(N-2)^2}{4} + 1\right]\frac{\rho^{\alpha-2}}{|x|^\alpha}\phi_{\delta,\rho} < \frac{\lambda}{|x|^\alpha}\phi_{\delta,\rho} \text{ in } \Omega^{\delta,\rho}.$$

Define

$$\underline{u}_{\delta,\varepsilon} = \begin{cases} \varepsilon\phi_{\delta,\rho}, & x \in \Omega^{\delta,\rho}, \\ 0, & x \in \Omega^\delta \backslash \Omega^{\delta,\rho}. \end{cases} \tag{2.8}$$

By (g_3), assuming $\varepsilon > 0$ is sufficiently small, the following two inequalities hold:

$$\varepsilon\left[\left(\frac{(N-2)^2}{4} + 1\right)\rho^{\alpha-2} - \lambda\right] + 4ms\varepsilon^p\rho^{\theta+\alpha}\phi_{\delta,\rho}^{p-1} \le 0,$$

$$\frac{s}{2}\underline{u}_{\delta,\varepsilon}^p \le g(\underline{u}_{\delta,\varepsilon}) \le 2s\underline{u}_{\delta,\varepsilon}^p \text{ for } x \in \Omega^{\delta,\rho}.$$

Through simple calculation, for any $x \in \Omega^{\delta,\rho}$, we have

$$\begin{aligned} -\Delta\underline{u}_{\delta,\varepsilon} - \tfrac{\lambda}{|x|^\alpha}\underline{u}_{\delta,\varepsilon} + b(x)g(\underline{u}_{\delta,\varepsilon}) &= -\Delta(\varepsilon\phi_{\delta,\rho}) - \tfrac{\lambda}{|x|^\alpha}(\varepsilon\phi_{\delta,\rho}) + b(x)g(\varepsilon\phi_{\delta,\rho}) \\ &\le \tfrac{\phi_{\delta,\rho}}{|x|^\alpha}\left\{\varepsilon\left[\left(\tfrac{(N-2)^2}{4} + 1\right)\rho^{\alpha-2} - \lambda\right] + 4ms\varepsilon^p\rho^{\theta+\alpha}\phi_{\delta,\rho}^{p-1}\right\}. \end{aligned}$$

Therefore

$$-\Delta\underline{u}_{\delta,\varepsilon} - \frac{\lambda}{|x|^\alpha}\underline{u}_{\delta,\varepsilon} + b(x)g(\underline{u}_{\delta,\varepsilon}) \le 0,\ x \in \Omega^{\delta,\rho}. \tag{2.9}$$

When $x \in \Omega^{\delta} \backslash \Omega^{\delta,\rho}$, by direct calculation,

$$-\Delta \underline{u}_{\delta,\varepsilon} - \frac{\lambda}{|x|^{\alpha}} \underline{u}_{\delta,\varepsilon} + b(x) g(\underline{u}_{\delta,\varepsilon}) = 0. \tag{2.10}$$

So, by (2.9) and (2.10), we have

$$-\Delta \underline{u}_{\delta,\varepsilon} - \frac{\lambda}{|x|^{\alpha}} \underline{u}_{\delta,\varepsilon} + b(x) g(\underline{u}_{\delta,\varepsilon}) \leq 0, \ x \in \Omega^{\delta}. \tag{2.11}$$

Define

$$\overline{u}_{\delta,M} = \begin{cases} M\phi_{\delta,\rho}, & x \in \Omega^{\delta,\rho}, \\ 0, & x \in \Omega^{\delta} \backslash \Omega^{\delta,\rho}. \end{cases} \tag{2.12}$$

By (g_2), when M is sufficiently large, we have

$$\frac{k}{2} \overline{u}_{\delta,M}^{q} \leq g(\overline{u}_{\delta,M}) \leq 2k \overline{u}_{\delta,M}^{q} \text{ for any } x \in \Omega^{\delta,\rho}$$

and

$$-\lambda M \phi_{\delta,\rho} + \frac{mk}{4} M^{q} \delta^{\theta+\alpha} \phi_{\delta,\rho}^{q} \geq 0 \text{ for any } x \in \Omega^{\delta,\rho}.$$

A quick calculation gives

$$\begin{aligned} -\Delta \overline{u}_{\delta,M} - \tfrac{\lambda}{|x|^{\alpha}} \overline{u}_{\delta,M} + b(x) g(\overline{u}_{\delta,M}) &= -\Delta(M\phi_{\delta,\rho}) - \tfrac{\lambda}{|x|^{\alpha}} (M\phi_{\delta,\rho}) + b(x) g(M\phi_{\delta,\rho}) \\ &\geq M\lambda_1 \left[\tfrac{1}{|x|^2}, \Omega^{\delta,\rho}\right] \tfrac{1}{|x|^2} \phi_{\delta,\rho} - \tfrac{\lambda}{|x|^{\alpha}} (M\phi_{\delta,\rho}) + \tfrac{mk}{4} M^{q} |x|^{\theta} \phi_{\delta,\rho}^{q} \\ &\geq \tfrac{1}{|x|^{\alpha}} \left[-\lambda M \phi_{\delta,\rho} + \tfrac{mk}{4} M^{q} \delta^{\theta+\alpha} \phi_{\delta,\rho}^{q}\right] \end{aligned}$$

for any $x \in \Omega^{\delta,\rho}$. Thus

$$-\Delta \overline{u}_{\delta,M} - \frac{\lambda}{|x|^{\alpha}} \overline{u}_{\delta,M} + b(x) g(\overline{u}_{\delta,M}) \geq 0, \ x \in \Omega^{\delta,\rho}. \tag{2.13}$$

When $x \in \Omega^{\delta} \backslash \Omega^{\delta,\rho}$, through rigorous computation,

$$-\Delta \overline{u}_{\delta,M} - \frac{\lambda}{|x|^{\alpha}} \overline{u}_{\delta,M} + b(x) g(\overline{u}_{\delta,M}) = 0. \tag{2.14}$$

So, by (2.13) and (2.14), we can obtain

$$-\Delta \overline{u}_{\delta,M} - \frac{\lambda}{|x|^{\alpha}} \overline{u}_{\delta,M} + b(x) g(\overline{u}_{\delta,M}) \geq 0, \ x \in \Omega^{\delta}. \tag{2.15}$$

From the above derivation, $\overline{u}_{\delta,M}(x)$ and $\underline{u}_{\delta,\varepsilon}(x)$ are the upper and lower solutions of the following equation:

$$\begin{cases} -\Delta u - \lambda |x|^{-\alpha} u + b(x) g(u) = 0, & x \in \Omega^{\delta}, \\ u = 0, & x \in \partial\Omega^{\delta}. \end{cases} \tag{2.16}$$

According to the upper and lower solution theorem [24] and Lemma 2.1, we know that equation (2.16) has a unique positive solution, denoted as $\underline{u}_{\delta}(x)$. So, we have $\underline{u}_{\delta,\varepsilon}(x) \leq \underline{u}_{\delta}(x) \leq \overline{u}_{\delta,M}(x)$. For any $0 \leq \delta_1 \leq \delta_2 < \rho$, note that for any $x \in \partial\Omega$, $\underline{u}_{\delta_1}(x) = 0 = \underline{u}_{\delta_2}(x)$. When $|x| = \delta_2$, $\underline{u}_{\delta_1}(x) > 0 = \underline{u}_{\delta_2}(x)$.

Thus, by Lemma 2.1, for any $x \in \Omega^{\delta_2}$, we have $\underline{u}_{\delta_1}(x) \geq \underline{u}_{\delta_2}(x)$. So, u_δ increases as δ decreases. When $\delta \to 0^+$, $u(x)$ is monotonically increasing with upper bound $U(x)$, so

$$\underline{u}(x) = \lim_{\delta \to 0^+} \underline{u}_\delta(x), \ x \in \Omega \backslash \{0\}.$$

By the regularity argument of the elliptic equation, $\underline{u}(x)$ is a solution of equation (2.4). Suppose that $v(x)$ is an arbitrary positive solution of (2.4). When $x \in \partial\Omega$, $v(x) = 0 = \underline{u}_\delta(x)$. When $|x| = \delta$, $v(x) > 0 = \underline{u}_\delta(x)$. Using Lemma 2.1, when $x \in \Omega^\delta$, we have $\underline{u}_\delta(x) \leq v(x)$. Letting $\delta \to 0^+$, we have

$$\underline{u}(x) \leq v(x), \ x \in \Omega \backslash \{0\}.$$

This shows that $\underline{u}(x)$ is the minimal positive solution of (2.4).

Lemma 2.6. *Suppose $b(x)$ satisfies the conditions (b_1) and (b_2) and $g(u)$ satisfies the conditions (g_1) and (g_2). For any $\alpha > 2$ and $\lambda > 0$, there exists a constant $C_3 > 0$ such that for any positive solution $u(x)$ of equation* (1.4),

$$u(x) \leq C_3 |x|^{\frac{-(\alpha+\theta)}{q-1}}, \quad \text{as } |x| \to 0. \tag{2.17}$$

Proof. According to Lemma 2.5, if $u(x)$ is a solution of equation (1.4), then $u(x) \leq U(x)$. In order to prove that all positive solutions $u(x)$ of equation (1.4) satisfy (2.17), it is sufficient to show that $U(x)$ satisfies (2.17) (replace $u(x)$ by $U(x)$ in (2.17)).

By (b_2) and (g_2), for any $\varepsilon > 0$, there exists $0 < \delta < 1$ such that $(m-\varepsilon)|x|^\theta \leq b(x) \leq (m+\varepsilon)|x|^\theta$ and $(k-\varepsilon)U(x)^q \leq g(U(x)) \leq (k+\varepsilon)U(x)^q$ for $x \in B_\delta(0)\backslash\{0\} \subset \Omega$.

For any $x_0 \in \Omega\backslash\{0\}$ and $|x_0| < \frac{\beta\delta}{\beta+1}$, where $\beta > 0$ is a constant and denote

$$D(x_0) = \left\{ x_0 + \frac{|x_0|^{\frac{\alpha}{2}}}{\beta} x : x \in B_1(0) \right\}.$$

Since $\alpha > 2$ and $\delta < 1$, a straightforward observation shows that

$$\frac{\beta-1}{\beta}|x_0| \leq \left| x_0 + \frac{|x_0|^{\frac{\alpha}{2}}}{\beta} x \right| \leq \frac{\beta+1}{\beta}|x_0|, \ x \in B_1(0).$$

From the above inequality, we can see that $D(x_0) \subset \Omega\backslash\{0\}$.

When $\theta \geq 0$ and $y \in D(x_0)$, we have

$$\begin{aligned} -\Delta U(y) &= \lambda |y|^{-\alpha} U(y) - b(y) g(U(y)) \\ &\leq \lambda |y|^{-\alpha} U(y) - (m-\varepsilon)(k-\varepsilon)|y|^\theta U(y)^q \\ &\leq \lambda \left(\tfrac{\beta-1}{\beta}\right)^{-\alpha} |x_0|^{-\alpha} U(y) - (m-\varepsilon)(k-\varepsilon) \cdot \left(\tfrac{\beta-1}{\beta}\right)^\theta |x_0|^\theta U(y)^q. \end{aligned}$$

Define

$$V(x) := |x_0|^{\frac{\alpha+\theta}{q-1}} U\left(x_0 + \frac{|x_0|^{\frac{\alpha}{2}}}{\beta} x \right), \ x \in B_1(0).$$

It is easy to see that

$$\begin{aligned} -\Delta V(x) &= -\beta^{-2} |x_0|^{\alpha + \frac{\alpha+\theta}{q-1}} \Delta U\left(x_0 + \tfrac{|x_0|^{\frac{\alpha}{2}}}{\beta} x \right) \\ &\leq \lambda \cdot \beta^{-2} \left(\tfrac{\beta-1}{\beta}\right)^{-\alpha} |x_0|^{\frac{\alpha+\theta}{q-1}} U\left(x_0 + \tfrac{|x_0|^{\frac{\alpha}{2}}}{\beta} x \right) - (m-\varepsilon)(k-\varepsilon) \cdot \beta^{-2} \left(\tfrac{\beta-1}{\beta}\right)^\theta |x_0|^{\frac{\alpha+\theta}{q-1} \cdot q} U^q \left(x_0 + \tfrac{|x_0|^{\frac{\alpha}{2}}}{\beta} x \right) \\ &= \lambda \cdot \beta^{-2} \left(\tfrac{\beta-1}{\beta}\right)^{-\alpha} V(x) - (m-\varepsilon)(k-\varepsilon) \cdot \beta^{-2} \left(\tfrac{\beta-1}{\beta}\right)^\theta V(x)^q. \end{aligned}$$

We now analyze the fundamental equation:

$$\begin{cases} -\Delta W_1 = \lambda \cdot \beta^{-2} \left(\frac{\beta-1}{\beta}\right)^{-\alpha} W_1 - (m-\varepsilon)(k-\varepsilon) \cdot \beta^{-2} \left(\frac{\beta-1}{\beta}\right)^{\theta} W_1^q, & x \in B_1(0), \\ W_1 = +\infty, & x \in \partial B_1(0). \end{cases} \tag{2.18}$$

By Theorem 6.15 in [26], the equation (2.18) has a unique positive solution $W_1(x)$. By Lemma 2.1, we have $V(x) \leq W_1(x)$ for any $x \in B_1(0)$. So, we get $|x_0|^{\frac{\alpha+\theta}{q-1}} U(x_0) \leq W_1(0)$.

That is,

$$U(x_0) \leq W_1(0)|x_0|^{-\frac{\alpha+\theta}{q-1}}.$$

By the arbitrariness of x_0, we know that there exist positive constants C_3 and δ_0 such that

$$U(x) \leq C_3|x|^{-\frac{\alpha+\theta}{q-1}}, \ \forall x \in B_{\delta_0}(0)\backslash\{0\},$$

which shows that when $\theta \geq 0$, $u(x)$ satisfies (2.17).

Similarly, we can prove that when $-\alpha < \theta < 0$, $u(x)$ also satisfies (2.17). This completes the proof.

Next, we will explore the blow-up behavior of a positive solution at the origin by considering the following ordinary differential equation(ODE):

$$-u'' - \frac{N-1}{r}u' = \lambda\frac{u}{r^\alpha} - b(r)g(u), \ r \in (0, \infty). \tag{2.19}$$

Lemma 2.7. *Suppose $b(x)$ is a radial function and satisfies the conditions (b_1) and (b_2) and $g(u)$ satisfies the conditions (g_1) and (g_2). Also suppose that $\lambda > H$, $\alpha > 2$, and $\theta + \alpha > 0$. Then there exists a small $\rho > 0$ such that $\underline{u}'(r) \leq 0$ in $(0, \rho)$ and $\underline{u}(x)$ blows up at the origin, where $\underline{u}(x)$ is the minimal positive solution $\underline{u}(x)$ of equation (2.4).*

Proof. From (b_2), for any $0 < \varepsilon < m$, there exists a $\tau(\varepsilon)(0 < \tau(\varepsilon) < 1)$, which is sufficiently small such that

$$(m-\epsilon)|x|^\theta \leq b(x) \leq (m+\epsilon)|x|^\theta, \text{ for } 0 < |x| < \tau(\varepsilon).$$

For any $0 < \rho < \tau(\varepsilon)$, let $\underline{u}_{\delta,\rho}(x)$ be a unique positive solution of the following equation:

$$\begin{cases} -\Delta u = \lambda\frac{u}{|x|^\alpha} - b(x)g(u), & x \in \Omega^{\delta,\rho}, \\ u = 0, & |x| = \rho \text{ or } \delta. \end{cases} \tag{2.20}$$

From Lemma 2.1, it is easy to see that, $\underline{u}(x) \geq \underline{u}_{\delta,\rho}(x)$. Let $\underline{u}_\rho(x) = \lim_{\delta\to0} \underline{u}_{\delta,\rho}(x)$. Obviously, $\underline{u}_\rho(x)$ is a radially symmetric function, and so we denote $\underline{u}_\rho(r) = \underline{u}_\rho(x)$ with $|x| = r$.

Firstly we claim that $\underline{u}_\rho'(r) \leq 0$ for $r \in (0, \rho)$. Define

$$r_0 = \inf\{r : \underline{u}_\rho'(s) \leq 0, \ s \in (r, \rho]\}.$$

Notice that $\underline{u}_\rho(r) > 0$ for $r \in (0, \rho)$ and $\underline{u}_\rho(\rho) = 0$, thus by virtue of the strong maximum principle [27], we see $\underline{u}_\rho'(r) < 0$, when $r(< \rho)$ is close to ρ. So, $r_0 < \rho$. Obviously, if $r_0 = 0$, then the claim is valid. Now we suppose that $r_0 > 0$ and divided the proof into two cases.

Case 1. $\underline{u}_\rho'(r) \geq 0$ for $r \in (0, r_0)$. This implies that

$$\lim_{r\to0} \underline{u}_\rho(r) \in [0, \infty).$$

By $\theta + \alpha > 0$, we have

$$\begin{aligned} -|x|^{\alpha}\Delta \underline{u}_{\rho} &= \lambda \underline{u}_{\rho} - |x|^{\alpha} b(x) g(\underline{u}_{\rho}) \\ &\geq \lambda \underline{u}_{\rho} - (m+\varepsilon)|x|^{\theta+\alpha} g(\underline{u}_{\rho}) \\ &= (\lambda + o(1))\underline{u}_{\rho}. \end{aligned} \tag{2.21}$$

Since $\lambda > H$, from the above inequality, it follows that there exists $\tau_1 \in (0, \rho)$ such that

$$-(|x|^{\alpha})\Delta \underline{u}_{\rho} > \frac{(\lambda + H)}{2}\underline{u}_{\rho},\ |x| \in (0, \tau_1).$$

Hence, for any $\epsilon \in (0, \tau_1)$, $\underline{u}_{\rho}$ satisfies

$$\begin{cases} -\Delta \underline{u}_{\rho} > \frac{\lambda+H}{2}\frac{\underline{u}_{\rho}}{|x|^{\alpha}}, & x \in \Omega^{\epsilon,\tau_1}, \\ \underline{u}_{\rho}(x) > 0, & |x| = \epsilon, \\ \underline{u}_{\rho}(x) > 0, & |x| = \tau_1. \end{cases} \tag{2.22}$$

Therefore, $\underline{u}_{\rho}$ is a positive strict supersolution of

$$\begin{cases} -\Delta u = \frac{\lambda+H}{2}\frac{u}{|x|^{\alpha}}, & x \in \Omega^{\epsilon,\tau_1}, \\ u(x) = 0, & x = \tau \ or\ \epsilon. \end{cases}$$

Applying Lemma 2.3, we have $\lambda_1\left[\frac{1}{|x|^{\alpha}}, \Omega^{\epsilon,\tau_1}\right] > \frac{\lambda+H}{2} > H$ for any $\epsilon \in (0, \tau_1)$. On the other hand, from Lemma 2.2, notice that $\alpha > 2$, $|x| < 1$, and we have

$$\lambda_1\left[\frac{1}{|x|^{\alpha}}, \Omega^{\epsilon,\tau_1}\right] < \lambda_1\left[\frac{1}{|x|^{2}}, \Omega^{\epsilon,\tau_1}\right].$$

By Lemma 2.4, $\lambda_1\left[\frac{1}{|x|^{2}}, \Omega^{\epsilon,\tau_1}\right] \to H$ as $\epsilon \to 0^+$, which yields a contradiction.

Case 2. There exist $r_* < r^* < r_0$ such that

$$\underline{u}'_{\rho}(r_*) < 0,\ \underline{u}'_{\rho}(r^*) > 0.$$

Then there exist $\overline{r} \in (r_*, r^*)$ satisfying

$$\underline{u}'_{\rho}(\overline{r}) = 0.$$

Without loss of generality, we further assume that

$$\underline{u}'_{\rho}(r) < 0,\ \text{for } r \in (r_*, \overline{r}) \cup (r_0, r_0 + \tau_2), \tag{2.23}$$

$$\underline{u}'_{\rho}(r) > 0,\ \text{for } r \in (\overline{r},\ r_0), \tag{2.24}$$

where $\tau_2 > 0$ is sufficiently small such that $r_0 + \tau_2 < \rho$.

By (2.23) and (2.24), we can choose $r_1 \in (\overline{r}, r_0)$ and $r_2 \in (r_0, r_0 + \tau_2)$ such that

$$\underline{u}_{\rho}(r_1) < \underline{u}_{\rho}(r_2),\ \underline{u}''_{\rho}(r_1) > 0,\ \underline{u}''_{\rho}(r_2) < 0. \tag{2.25}$$

In fact, we only need to choose r_1 sufficiently close to $\overline{r}$ and r_2 sufficiently close to r_0.

From (2.23) and (2.25), we have

$$-\underline{u}_\rho''(r) - \frac{N-1}{r}\underline{u}'(r) > 0,\ r = r_2.$$

Then by (2.19), we obtain

$$\frac{\lambda}{r_2^\alpha}\underline{u}_\rho(r_2) - b(r_2)g(\underline{u}_\rho(r_2)) > 0.$$

Thus,

$$\frac{g(\underline{u}_\rho(r_2))}{\underline{u}_\rho(r_2)} < \frac{\lambda}{r_2^\alpha b(r_2)}. \tag{2.26}$$

Similarly, we can obtain

$$\frac{g(\underline{u}_\rho(r_1))}{\underline{u}_\rho(r_1)} > \frac{\lambda}{r_1^\alpha b(r_1)}. \tag{2.27}$$

Notice that $\theta + \alpha > 0$, then (2.26) and (2.27) lead to

$$\frac{g(\underline{u}_\rho(r_2))}{\underline{u}_\rho(r_2)} < \frac{g(\underline{u}_\rho(r_1))}{\underline{u}_\rho(r_1)}.$$

In view of $\underline{u}_\rho(r_2) > \underline{u}_\rho(r_1)$ and condition (g_1), this is a contradiction.

By the regularity argument of the quasilinear elliptic equations, $\underline{u}_\rho(r)$ converges to $\underline{u}(r)$ in $C^1_{loc}(0,\rho]$. So, $\underline{u}_\rho'(r) \le 0$ for any $\rho > 0$ implies $\underline{u}'(r) \le 0$ for all $r \in (0,\rho)$, and $\lim_{r\to 0}\underline{u}(r) \in (0,\infty]$ is well–defined. Similar to that of Case 1, we can derive $\underline{u}(r) \to \infty$ as $r \to 0$, i.e., $\underline{u}(x)$ blows up at the origin.

Remark. During the proof of Lemma 2.7, if we first apply conditions (b_1) and (b_2) on $b(x)$ to estimate equation (2.20), obtaining inequality (2.20), and then use the comparison principle on the solution of inequality (2.20), we can remove the constraint that $b(x)$ must be a radial function. Hence, we have: In Lemma 2.7, if we remove the restriction that $b(x)$ is a radial function, the conclusion that $\underline{u}(x)$ blows up at the origin in Lemma 2.7 still holds true (note that $\underline{u}(x)$ is not necessarily radial at this point).

Lemma 2.8. *Suppose that $b(x)$ satisfies the conditions (b_1) and (b_2) and $g(u)$ satisfies the conditions (g_1) and (g_2). If $\lambda > \frac{\alpha^\alpha}{4(\alpha-2)^{\alpha-2}}\lambda_1[B_1(0)]$, then there exists a constant $C_4 > 0$ such that*

$$u(x) \ge C_4|x|^{\frac{-(\alpha+\theta)}{q-1}}, \quad as\ |x| \to 0 \tag{2.28}$$

for any positive solution $u(x)$ of equation (2.4).

Proof. Notice that $\lambda > \frac{\alpha^\alpha}{4(\alpha-2)^{\alpha-2}}\lambda_1[B_1(0)]$ implies that $\lambda > H$. According to Lemma 2.7, $\underline{u}(x)$ blows up at the origin. By (g_2) and (b_2), there exists a $\delta_0(0 < \delta_0 < 1)$ such that $\frac{k}{2}\underline{u}^q(x) \le g(\underline{u}(x)) \le 2k\underline{u}^q(x)$ and $\frac{m}{2}|x|^\theta \le b(x) \le 2m|x|^\theta$ for $x \in B_{\delta_0}(0)\backslash\{0\} \subset \Omega\backslash\{0\}$. Similarly, when $\theta \ge 0$, for any $y \in D(x_0)$ and $|x_0| < \frac{\beta\delta_0}{\beta+1}$, where $\beta > 0$ is a undetermined constant, we have

$$\begin{aligned} -\Delta\underline{u}(y) &= \lambda|y|^{-\alpha}\underline{u}(y) - b(y)g(\underline{u}(y)) \\ &\ge \lambda|y|^{-\alpha}\underline{u}(y) - 4mk|y|^\theta\underline{u}^q(y) \\ &\ge \left(\frac{\beta+1}{\beta}\right)^{-\alpha}\lambda|x_0|^{-\alpha}\underline{u}(y) - 4\left(\frac{\beta+1}{\beta}\right)^\theta mk|x_0|^\theta\underline{u}^q(y). \end{aligned}$$

Define

$$V_1(x) := |x_0|^{\frac{\alpha+\theta}{q-1}} \underline{u}\left(x_0 + \frac{|x_0|^{\frac{\alpha}{2}}}{\beta} x\right), \ x \in B_1(0).$$

Then for any $x \in B_1(0)$, we have

$$\begin{aligned} -\Delta V_1(x) &= -\beta^{-2}|x_0|^{\alpha+\frac{\alpha+\theta}{q-1}} \Delta\underline{u}\left(x_0 + \tfrac{|x_0|^{\frac{\alpha}{2}}}{\beta}x\right) \\ &\geq \lambda \cdot \beta^{-2}\left(\tfrac{\beta+1}{\beta}\right)^{-\alpha} |x_0|^{\frac{\alpha+\theta}{q-1}} \underline{u}\left(x_0 + \tfrac{|x_0|^{\frac{\alpha}{2}}}{\beta}x\right) - 4\beta^{-2}\left(\tfrac{\beta+1}{\beta}\right)^{\theta} mk|x_0|^{\frac{\alpha+\theta}{q-1}\cdot q}\underline{u}^q\left(x_0 + \tfrac{|x_0|^{\frac{\alpha}{2}}}{\beta}x\right) \\ &= \lambda \cdot \beta^{-2}\left(\tfrac{\beta+1}{\beta}\right)^{-\alpha} V_1(x) - 4\beta^{-2}\left(\tfrac{\beta+1}{\beta}\right)^{\theta} mkV_1^q(x). \end{aligned}$$

Consider the following equation:

$$\begin{cases} -\Delta W_3 = \lambda \cdot \beta^{-2}\left(\frac{\beta+1}{\beta}\right)^{-\alpha} W_3 - 4\beta^{-2}\left(\frac{\beta+1}{\beta}\right)^{\theta} mkW_3^q, & x \in B_1(0), \\ W_3 = 0, & x \in \partial B_1(0). \end{cases} \tag{2.29}$$

By Theorem 5.1 in [26], we know that when $\lambda > \frac{(\beta+1)^\alpha \lambda_1[B_1(0)]}{\beta^{\alpha-2}}$, the equation (2.29) has a unique positive solution $W_3(x)$. By Lemma 2.1, we have $W_3(x) \leq V_1(x)$ for $x \in B_1(0)$. Therefore, we conclude $W_3(0) \leq V_1(0)$, that is,

$$\underline{u}(x_0) \geq W_3(0)|x_0|^{-\frac{\alpha+\theta}{q-1}}.$$

By the arbitrariness of x_0, there exists a positive constant C_4. For any positive solution $u(x)$ of (2.4), we obtain

$$u(x) \geq C_4|x|^{-\frac{\alpha+\theta}{q-1}}, \ \forall x \in B_{\delta_0}(0)\backslash\{0\}.$$

Notice that

$$\inf_{\beta>0} \frac{(\beta+1)^\alpha}{\beta^{\alpha-2}} = \frac{\alpha^\alpha}{4(\alpha-2)^{\alpha-2}},$$

which shows that when $\lambda > \frac{\alpha^\alpha}{4(\alpha-2)^{\alpha-2}}\lambda_1\left[B_1(0)\right]$ holds, $u(x)$ satisfies (2.28). This completes the proof.

3. Proof of the main result

3.1. Proof of (i) in Theorem 1.3:

Lemma 3.1. *Suppose that $\alpha > 2$, $b(x)$ satisfies the condition (b_1) and (b_2) and $g(u)$ satisfies the condition (g_1), (g_2), and (g_3). If $q > \frac{2(\alpha+\theta)}{N-2} + 1$, $\lambda \leq 0$, then the following Dirichlet problem*

$$\begin{cases} -\Delta u = \lambda|x|^{-\alpha}u - b(x)g(u), & x \in \Omega\backslash\{0\}, \\ u > 0, & x \in \Omega\backslash\{0\}, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{3.1}$$

has no solution.

Proof. Assume that the equation (3.1) has a positive solution $u(x)$. Multiplying both sides of the equation (3.1) by u and integrating over $\Omega\backslash B_r(0)$ gives

$$\int_{\Omega\backslash B_r(0)} -\Delta u \cdot u dx = \int_{\Omega\backslash B_r(0)} \left[\lambda|x|^{-\alpha}u^2 - b(x)g(u)u\right] dx, \tag{3.2}$$

where $0 < r < 1/2$ sufficiently small such that $B_{2r}(0) \subset \Omega$. Then by the divergence formula, we obtain

$$\int_{\Omega\backslash B_r(0)} |\nabla u|^2 dx - \int_{\partial B_r(0)} u\frac{\partial u}{\partial \nu} dS = \int_{\Omega\backslash B_r(0)} \left[\lambda|x|^{-\alpha}u^2 - b(x)g(u)u\right] dx, \tag{3.3}$$

where ν represents the unit outer normal vector on $\partial B_r(0)$.

First, notice that $g(u)$ satisfies (g_1) and (g_3), then $g(0) = 0$ and $g(u) > 0$ for $u > 0$. Also notice that $b(x) > 0$ in $\Omega\backslash B_r(0)$, which implies that

$$-\int_{\Omega\backslash B_r(0)} b(x)g(u)u dx < 0, \tag{3.4}$$

therefore by (3.3), we have

$$\int_{\Omega\backslash B_r(0)} |\nabla u|^2 dx - \int_{\partial B_r(0)} u\frac{\partial u}{\partial \nu} dS < \int_{\Omega\backslash B_r(0)} \lambda|x|^{-\alpha}u^2 dx. \tag{3.5}$$

Notice that $q > \frac{2(\alpha+\theta)}{N-2} + 1$ implies that $u \in H_0^1(\Omega)$ (notice that Lemma 2.6 also holds when $\lambda \le 0$), thus

$$\lim_{r\to 0^+} \int_{\partial B_r(0)} u\frac{\partial u}{\partial \nu} dS = \lim_{r\to 0^+} \int_{\partial B_r(0)} u\nabla u \cdot \frac{x}{r} dS = 0.$$

Since $u(x)$ is a positive continuous function, the first term of the left side for equation (3.3) is greater than zero, but when $\lambda \le 0$, the right side of equation (3.5) is less than zero. This contradiction shows that the equation (3.1) has no solution when $\lambda \le 0$.

From Lemma 3.1, the conclusion of Theorem 1.3 (i) holds true.

Remark. (1) If the solution u to (3.1) satisfies $u \in C^2(\Omega\backslash\{0\}) \cap H_0^1(\Omega)$, then condition $q > \frac{2(\alpha+\theta)}{N-2} + 1$ in the above lemma can be deleted.

(2) When $q \le \frac{2(\alpha+\theta)}{N-2} + 1$, it would be a highly meaningful research topic to provide specific examples illustrating that equation (3.1) admits a classical solution($u \in C^2(\Omega\backslash\{0\})$) but not a weak solution($u \in H_0^1(\Omega)$).

3.2. Proof of (ii) in Theorem 1.3

The assertion of Theorem 1.3 (ii) follows by combining Lemma 3.2 - Lemma 3.4.

Lemma 3.2. *Assume that $\lambda > \frac{\alpha^\alpha}{4(\alpha-2)^{\alpha-2}}\lambda_1 [B_1(0)]$, $b(x)$ satisfies the conditions (b_1) and (b_2) and $g(u)$ satisfies the conditions (g_1), (g_2) , and (g_3). If $u(x)$ is any positive solution of the problem* (3.1)*, then we have*

$$\liminf_{|x|\to 0^+} |x|^{\frac{\alpha+\theta}{q-1}} u(x) \ge \left(\frac{\lambda}{mk}\right)^{\frac{1}{q-1}}. \tag{3.6}$$

Proof. By (b_2), we know for any $\varepsilon > 0$, there exists a $\delta_1(\delta_1 > 0)$ such that when $0 < |x| < \delta_1$,

$$(m - \varepsilon)|x|^\theta \le b(x) \le (m + \varepsilon)|x|^\theta.$$

For any sufficiently small $\tau \in (0, \frac{\lambda}{(m+\varepsilon)(k+\varepsilon)})$, let

$$\xi_\tau = \left[\frac{\lambda}{(m + \varepsilon)(k + \varepsilon)} - \tau\right]^{\frac{1}{q-1}}.$$

Define $\varphi(r) = \frac{1}{Cr+1}$, $r \le R$, where $R \in (0,1)$ and C is a positive constant, then

$$\varphi'(r) = -\frac{C}{(Cr+1)^2},\ \varphi''(r) = \frac{2C^2}{(Cr+1)^3}.$$

Let $l = \frac{\alpha+\theta}{q-1}$, and define a function $v(x) = \xi_\tau |x|^{-l}\varphi(|x|)$. Obviously $v(x)$ is a radial function, and denote $v(r) = v(x)$. Then

$$v'(r) = -\xi_\tau l r^{-l-1}\varphi(r) + \xi_\tau r^{-l}\varphi'(r),\ 0 < r < R$$

and

$$v''(r) = \xi_\tau (l+1) l r^{-l-2}\varphi(r) - 2\xi_\tau l r^{-l-1}\varphi'(r) + \xi_\tau r^{-l}\varphi''(r),\ 0 < r < R.$$

Notice that

$$\lim_{|x|\to 0^+} v(x) = +\infty.$$

By (g_2), we can obtain that there exists a positive constant $\delta_2(\delta_2 < \delta_1)$ such that

$$(k-\varepsilon)v^q(x) \le g(v(x)) \le (k+\varepsilon)v^q(x),\ \text{for } 0 < |x| < \delta_2.$$

From the above inequalities, we deduce

$$\begin{aligned}
&-\Delta v - \lambda|x|^{-\alpha}v + b(x)g(v)\\
\le& -\left(v''(r) + \tfrac{N-1}{r}v'(r)\right) - \lambda|x|^{-\alpha}v + (m+\varepsilon)(k+\varepsilon)|x|^{\theta}v^q\\
=& -\xi_\tau(l+1)l r^{-l-2}\varphi(r) + 2\xi_\tau l r^{-l-1}\varphi'(r) - \xi_\tau r^{-l}\varphi''(r) + \xi_\tau(N-1)l r^{-l-2}\varphi(r)\\
&-\xi_\tau(N-1)r^{-l-1}\varphi'(r) - \lambda\xi_\tau r^{-\alpha-l}\varphi(r) + (m+\varepsilon)(k+\varepsilon)r^{\theta-lq}\xi_\tau^q\varphi^q(r)\\
\le& \xi_\tau r^{-l-2}\varphi(r)\left[-(l+1)l - \tfrac{C(2l-N+1)r}{1+Cr} + (N-1)l - \lambda r^{2-\alpha} + (m+\varepsilon)(k+\varepsilon)\xi_\tau^{q-1}r^{2-\alpha}\right]\\
\le& \xi_\tau l r^{-l-2}\varphi(r)\left[(N-l-2)l + N - 1 - (m+\varepsilon)(k+\varepsilon)\tau r^{2-\alpha}\right]\\
\le& \xi_\tau l r^{-l-2}\varphi(r)\left[(N-l-1)(l+1) - (m+\varepsilon)(k+\varepsilon)\tau r^{2-\alpha}\right].
\end{aligned}$$

Since $\alpha > 2$, we can take R small enough (R is independent of C) such that

$$(N-1)l + 1 - (m+\varepsilon)(k+\varepsilon)\tau r^{2-\alpha} < 0,\ r \in (0,R).$$

Thus,

$$-\Delta v - \lambda|x|^{-\alpha}v + b(x)g(v) < 0,\ x \in B_R(0)\backslash\{0\}.$$

Define a new function v_ϵ, where $\epsilon \in (0,R)$ and

$$v_\epsilon(x) = v(r+\epsilon),\ r = |x| \in (0, R-\epsilon).$$

So,

$$-\Delta v_\epsilon(x) - \lambda|x|^{-\alpha}v_\epsilon(x) + b(x)g(v_\epsilon(x)) < 0,\ \text{for } x \in B_{R-\epsilon}(0)\backslash\{0\}.$$

Since $u(x)$ blows up at the origin, for any $\epsilon > 0$ sufficiently small, when $r \to 0$,

$$v_\epsilon(x) < u(x).$$

Conversely, by selecting C sufficiently large, we ensure that

$$v_\epsilon(x) < u(x),\ \text{for } r = R-\epsilon.$$

Applying Lemma 2.1, we obtain

$$v_\epsilon(x) \le u(x),\ \ x \in B_{R-\epsilon}(0)\backslash\{0\}.$$

So when $x \in B_R(0)\backslash\{0\}$, we have

$$u(x) \ge v(x),\ \ x \in B_R(0)\backslash\{0\}.$$

By the arbitrariness of ε and τ, we know

$$\liminf_{|x|\to 0^+} |x|^{\frac{\alpha+\theta}{q-1}} u(x) \ge \left(\frac{\lambda}{mk}\right)^{\frac{1}{q-1}}.$$

Lemma 3.3. *Assume that $b(x)$ satisfies the conditions (b_1) and (b_2) and $g(u)$ satisfies the conditions (g_1), (g_2), and (g_3). Moreover, assume that $g(u)$ is convex in u for $u > 0$. If $\lambda > \frac{\alpha^\alpha}{4(\alpha-2)^{\alpha-2}}\lambda_1\,[B_1(0)]$, then equation* (3.1) *has a unique solution.*

Proof. Combining Lemma 2.5, Lemma 2.6, and Lemma 2.8, we conclude that there exist $\delta(\delta > 0)$ and $C_5, C_6(C_5 < C_6)$ such that for any $x \in B_\delta(0)\backslash\{0\}$,

$$C_5|x|^{-\frac{\alpha+\theta}{q-1}} \le \underline{u}(x) \le U(x) \le C_6|x|^{-\frac{\alpha+\theta}{q-1}}.$$

If $\underline{u}(x) \neq U(x)$, a direct application of the strong maximum principle gives $\underline{u}(x) < U(x)$ in $\Omega\backslash\{0\}$. Define

$$v = \underline{u} - \frac{1}{2c}(U - \underline{u}),$$

where $c > 0$ is a constant such that $U \le c\underline{u}$. We have

$$\underline{u} > v \ge \frac{c+1}{2c}\underline{u},\ \ \frac{2c}{2c+1}v + \frac{1}{2c+1}U = \underline{u}.$$

Denote $f(x,t) = -\lambda|x|^{-\alpha}t + b(x)g(t)$. Since $g(u)$ is convex in u for $u > 0$, then for any $x \in \Omega\backslash\{0\}$, $f(x,t)$ is convex in t for $t > 0$. Therefore,

$$f(x,\underline{u}) \le \frac{2c}{2c+1}f(x,v) + \frac{1}{2c+1}f(x,U).$$

Thus,

$$-\Delta v = -\frac{2c+1}{2c}f(x,\underline{u}) + \frac{1}{2c}f(x,U) \ge -f(x,v),$$

so,

$$-\Delta v \ge \lambda|x|^{-\alpha}v - b(x)g(v),\ \ x \in \Omega.$$

When $x \in \Omega^\delta$, we have $-\Delta v - \lambda|x|^{-\alpha}v + b(x)g(v) \ge 0 = -\Delta\underline{u}_\delta - \lambda|x|^{-\alpha}\underline{u}_\delta + b(x)g(\underline{u}_\delta)$. When $x \in \partial\Omega$, we have $v(x) = 0 = \underline{u}_\delta(x)$. When $|x| = \delta$, we have $v(x) \ge \frac{c+1}{2c}\underline{u}(x) > 0 = \underline{u}_\delta(x)$, where $\underline{u}_\delta(x)$ is the unique positive solution of the equation (2.16). Using Lemma 2.1, when $x \in \Omega^\delta$, we have $v(x) \ge \underline{u}_\delta(x)$. Letting $\delta \to 0^+$, we get $\underline{u}(x) \le v(x)$, which is a contradiction with $\underline{u}(x) > v(x)$. So, the equation (3.1) has a unique positive solution.

Lemma 3.4. *Let $\lambda > 0$, $\alpha > 2$, and $\alpha + \theta > 0$. $b(x)$ satisfies the conditions (b_1) and (b_2) and $g(u)$ satisfies the conditions (g_1) , (g_2), and (g_3). Suppose that $u(x)$ is the unique positive solution of the problem* (3.1)*. Then we have*

$$\limsup_{|x|\to 0^+} |x|^{\frac{\alpha+\theta}{q-1}} u(x) \le \left(\frac{\lambda}{mk}\right)^{\frac{1}{q-1}}.$$

The proof is similar to that of Lemma 3.2; we omit it here.

4. Numerical example

Next, we provide numerical examples to validate Theorem 1.1(ii). For convenience, we assume that $N = 4$, $\Omega = B_1(0)$ (in this case, $\lambda_1[B_1(0)] \approx (3.8317)^2 < 16$), $b(x) = m|x|^\theta$ and $g(u) = ku^q$, which shows that $b(x)$ satisfies the conditions (b_1) and (b_2) and $g(u)$ satisfies the conditions (g_1), (g_2), and (g_3). Let $u(x) = u(\rho)$ and $|x| = \rho$, then (3.1) can be transformed into the following second–order ODEs with singular coefficients:

$$\begin{cases} -u''(\rho) - \frac{N-1}{\rho}u'(\rho) = \frac{\lambda}{\rho^\alpha}u(\rho) - mk\rho^\theta u^q, & \rho \in (0,1), \\ u(\rho) > 0, & \rho \in (0,1), \\ u(1) = 0. \end{cases} \tag{4.1}$$

To utilize numerical approaches in solving ODEs, we assume that $\rho = 1 - t$ and set $u(\rho) = y_1(t)$, $u'(\rho) = -y_2(t)$, then (4.1) can be transformed into

$$\begin{cases} y_1'(t) = y_2(t), \\ y_2'(t) = \frac{3}{1-t}y_2(t) - \frac{\lambda}{(1-t)^\alpha}y_1(t) + mk(1-t)^\theta y_1(t)^q, & t \in (0,1), \\ y_1(0) = 0. \end{cases} \tag{4.2}$$

Example 1. Assume that $\lambda = 108$, $\theta = 1$, $q = 2$, $m = 1$, $k = 2$ and $\alpha = 3$. A simple verification shows that the hypotheses of Theorem 1.3 are satisfied. Based on Theorem 1.3, we have

$$\lim_{t\to 1^-} \frac{y_1(t)}{(1-t)^{-\frac{\alpha+\theta}{q-1}}(\frac{\lambda}{mk})^{\frac{1}{q-1}}} = 1. \tag{4.3}$$

Let

$$v(t) = (1-t)^{-\frac{\alpha+\theta}{q-1}}\left(\frac{\lambda}{mk}\right)^{\frac{1}{q-1}}. \tag{4.4}$$

In order to plot the graph of the numerical solution $y_1(t)$ of (4.2), we still need to know $y_2(0)$. However, from the original equation (3.1), we cannot obtain any information about it. To address this problem, we will adopt an alternative approach: Set $y_2(0) = -u'(1) = -a$, and regard a as a parameter. We shall then investigate the following question: For which a does the solution u of Eq (4.1) with initial condition $u(1) = 0$ and $u'(1) = -a$ possess a maximal interval of existence $(0, 1]$ and satisfy (4.3)?

First, using numerical observation, we examine how the solution to equation (4.1) changes with $u'(1)$(i.e., $-a$) near the origin. The graph of the numerical solution $u(t)$ for Eq (4.1) can be found in Figure 1 with initial condition $u(1) = 0$ and $u'(1) = -500$. Under this initial condition, we find the solution oscillating near the origin. While $u'(1)$ is sufficiently small, the numerical solution $u(t)$ of Eq (4.1) will blow up at $\rho_0 > 0$. As can be seen from Figure 2, the maximal existence interval of the solution decreases as a decreases. By the continuous dependence of solutions on initial data, there exists a unique $a_0 < 0$ such that the maximal existence interval of the solution to Eq (4.1) is $(0, 1]$, and the solution experiences blow-up at the origin. Obviously, if we denote

$$A = \{a | \text{the solution } u \text{ of Eq (4.1) with } u'(1) = a < 0 \text{ oscillating near the origin}\},$$

$$B = \{a | \text{the solution } u \text{ of Eq (4.1) with } u'(1) = a < 0 \text{ blow up at } \rho_0 > 0\}$$

from Figure 2, we can see that

$$a_0 = \inf A = \sup B.$$

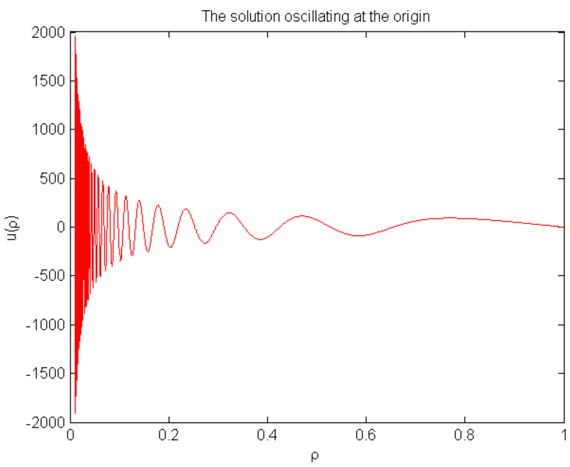

Figure 1. The solution of (4.1) oscillating near the origin for $u'(1) = -500$.

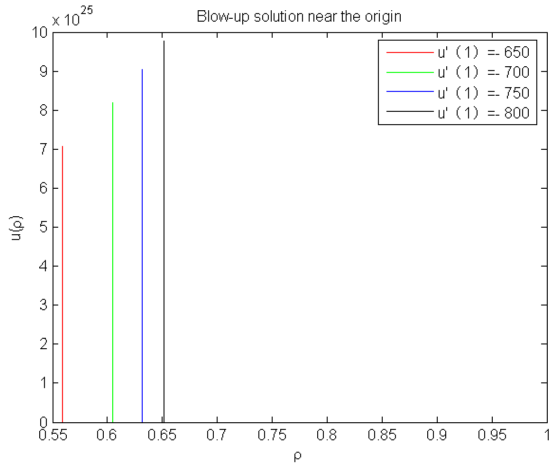

Figure 2. Comparison diagram of the solution of (4.1) blow-up near the origin when $u'(1) = -650, u'(1) = -700, u'(1) = -750$ and $u'(1) = -800$, respectively.

In the following, we will present a method for determining the aforementioned critical value of a_0. Let $w(t) = y_1(t)(1-t)^4$, then $w(t)$ satisfies the following ODE:

$$(1-t)^3 w'' + 5(1-t)^2 w' + [8(1-t) + 108]\, w - 2w^2 = 0. \tag{4.5}$$

From (4.2) and (4.3), we know that

$$w(0) = 0 \text{ and } w(1) = 54. \tag{4.6}$$

Using numerical methods for solving two–point boundary value problems (BVPs) of ODEs, we readily plot the solution of Eq (4.5) under boundary conditions (4.6) as shown in Figure 3. Simultaneously, we can obtain $w'(0) \approx 612.9906313105$.

On the other hand,

$$w'(0) = y_1'(0) = y_2(0) = -a_0,$$

which implies that

$$a_0 \approx -612.9906313105 \approx -613. \tag{4.7}$$

To verify the correctness of (4.7), we conversely utilize Eq (4.2) along with the initial conditions $y_2(0) = -a_0 \approx 613$ to generate its numerical solution shown in Figure 4. From Figure 4, it is evident that this parameter a_0 satisfies the required criteria.

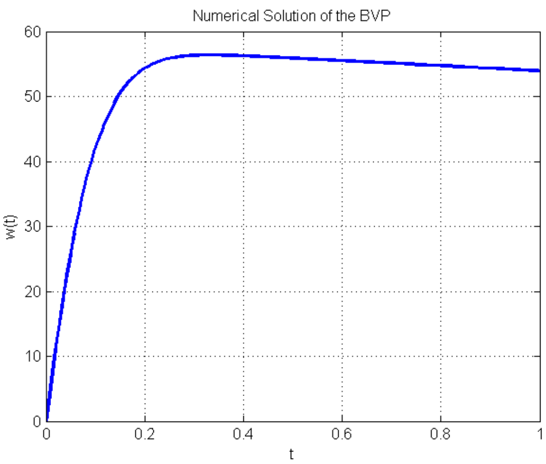

Figure 3. The numerical solution of BVP (4.5) with (4.6).

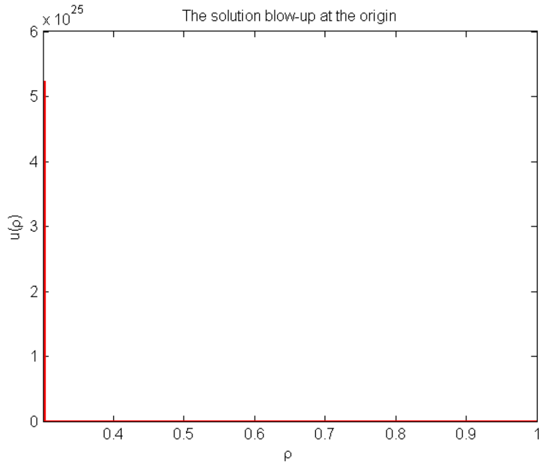

Figure 4. The numerical solution $u(\rho)$ of (4.1) for $a_0 \approx -613$.

Now we can use the same methods mentioned above to consider the general case.

Let $w_1(t) = y_1(t)(1-t)^{\frac{\alpha+\theta}{q-1}}$, then $w(t)$ satisfies the following ODE:

$$(1-t)^2 w_1'' + (2l - N + 1)(1-t)w_1' + \left[l^2 + 2 - N + \lambda(1-t)^{2-\alpha}\right] w_1 - mk(1-t)^{2-\alpha} w_1^q = 0, \tag{4.8}$$

where $l = \frac{\alpha+\theta}{q-1}$. From (4.2) and (4.3), we know that

$$w_1(0) = 0 \text{ and } w_1(1) = \left(\frac{\lambda}{mk}\right)^{\frac{1}{q-1}}. \tag{4.9}$$

Example 2. Assume that $\theta = -1$, $\lambda = 128$, $q = 2$, $m = 1$, $k = 2$, $\alpha = 3$ and $N = 4$. It is easy to see that Theorem 1.3 still holds. The comparative diagram of $u(\rho)$ and $v(\rho)$ is provided in Figure 5. From Figure 5, we can see that when $\rho \to 0$, $u(\rho)$ and $v(\rho)$ are equivalent infinitely large quantities. This provides further corroboration of the theoretical results.

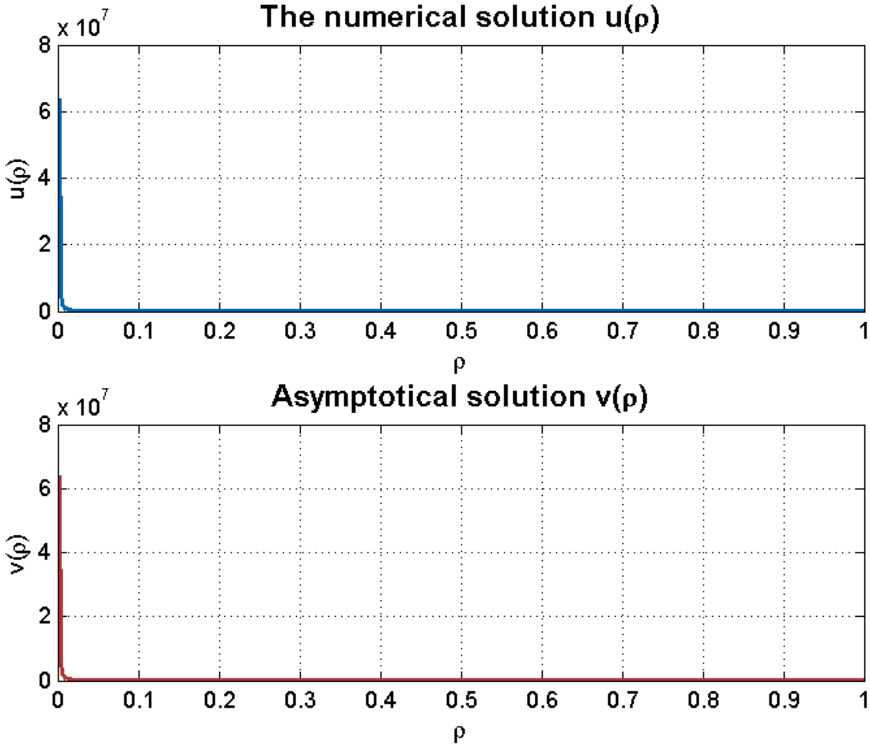

Figure 5. The comparative diagram of $u(\rho)$ and $v(\rho)$.

5. Conclusions

In this paper, we study the nonlinear elliptic equations with generalized Hardy potential function $|x|^{-\alpha}$. Using the method of upper and lower solutions and the principle of comparison, we obtain the existence of maximum and minimum solutions with the zero Dirichlet boundary condition. Furthermore, the blow-up behavior of the solution near the origin has been obtained. Also, the uniqueness of the positive solution has been obtained. The above method can be also used for the case when $\alpha = 2$, $m = 1$, and $k = 1$.

By Lemma 3.2 to Lemma 3.4, we can easily see that the asymptotic behavior of the solution for (3.1) at the origin does not depend on N; it only depends on the asymptotic behavior of $b(x)$ at the origin and the asymptotic behavior of $g(u)$ at infinity and λ. This conclusion differs fundamentally from the case when $\alpha = 2$ (see [19] and [21] for details).

Now, we discuss how the solution of equation (4.2) changes at the origin when the parameters vary. We only consider the single-parameter variation case.

First, from (4.3), we can see that $u(x)$ increases near the origin as λ increases.

Second, we consider the variation of the solution with the variation of parameter q under each of the following three cases: $0 < \frac{\lambda}{mk} < 1$, $\frac{\lambda}{mk} = 1$, and $\frac{\lambda}{mk} > 1$, respectively. In each case, notice that (4.3) holds and

$$\frac{\partial v}{\partial q} = (1-t)^{-\frac{\alpha+\theta}{q-1}} \left(\frac{\lambda}{mk}\right)^{\frac{1}{q-1}} \frac{1}{(q-1)^2} \left[(\alpha+\theta)\ln(1-t) - \ln\frac{\lambda}{mk}\right] < 0, \text{ as } t \to 1^-,$$

thus the blow-up rate of the solution decreases as q increases.

From Lemma 2.7, we see that when $\lambda > H$, the minimal positive solution $\underline{u}(x)$ of equation (2.4) blows up at the origin. In order to obtain sharper estimates of the solution at the origin, while keeping all other assumptions unchanged, we impose additional restrictions on λ (see Lemma 2.8). However, if we only replace $\lambda = 108$ by $\lambda = 4$ in Example 1 (see Figure 6), the numerical computations demonstrate that such restrictions can potentially be unnecessary, thus we may have:

Conjecture: *Suppose that $\alpha > 2$ and $\theta + \alpha > 0$, $b(x)$ satisfies the conditions (b_1) and (b_2), and $g(u)$ satisfies the conditions (g_1) , (g_2), and (g_3). Moreover, assume that $g(u)$ is convex in u for $u > 0$. If $\lambda > H$, then equation (1.4) with the zero Dirichlet boundary condition has a unique positive solution $u(x)$ satisfying (1.5).*

On the other hand, from our analysis of the numerical solution in Example 1, we can arrive at the following conjecture:

Conjecture: *Suppose that $\alpha > 2$ and $\theta + \alpha > 0$, $b(x)$ satisfies the conditions (b_1) and (b_2), and $g(u)$ satisfies the conditions (g_1) , (g_2), and (g_3). Moreover, assume that $g(u)$ is convex in u for $u > 0$. If $\lambda > H$, then there exists a unique $a_0 < 0$ such that*

(1) *For $a > a_0$, the solution to radial equation (4.1) with $u'(1) = -a$ is oscillatory about $u = 0$ on its maximal existence interval* $(0, 1]$.

(2) *For $a \leq a_0$, the solution of radial equation (4.1) with $u'(1) = -a$ exhibits singular behavior near the origin, that is, $\exists \rho_0 \geq 0$ satisfies*

$$\lim_{\rho \to \rho_0^+} u(\rho) = +\infty,$$

so its maximal existence interval is $(\rho_0, 1]$.

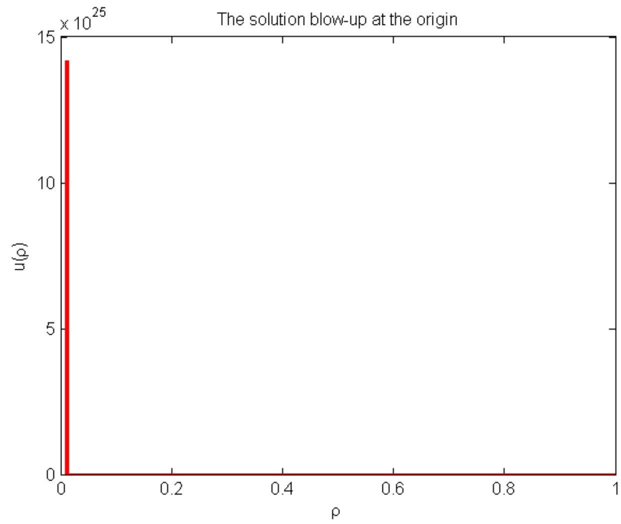

Figure 6. The numerical solution $u(\rho)$ of (4.1) for $a_0 = -27.942090834$.

In this paper, we restrict our discussion to the case $\alpha > 2$. For the case $0 < \alpha < 2$, we conjecture that the asymptotic behavior of solutions near the origin will be more complex; nevertheless, the methodology developed above can be adapted to offer a viable approach for such investigations. We designate the conjectures mentioned above as open problems for further study.

Author contributions

L. L. Wang and Y. H. Fan designed the study. Y. H. Fan performed the experiments and analyzed the data. L. L. Wang provided reagents and analytical tools. The manuscript was written by L. L. Wang and J. J. Liu, and revised by Y. H. Fan. All authors approved the final version.

Use of Generative-AI tools declaration

The authors did not use any generative AI or AI-assisted technologies in the preparation of this manuscript.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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