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Research article

A non-parametric test of homogeneity for umbrella ordering alternative based on an isotonic estimators

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Abstract: Although there are many methods to test of homogeneity under umbrella ordering alternatives, most of these tests are aimed at the case of multinomial distributions or parametric families of continuous distributions. In this paper, we focus on testing homogeneity for umbrella ordering alternatives under nonparametric continuous distributions, considering both scenarios in which the peak of the umbrella ordering is known and unknown. The test statistics are constructed using isotonic estimates of the distribution functions, and the asymptotic distributions of the test statistics are derived. Critical values are obtained via a bootstrap procedure, and simulation results are provided to illustrate the performance of the proposed testing method.

Keywords: nonparametric test; umbrella ordering; isotonic estimator; asymptotic distribution; bootstrap critical value
Mathematics Subject Classification: 62G10, 62G20

1. Introduction

Since stochastic ordering was proposed by Lehmann [1], it has been widely applied across many fields. For example, in risk comparison within mathematical finance, increasing convex ordering is commonly used. In dose-response experiments, however, the relationship between dose and patient survival time can be described by an umbrella ordering, in which patient survival time first increases and then decreases stochastically with the dose level. Consequently, various types of stochastic orderings (such as simple stochastic ordering, increasing convex ordering, and umbrella ordering) have been extensively studied in the literature for different purposes. In this paper, we focus on testing for umbrella ordering. For convenience, we first recall the definition of umbrella ordering below.

Definition 1. k populations $X_1, X_2, \cdots, X_k$ with cumulative distribution functions (CDFs) $F_1(x), F_2(x), \cdots, F_k(x)$ are said to be in umbrella ordering, if

$$F_1(x) \leq F_2(x) \leq \cdots \leq F_h(x) \geq F_{h+1}(x) \geq \cdots \geq F_k(x), \quad \forall x \in R,$$

where h is called the peak of the umbrella ordering.

The hypotheses are defined as

$$H_0: \ F_1 = F_2 = \cdots = F_k;$$
$$H_1: \ F_1(x) \leq F_2(x) \leq \cdots \leq F_h(x) \geq F_{h+1}(x) \geq \cdots \geq F_k(x), \quad \text{with at least one strict inequality.}$$

In this paper, we consider testing homogeneity against an umbrella ordering alternative with at least one strict inequality; that is, we test H_0 versus H_1.

Testing against umbrella ordering has a rich history and has been studied extensively by many authors. For example, in the context of location parameters with a known peak, Puri [2] proposed a distribution-free k-sample rank test for homogeneity against ordered alternatives, laying the theoretical foundation for nonparametric inference. Tryon and Hettmansperger [3] constructed nonparametric tests based on linear combinations of Chernoff–Savage-type two-sample statistics. Building on these studies, Mack and Wolfe [4] proposed a classical nonparametric test. Further contributions include the linear rank statistic framework developed by Hettmansperger and Norton [5] and the adaptive tests introduced by Büning and Kössler [6,7]. Chen and Wolfe [8] further enriched the known-peak literature by proposing refined homogeneity tests against umbrella alternatives with at least one strict inequality. For the unknown-peak case, Hettmansperger and Norton [5] extended their framework by incorporating a peak-search procedure. Shi [9] proposed a likelihood ratio test under normality for both known and unknown peaks. Chen and Wolfe [10] advanced this line of research with systematic distribution-free testing procedures. Chen [11] further refined this distribution-free framework for unknown-peak umbrella alternatives by deriving asymptotic critical values for the standardized rank-based test statistic. Later developments continued with Magel and Qin [12], Kössler [13], Alvo [14], Chang and Yen [15], and Gökpinar and Gökpinar [16]. Beyond location parameters, analogous hypotheses for scale parameters have been examined by Singh and Liu [17], Carpenter and Singh [18], Gaur et al. [19], and Gaur [20,21], while Shi [22] extended the framework to multinomial parameters. Although Basso and Salmaso [23], Xiong and Ding [24] considered tests of homogeneity under umbrella alternatives, they did not address test consistency. In this paper, the proposed test is consistent.

The remaining part of this paper is organized as follows. In Section 2, we present the definition of isotonic estimates for distribution functions and a discussion of their consistency. In Section 3, we construct test statistics for both the known and unknown peak cases of the umbrella ordering, and present the asymptotic properties of these tests. In Section 4, a bootstrap procedure is established for the implementation of the proposed tests. In Section 5, we present numerical results that demonstrate the performance of the proposed methods. Section 6 concludes the paper, and the Appendix contains the proofs of the theorems.

In this paper, we use " $\stackrel{w}{\to}$ ", " $\stackrel{\text{a.e.}}{\to}$ ", and " $\stackrel{d}{=}$ " denote convergence in distribution, almost sure convergence, and equivalence in distribution, respectively.

2. Estimation

Let $X_{i1}, X_{i2}, \cdots, X_{in_i}, i = 1, 2, \cdots, k$ be independent random samples from k distributions $F_1(x), F_2(x), \cdots, F_k(x)$, and these samples are defined on the same probability space $(\mathbf{\Omega}, \mathcal{A}, \mathbf{P})$. Furthermore, we assume that the CDF's $F_i(x), i = 1, 2, \cdots, k$ are continuous and satisfy the monotone inequality

$$F_1(x) \le F_2(x) \le \cdots \le F_h(x) \ge F_{h+1}(x) \ge \cdots \ge F_k(x), \quad \forall x \in R. \tag{2.1}$$

$\hat{F}_i(x)$ is the empirical distribution function of $F_i(x)$, that is

$$\hat{F}_i(x) = \frac{1}{n_i}\sum_{j=1}^{n_i} I_{[X_{ij},\infty)}(x),$$

where $I_A(\cdot)$ denotes the indicator function associated with set A. Obviously, the estimates $\hat{F}_i(x), i = 1, 2, \cdots, k$ are not guaranteed to satisfy inequality (2.1), even if the true CDFs satisfy it. To get such estimators, we give the following estimation method. The specific steps are as follows.

For the convenience of the following description, we define

$$N_{rs} = \sum_{j=r}^{s} n_j \text{ and } Av_n[\hat{F}(x), r, s] = \sum_{j=r}^{s} n_j \hat{F}_j(x)/N_{rs}, \text{ for } 1 \le r \le s \le k,$$

Step 1. Define the estimator of $F_i(x)$ by

$$\hat{F}_i^*(x) = \max_{r\le i}\min_{i\le s\le h} Av_n[\hat{F}(x), r, s],\ i = 1, 2, \cdots, h-1.$$

Obviously, $\hat{F}_1^*(x) \le \hat{F}_2^*(x) \le \cdots \le \hat{F}_{h-1}^*(x)$ (see p19 of Barlow et al. [25]) ;
Step 2. Define the estimator of $F_i(x)$ by

$$\hat{F}_i^*(x) = \max_{s\ge i}\min_{h\le r\le i} Av_n[\hat{F}(x), r, s], \quad i = h+1, h+2, \cdots, k.$$

Similar to Step 1, we have $\hat{F}_{h+1}^*(x) \ge \hat{F}_{h+2}^*(x) \ge \cdots \ge \hat{F}_k^*(x)$;
Step 3. Define the estimator of $F_h(x)$ by

$$\hat{F}_h^*(x) = \max_{r\le h}\max_{s\ge h} Av_n[\hat{F}(x), r, s]. \tag{2.2}$$

Because

$$\begin{aligned}
\hat{F}_{h-1}^*(x) &= \max_{r\le h-1}\min_{h-1\le s\le h} Av_n[\hat{F}(x), r, s] \\
&= \max\{\min\{Av_n[\hat{F}(x), 1, h-1], Av_n[\hat{F}(x), 1, h]\}, \cdots, \\
&\quad \min\{Av_n[\hat{F}(x), h-1, h-1], Av_n[\hat{F}(x), h-1, h]\}\}, \\
\hat{F}_h^*(x) &= \max_{r\le h}\max_{s\ge h} Av_n[\hat{F}(x), r, s] \\
&= \max\{\max\{Av_n[\hat{F}(x), 1, h], \cdots, Av_n[\hat{F}(x), 1, k]\}, \cdots, \\
&\quad \max\{Av_n[\hat{F}(x), h-1, h], \cdots, Av_n[\hat{F}(x), h-1, k]\}, \\
&\quad \max\{Av_n[\hat{F}(x), h, h], \cdots, Av_n[\hat{F}(x), h, k]\}\}.
\end{aligned}$$

Obviously, $\hat{F}_{h-1}^*(x) \le \hat{F}_h^*(x)$. Similarly, $\hat{F}_{h+1}^*(x) \le \hat{F}_h^*(x)$. Therefore, for each x, $\hat{F}_i^*(x)$ satisfy inequality (2.1).

Let $\|\cdot\|$ denote the sup norm. The following theorem gives the consistency of $\hat{F}_i^*(x), i = 1, 2, \cdots, k$ by Theorem 1 of EI Barmi and Mukerjee [26] and the properties of the isotonic regression (see p42 of Robertson et al. [27]).

Theorem 1. Given that the CDFs satisfy inequality (2.1), then

$$P[\| \hat{F}_i^* - F_i \| \to 0, \quad n_i \to \infty, \ i = 1, \cdots, k] = 1.$$

3. Test statistics and asymptotic theory

In this section, we will give test statistics of H_0 versus H_1 by $\hat{F}_i^*(x), i = 1, 2, \cdots, k$, and discuss the asymptotic distributions of the proposed test statistics. Next, we first discuss the case where the peak h is known.

3.1. A test when h is known

First, we define the test statistic T_n of testing problem H_0 versus H_1 by

$$T_n = \sqrt{n} \sum_{i=1}^{h-1} \sup_{x \in R} (\hat{F}_{i+1}^*(x) - \hat{F}_i^*(x)) + \sqrt{n} \sum_{i=h+1}^{k} \sup_{x \in R} (\hat{F}_{i-1}^*(x) - \hat{F}_i^*(x)).$$

Then, we introduce some additional notations to facilitate studying the asymptotic distribution of T_n. Let

$$n = \sum_{i=1}^{k} n_i, \quad \text{and} \quad a_{in} = \frac{n_i}{n},$$

and define

$$\begin{aligned} S_i &= \{ j : F_j(x) = F_i(x), j = 1, \cdots, k \}, \\ Z_{in_i}(x) &= \sqrt{n_i} [\hat{F}_i(x) - F_i(x)], \\ Z_{in_i}^*(x) &= \sqrt{n_i} [\hat{F}_i^*(x) - F_i(x)], \ i = 1, 2, \cdots, k. \end{aligned}$$

As is well known, when $\min_{i=1,\cdots,k} n_i \to \infty$, we have

$$(Z_{1n_1}(x), Z_{2n_2}(x), \cdots, Z_{kn_k}(x))^T \xrightarrow{w} (Z_1(x), Z_2(x), \cdots, Z_k(x))^T$$

where the right-hand side denotes a k-variate Gaussian process with independent components (see Theorem 16.4 in Billingsley [28]).

For the weak convergence process of the isotonic estimators to hold, we specify the following assumptions:

$$\lim_{n \to \infty} a_{in} = a_i > 0, \quad i = 1, 2, \cdots, k, \tag{3.1}$$

and

$$\inf_{c_i + \eta \le x \le d_i - \eta} [F_j(x) - F_i(x)] > 0, \tag{3.2}$$

for$\forall\eta > 0$, $j > \max\{l, l \in S_i\}$, $i = 1, 2, \cdots, h-1$, and $j < \min\{l, l \in S_i\}$, $i = h+1, h+2, \cdots, k$, where $\inf_{\emptyset}(.) = \infty$, (c_i, d_i) is a subset of the support of $F_i(x), i = 1, \cdots, k$.

The formula (3.2) is equivalent to dividing k distributions into several bundles; each bundle is completely coincident, and the various bundles are completely disconnected beyond the two endpoints, thus ensuring the tightness of the isotonic estimators.

Finally, we give the asymptotic distribution of T_n by the following theorem.

Theorem 2. Assume that the assumptions (3.1) and (3.2) hold, then under H_0, it holds that

$$\begin{aligned} T_n \xrightarrow{w} T &= \sum_{i=1}^{h-1} \sup_{x\in R}\Big(\sqrt{\frac{1}{a_{i+1}}} Z_{i+1}^*(x) - \sqrt{\frac{1}{a_i}} Z_i^*(x)\Big) \\ &+ \sum_{i=h+1}^{k} \sup_{x\in R}\Big(\sqrt{\frac{1}{a_{i-1}}} Z_{i-1}^*(x) - \sqrt{\frac{1}{a_i}} Z_i^*(x)\Big) \end{aligned}$$

where $Z_i^*(x), i = 1, 2, \cdots, k$ are the same as Lemma 1 in the Appendix.

Theorem 2 establishes the theoretical feasibility of the test. The theorem that follows demonstrates the consistency of the proposed test.

Theorem 3. If the assumptions (3.1) and (3.2) hold, then under H_1, we have $P(T_n \to \infty) = 1$.

3.2. A test when h is unknown

In this subsection, we consider the case that h is unknown. Let $H_1^{(i)}$ denote an umbrella hypothesis with its peak at $i, i = 1, ..., k$, that is,

$$H_1^{(i)}: \ F_1(x) \le F_2(x) \le \cdots \le F_i(x) \ge F_{i+1}(x) \ge \cdots \ge F_k(x), \quad \forall x \in R, i = 1, \cdots, k,$$

then H_1 can be written as

$$H_1 = H_1^{(1)} \cup H_1^{(2)} \cup \cdots \cup H_1^{(k)}.$$

Therefore, we can get the estimator $\hat{h}$ of h by the following equation:

$$\sup_{x\in R} \sum_{j=1}^{k} (\hat{\hat{F}}_j^{\hat{h}}(x) - \hat{F}_j(x))^2 = \min_{i=1,2,\cdots,k} \sup_{x\in R} \sum_{j=1}^{k} (\hat{\hat{F}}_j^{i}(x) - \hat{F}_j(x))^2 \tag{3.3}$$

where $\hat{\hat{F}}_j^i(x)$ is the isotonic estimate of $F_j(x), j = 1, 2, \cdots, k$ under the assumption $H_1^{(i)}, i = 1, 2, \cdots, k$.

Sometimes the solution of Eq (3.3) is not unique. In this case, we take the estimator as the maximum of the solution set.

Theorem 4. If the assumptions (2.1) and (3.2) hold, and h_0 is the true peak of the umbrella ordering, then we have

$$\hat{h} \xrightarrow{\text{a.e.}} h_0 \quad \text{as } n_j \to \infty, \ j = 1, 2, \cdots, k.$$

In the present case, the test statistic T_{1n} of testing problem H_0 versus H_1 is given by

$$T_{1n} = \sqrt{n} \sum_{i=1}^{\hat{h}-1} \sup_{x\in R}(\hat{F}_{i+1}^*(x) - \hat{F}_i^*(x)) + \sqrt{n} \sum_{i=\hat{h}+1}^{k} \sup_{x\in R}(\hat{F}_{i-1}^*(x) - \hat{F}_i^*(x)).$$

Obviously, by Theorems 2 and 4, we have

$$P(\hat{T}_{1n} \xrightarrow{w} T) = 1$$

where T is the same as in Theorem 2.

4. Bootstrap critical values

For practical decision-making purposes, we require the $p-$value associated with the test for the observed samples, or the critical value $c = c(\alpha)$ that depends on the probability α of wrongly rejecting the null hypothesis. Although the asymptotic null distribution of T_n is given, it is difficult to use directly to compute the $p-$value, or the $c(\alpha)$ since it is very complicated, and depends on the underlying unknown distributions F_i. Therefore, in this section, we will propose a bootstrap procedure for calculating the $p-$value of T_n, and subsequently provide the theoretical validation of the bootstrap method.

First, we give the bootstrap version of T_n.

Recall that $\hat{F}_i(x)$ is the empirical distribution function associated with the sample $X_{i1}, \cdots, X_{in_i}$ from $F_i(x), i = 1, \cdots, k$. Given the initial samples, let $\zeta_{n,1} \cdots, \zeta_{n,n}$ be a sample of size n from the (random) distribution function

$$H_n(x) = \frac{n_1}{n}\hat{F}_1(x) + \frac{n_2}{n}\hat{F}_2(x) + \cdots + \frac{n_k}{n}\hat{F}_k(x).$$

For convenience, we introduce some concerned notations as follows.

$$\begin{aligned}
\hat{F}_{n,n_i}(x) &= \frac{1}{n_i}\sum_{j=n_1+\cdots+n_{i-1}+1}^{n_1+\cdots+n_i} I_{[\hat{\zeta}_{n,j},\infty)}(x),\\
\hat{F}^*_{n,n_i}(x) &= \max_{r\le i}\min_{i\le s\le h} Av_n[\hat{F}_{n,n_i}(x), r, s],\ i = 1, 2, \cdots, h-1,\\
\hat{F}^*_{n,n_h}(x) &= \max_{r\le h}\max_{h\le s} Av_n[\hat{F}_{n,n_i}(x), r, s],\\
\hat{F}^*_{n,n_i}(x) &= \max_{s\ge i}\min_{h\le r\le i} Av_n[\hat{F}_{n,n_i}, r, s],\ i = h+1, h+2, \cdots, k.
\end{aligned}$$

And the bootstrap version of T_n is given by

$$\begin{aligned}
\hat{T}_n = \sqrt{n}\sum_{i=1}^{h-1}\sup_{x\in R}(\hat{F}^*_{n,n_{i+1}}(x) - \hat{F}^*_{n,n_i}(x))\\
+ \sqrt{n}\sum_{i=h+1}^{k}\sup_{x\in R}(\hat{F}^*_{n,n_{i-1}}(x) - \hat{F}^*_{n,n_i}(x))
\end{aligned} \tag{4.1}$$

Then, to obtain the proposed bootstrap approximation for the p-value of the test, we give the following specific steps.

Step 1. Calculate the test statistic T_n using the original samples $X_{i1}, \cdots, X_{in_i}, i = 1, 2, \cdots, k$;

Step 2. Draw $\zeta_{n,i}, i = 1, 2, \cdots, n$ from $H_n(x)$ and calculate a bootstrap version of the test statistic, denoted as $\hat{T}_n$, via Eq (4.1);

Step 3. Repeat step 2 a large number of times, say B times, which results in B bootstrap test statistics $\hat{T}_n^{(b)}$, where $b = 1, \cdots, B$;

Step 4. The p-value for the proposed test can be expressed as $p = Card\{b : \hat{T}_n^{(b)} > T_n, b = 1, \cdots, B\}/B$.

Finally, the theorem presented below demonstrates that, with probability 1, the limit distribution of the $\hat{T}_n$ coincides with that given in Theorem 2, which establishes the validity of the bootstrap test theoretically. For the proof of the following theorem, we treat $X_{i1}, \cdots, X_{in_i}, i = 1, 2, \cdots, k$ as the initial segments of k infinite sequences $(X_{ij})_{j\in\mathbb{N}}$ of random variables defined on a certain background probability space $(\Omega, \mathcal{A}, P)$, and the almost sure statements below refer to P.

Theorem 5. Suppose that the H_0 is true and the assumptions (3.1) and (3.2) hold. Then as $n_j \to \infty,\ j = 1, 2, \cdots, k$, we have

$$P(\hat{T}_n \xrightarrow{w} T) = 1$$

where T is the same as in Theorem 2.

5. Numerical results

5.1. Simulation study

In this section, we show the performances of the proposed nonparametric test by simulations. Five distribution families, namely, Uniform, Normal, Exponential, Cauchy, and Logistic distributions, are used in the simulations. We set $k = 4$, and consider four settings of the sample sizes: (A) $n_1 = n_2 = n_3 = n_4 = 20$, (B) $n_1 = n_2 = n_3 = n_4 = 30$, (C) $n_1 = n_2 = n_3 = n_4 = 50$, (D) $n_1 = 20, n_2 = 30, n_3 = 50, n_4 = 70$.

The samples are drawn from several distribution families, including Uniform distributions $U(\theta)$ on finite intervals $(\theta - 0.5, \theta + 0.5)$, Exponential distributions $E(\lambda)$, Normal distributions $N(\mu, \sigma^2)$ with expectation μ and variance σ^2, Cauchy distributions $C(\mu, \sigma)$ with location parameter μ and scale parameter σ, Logistic distributions $L(\mu, \sigma)$ with location parameter μ and scale parameter σ. The simulation results are summarized in Tables 1–4.

The tests proposed in this paper are with a known peak (IRB) and with an unknown peak (IRBU), Chen–Wolfe's test(CW) [10], and Mack–Wolfe's test (MW) [4], where the MW is for a known peak, the CW is for an unknown peak. The significance level is taken as $\alpha = 0.05$. In each case, we carry out 1000 replications to get the empirical size or power of the test. The number B of the bootstrap resampling is 1000 to get the $p-$values.

Table 1. Empirical rejection rates under H_0 with known peak.

Distributions							
F_1	F_2	F_3	F_4	(A)	(B)	(C)	(D)
U(0)	U(0)	U(0)	U(0)	0.040	0.043	0.045	0.046
E(1)	E(1)	E(1)	E(1)	0.042	0.045	0.046	0.044
N(0,1)	N(0,1)	N(0,1)	N(0,1)	0.057	0.055	0.054	0.045
C(0,1)	C(0,1)	C(0,1)	C(0,1)	0.059	0.057	0.055	0.056
L(0,1)	L(0,1)	L(0,1)	L(0,1)	0.043	0.045	0.047	0.054

Table 2. Empirical rejection rates under H_1 with known peak.

Distributions				(A)		(B)		(C)		(D)	
F_1	F_2	F_3	F_4	*IRB*	*MW*	*IRB*	*MW*	*IRB*	*MW*	*IRB*	*MW*
U(1.35)	U(1.5)	U(1.4)	U(1.35)	0.418	0.420	0.528	0.532	0.724	0.729	0.568	0.566
U(0.85)	U(0.9)	U(1)	U(0.95)	0.350	0.353	0.462	0.465	0.709	0.723	0.505	0.513
E(1)	E(1.5)	E(1.3)	E(1)	0.287	0.287	0.453	0.462	0.610	0.610	0.512	0.521
E(0.3)	E(0.5)	E(1)	E(0.5)	0.363	0.365	0.600	0.607	0.714	0.715	0.672	0.678
N(1,1)	N(1.5,1)	N(1.3,1)	N(1,1)	0.487	0.499	0.647	0.653	0.813	0.816	0.810	0.836
N(0.3,1)	N(0.5,1)	N(1,1)	N(0.5,1)	0.695	0.712	0.832	0.830	0.921	0.920	0.915	0.930
C(1,1)	C(1.5,1)	C(1.3,1)	C(1,1)	0.220	0.223	0.302	0.307	0.436	0.457	0.417	0.423
C(0.3,1)	C(0.5,1)	C(1,1)	C(0.5,1)	0.368	0.372	0.386	0.397	0.589	0.597	0.576	0.582
L(1,1)	L(1.5,1)	L(1.3,1)	L(1,1)	0.233	0.235	0.356	0.367	0.476	0.500	0.469	0.473
L(0.3,1)	L(0.5,1)	L(1,1)	L(0.5,1)	0.380	0.383	0.424	0.432	0.673	0.688	0.634	0.641

Table 3. Empirical rejection rates under H_0 with estimated peak.

Distributions							
F_1	F_2	F_3	F_4	(A)	(B)	(C)	(D)
U(0,1)	U(0,1)	U(0,1)	U(0,1)	0.056	0.045	0.053	0.054
E(1)	E(1)	E(1)	E(1)	0.043	0.045	0.055	0.055
N(0,1)	N(0,1)	N(0,1)	N(0,1)	0.056	0.054	0.052	0.047
C(0,1)	C(0,1)	C(0,1)	C(0,1)	0.045	0.056	0.047	0.054
L(0,1)	L(0,1)	L(0,1)	L(0,1)	0.057	0.056	0.055	0.054

Table 4. Empirical rejection rates under H_1 with estimated peak.

Distributions				(A)		(B)		(C)		(D)	
F_1	F_2	F_3	F_4	*IRBU*	*CW*	*IRBU*	*CW*	*IRBU*	*CW*	*IRBU*	*CW*
U(1.35)	U(1.5)	U(1.4)	U(1.35)	0.156	0.297	0.190	0.321	0.387	0.541	0.3170	0.419
U(0.85)	U(0.9)	U(1)	U(0.95)	0.173	0.290	0.209	0.318	0.353	0.547	0.305	0.410
U(1)	U(1.5)	U(1.3)	U(1)	0.786	0.787	0.892	0.893	1.000	1.000	1.000	1.000
U(0.3)	U(0.5)	U(1)	U(0.5)	0.790	0.790	0.895	0.898	1.000	1.000	1.000	1.000
E(1)	E(1.5)	E(1.3)	E(1)	0.245	0.249	0.378	0.385	0.516	0.509	0.412	0.424
E(0.3)	E(0.5)	E(1)	E(0.5)	0.261	0.278	0.383	0.397	0.600	0.603	0.530	0.543
N(1,1)	N(1.5,1)	N(1.3,1)	N(1,1)	0.400	0.404	0.542	0.549	0.836	0.840	0.755	0.756
N(0.3,1)	N(0.5,1)	N(1,1)	N(0.5,1)	0.503	0.511	0.751	0.763	0.869	0.876	0.853	0.864
C(1,1)	C(1.5,1)	C(1.3,1)	C(1,1)	0.171	0.178	0.204	0.211	0.436	0.467	0.354	0.360
C(0.3,1)	C(0.5,1)	C(1,1)	C(0.5,1)	0.206	0.212	0.295	0.309	0.470	0.504	0.412	0.418
L(1,1)	L(1.5,1)	L(1.3,1)	L(1,1)	0.170	0.176	0.199	0.210	0.498	0.511	0.286	0.300
L(0.3,1)	L(0.5,1)	L(1,1)	L(0.5,1)	0.168	0.171	0.180	0.181	0.578	0.601	0.306	0.308

Tables 1 and 3 show the results of the cases in which H_0 is true, and the peak is known and unknown, respectively. In both cases, the empirical rejection rates are close to the significance level α and get closer as the increasing of sample sizes.

From the results of Tables 2 and 4, we observe that the empirical rejection rates are consistently greater than the significance level α and increase monotonically with the sample sizes. In most scenarios, the power of the IRB/IRBU tests is comparable to that of the classical MW/CW tests, with differences largely confined to within 0.03. A notable exception is the uniform distribution case in Table 4, where the IRBU test exhibits a measurable power deficit relative to the CW test. This observation confirms that when the data distribution provides weak local contrast around the peak (i.e., in the presence of a "flat peak"), the accuracy of peak identification is compromised, which in turn reduces the test's power.

In summary, although the above simulations are carried out in several parametric distribution families, we can inspect the performance of the proposed test through the above important cases. All in all, the simulation results show that the theoretical results in Section 3 and Section 4 are a feasible solution for the homogeneity test problem under umbrella ordering.

5.2. *Example*

We consider the example in Alvo [14] on the Wechsler adult intelligence scale scores on males by age groups. We reproduce the data in Table 5.

Table 5. Example date set.

Age group				
16 – 19	20 – 34	35 – 54	55 – 69	> 70
8.62	9.85	9.98	9.12	4.80
9.94	10.43	10.69	9.89	9.18
10.06	11.31	11.40	10.57	9.27

Figure 1 is the empirical distribution function diagram for the five groups of data.

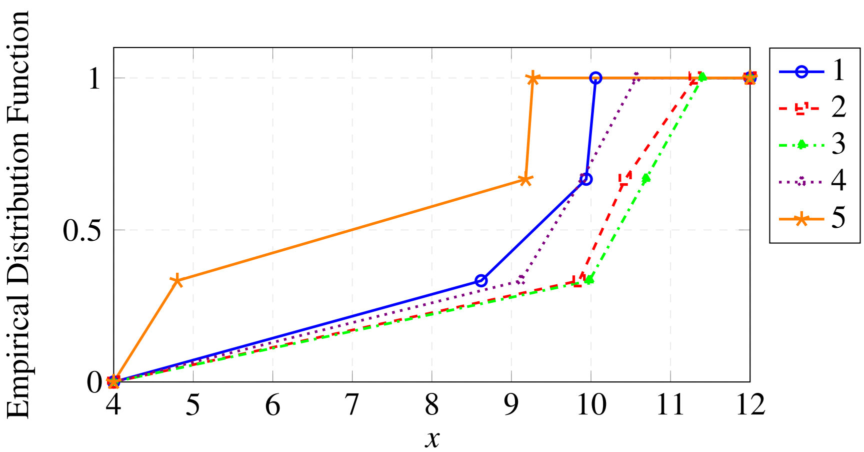

Figure 1. Empirical distribution functions of five groups.

From the empirical distribution function diagram, we observe that the empirical distribution functions satisfy $\hat{F}_1(x) \geq \hat{F}_2(x) \geq \hat{F}_3(x) \leq \hat{F}_4(x) \leq \hat{F}_5(x)$, which corresponds to an inverted umbrella ordering. In contrast, the complementary functions satisfy $1 - \hat{F}_1(x) \leq 1 - \hat{F}_2(x) \leq 1 - \hat{F}_3(x) \geq 1 - \hat{F}_4(x) \geq 1 - \hat{F}_5(x)$, a pattern consistent with an umbrella ordering.

We use the testing method proposed in this paper for analysis, so as to confirm whether the test result is consistent with the intuitively observed order characteristics of the distribution functions, and compare our results with the results obtained from the permutation test of Basso and Salmaso [23].

Assuming that the location of the peak is unknown, based on the above data and significance level of $\alpha = 0.05$, we obtain the peak estimate $\hat{h} = 3$, which is consistent with the result in Basso and Salmaso [23]. To get the $p-$values, we carry out 100000 bootstrap resamplings and obtain a $p-$value of 0.01053, and the resulting $p-$value of Basso and Salmaso [23] is 0.00299. Both tests reject the null hypothesis.

6. Concluding remarks

In this paper, we propose tests of homogeneity against the umbrella ordering alternative for k continuous distribution functions, considering both cases in which the peak of the umbrella ordering is known and unknown, respectively. We construct the test statistics through the isotonic estimators and establish asymptotic null distributions of the test statistics. A bootstrap procedure is employed to give the p-value of the proposed tests. Numerical results indicate that the empirical sizes of the proposed tests are close to the nominal significance level α under H_0, and that the empirical powers of the tests are satisfactory.

Compared with the classical MW and CW tests, the IRB/IRBU methods proposed in this paper establish a more comprehensive theoretical framework. Supported by Theorems 1–5, this framework guarantees well-defined asymptotic properties, effective testing power, and reliable practical implementation. It should be noted, however, that when the peak is flat (i.e., the data distribution around the peak shows weak differentiation), the accuracy of peak localization may deteriorate. Future methodological refinements could explore adaptive mechanisms to enhance robustness in such low-contrast scenarios.

Use of generative-AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest.

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Appendix: Proofs

To prove Theorem 2, we first give the following lemmas.

Lemma 1. Suppose that the umbrella ordering condition (2.1) and assumptions (3.1) and (3.2) are satisfied. Then

$$(Z^*_{1n_1}(x), Z^*_{2n_2}(x), \cdots, Z^*_{kn_k}(x))^T \xrightarrow{w} (Z^*_1(x), Z^*_2(x), \cdots, Z^*_k(x))^T,$$

as $n_i \to \infty,\ i = 1, 2, \cdots, k$, where

$$Z^*_i(x) = \sqrt{a_i} \max_{r\le i} \min_{i\le s\le h} \frac{\sum_{\{r\le j\le s, r,s\in S_i\}} \sqrt{a_j} Z_j(x)}{\sum_{j=r}^{s} a_j}, i = 1, 2, \cdots, h-1;$$

$$Z^*_h(x) = \sqrt{a_h} \max_{r\le h} \max_{h\le s} \frac{\sum_{\{r\le j\le s, r,s\in S_i\}} \sqrt{a_j} Z_j(x)}{\sum_{j=r}^{s} a_j};$$

$$Z^*_i(x) = \sqrt{a_i} \min_{h\le r\le i} \max_{i\le s} \frac{\sum_{\{r\le j\le s, r,s\in S_i\}} \sqrt{a_j} Z_j(x)}{\sum_{j=r}^{s} a_j}, i = h+1, \cdots, k.$$

The proof of Lemma 1 is straightforward from Theorem 4 in El Barmi and Mukerjee [26] and is omitted.

Based on the conclusions of Lemma 1 and Lemma 1 in Baringhaus and Gr*ü*bel [29], we give now the proof of Theorem 2.

Proof of Theorem 2. It is easy to see that

$$\begin{aligned}
T_n &= \sqrt{n} \sum_{i=1}^{h-1} \sup_{x\in R} (\hat{F}^*_{i+1}(x) - \hat{F}^*_i(x)) + \sqrt{n} \sum_{i=h+1}^{k} \sup_{x\in R} (\hat{F}^*_{i-1}(x) - \hat{F}^*_i(x)) \\
&= \sum_{i=1}^{h-1} \sup_{x\in R} [\sqrt{\frac{n}{n_{i+1}}} \sqrt{n_{i+1}} (\hat{F}^*_{i+1}(x) - F_{i+1}(x)) - \sqrt{\frac{n}{n_i}} \sqrt{n_i} (\hat{F}^*_i(x) - F_i(x)) \\
&+ \sqrt{n}(F_{i+1}(x) - F_i(x))] + \sum_{i=h+1}^{k} \sup_{x\in R} [\sqrt{\frac{n}{n_{i-1}}} \sqrt{n_{i-1}} (\hat{F}^*_{i-1}(x) - F_{i-1}(x)) \\
&- \sqrt{\frac{n}{n_i}} \sqrt{n_i} (\hat{F}^*_i(x) - F_i(x)) + \sqrt{n}(F_{i-1}(x) - F_i(x))] \\
&= \sum_{i=1}^{h-1} \sup_{x\in R} [\sqrt{\frac{n}{n_{i+1}}} Z^*_{i+1 n_{i+1}}(x) - \sqrt{\frac{n}{n_i}} Z^*_{in_i}(x) + \sqrt{n}(F_{i+1}(x) - F_i(x))] \\
&+ \sum_{i=h+1}^{k} \sup_{x\in R} [\sqrt{\frac{n}{n_{i-1}}} Z^*_{i-1 n_{i-1}}(x) - \sqrt{\frac{n}{n_i}} Z^*_{in_i}(x) + \sqrt{n}(F_{i-1}(x) - F_i(x))] \qquad \text{(A.1)}
\end{aligned}$$

Under H_0, the third term in both the brackets on the right-hand side of (A.1) is just zero. Therefore, we have

$$\begin{aligned}
T_n = &\sum_{i=1}^{h-1} \sup_{x\in R} [\sqrt{\frac{n}{n_{i+1}}} Z^*_{i+1 n_{i+1}}(x) - \sqrt{\frac{n}{n_i}} Z^*_{in_i}(x)] \\
&+ \sum_{i=h+1}^{k} \sup_{x\in R} [\sqrt{\frac{n}{n_{i-1}}} Z^*_{i-1 n_{i-1}}(x) - \sqrt{\frac{n}{n_i}} Z^*_{in_i}(x)].
\end{aligned}$$

By Lemma 1, Lemma 1 in Baringhaus and Grübel [29] and the Slutsky theorem, we have

$$\begin{aligned} T_n \xrightarrow{w} T &= \sum_{i=1}^{h-1} \sup_{x\in R}\Big(\sqrt{\frac{1}{a_{i+1}}}Z_{i+1}^*(x) - \sqrt{\frac{1}{a_i}}Z_i^*(x)\Big) \\ &+ \sum_{i=h+1}^{k} \sup_{x\in R}\Big(\sqrt{\frac{1}{a_{i-1}}}Z_{i-1}^*(x) - \sqrt{\frac{1}{a_i}}Z_i^*(x)\Big). \end{aligned}$$

Proof of Theorem 3. The first two terms in the first bracket and the first two terms in the second bracket on the right-hand side of (A.1) are stochastically bounded uniformly for all $i = 1, \cdots, k$. We now consider the third term in the two brackets.

If H_1 does hold, then there is at least one $i \in \{1, 2, \cdots, h-1\}(i \in \{h+1, h+2, \cdots, k\})$ which satisfies $F_i(x) < F_{i+1}(x)(F_{i-1}(x) > F_i(x))$ for all x in some non-empty interval $(a, b) \subset R$. We have

$$\begin{aligned} &\sup_{x\in[a,b]} (F_{i+1}(x) - F_i(x)) > 0, i \in \{1, 2, \cdots, h-1\} \\ &(\sup_{x\in[a,b]} (F_{i-1}(x) - F_i(x)) > 0, i \in \{h+1, h+2, \cdots, k\}). \end{aligned}$$

Therefore, we have $T_n \to \infty$ with probability one.

Proof of Theorem 4. We first prove $P(\hat{h} > h_0) \to 0$ as $n \to \infty$. If $h_0 = k$, it is obviously true. Now consider the case of $h_0 < k$. By (3.3), we have

$$\begin{aligned} P(\hat{h} > h_0) &= P\Big(\Big[\sup_{x\in R}\sum_{j=1}^{k}(\hat{\hat{F}}_j^{h_0+1}(x) - \hat{F}_j(x))^2 < \sup_{x\in R}\sum_{j=1}^{k}(\hat{\hat{F}}_j^{h_0}(x) - \hat{F}_j(x))^2\Big] \\ &\quad \cup \cdots \cup \Big[\sup_{x\in R}\sum_{j=1}^{k}(\hat{\hat{F}}_j^{k}(x) - \hat{F}_j(x))^2 < \sup_{x\in R}\sum_{j=1}^{k}(\hat{\hat{F}}_j^{h_0}(x) - \hat{F}_j(x))^2\Big]\Big) \\ &\le \sum_{l=h_0+1}^{k} P\Big(\sup_{x\in R}\sum_{j=1}^{k}(\hat{\hat{F}}_j^{l}(x) - \hat{F}_j(x))^2 < \sup_{x\in R}\sum_{j=1}^{k}(\hat{\hat{F}}_j^{h_0}(x) - \hat{F}_j(x))^2\Big) \end{aligned} \tag{A.2}$$

Under H_1, by Theorem 1 and $|\ \hat{F}_j(x) - F_j(x)\ | \xrightarrow{\text{a.e.}} 0$, we have

$$P[\|\ \hat{\hat{F}}_j^{h_0}(x) - \hat{F}_j(x)\ \| \to 0,\ n_j \to \infty,\ j = 1, \cdots, k] = 1.$$

Obviously,

$$P[\sup_{x\in R}\sum_{j=1}^{k}(\hat{\hat{F}}_j^{h_0}(x) - \hat{F}_j(x))^2 \to 0, n_j \to \infty,\ j = 1, 2, \cdots, k] = 1. \tag{A.3}$$

By assumption (3.2) and the definition of the peak, there must be at least one $i_0 > h_0$, such that $F_{i_0-1}(x) > F_{i_0}(x)$ on some non-empty subinterval $[a, b] \subset R$. Denote $\varepsilon_0 = \inf_{x\in[a,b]}[F_{i_0-1}(x) - F_{i_0}(x)]$. Then by the definition of $\hat{\hat{F}}_{i_0-1}^{i_0}(x)$, we have

$$[\hat{\hat{F}}_{i_0-1}^{i_0}(x) - \hat{F}_{i_0-1}(x)]^2 \xrightarrow{\text{a.e.}} \beta^2[F_{i_0}(x) - F_{i_0-1}(x)]^2 \ge \beta^2\varepsilon_0^2$$

as $n_j \to \infty,\ j = 1, 2, \cdots, k$, where $\beta = \lim_{n\to\infty} \frac{n_{i_0}}{n_{i_0-1}+n_{i_0}} > 0$. Combining(A.2) and (A.3), this implies that $P(\hat{h} > h_0) \to 0$ as $n_j \to \infty,\ j = 1, 2, \cdots, k$.

Similarly, we can get $P(\hat{h} < h_0) \to 0$. Thus, Theorem 4 is proved.

To prove Theorem 5, we first give the following Lemma 2 (see Theorem 3.7.7 of van der Vaart and Wellner [30]).

Lemma 2. Suppose that $X_1, X_2, \cdots, X_m$ is an *i.i.d.* sample from the probability measure P, and $Y_1, Y_2, \cdots, Y_n$ is an *i.i.d.* sample from the probability measure Q. Let $\mathcal{F}$ be a class of measurable functions that is Donsker under both P and Q and possesses an envelope function F with both $P^*F^2 < \infty$ and $Q^*F^2 < \infty$ ($*$ denotes the outer probability). Denote $N = m + n$. If $m, n \to \infty$ such that $m/N \to \lambda \in (0, 1)$, then $\sqrt{m}(\hat{\mathbb{P}}_{m,N} - \mathbb{H}_N) \xrightarrow{w} \mathbb{G}_H$ given almost every sequence $X_1, X_2 \cdots, Y_1, Y_2, \cdots$. Here

$$\hat{\mathbb{P}}_{m,N} = \frac{1}{m}\sum_{i=1}^{m} \delta_{\hat{Z}_{Ni}},\ \hat{\mathbb{Q}}_{n,N} = \frac{1}{n}\sum_{i=1}^{n} \delta_{\hat{Z}_{N,m+i}},$$

where $\hat{Z}_{N1}, \cdots, \hat{Z}_{NN}$ is an *i.i.d.* sample from the pooled empirical measure $\mathbb{H}_N$ and $\mathbb{G}_H$ is a tight Brownian bridge process corresponding to the measure $H = \lambda P + (1 - \lambda)Q$.

The conclusion can be easily extended to $k(> 2)$ distributions. The only change in the conditions is $n_i / \sum_{j=1}^{k} \to a_i \in (0, 1), i = 1, \cdots, k$, which is satisfied by the assumptions in Theorem 5. In our case, $\mathcal{F} = \{1_{(-\infty,t]}(x) : t \in R\}$, and all the conditions on $\mathcal{F}$ in Lemma 2 are satisfied.

Proof of Theorem 5. Let

$$\begin{aligned} \hat{Z}_{n,n_i}(x) &= \sqrt{n_i}(\hat{F}_{n,n_i}(x) - H_n(x)), \\ \hat{Z}^*_{n,n_i}(x) &= \sqrt{n_i}(\hat{F}^*_{n,n_i}(x) - H_n(x)), \quad i = 1, \cdots, k. \end{aligned}$$

$$\begin{aligned} \hat{T}_n &= \sqrt{n}\sum_{i=1}^{h-1} \sup_{x\in R}[(\hat{F}^*_{n,n_{i+1}}(x) - H_n(x)) - (\hat{F}^*_{n,n_i}(x) - H_n(x))] \\ &+ \sqrt{n}\sum_{i=h+1}^{k} \sup_{x\in R}[(\hat{F}^*_{n,n_{i-1}}(x) - H_n(x)) - (\hat{F}^*_{n,n_i}(x) - H_n(x))] \\ &= \sum_{i=1}^{h-1} \sup_{x\in R}[\sqrt{\frac{n}{n_{i+1}}}\sqrt{n_{i+1}}\hat{Z}^*_{n,n_{i+1}}(x) - \sqrt{\frac{n}{n_i}}\sqrt{n_i}\hat{Z}^*_{n,n_i}(x)] \\ &+ \sum_{i=h+1}^{k} \sup_{x\in R}[\sqrt{\frac{n}{n_{i-1}}}\sqrt{n_{i-1}}\hat{Z}^*_{n,n_{i-1}}(x) - \sqrt{\frac{n}{n_i}}\sqrt{n_i}\hat{Z}^*_{n,n_i}(x)] \end{aligned}$$

According to Lemma 2, with probability 1, we have

$$\hat{Z}_{n,n_i}(x) \xrightarrow{w} Z_i^H(x),\ i = 1, \cdots, k,$$

where $Z_i^H(x) \stackrel{d}{=} B_i(H(x)), B_i(t),\ i = 1, \cdots, k$ are independent standard Brownian bridges, $H(x) = a_1F_1(x) + \cdots + a_kF_k(x)$ is the pooled distribution. By condition (2.1), assumption (3.2), and continuous mapping theorem, we have

$$\hat{Z}^*_{n,n_i}(x) \xrightarrow{w} Z_i^{*H}(x),$$

for given almost every sample sequences, as $n_i \to \infty$, $i = 1, 2, \cdots, k$, where

$$Z_i^{*H}(x) = \sqrt{a_i} \max_{r \le i} \min_{i \le s \le h} \frac{\sum_{\{r \le j \le s, r, s \in S_i\}} \sqrt{a_j} Z_j^H(x)}{\sum_{j=r}^{s} a_j}, i = 1, 2, \cdots, h-1;$$

$$Z_h^{*H}(x) = \sqrt{a_h} \max_{r \le h} \max_{h \le s} \frac{\sum_{\{r \le j \le s, r, s \in S_i\}} \sqrt{a_j} Z_j^H(x)}{\sum_{j=r}^{s} a_j};$$

$$Z_i^{*H}(x) = \sqrt{a_i} \min_{h \le r \le i} \max_{i \le s} \frac{\sum_{\{r \le j \le s, r, s \in S_i\}} \sqrt{a_j} Z_j^H(x)}{\sum_{j=r}^{s} a_j}, i = h+1, \cdots, k.$$

By Slutsky theorem, we have

$$\begin{aligned} \hat{T}_n \xrightarrow{w} & \sum_{i=1}^{h-1} \sup_{x \in R} \left(\sqrt{\frac{1}{a_{i+1}}} Z_{i+1}^{*H}(x) - \sqrt{\frac{1}{a_i}} Z_i^{*H}(x) \right) \\ & + \sum_{i=h+1}^{k} \sup_{x \in R} \left(\sqrt{\frac{1}{a_{i-1}}} Z_{i-1}^{*H}(x) - \sqrt{\frac{1}{a_i}} Z_i^{*H}(x) \right) \end{aligned} \tag{A.4}$$

with probability 1.

Under H_0, we have $Z_i^H(x) = Z_i(x)$, $Z_i^{*H}(x) = Z_i^*(x)$, $i = 1, \cdots, k$. By (A.4), we have $P(\hat{T}_n \xrightarrow{w} T) = 1$.