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Research article

Pullback attractors and statistical solutions for the lattice Zakharov equations on time-dependent spaces

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Abstract: In this paper, the authors investigate the probability distribution of solutions within the time-dependent phase spaces for the lattice Zakharov equations with varying coefficients via the pullback attractors and the notion of generalized Banach limits. They firstly show that the addressed initial value problem is globally well-posed and prove that the related evolution process has a time-dependent pullback attractor on the time-dependent phase spaces. Then they construct a family of invariant Borel probability measures with supports contained in the pullback attractor. Furthermore, they prove that the constructed family of invariant measures is a statistical solution for the addressed lattice Zakharov equations and that Liouville's theorem holds true.

Keywords: time-dependent pullback attractor; lattice Zakharov equations; varying coefficient; invariant Borel probability measure; statistical solution
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1. Introduction

In [1], the authors combined the ideas of [2–5] and developed an approach for constructing statistical solutions on time-dependent phase spaces for second-order lattice systems. The goal here is to follow the approaches of [1, 6] by presenting a certain scheme to construct statistical solutions for a coupled system containing a lattice nonlinear Schrödinger equation and a lattice wave equation with varying coefficients. We will restrict our attention to the existence of statistical solutions and the correctness of Liouville's theorem on time-dependent phase spaces. Some further issues such as the singular limits of statistical solutions will be investigated in a forthcoming work.

Consider the following lattice Zakharov equations with time-dependent coefficient:

$$i\dot{\varphi}_n + (A\varphi)_n - h^2(D\varphi)_n - u_n\varphi_n + i\gamma\varphi_n = f_n(t), \tag{1.1}$$

$$\epsilon(t)\ddot{u}_n + \lambda\dot{u}_n - (Au)_n + h^2(Du)_n - (A|\varphi|^2)_n + \mu u_n = g_n(t), \tag{1.2}$$

$$\varphi_n(\tau) = \varphi_{n,\tau}, \quad u_n(\tau) = u_{n,\tau}, \quad \dot{u}_n(\tau) = u_{1n,\tau}, \tag{1.3}$$

where $n \in \mathbb{Z}$, $t > \tau$, unknowns $\varphi_n(\cdot) \in \mathbb{C}$ and $u_n(\cdot) \in \mathbb{R}$, $h, \gamma, \alpha, \mu > 0$ are constants, $\epsilon(\cdot) > 0$ is the varying coefficient function, and the real-valued functions g_n and the complex-valued functions f_n represent external forces. In addition, $|\varphi|^2 = (|\varphi_n|^2)_{n\in\mathbb{Z}}$, A and D are linear operators defined as

$$(Au)_n = u_{n+1} - 2u_n + u_{n-1}, \quad (Du)_n = u_{n+2} - 4u_{n+1} + 6u_n - 4u_{n-1} + u_{n-2}, \ \forall\, u = (u_m)_{m\in\mathbb{Z}}.$$

Equations (1.1) and (1.2) can be viewed as a discrete approximation for the spatial variable x on the one-dimensional real line $\mathbb{R}$ with step length one of the following quantum Zakharov equations with time-dependent coefficients:

$$\begin{cases} i\varphi_t + \varphi_{xx} - h^2\varphi_{xxxx} - \varphi u + i\gamma\varphi = f(x,t), \\ \epsilon(t)u_{tt} - u_{xx} + h^2 u_{xxxx} - |\varphi|^2_{xx} + \lambda u_t + \mu u = g(x,t), \end{cases} \tag{1.4}$$

where the unknown function $\psi : \mathbb{R}\times\mathbb{R} \mapsto \mathbb{C}$ represents the envelope of the high-frequency electric field, and the unknown function $u : \mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}$ denotes the plasmas density measured relative to the equilibrium value ([7]). When $\epsilon(t) \equiv 1$ for $t \in \mathbb{R}$, both the discrete system (1.1)–(1.2) and the continuous one (1.4) were extensively studied. We can refer to chronologically [7–14] and references therein.

This paper focuses on the probability distribution of solutions on the chosen time-dependent phase spaces for the system (1.1)–(1.2). The probability distribution of solutions for different types of evolutionary equations is now usually characterized by statistical solutions. We first recall some published results. In terms of invariant measures, reference [2] proved a sufficient condition for the existence for a broad class of dissipative continuous semigroup, and [5] extended the result of [2] to the non-autonomous situation by presenting a sufficient condition for the existence of invariant measures for dissipative continuous process. In terms of statistical solutions, references [15, 16] studied the existence and properties of the statistical solutions for the two-dimensional (2D) and three-dimensional (3D) incompressible Navier-Stokes equations; reference [17] presented an abstract framework for the theory of statistical solutions and trajectory statistical solutions for general evolutionary equations; reference [18] established sufficient conditions for the existence of trajectory statistical solutions for general autonomous evolutionary equations. Very recently, Zhao investigated in [19] the existence of trajectory statistical solutions for the second-order elliptic equations in half-cylindrical domains. Note that all the equations aforementioned were investigated in fixed phase spaces or fixed trajectory spaces.

There are two results within the current article. The first one shows that system (1.1)–(1.2) has a time-dependent pullback attractor within the family of time-dependent phase spaces. The second one proves that system (1.1)–(1.2) admits invariant measures and statistical solutions. The pullback attractor is essential for the construction of statistical solution for Eqs (1.1) and (1.2). On the one hand, these results can characterize the pullback asymptotic behavior of solutions to the lattice equations under investigation, as well as their probability distributions in the phase space. On the other hand, these findings can provide a reference for the numerical computation of partial differential equations. Nowadays, modern computational techniques for addressing complexity challenges in diverse application scenarios is an important research topic in the fields of differential equations and computer simulations. For instance, reference [20] proposed a dual-level fast direct solver for efficiently solving the dense and ill-conditioned linear equations arising from acoustic scattering problems. This solver outperforms the Generalized Minimal Residual Method (GMRES) and has been validated by cases such as the acoustic scattering simulations of the A-320 aircraft and the human head, thereby providing an efficient numerical solution scheme for the rapid calculation of large-scale acoustic scattering from complex targets.

Compared to reference [14], the main difficulty we encounter here comes from the varying coefficient $\epsilon(\cdot)$. As will be seen, it is more convenient to settle problem (1.1)–(1.3) in time-dependent phase spaces than in fixed ones. The theory of time-dependent attractors has been well developed. Firstly, Flandoli and Schmalfuss presented the approach to deal with time dependent phase spaces when they studied the random attractors for the 3D stochastic Navier-stokes equation with multiplicative white noise (see [4]). Later, Temam and his group formulated the theory of time-dependent attractors during the study of non-autonomous oscillon equation and wave equation with varying coefficient [3, 21], and this theory was developed by [22–26] to study the dynamical behavior of reaction-diffusion equations, as well as that of wave equations with varying coefficients.

Although we can borrow the theory of time-dependent attractor and employ the sufficient and necessary condition guaranteeing the existence of the time-dependent attractor for dissipative lattice systems with varying coefficients (see [1, Lemma 3.1]) in our study, there are some additional difficulties, in comparison with [6] where the lattice Klein-Gordon-Schrödinger equations with varying coefficient were investigated, when we estimate the solutions and establish the pullback asymptotic compactness of the generated evolution process. In fact, the nonlinear term $A|\psi|^2$, and the additional terms $D\psi$ and Du corresponding to the higher-order derivative terms $|\psi|^2_{xx}$, ψ_{xxxx} and u_{xxxx} will produce some addition difficulties when we verify the uniform estimates on "Tail End" of solutions. Because that these terms require us to construct subtle time-dependent phase spaces where to establish a certain coercive property of the operator corresponding to the linear principle part extracted from Eqs (1.1) and (1.2).

The paper is arranged as follows. Section 2 is some preparation work including some notations and lemmas concerning the global well-posedness of problem (1.1)–(1.3). Section 3 shows that the generated evolution process admits a time-dependent pullback attractor. Section 4 first constructs a family of invariant Borel probability measures and then proves that the constructed invariant measures are a statistical solution for system (1.1)–(1.2) and that Liouville's theorem holds true. The last section is a short summary and some discussions. The usual symbols frequently used throughout the article are compiled in the Appendix.

2. Global well-posedness

In this article, for simplicity, we use X to denote ℓ^2 or l^2 defined as

$$\ell^2 = \{u = (u_k)_{k\in\mathbb{Z}} : u_k \in \mathbb{C}, \Sigma_{k\in\mathbb{Z}}|u_k^2| < +\infty\}, \;\; l^2 = \{u = (u_k)_{k\in\mathbb{Z}} : u_k \in \mathbb{R}, \Sigma_{k\in\mathbb{Z}}u_k^2 < +\infty\},$$

and equip it with the inner product and norm as $(u, v) = \sum_{k\in\mathbb{Z}} u_k\bar{v}_k$, $\|u\|^2 = (u, u)$, $u = (u_k)_{k\in\mathbb{Z}}$, $v = (v_k)_{k\in\mathbb{Z}} \in X$. It is evident that $(X, (\cdot, \cdot))$ is a Hilbert space. Let B and B^* be two linear operators on X, given by $(Bu)_k = u_{k+1} - u_k$, $(B^*u)_k = u_{k-1} - u_k$, where $u = (u_k)_{k\in\mathbb{Z}}$. Obviously, the operators B and B^*, which map X into itself, are bounded. Furthermore,

$$(Au,v) = -(Bu,Bv) = -(B^*Bu,v),\ \ (Du,v) = (Au,Av),\ \forall\, u,v \in X,$$
$$\|Bu\| = \|B^*u\| \leqslant 2\|u\|,\ \ \|Au\| \leqslant 4\|u\|,\ \|Du\| \leqslant 16\|u\|,\ \forall\, u \in X.$$

In addition, we pick throughout this article

$$\sigma_0 = \frac{\mu\lambda}{\sqrt{\lambda^2+\mu\lambda}(\lambda+\sqrt{\lambda^2+\mu\lambda})},\quad \sigma = \min\{\frac{\sigma_0}{2},\frac{\gamma}{4}\}.$$

Write

$$\varphi = (\varphi_n)_{n\in\mathbb{Z}},\ u = (u_n)_{n\in\mathbb{Z}},\ \varphi u = (\varphi_n u_n)_{n\in\mathbb{Z}},\ \epsilon(t)u = (\epsilon(t)u_n)_{n\in\mathbb{Z}},$$
$$f(t) = (f_n(t))_{n\in\mathbb{Z}},\ g(t) = (g_n(t))_{n\in\mathbb{Z}}.$$

Then we can put problem (1.1)–(1.3) as

$$i\dot{\varphi} + A\varphi - h^2D\varphi - u\varphi + i\gamma\varphi = f(t), \tag{2.1}$$
$$\epsilon(t)\ddot{u} + \lambda\dot{u} - Au + h^2Du - A|\varphi|^2 + \mu u = g(t), \tag{2.2}$$
$$\varphi(\tau) = \varphi_\tau,\ \ u(\tau) = u_\tau,\ \ \dot{u}(\tau) = u_{1,\tau}. \tag{2.3}$$

In the following research on problem (2.1)–(2.3), we need following assumptions:

(A1) Assume that function $\epsilon(\cdot) \in C^1(\mathbb{R})$ is bounded and monotonically decreasing, satisfying for any $t \in \mathbb{R}$,

$$\begin{cases} \lim_{t\to+\infty}\epsilon(t) = 0 \ \text{ and } \ \epsilon(t) - \epsilon'(t) \leqslant \frac{\lambda}{4}, \\ |\frac{\epsilon'(t)}{\epsilon(t)}| \leqslant \sqrt{\frac{16\mu^2}{\lambda^2} + \frac{16\mu\sigma}{\lambda}} - \frac{4\mu}{\lambda}, \\ 16h^2(\lambda^2 + 4\mu\epsilon(t))^2 \leqslant 4\mu^2\lambda\epsilon'(t)\epsilon(t) + \mu^2\lambda^2\epsilon(t). \end{cases} \tag{2.4}$$

(A2) Assume that $f(\cdot) \in C(\mathbb{R},\ell^2)$, $g(\cdot) \in C(\mathbb{R},l^2)$. Moreover, suppose that

$$\int_{-\infty}^{t} e^{\sigma s}\|g(s)\|^2 \mathrm{d}s < +\infty,\ t \in \mathbb{R}, \tag{2.5}$$

and that there exists a continuous function $J(\cdot)$ defined on the real line, bounded on every interval of the form $(-\infty,t)$, such that

$$\int_{-\infty}^{t} e^{\sigma s}\|f(s)\|^2 \mathrm{d}s \leqslant e^{(\frac{\sigma}{2}+\varrho)t}J(t) < +\infty,\ \text{for every } t \in \mathbb{R},\ \text{where } 0 < \varrho < \frac{\sigma}{2}. \tag{2.6}$$

We want to remark that there indeed exist functions $\epsilon(t)$, $f(t)$ and $g(t)$ satisfying Assumptions (A1)–(A2). In fact, we can check directly that

$$\epsilon(t) = (\frac{8}{\lambda} + e^{(\sqrt{\frac{16\mu^2}{\lambda^2}+\frac{16\mu\sigma}{\lambda}}-\frac{4\mu}{\lambda})t})^{-1},\ t \in \mathbb{R},$$

satisfies Assumption (A1), and can refer to [27, Example 3.1] for the existence of functions $f(\cdot)$ and $g(\cdot)$ satisfying assumption (A2).

Next, for each $t \in \mathbb{R}$, we set $\mathfrak{s}(t) = \dfrac{\lambda\mu\epsilon(t)}{\lambda^2 + 4\mu\epsilon(t)}$ and

$$v(t) = \dot{u}(t) + \frac{\mathfrak{s}(t)}{\epsilon(t)}u(t). \tag{2.7}$$

Let $z = (\varphi, u, v)^T$, $\Upsilon(z, \cdot) = (-i\varphi u - if(\cdot), 0, \frac{A|\varphi|^2 + g(\cdot)}{\epsilon(\cdot)})^T$ and

$$\Theta(\cdot) = \begin{pmatrix} \gamma I - iA + ih^2 D & 0 & 0 \\ 0 & \frac{\mathfrak{s}(\cdot)}{\epsilon(\cdot)}I & -I \\ 0 & \frac{4\mu}{\lambda^2}\mathfrak{s}'(\cdot)I - \frac{A}{\epsilon(\cdot)} + \frac{h^2 D}{\epsilon(\cdot)} + \frac{\mu I}{\epsilon(\cdot)} - \frac{(\lambda - \mathfrak{s}(\cdot))\mathfrak{s}(\cdot)}{\epsilon^2(\cdot)}I & \frac{\lambda - \mathfrak{s}(\cdot)}{\epsilon(\cdot)}I \end{pmatrix}. \tag{2.8}$$

Thus, we can equivalently rewrite the problem (2.1)–(2.3) as

$$\dot{z} + \Theta(t)z = \Upsilon(z, t), \quad t > \tau, \tag{2.9}$$

$$z(\tau) = z_\tau = (\varphi_\tau, u_\tau, v_\tau)^T, \tag{2.10}$$

where $v_\tau = u_{1\tau} + \frac{\mathfrak{s}(t)}{\epsilon(t)}u_\tau$.

We now introduce the time-dependent phase spaces by endowing l^2 with time-dependent norms. Firstly define $(u, v)_\mu = (Bu, Bv) + \mu(u, v)$, $u, v \in l^2$. It is easy to check that

$$\mu\|u\|^2 \leqslant \|u\|_\mu^2 = (u, u)_\mu = \|Bu\|^2 + \mu\|u\|^2 \leqslant (4 + \mu)\|u\|^2, \ \forall\, u \in l^2, \tag{2.11}$$

and that $(\cdot, \cdot)_\mu$ is an inner product in l^2 which induce a norm $\|\cdot\|_\mu$ equivalent to $\|\cdot\|$. Then for any $u, v \in l^2$ we define

$$(u, v)_{\epsilon(t)} = \frac{1}{\epsilon(t)}(u, v)_\mu = \frac{1}{\epsilon(t)}(Bu, Bv) + \frac{1}{\epsilon(t)}\mu(u, v), \tag{2.12}$$

where $\epsilon(\cdot)$ is the varying coefficient function in Eq (1.2). Consequently,

$$\frac{\mu}{\epsilon(t)}\|u\|^2 \leqslant \|u\|_{\epsilon(t)}^2 = (u, u)_{\epsilon(t)} = \frac{1}{\epsilon(t)}\|u\|_\mu^2 \leqslant \frac{4 + \mu}{\epsilon(t)}\|u\|^2, \ \forall\, u \in l^2, \tag{2.13}$$

implying that, for any $t \in \mathbb{R}$, the inner product $(\cdot, \cdot)_{\epsilon(t)}$ induces a norm $\|\cdot\|_{\epsilon(t)}$ equivalent to $\|\cdot\|$ in l^2, and that $l^2_{\epsilon(t)} = (l^2, (\cdot, \cdot)_{\epsilon(t)})$ is a Hilbert space. From now on, we write $E = \ell^2 \times l^2 \times l^2$ and define the time-dependent spaces $E_t = \ell^2 \times l^2_{\epsilon(t)} \times l^2$. We equip E and E_t respectively with inner products and norms as $(z_1, z_2)_E = (\varphi_1, \varphi_2) + (u_1, u_2) + (v_1, v_2)$, $\|z\|_E^2 = (z, z)_E$, $(z_1, z_2)_{E_t} = (\varphi_1, \varphi_2) + (u_1, u_2)_{\epsilon(t)} + (v_1, v_2)$, $\|z\|_{E_t}^2 = (z, z)_{E_t}$.

Straightforward computations give

$$\begin{cases} \min\{\frac{\mu}{\epsilon(t)}, 1\}\|z\|_E^2 \leqslant \|z\|_{E_t}^2 \leqslant \max\{\frac{4+\mu}{\epsilon(t)}, 1\}\|z\|_E^2, \ \forall\, z \in E, \ \forall\, t \in \mathbb{R}, \\ \|z\|_{E_\tau}^2 \leqslant \|z\|_{E_t}^2 \leqslant \frac{\epsilon(\tau)}{\epsilon(t)}\|z\|_{E_\tau}^2, \ \forall\, z \in E, \ \forall\, t \geqslant \tau \in \mathbb{R}. \end{cases} \tag{2.14}$$

Thus, we get that $\|\cdot\|_E$, $\|\cdot\|_{E_t}$ and $\|\cdot\|_{E_\tau}$ are equivalent to each other. In addition, we use $\int_{E_t} \varphi_t(z)\mathrm{d}\mathfrak{m}_t(z)$ to represent the Bochner integral for given $\mathfrak{m}_t \in \mathcal{P}(E_t)$ and $\varphi_t \in C(E_t)$.

We now begin to investigate the well-posedness of problem (2.9)–(2.10).

Lemma 2.1. *Suppose that* (A1)–(A2) *hold. Then, for any $\tau \in \mathbb{R}$ and initial value $z_\tau = (\varphi_\tau, u_\tau, v_\tau)^T \in E$, system* (2.9)–(2.10) *admits a unique local solution $z(\cdot) = (\varphi(\cdot), u(\cdot), v(\cdot))^T \in E$ such that for some $T > \tau$, $z(\cdot) \in C([\tau, T), E) \cap C^1((\tau, T), E)$. Furthermore, if $T < +\infty$, then $\|z(t)\|_E \to +\infty,\ \ t \to T^-$.*

Proof. Obviously, the operator $\Theta(t) : E_t \to E_t$ is linear for every $t \in \mathbb{R}$. Direct computations give

$$\begin{aligned}
&\|\Theta(t)z\|_{E_t}^2 = \|(\gamma I - iA + ih^2 D)\varphi\|^2 + \|\frac{\mathfrak{s}(t)}{\epsilon(t)}u - v\|_{\epsilon(t)}^2 \\
&+ \|(\frac{4\mu}{\lambda^2}\mathfrak{s}'(t) - \frac{A}{\epsilon(t)} + \frac{h^2 D}{\epsilon(t)} + \frac{\mu I}{\epsilon(t)} - \frac{(\lambda - \mathfrak{s}(t))\mathfrak{s}(t)}{\epsilon^2(t)})u + \frac{\lambda - \mathfrak{s}(t)}{\epsilon(t)}v\|^2 \\
\lesssim&\|\varphi\|^2 + \frac{\mathfrak{s}^2(t)}{\epsilon^2(t)}\|u\|_{\epsilon(t)}^2 + (\frac{1}{\epsilon(t)} + \frac{(\lambda - \mathfrak{s}(t))^2\mathfrak{s}^2(t)}{\epsilon^3(t)} + \epsilon(t))\frac{\mu}{\epsilon(t)}\|u\|^2 + \frac{1 + \frac{(\lambda - \mathfrak{s}(t))^2}{\epsilon(t)}}{\epsilon(t)}\|v\|^2 \\
\lesssim&c_1(t)\|z\|_{E_t}^2,\ \forall z = (\varphi, u, v)^T \in E_t,
\end{aligned} \tag{2.15}$$

where

$$c_1(\cdot) = \max\Big\{\frac{\mathfrak{s}^2(\cdot)}{\epsilon^2(\cdot)}, \frac{1}{\epsilon(\cdot)} + \frac{(\lambda - \mathfrak{s}(\cdot))^2\mathfrak{s}^2(\cdot)}{\epsilon^3(\cdot)} + \epsilon(\cdot), \frac{1 + \frac{(\lambda - \mathfrak{s}(\cdot))^2}{\epsilon(\cdot)}}{\epsilon(\cdot)}\Big\}. \tag{2.16}$$

For each $t \in \mathbb{R}$, the boundedness of the linear operator $\Theta(t) : E_t \to E_t$ is proved.

Next, let $\mathcal{B} \in B_{E_t}[0, r]$ with any given $r > 0$, $z_i = (\varphi_i, u_i, v_i)^T \in \mathcal{B}$, $i = 1, 2$. Then

$$\begin{aligned}
&\|\Upsilon(z_1, t) - \Upsilon(z_2, t)\|_{E_t}^2 = \| - i\varphi_1 u_1 + i\varphi_2 u_2\|^2 + \|\frac{1}{\epsilon(t)}(A|\varphi_1|^2 - A|\varphi_2|^2)\|^2 \\
\leqslant&\|\varphi_1(u_1 - u_2) + u_2(\varphi_1 - \varphi_2)\|^2 + \frac{16}{\epsilon^2(t)}\||\varphi_1| - |\varphi_2|\|^2\||\varphi_1| + |\varphi_2|\|^2 \\
\leqslant&(\frac{4\epsilon(t)}{\mu} + \frac{64}{\epsilon^2(t)})r^2\|z_1 - z_2\|_{E_t}^2.
\end{aligned} \tag{2.17}$$

Therefore, $\Upsilon(\cdot, t)$ is a locally Lipschitz function on E_t. Notice that $\|\cdot\|_E$ and $\|\cdot\|_{E_t}$ are equivalent for every given $t \in \mathbb{R}$, we conclude from above estimates that $\Theta(t) + \Upsilon(\cdot, t)$ is locally Lipschitz from E to itself. According to the classical theory of ODE, we get the result of Lemma 2.1. □

In what follows, we verify a certain coercivity of $\Theta(\cdot)$.

Lemma 2.2. *For every $t \in \mathbb{R}$ and each $z = (\varphi, u, v)^T \in E_t$, the following inequality holds:*

$$\mathbf{Re}(\Theta(t)z, z)_{E_t} \geqslant \frac{\delta(t)}{\epsilon(t)}(\|u\|_{\epsilon(t)}^2 + \|v\|^2) + \frac{\lambda}{2\epsilon(t)}\|v\|^2 + \gamma\|\varphi\|^2, \tag{2.18}$$

where

$$\delta(t) = \frac{\lambda\mu\epsilon(t)}{\sqrt{\lambda^2 + 4\mu\epsilon(t)}(\lambda + \sqrt{\lambda^2 + 4\mu\epsilon(t)})} \in (0, \mathfrak{s}(t)). \tag{2.19}$$

Proof. Direct computations gives

$$\begin{aligned}
\mathbf{Re}(\Theta(t)z, z)_{E_t} =&\gamma\|\varphi\|^2 + (\frac{\mathfrak{s}(t)}{\epsilon(t)}u - v, u)_{\epsilon(t)} + \frac{4\mu}{\lambda^2}\mathfrak{s}'(t)(u, v) + \frac{1}{\epsilon(t)}(Bu, Bv) + \frac{h^2}{\epsilon(t)}(Du, v) \\
&+ \frac{\mu}{\epsilon(t)}(u, v) - \frac{(\lambda - \mathfrak{s}(t))\mathfrak{s}(t)}{\epsilon^2(t)}(u, v) + \frac{\lambda - \mathfrak{s}(t)}{\epsilon(t)}\|v\|^2,
\end{aligned} \tag{2.20}$$

and

$$
\begin{aligned}
&\Big|\frac{4\mu}{\lambda^2}\mathfrak{s}'(t)(u,v)+\frac{h^2}{\epsilon(t)}(Du,v)-\frac{(\lambda-\mathfrak{s}(t))\mathfrak{s}(t)}{\epsilon^2(t)}(u,v)\Big|\\
\leqslant&-\frac{4\mu}{\lambda^2}\mathfrak{s}'(t)\|u\|\|v\|+\frac{16h^2}{\epsilon(t)}\|u\|\|v\|+\frac{(\lambda-\mathfrak{s}(t))\mathfrak{s}(t)}{\epsilon^2(t)}\|u\|\|v\|\\
\leqslant&\frac{\lambda\mathfrak{s}(t)}{\epsilon^2(t)}\|u\|\|v\|+(\frac{16h^2}{\epsilon(t)}-\frac{4\mu}{\lambda^2}\mathfrak{s}'(t)-\frac{\mathfrak{s}^2(t)}{\epsilon^2(t)})\|u\|\|v\|\leqslant\frac{\lambda\mathfrak{s}(t)}{\epsilon^2(t)}\|u\|\|v\|,
\end{aligned}\tag{2.21}
$$

where we have used the fact that $\frac{16h^2}{\epsilon(t)}-\frac{4\mu}{\lambda^2}\mathfrak{s}'(t)-\frac{\mathfrak{s}^2(t)}{\epsilon^2(t)}\leqslant 0$. Thus,

$$
\mathbf{Re}(\Theta(t)z,z)_{E_t}\geqslant\gamma\|\varphi\|^2+\frac{\mathfrak{s}(t)}{\epsilon(t)}\|u\|^2_{\epsilon(t)}-\frac{\lambda\mathfrak{s}(t)}{\epsilon^2(t)}\|u\|\|v\|+\frac{\lambda-\mathfrak{s}(t)}{\epsilon(t)}\|v\|^2.\tag{2.22}
$$

Owing to for any $t\in\mathbb{R}$, $\|u\|\leqslant(\frac{\epsilon(t)}{\mu})^{\frac{1}{2}}\|u\|_{\epsilon(t)}$ and

$$
4(\mathfrak{s}(t)-\delta(t))(\frac{\lambda}{2}-\mathfrak{s}(t)-\delta(t))=\frac{\lambda^2\mathfrak{s}^2(t)}{\mu\epsilon(t)}\tag{2.23}
$$

holds, it is not difficult to obtain (2.18). This completes the proof. □

Lemma 2.3. *Suppose that* (A1)–(A2) *hold. Then the solution* $z(\cdot)=(\varphi(\cdot),u(\cdot),v(\cdot))^T$ *(with the initial data* $z_\tau=(\varphi_\tau,u_\tau,v_\tau)^T\in E$ *at the initial time* τ*) satisfies for any* $t\geqslant\tau$*,*

$$
\|z(t)\|^2_{E_t}\leqslant\|z_\tau\|^2_{E_\tau}e^{-\sigma(t-\tau)}+c_2(t)e^{-\sigma t}\int_\tau^t e^{\sigma s}(\|\varphi(s)\|^4+\|f(s)\|^2+\|g(s)\|^2)\mathrm{d}s,\tag{2.24}
$$

and

$$
\begin{aligned}
\|z(t)\|^2_E\leqslant&\frac{\max\{(4+\mu)\epsilon^{-1}(\tau),1\}}{\min\{\mu\epsilon^{-1}(t),1\}}\|z_\tau\|^2_Ee^{-\sigma(t-\tau)}\\
&+\frac{c_2(t)e^{-\sigma t}}{\min\{\mu\epsilon^{-1}(t),1\}\epsilon(t)}\int_\tau^t e^{\sigma s}(\|\varphi(s)\|^4+\|f(s)\|^2+\|g(s)\|^2)\mathrm{d}s,
\end{aligned}\tag{2.25}
$$

hereinafter $c_2(\cdot)=\max\{\frac{2}{\gamma},\frac{32}{\lambda\epsilon(\cdot)}\}$.

Proof. Taking the inner product in E_t of $z(\cdot)$ with Eq (2.9) and extracting the real part of the obtained equation, we get

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z\|^2_{E_t}+\frac{1}{2}\frac{\epsilon'(t)}{\epsilon(t)}\|u\|^2_{\epsilon(t)}+\mathbf{Re}(\Theta(t)z,z)_{E_t}=\mathbf{Re}(\Upsilon(z,t),z)_{E_t},\ \forall\, t\geqslant\tau.\tag{2.26}
$$

Using Cauchy's inequality, we obtain

$$
\begin{cases}
\mathbf{Re}(\Upsilon(z,t),z)_{E_t}\leqslant\mathbf{Im}(f,\varphi)+(\frac{A|\varphi|^2}{\epsilon(t)},v)+(\frac{g}{\epsilon(t)},v),\\
\mathbf{Im}(f,\varphi)\leqslant\frac{1}{\gamma}\|f\|^2+\frac{\gamma}{2}\|\varphi\|^2,\\
(\frac{A|\varphi|^2}{\epsilon(t)},v)\leqslant\frac{16}{\lambda\epsilon(t)}\|\varphi\|^4+\frac{\lambda}{4\epsilon(t)}\|v\|^2,\\
(\frac{g}{\epsilon(t)},v)\leqslant\frac{1}{\lambda\epsilon(t)}\|g\|^2+\frac{\lambda}{4\epsilon(t)}\|v\|^2.
\end{cases}\tag{2.27}
$$

Inserting (2.18) and (2.27) into (2.26) yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|_{E_t}^2+\frac{1}{2}\frac{\epsilon'(t)}{\epsilon(t)}\|u\|_{\epsilon(t)}^2+\frac{\delta(t)}{\epsilon(t)}(\|u\|_{\epsilon(t)}^2+\|v\|^2)+\frac{\gamma}{2}\|\varphi\|^2\\ =&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|_{E_t}^2+[\frac{1}{2}\frac{\epsilon'(t)}{\epsilon(t)}\|u\|_{\epsilon(t)}^2+\frac{1}{2}\frac{\delta(t)}{\epsilon(t)}(\|u\|_{\epsilon(t)}^2+\|v\|^2)]+\frac{1}{2}\frac{\delta(t)}{\epsilon(t)}(\|u\|_{\epsilon(t)}^2+\|v\|^2)+\frac{\gamma}{2}\|\varphi\|^2\\ \leqslant&\frac{1}{\gamma}\|f\|^2+\frac{16}{\lambda\epsilon(t)}\|\varphi\|^4+\frac{1}{\lambda\epsilon(t)}\|g\|^2.\end{aligned}\tag{2.28}$$

By $(2.4)_1$, we know that $4\epsilon(t)\leqslant\lambda,\ \ \forall\, t\in\mathbb{R}$. Thus

$$\frac{\delta(t)}{\epsilon(t)}=\frac{\mu\lambda}{\sqrt{\lambda^2+4\mu\epsilon(t)}(\lambda+\sqrt{\lambda^2+4\mu\epsilon(t)})}\geqslant\frac{\mu\lambda}{\sqrt{\lambda^2+\mu\lambda}(\lambda+\sqrt{\lambda^2+\mu\lambda})}=\sigma_0.\tag{2.29}$$

Noticing $\sigma\leqslant\frac{\sigma_0}{2}$, we can derive from $(2.4)_2$ that

$$\frac{\lambda(\frac{\epsilon'(t)}{\epsilon(t)})^2}{16\mu}+\frac{|\frac{\epsilon'(t)}{\epsilon(t)}|}{2}\leqslant\frac{\sigma_0}{2},\ \forall\, t\in\mathbb{R},$$

which gives that $\frac{\epsilon'(t)}{\epsilon(t)}\geqslant-\frac{\sigma_0}{2}$ and $\dfrac{\delta(t)}{\epsilon(t)}+\dfrac{\epsilon'(t)}{\epsilon(t)}\geqslant\sigma_0-\dfrac{\sigma_0}{2}\geqslant0$. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z\|_{E_t}^2+\sigma\|z\|_{E_t}^2\leqslant c_2(t)(\|\varphi\|^4+\|f\|^2+\|g\|^2),\ \forall\, t\in\mathbb{R}.\tag{2.30}$$

It then follows from Gronwall's inequality that (2.24) holds. Estimate (2.25) is a consequence of (2.24) and (2.14). This completes the proof. □

With respect to Lemmas 2.1 and 2.3, we assert that for every initial data $z_\tau\in E$, the solution $z(\cdot)\in E$ corresponding to problem (2.9)–(2.10) is globally defined on $[\tau,+\infty)$. Moreover, the solution mappings

$$V(t,\tau):z_\tau=(\varphi_\tau,u_\tau,v_\tau)^T\in E_\tau\mapsto z(t)=V(t,\tau)z_\tau=(\varphi(t),u(t),v(t))^T\in E_t,\tag{2.31}$$

form an evolution process $\{V(t,\tau):E_\tau\to E_t,(t,\tau)\in\mathbb{R}_d^2\}$ on the family $\{E_s,\|\cdot\|_{E_s}\}_{s\in\mathbb{R}}$.

Subsequently, we verify that the evolution process $\{V(t,\tau):E_\tau\to E_t,(t,\tau)\in\mathbb{R}_d^2\}$ is continuous.

Lemma 2.4. *Suppose that* (A1)–(A2) hold. *Then for any given $r>0$, $(t,\tau)\in\mathbb{R}_d^2$, there is some $c_3=c_3(r,t,\tau)>0$ such that*

$$\|V(t,\tau)z_{\tau,1}-V(t,\tau)z_{\tau,2}\|_{E_t}^2\lesssim c_3\|z_{\tau,1}-z_{\tau,2}\|_{E_\tau}^2,\tag{2.32}$$

where $z_{\tau,k}=(\varphi_{\tau,k},u_{\tau,k},v_{\tau,k})\in B_{E_\tau}[0,r]$, $k=1,2$.

Proof. Let $r>0$, $(t,\tau)\in\mathbb{R}_d^2$ be given. Consider any $z_{\tau,k}=(\varphi_{\tau,k},u_{\tau,k},v_{\tau,k})^T\in B_{E_\tau}[0,r]$, $k=1,2$, and denote $z_k(\cdot)=V(\cdot,\tau)z_{\tau,k}=(\varphi_k(\cdot),u_k(\cdot),v_k(\cdot))^T$. Then

$$\tilde{z}(\cdot)=(\tilde{\varphi}(\cdot),\tilde{u}(\cdot),\tilde{v}(\cdot))^T=V(\cdot,\tau)z_{\tau,1}-V(\cdot,\tau)z_{\tau,2}$$

fulfills

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{z}(t)+\Theta(t)\tilde{z}(t)=\Upsilon(z_1(t),t)-\Upsilon(z_2(t),t), \tag{2.33}$$
$$\tilde{z}(\tau)=\tilde{z}_\tau=z_{\tau,1}-z_{\tau,2}. \tag{2.34}$$

Taking the inner product of $\tilde{z}(\cdot)$ with (2.33) in E_t and extracting the real part of the obtained equation, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{z}\|_{E_t}^2+\frac{1}{2}\frac{\epsilon'(t)}{\epsilon(t)}\|\tilde{u}\|_{\epsilon(t)}^2+\mathbf{Re}(\Theta(t)\tilde{z},\tilde{z})_{E_t}\\ =&\mathbf{Re}(\Upsilon(z_1,t)-\Upsilon(z_2,t),\tilde{z})_{E_t}.\end{aligned} \tag{2.35}$$

Direct computations with using of Cauchy's inequality show that

$$\begin{aligned}&\mathbf{Re}(\Upsilon(z_1,t)-\Upsilon(z_2,t),\tilde{z})_{E_t}\\ =&\mathbf{Re}(-i(\varphi_1u_1-\varphi_2u_2),\tilde{\varphi})+\mathbf{Re}(\frac{1}{\epsilon(t)}(A|\varphi_1|^2-A|\varphi_2|^2),\tilde{v})\\ \leqslant&(\frac{2\epsilon(t)}{\gamma\mu}+\frac{32}{\lambda\epsilon(t)})(\|\varphi_1\|^2+\|\varphi_2\|^2+\|u_2\|_{\epsilon(t)}^2)\|\tilde{z}\|_{E_t}^2+\frac{\gamma}{2}\|\tilde{\varphi}\|^2+\frac{\lambda}{2\epsilon(t)}\|\tilde{v}\|^2.\end{aligned} \tag{2.36}$$

Now, by using the similar derivations as those as (2.28)–(2.30) and taking (2.18), (2.35) and (2.36) into account, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{z}(t)\|_{E_t}^2\leqslant(\frac{4\epsilon(t)}{\gamma\mu}+\frac{64}{\lambda\epsilon(t)})(\|\varphi_1\|^2+\|\varphi_2\|^2+\|u_2\|_{\epsilon(t)}^2)\|\tilde{z}\|_{E_t}^2. \tag{2.37}$$

It then follows from Gronwall's inequality that

$$\begin{aligned}&\|V(t,\tau)z_{\tau,1}-V(t,\tau)z_{\tau,2}\|_{E_t}^2\\ \lesssim&\exp\Big\{\int_\tau^t(\frac{1}{\epsilon(s)}+\epsilon(s))(\|\varphi_1(s)\|^2+\|\varphi_2(s)\|^2+\|u_2(s)\|_{\epsilon(s)}^2)\mathrm{d}s\Big\}\|z_{\tau.1}-z_{\tau,2}\|_{E_\tau}^2.\end{aligned} \tag{2.38}$$

By [14, Lemma 2.2], we have

$$\|\varphi(t)\|^2\leqslant\|\varphi_\tau\|^2e^{-\gamma(t-\tau)}+\frac{e^{-\gamma t}}{\gamma}\int_\tau^t e^{\gamma s}\|f(s)\|^2\mathrm{d}s,\ \forall\, t\geqslant\tau. \tag{2.39}$$

It then follows from (2.24), (2.25) and (2.39) that there exists a function $c_3=c_3(r,t,\tau)$ which depends continuously on τ and satisfies

$$(\frac{1}{\epsilon(s)}+\epsilon(s))(\|\varphi_1(s)\|^2+\|\varphi_2(s)\|^2+\|u_2(s)\|_{\epsilon(s)}^2)\leqslant c_3(r,t,\tau),\ \forall\, s\in[\tau,t]. \tag{2.40}$$

Substituting (2.40) into (2.38), we can obtain (2.32). This completes the proof. □

3. A time-dependent pullback attractor

In this section, we aim to verify that the evolution process $\{V(t,\tau): E_\tau \to E_t, (t,\tau) \in \mathbb{R}^2_d\}$ admits a time-dependent pullback attractor in the family of phase spaces $\{E_s, \|\cdot\|_{E_s}\}_{s\in\mathbb{R}}$.

We first select some definitions from [1]. We denote the Hausdorff semi-distance between two nonempty sets $B_s, C_s \subset E_s$ by

$$\mathrm{Dist}_{E_s}(B_s, C_s) = \sup_{z\in B_s} \inf_{y\in C_s} \|z - y\|_{E_s}.$$

Definition 3.1. *Let* $\{V(t,\tau): E_\tau \to E_t, (t,\tau) \in \mathbb{R}^2_d\}$ *be the evolution process defined by* (2.31).

(1) *A bounded family* $\hat{\mathcal{B}} = \{\mathcal{B}_s \subset E_s, \|\cdot\|_{E_s}\}_{s\in\mathbb{R}}$ *is a bounded pullback absorbing set for the evolution process* $\{V(t,\tau): E_\tau \to E_t, (t,\tau) \in \mathbb{R}^2_d\}$, *if for any* $r > 0$ *and* $(t,\tau) \in \mathbb{R}^2_d$ *there corresponds a* $t_0 = t_0(t,r) \leqslant t$ *yielding*

$$V(t,\tau)B_{E_\tau}[0,r] \subset \mathcal{B}_t, \quad \forall\, \tau \leqslant t_0. \tag{3.1}$$

(2) *The evolution process* $\{V(t,\tau): E_\tau \to E_t, (t,\tau) \in \mathbb{R}^2_d\}$ *is pullback asymptotically compact if* $\hat{\mathcal{K}} = \{\mathcal{K}_s, \|\cdot\|_{E_s}\}_{s\in\mathbb{R}} \neq \varnothing$, *where* $\mathcal{K}_s \subset E_s$ *is compact, and* $\hat{\mathcal{K}}$ *is pullback attracting for* $\{V(t,\tau): E_\tau \to E_t, (t,\tau) \in \mathbb{R}^2_d\}$.

(3) *A family* $\hat{\mathcal{A}} = \{\mathcal{A}_s \subset E_s, \|\cdot\|_{E_s}\}_{s\in\mathbb{R}}$ *is a time-dependent pullback attractor for the evolution process* $\{V(t,\tau): E_\tau \to E_t, (t,\tau) \in \mathbb{R}^2_d\}$, *if it satisfies:*

 (a) Compactness: *for any* $s \in \mathbb{R}$, $\mathcal{A}_s$ *is a nonempty compact subset of* E_s;
 (b) Invariance: $V(t,\tau)\mathcal{A}_\tau = \mathcal{A}_t, \ \forall\, (t,\tau) \in \mathbb{R}^2_d$;
 (c) Pullback attraction: *for any bounded family* $\hat{C} = \{C_s \subset E_s, \|\cdot\|_{E_s}\}_{s\in\mathbb{R}}$,

$$\lim_{\tau\to-\infty} \mathrm{Dist}_{E_t}\big(V(t,\tau)C_t, \mathcal{A}_t\big) = 0, \quad \forall\, t \in \mathbb{R}.$$

Lemma 3.1. *Suppose that* (A1)–(A2) *hold. Then the family* $\mathfrak{B} = \{B_{E_t}[0, r_\sigma^{1/2}(t)] \subset E_t\}_{t\in\mathbb{R}}$ *is a bounded pullback absorbing set of the evolution process* $\{V(t,\tau): E_\tau \to E_t, (t,\tau) \in \mathbb{R}^2_d\}$, *where*

$$\begin{aligned} r_\sigma(t) =& 1 + c_2(t)e^{-\sigma t}\int_{-\infty}^{t} e^{\sigma s}(\|f(s)\|^2 + \|g(s)\|^2)\mathrm{d}s \\ &+ c_2(t)e^{-\sigma t}\int_{-\infty}^{t} e^{(\sigma-2\gamma)s}\Big(\int_{-\infty}^{s} e^{\gamma\vartheta}\|f(\vartheta)\|^2\mathrm{d}\vartheta\Big)^2\mathrm{d}s < +\infty,\ t \in \mathbb{R}. \end{aligned} \tag{3.2}$$

Proof. Let $\tau \in \mathbb{R}$ be given, we consider any $r > 0$, $z_\tau \in B_{E_\tau}[0,r]$ and (2.24). Now by Assumption (A2), for each $t \geqslant \tau$, we have

$$\lim_{\tau\to-\infty} \|z_\tau\|^2_{E_\tau} e^{-\sigma(t-\tau)} = 0, \tag{3.3}$$

$$c_2(t)e^{-\sigma t}\int_{-\infty}^{t} e^{\sigma s}(\|f(s)\|^2 + \|g(s)\|^2)\mathrm{d}s < +\infty. \tag{3.4}$$

For the term $\int_\tau^t e^{\sigma s}\|\varphi(s)\|^4\mathrm{d}s$ in (2.24), we get from (2.39) that

$$\int_\tau^t e^{\sigma s}\|\varphi(s)\|^4\mathrm{d}s \leqslant K_1 + K_2 + K_3, \tag{3.5}$$

where

$$\begin{cases} K_1 = \int_\tau^t e^{\sigma s}\|\varphi_\tau\|^4 e^{-2\gamma(s-\tau)}\mathrm{d}s, \\ K_2 = \int_\tau^t e^{(\sigma-2\gamma)s}(\int_\tau^s e^{\gamma\vartheta}\|f(\vartheta)\|^2\mathrm{d}\vartheta)^2\mathrm{d}s, \\ K_3 = 2\int_\tau^t e^{\sigma s}\|\varphi_\tau\|^2 e^{\gamma\tau}e^{-2\gamma s}\int_\tau^s e^{\gamma\vartheta}\|f(\vartheta)\|^2\mathrm{d}\vartheta\mathrm{d}s. \end{cases}$$

Firstly, for K_1, we obtain for any $t \geqslant \tau$ that

$$\begin{aligned} K_1 &= \|\varphi_\tau\|^4 e^{2\gamma\tau}\int_\tau^t e^{(\sigma-2\gamma)s}\mathrm{d}s \\ &= \frac{1}{\sigma-2\gamma}(e^{(\sigma-2\gamma)t} - e^{(\sigma-2\gamma)\tau})e^{2\gamma\tau}\|\varphi_\tau\|^4 \leqslant \frac{r^4 e^{\sigma\tau}}{2\gamma-\sigma} \to 0, \text{ as } \tau \to -\infty. \end{aligned} \tag{3.6}$$

Next, for K_2, we obtain from (2.6) that

$$\int_{-\infty}^s e^{\gamma\vartheta}\|f(\vartheta)\|^2\mathrm{d}\vartheta \leqslant e^{(\gamma-\sigma)s}\int_{-\infty}^s e^{\sigma\vartheta}\|f(\vartheta)\|^2\mathrm{d}\vartheta \leqslant e^{(\gamma-\frac{\sigma}{2}+\varrho)s}J(s). \tag{3.7}$$

Hence,

$$\begin{aligned} K_2 =& \int_\tau^t e^{(\sigma-2\gamma)s}e^{(\sigma-2\gamma-2\varrho)s}(\int_\tau^s e^{\gamma\vartheta}\|f(\vartheta)\|^2\mathrm{d}\vartheta)^2 e^{(-\sigma+2\gamma+2\varrho)s}\mathrm{d}s \\ \leqslant& \tilde{J}_1(t)\int_{-\infty}^t e^{2\varrho s}\mathrm{d}s \leqslant \frac{e^{2\varrho t}}{2\varrho}\tilde{J}_1(t) < +\infty, \end{aligned} \tag{3.8}$$

where the bounded function $\tilde{J}_1(\cdot)$ depends only on $J(\cdot)$. Lastly, for K_3, we proceed similarly to (3.8) to get

$$\begin{aligned} K_3 =& 2\|\varphi_\tau\|^2 e^{\frac{\sigma}{2}\tau}e^{(\gamma-\frac{\sigma}{2})\tau}\int_\tau^t e^{-(2\gamma-\sigma)s}\int_\tau^s e^{\gamma\vartheta}\|f(\vartheta)\|^2\mathrm{d}\vartheta\mathrm{d}s \\ \leqslant& 2r^2 e^{\frac{\sigma}{2}\tau}e^{\varrho\tau}e^{(\gamma-\frac{\sigma}{2}-\varrho)\tau}\int_\tau^t e^{(\frac{\sigma}{2}-\gamma+\varrho)s}J(s)\mathrm{d}s \leqslant \frac{4r^2 e^{\frac{\sigma}{2}\tau}e^{\varrho\tau}\tilde{J}_2(t)}{2\gamma-\sigma-2\varrho} \to 0, \text{ as } \tau \to -\infty, \end{aligned} \tag{3.9}$$

where the bounded function $\tilde{J}_2(\cdot)$ depends only on $J(\cdot)$. By (3.5), (3.6), (3.8) and (3.9), we obtain

$$\int_\tau^t e^{\sigma s}\|\varphi(s)\|^4\mathrm{d}s \leqslant \frac{e^{2\varrho t}}{2\varrho}\tilde{J}_1(t) < +\infty, \text{ as } \tau \to -\infty. \tag{3.10}$$

Now pick $r_\sigma(t)$ as (3.2) for each $t \in \mathbb{R}$. Then through the above analyses, we claim that for any given $t \geqslant \tau$ and $z_\tau \in B_{E_\tau}[0, r]$ with $r > 0$, there is a $t_0 = t_0(t, r) \leqslant t$ yielding for any $\tau \leqslant t_0$, $\|V(t, \tau)z_\tau\|^2_{E_t} \leqslant r_\sigma(t)$, i.e.,

$$V(t, \tau)B_{E_\tau}[0, r] \subset B_{E_t}[0, r_\sigma^{1/2}(t)], \ \forall\, \tau \leqslant t_0. \tag{3.11}$$

This completes the proof. □

We can refer to [1, Lemma 3.1(2)] for the definition of pullback asymptotically nullness.

Lemma 3.2. *Suppose that* (A1)–(A2) hold. *Then the evolution process* $\{V(t,\tau) : E_\tau \to E_t, (t,\tau) \in \mathbb{R}_d^2\}$ *possesses pullback asymptotically nullness.*

Proof. By Urysohn's lemma, we can choose a smooth function $\xi(\cdot) \in C^1(\mathbb{R}_+;\mathbb{R}_+)$ that satisfies

$$\begin{cases} \xi(\vartheta) = 0, & 0 \leqslant \vartheta \leqslant 1, \\ 0 \leqslant \xi(\vartheta) \leqslant 1, & 1 \leqslant \vartheta \leqslant 2, \\ \xi(\vartheta) = 1, & \vartheta \geqslant 2, \\ |\xi'(\vartheta)| \leqslant \xi_0, & \vartheta \geqslant 0, \end{cases} \tag{3.12}$$

where $\xi_0 > 0$. For any given $r > 0$, $\tau \in \mathbb{R}$ and $z_\tau = (\varphi_\tau, u_\tau, v_\tau)^T \in B_{E_\tau}[0,r]$, let $z = z(\cdot) = z(\cdot,\tau;z_\tau) = (\varphi(\cdot), u(\cdot), v(\cdot))^T = (\varphi_n(\cdot), u_n(\cdot), v_n(\cdot))^T_{n\in\mathbb{Z}} \in E.$ be the solution of problem (2.9)–(2.10) and z_τ be its initial value. Define

$$\begin{cases} y_n = y_n(\cdot) = \xi(\frac{|n|}{\zeta})\varphi_n(\cdot),\ \omega_n = \omega_n(\cdot) = \xi(\frac{|n|}{\zeta})u_n(\cdot),\ p_n = p_n(\cdot) = \xi(\frac{|n|}{\zeta})v_n(\cdot), \\ y = y(\cdot) = (y_n)_{n\in\mathbb{Z}},\ \omega = \omega(\cdot) = (\omega_n)_{n\in\mathbb{Z}},\ p = p(\cdot) = (p_n)_{n\in\mathbb{Z}}, \\ \psi_n = \psi_n(\cdot) = (y_n, \omega_n, p_n)^T,\ \psi = \psi(\cdot) = (\psi_n)_{n\in\mathbb{Z}}, \end{cases}$$

where $\zeta \in \mathbb{Z}_+$. Taking the inner product of $\psi(\cdot)$ with Eq (2.9) in E_t and extracting the real part of the obtained equation, we obtain

$$\mathbf{Re}(\dot{z}(t), \psi(t))_{E_t} + \mathbf{Re}(\Theta(t)z(t), \psi(t))_{E_t} = \mathbf{Re}(\Upsilon(z(t),t), \psi(t))_{E_t},\ \forall\, t > \tau. \tag{3.13}$$

Next, we calculate each of the three terms in (3.13) individually.

Firstly, for the term $\mathbf{Re}(\dot{z}(t), \psi(t))_{E_t}$, we have

$$\begin{aligned} \mathbf{Re}(\dot{z}(t), \psi(t))_{E_t} =& \mathbf{Re}\Big\{\sum_{n\in\mathbb{Z}} \dot{\varphi}_n\bar{y}_n + \frac{1}{\epsilon(t)}\sum_{n\in\mathbb{Z}}(B\dot{u})_n(B\omega)_n + \frac{1}{\epsilon(t)}\sum_{n\in\mathbb{Z}}\mu\dot{u}_n\omega_n + \sum_{n\in\mathbb{Z}}\dot{v}_n p_n\Big\} \\ =& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})|z_n(t)|^2_{E_t} + \frac{\epsilon'(t)}{2\epsilon^2(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})((Bu)_n^2 + \mu u_n^2) \\ &+ \frac{1}{\epsilon(t)}\sum_{n\in\mathbb{Z}}(B\dot{u})_n(B\omega)_n - \frac{1}{\epsilon(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})(B\dot{u})_n(Bu)_n. \end{aligned} \tag{3.14}$$

According to Lemma 3.1, there is some $t_0 = t_0(t,r) \leqslant t$ such that

$$\begin{aligned} &\frac{1}{\epsilon(t)}\sum_{n\in\mathbb{Z}}(B\dot{u})_n(B\omega)_n - \frac{1}{\epsilon(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})(B\dot{u})_n(Bu)_n \\ =& \frac{1}{\epsilon(t)}\sum_{n\in\mathbb{Z}}(B\dot{u})_n[\xi(\frac{|n+1|}{\zeta}) - \xi(\frac{|n|}{\zeta})]u_{n+1} \geqslant -\frac{\xi_0}{\zeta\epsilon(t)}\sum_{n\in\mathbb{Z}}|\dot{u}_{n+1} - \dot{u}_n||u_{n+1}| \\ \gtrsim& \gtrsim -\frac{1}{\zeta\epsilon(t)}(1 + \epsilon(t) + \frac{\mathfrak{s}^2(t)}{\epsilon(t)})r_\sigma(t). \end{aligned} \tag{3.15}$$

Thus,

$$\mathbf{Re}(\dot{z}(t), \psi(t))_{E_t} \gtrsim \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})|z_n(t)|^2_{E_t} + \frac{\epsilon'(t)}{2\epsilon(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})|z_n|^2_{E_t}$$

$$-\frac{1+\epsilon(t)+\frac{\mathfrak{s}^2(t)}{\epsilon(t)}}{\zeta\epsilon(t)}r_\sigma(t),\ \forall\,\tau\leqslant t_0. \tag{3.16}$$

Secondly, for $\mathbf{Re}(\Theta(t)z(t),\psi(t))_{E_t}$, we carry out direct computations to obtain

$$\begin{aligned}
&\mathbf{Re}(\Theta(t)z(t),\psi(t))_{E_t}\\
=&\gamma(\varphi,y)+\mathbf{Re}(i(B\varphi,By))+\mathbf{Re}(ih^2(A\varphi,Ay))+\frac{\mathfrak{s}(t)}{\epsilon(t)}[\frac{1}{\epsilon(t)}(Bu,B\omega)+\frac{1}{\epsilon(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})\mu u_n^2]\\
&+\frac{1}{\epsilon(t)}[(Bu,Bp)-(Bv,B\omega)]+\frac{4\mu\mathfrak{s}'(t)}{\lambda^2}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})u_nv_n+\frac{h^2}{\epsilon(t)}(Du,p)\\
&-\frac{(\lambda-\mathfrak{s}(t))\mathfrak{s}(t)}{\epsilon^2(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})u_nv_n+\frac{\lambda-\mathfrak{s}(t)}{\epsilon(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})v_n^2,
\end{aligned}\tag{3.17}$$

$$\begin{aligned}
\mathbf{Re}(i(B\varphi,By))=&-\mathbf{Im}(B\varphi,By)\\
=&\mathbf{Im}\sum_{n\in\mathbb{Z}}\xi(\frac{|n+1|}{\zeta})\varphi_n\bar{\varphi}_{n+1}+\mathbf{Im}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})\varphi_{n+1}\bar{\varphi}_n\\
\geqslant&-\frac{\xi_0}{\zeta}\sum_{n\in\mathbb{Z}}|\varphi_n\bar{\varphi}_{n+1}|\gtrsim-\frac{r_\sigma(t)}{\zeta},\ \forall\,\tau\leqslant t_0,
\end{aligned}\tag{3.18}$$

$$\begin{aligned}
&\mathbf{Re}(ih^2(A\varphi,Ay))=-\mathbf{Im}(h^2(A\varphi,Ay))\\
=&h^2\mathbf{Im}\sum_{n\in\mathbb{Z}}[2\xi(\frac{|n|}{\zeta})\varphi_{n+1}\bar{\varphi}_n-\xi(\frac{|n-1|}{\zeta})\varphi_{n+1}\bar{\varphi}_{n-1}+2\xi(\frac{|n+1|}{\zeta})\varphi_n\bar{\varphi}_{n+1}\\
&+2\xi(\frac{|n-1|}{\zeta})\varphi_n\bar{\varphi}_{n-1}-\xi(\frac{|n+1|}{\zeta})\varphi_{n-1}\bar{\varphi}_{n+1}+2\xi(\frac{|n|}{\zeta})\varphi_{n-1}\bar{\varphi}_n]\\
\geqslant&-\frac{h^2\xi_0}{\zeta}\sum_{n\in\mathbb{Z}}(2|\bar{\varphi}_{n+1}\varphi_n|+|\varphi_{n+1}\bar{\varphi}_{n-1}|+2|\bar{\varphi}_n\varphi_{n-1}|)\gtrsim-\frac{r_\sigma(t)}{\zeta},\ \forall\,\tau\leqslant t_0,
\end{aligned}\tag{3.19}$$

$$\begin{aligned}
(Bu,B\omega)=&\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})(Bu)_n^2+\sum_{n\in\mathbb{Z}}(Bu)_n[(B\omega)_n-\xi(\frac{|n|}{\zeta})(Bu)_n]\\
\gtrsim&\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})|(Bu)_n|^2-\frac{\epsilon(t)r_\sigma(t)}{\zeta},\ \forall\,\tau\leqslant t_0,
\end{aligned}\tag{3.20}$$

$$\begin{aligned}
&[(Bu,Bp)-(Bv,B\omega)]=\sum_{n\in\mathbb{Z}}[(Bu)_n(Bp)_n-(Bv)_n(B\omega)_n]\\
=&\sum_{n\in\mathbb{Z}}[(\xi(\frac{|n+1|}{\zeta})-\xi(\frac{|n|}{\zeta}))u_{n+1}v_n-(\xi(\frac{|n+1|}{\zeta})-\xi(\frac{|n|}{\zeta}))u_nv_{n+1}]\\
\gtrsim&-\frac{1+\epsilon(t)}{\zeta}r_\sigma(t),\ \forall\,\tau\leqslant t_0,
\end{aligned}\tag{3.21}$$

and

$$\begin{aligned}\frac{4\mu\mathfrak{s}'(t)}{\lambda^2}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})u_n v_n \geqslant & -\frac{\lambda}{4\epsilon(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})v_n^2-\frac{\epsilon(t)}{\lambda}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})\frac{16\mu^2}{\lambda^4}|\mathfrak{s}'(t)|^2u_n^2\\ \geqslant & -\frac{\lambda}{4\epsilon(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})v_n^2-\frac{\lambda(\epsilon'(t))^2}{16\mu\epsilon^2(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})|z_n|^2_{E_t}.\end{aligned} \tag{3.22}$$

For the term $\frac{h^2}{\epsilon(t)}(Du,p)$, we proceed as follows

$$\frac{h^2}{\epsilon(t)}(Du,p)=\frac{h^2}{\epsilon(t)}(Du,\dot{\omega})+\frac{h^2\mathfrak{s}(t)}{\epsilon^2(t)}(Du,\omega), \tag{3.23}$$

where

$$\begin{aligned}\frac{h^2}{\epsilon(t)}(Du,\dot{\omega})=&\frac{h^2}{\epsilon(t)}(Au,A\dot{\omega})\\ =&\frac{h^2}{\epsilon(t)}[\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})(Au)_n^2+\sum_{n\in\mathbb{Z}}(Au)_n((A\dot{\omega})_n-\xi(\frac{|n|}{\zeta})(A\dot{u})_n)].\end{aligned}$$

Since

$$\begin{aligned}&\sum_{n\in\mathbb{Z}}(Au)_n[(A\dot{\omega})_n-\xi(\frac{|n|}{\zeta})(A\dot{u})_n]\\ &\gtrsim-\frac{1}{\zeta}\sum_{n\in\mathbb{Z}}|u_{n+1}(t)-2u_n+u_{n-1}|(|v_{n+1}-\frac{\mathfrak{s}(t)}{\epsilon(t)}u_{n+1}|+|v_{n-1}-\frac{\mathfrak{s}(t)}{\epsilon(t)}u_{n-1}|)\\ &\gtrsim-\frac{r_\sigma(t)}{\zeta},\ \forall\,\tau\leqslant t_0,\end{aligned}$$

we have

$$\frac{h^2}{\epsilon(t)}(Du,\dot{\omega})-\frac{h^2}{2\epsilon(t)}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})(Au)_n^2\gtrsim-\frac{r_\sigma(t)}{\zeta\epsilon(t)},\ \forall\,\tau\leqslant t_0. \tag{3.24}$$

Similarly,

$$\begin{aligned}\frac{h^2\mathfrak{s}(t)}{\epsilon^2(t)}(Du,\omega)=&\frac{h^2\mathfrak{s}(t)}{\epsilon^2(t)}(Au,A\omega)\\ \gtrsim&\frac{h^2\mathfrak{s}(t)}{2\epsilon^2(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})(Au)_n^2-\frac{\frac{\mathfrak{s}(t)}{\epsilon(t)}}{\zeta\epsilon(t)}r_\sigma(t),\ \forall\,\tau\leqslant t_0.\end{aligned} \tag{3.25}$$

Then estimates (3.23)–(3.25) give

$$\frac{h^2}{\epsilon(t)}(Du,p)\gtrsim\frac{1}{2\epsilon(t)}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})(Au)_n^2+\frac{\mathfrak{s}(t)}{2\epsilon^2(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})(Au)_n^2-\frac{1+\frac{\mathfrak{s}(t)}{\epsilon(t)}}{\zeta\epsilon(t)}r_\sigma(t),\ \forall\,\tau\leqslant t_0. \tag{3.26}$$

Now, from (3.17)–(3.26), we obtain

$$\mathbf{Re}(\Theta(t)z(t),\psi(t))_{E_t}-\frac{\delta(t)}{\epsilon(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})[\frac{1}{\epsilon(t)}|(Bu)_n|^2$$

$$
\begin{aligned}
&+\frac{\mu}{\epsilon(t)}u_n^2+v_n^2]-\frac{\lambda}{2\epsilon(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})v_n^2-\gamma\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})|\varphi_n|^2\\
&-\frac{1}{2\epsilon(t)}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})(Au)_n^2-\frac{\mathfrak{s}(t)}{2\epsilon^2(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})(Au)_n^2\\
\gtrsim&\frac{1}{\epsilon(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})[(\mathfrak{s}(t)-\delta(t))|u_n|^2_{\epsilon(t)}+(\frac{\lambda}{2}-\mathfrak{s}(t)-\delta(t))v_n^2\\
&-\frac{\mathfrak{s}(t)(\lambda-\mathfrak{s}(t))}{\epsilon(t)}|u_n||v_n|]-\frac{\lambda}{4\epsilon(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})v_n^2-\frac{\lambda(\epsilon'(t))^2}{16\mu\epsilon^2(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})|z_n|^2_{E_t}\\
&-\frac{1}{\zeta\epsilon(t)}(1+\epsilon(t)+\mathfrak{s}(t))r_\sigma(t),\ \forall\,\tau\leqslant t_0,
\end{aligned}\tag{3.27}
$$

where $|u_n|^2_{\epsilon(t)}=\frac{1}{\epsilon(t)}(|(Bu)_n|^2+\mu u_n^2)$. Since $4(\mathfrak{s}(t)-\delta(t))(\frac{\lambda}{2}-\mathfrak{s}(t)-\delta(t))=\frac{\lambda^2\mathfrak{s}^2(t)}{\mu\epsilon(t)},\ \ \forall\,t\in\mathbb{R}$, we obtain

$$
(\mathfrak{s}(t)-\delta(t))|u_n|^2_{\epsilon(t)}+(\frac{\lambda}{2}-\mathfrak{s}(t)-\delta(t))v_n^2-\frac{\mathfrak{s}(t)(\lambda-\mathfrak{s}(t))}{\epsilon(t)}|u_n||v_n|\geqslant 0,\ \ \forall\,n\in\mathbb{Z}.\tag{3.28}
$$

Therefore, inequality (3.27) gives

$$
\begin{aligned}
&\mathbf{Re}(\Theta(t)z(t),\psi(t))_{E_t}-\frac{\delta(t)}{\epsilon(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})[\frac{1}{\epsilon(t)}|(Bu)_n|^2+\frac{\mu}{\epsilon(t)}u_n^2+v_n^2]\\
&-\frac{\lambda}{2\epsilon(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})v_n^2-\gamma\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})|\varphi_n|^2-\frac{1}{2\epsilon(t)}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})(Au)_n^2\\
&-\frac{\mathfrak{s}(t)}{2\epsilon^2(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})(Au)_n^2\\
\gtrsim&-\frac{\lambda}{4\epsilon(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})v_n^2-\frac{\lambda(\epsilon'(t))^2}{16\mu\epsilon^2(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})|z_n|^2_{E_t}\\
&-\frac{1}{\zeta\epsilon(t)}(1+\epsilon(t)+\mathfrak{s}(t))r_\sigma(t),\ \forall\,\tau\leqslant t_0.
\end{aligned}\tag{3.29}
$$

Lastly, for $\mathbf{Re}(\Upsilon(z,t),\psi)_{E_t}$, we estimate as follows

$$
\mathbf{Re}(\Upsilon(z,t),\psi)_{E_t}=\mathbf{Im}(f,y)+(\frac{A|\varphi|^2}{\epsilon(t)},p)+(\frac{g(t)}{\epsilon(t)},p),\tag{3.30}
$$

$$
\mathbf{Im}(f,y)\leqslant\frac{\gamma}{2}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})|\varphi_n|^2+\frac{1}{\gamma}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})|f_n|^2,\tag{3.31}
$$

$$
\begin{aligned}
(\frac{A|\varphi|^2}{\epsilon(t)},p)\leqslant&\frac{\lambda}{8\epsilon(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})v_n^2+\frac{32}{\lambda\epsilon(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})|\varphi_n|^4\\
\leqslant&\frac{\lambda}{8\epsilon(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})v_n^2+\frac{32r_\sigma(t)}{\lambda\epsilon(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})|\varphi_n|^2,
\end{aligned}\tag{3.32}
$$

$$
(\frac{g}{\epsilon(t)},p)\leqslant\frac{\lambda}{8\epsilon(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})v_n^2+\frac{2}{\lambda\epsilon(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})g_n^2.\tag{3.33}
$$

At this stage, using (2.29), (3.13), (3.16), (3.29) and (3.30)–(3.33), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})[|z_n|_{E_t}^2+(Au)_n^2]\\&+[2\sigma+\frac{\epsilon'(t)}{2\epsilon(t)}-\frac{\lambda}{16\mu}|\frac{\epsilon'(t)}{\epsilon(t)}|^2]\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})|z_n|_{E_t}^2+\frac{\mathfrak{s}(t)}{2\epsilon(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})(Au)_n^2\\\lesssim&\frac{r_\sigma(t)}{\epsilon(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})|\varphi_n|^2+\frac{1}{\epsilon(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})g_n^2\\&+\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})|f_n|^2+\frac{(1+\epsilon(t))r_\sigma(t)}{\zeta\epsilon(t)},\ \forall\,\tau\leqslant t_0.\end{aligned}\tag{3.34}$$

Notice that we can infer from $(2.4)_2$ that $2\sigma+\frac{\epsilon'(t)}{2\epsilon(t)}-\frac{\lambda}{16\mu}|\frac{\epsilon'(t)}{\epsilon(t)}|^2\geqslant\sigma$, and that $\frac{\mathfrak{s}(t)}{2\epsilon(t)}\geqslant\sigma$ for any $t\in\mathbb{R}$. Hence, the differential inequality (3.34) improves to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})[|z_n|_{E_t}^2+(Au)_n^2]+\sigma\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})[|z_n|_{E_t}^2+(Au)_n^2]\\\lesssim&\frac{1+\epsilon(t)}{\zeta\epsilon(t)}r_\sigma(t)+\frac{r_\sigma(t)}{\epsilon(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})|\varphi_n|^2+\frac{1}{\epsilon(t)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})g_n^2\\&+\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})|f_n|^2,\ \forall\,\tau\leqslant t_0.\end{aligned}\tag{3.35}$$

Taking the inner product of $(\xi(\frac{|n|}{\zeta})\bar{\varphi}_n)_{n\in\mathbb{Z}}$ with Eq (2.1) and extracting the imaginary part of the obtained equation, we get

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})|\varphi_n(t)|^2\lesssim&e^{-\gamma t}\int_\tau^t e^{\gamma s}(\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})|f_n(s)|^2+\frac{r_\sigma(s)}{\zeta})\mathrm{d}s\\&+e^{-\gamma(t-\tau)}\sum_{n\in\mathbb{Z}}\xi(\frac{|n|}{\zeta})|\varphi_n(\tau)|^2,\ \forall\,\tau\leqslant t_0.\end{aligned}\tag{3.36}$$

In the light of (A2), we have

$$e^{-\gamma t}\int_\tau^t e^{\gamma s}\sum_{n\in\mathbb{Z}}|f_n(s)|^2\mathrm{d}s\leqslant e^{(\varrho-\frac{\sigma}{2})t}J(t)<+\infty,\quad\forall\,t\in\mathbb{R}.$$

Thus, there is a $\zeta_1=\zeta_1(t,\varepsilon)\in\mathbb{Z}_+,\ \ \forall\,\varepsilon>0$ yielding

$$\frac{r_\sigma(t)}{\epsilon(t)}e^{-\gamma t}\int_\tau^t e^{\gamma s}\sum_{|n|\geqslant\zeta_1}|f_n(s)|^2\mathrm{d}s<\frac{\sigma\varepsilon^2}{36},\ \forall\,\zeta\geqslant\zeta_1.\tag{3.37}$$

Notice that we can get from (3.2)

$$\frac{e^{-\gamma t}}{\zeta}\int_\tau^t e^{\gamma s}r_\sigma(s)\mathrm{d}s\lesssim\frac{1}{\zeta}[1+c_2(t)\int_{-\infty}^t e^{\sigma\eta}(\|f(\eta)\|^2+\|g(\eta)\|^2)\mathrm{d}\eta]+\frac{e^{(2\varrho-\sigma)t}}{\zeta}c_2(t)\tilde{J}_1(t)<+\infty.$$

Therefore, for above ε and every given $t(\geqslant \tau)$, there is a $\zeta_2 = \zeta_2(t,\varepsilon) \in \mathbb{Z}_+$ yielding

$$\frac{r_\sigma(t)}{\epsilon(t)\zeta} e^{-\gamma t} \int_\tau^t e^{\gamma s} r_\sigma(t) \mathrm{d}s < \frac{\sigma\varepsilon^2}{36}, \ \forall \zeta \geqslant \zeta_2. \tag{3.38}$$

Obviously, for above t and ε, there is a $t_1 = t_1(t,\varepsilon,r) \leqslant t$ and some $\zeta_3 = \zeta_3(t,\varepsilon) \in \mathbb{Z}_+$ yielding

$$\frac{r_\sigma(t)}{\epsilon(t)} e^{-\gamma(t-\tau)} \sum_{n\in\mathbb{Z}} \xi(\frac{|n|}{\zeta}) |\varphi_n(\tau)|^2 \leqslant \frac{r_\sigma(t)}{\epsilon(t)} e^{-\gamma(t-\tau)} r^2 < \frac{\sigma\varepsilon^2}{36}, \ \forall \tau \leqslant t_1, \tag{3.39}$$

$$\frac{(1+\epsilon(t)) r_\sigma(t)}{\zeta\epsilon(t)} < \frac{\sigma\varepsilon^2}{12}, \ \forall \zeta \geqslant \zeta_3. \tag{3.40}$$

Using (3.35)–(3.40), Gronwall's inequality and the fact

$$\sum_{n\in\mathbb{Z}} \xi(\frac{|n|}{\zeta})((Au(\tau))_n)^2 \lesssim \|u_\tau\|^2 \leqslant \|z_\tau\|^2_{E_t},$$

we claim that for any $\zeta \geqslant \max\{\zeta_1,\zeta_2,\zeta_3\}$ and $\tau \leqslant \min\{t_0,t_1\}$ there holds

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \xi(\frac{|n|}{\zeta})[|z_n(t)|^2_{E_t} + (A(u(t)))^2_n] \lesssim& \|z_\tau\|^2_{E_\tau} e^{-\sigma(t-\tau)} + e^{-\sigma t}\int_\tau^t e^{\sigma s}[\frac{1}{\epsilon(s)}\sum_{n\in\mathbb{Z}} \xi(\frac{|n|}{\zeta}) g_n^2(s) \\ &+ \sum_{n\in\mathbb{Z}} \xi(\frac{|n|}{\zeta}) |f_n(s)|^2] \mathrm{d}s + \frac{\varepsilon^2}{6}.\end{aligned} \tag{3.41}$$

Also, by assumption (A2), there exists some $\zeta_4 = \zeta_4(t,\varepsilon) \in \mathbb{Z}_+$ such that

$$\begin{aligned}& e^{-\sigma t}\int_\tau^t e^{\sigma s}[\frac{1}{\epsilon(s)}\sum_{n\in\mathbb{Z}} \xi(\frac{|n|}{\zeta}) g_n^2(s) + \sum_{n\in\mathbb{Z}} \xi(\frac{|n|}{\zeta})|f_n(s)|^2]\mathrm{d}s \\ \leqslant& \frac{e^{-\sigma t}}{\epsilon(t)} \int_{-\infty}^t e^{\sigma s} \sum_{n\in\mathbb{Z}} \xi(\frac{|n|}{\zeta}) g_n^2(s)\mathrm{d}s + e^{-\sigma t}\int_{-\infty}^t e^{\sigma s}\sum_{n\in\mathbb{Z}} \xi(\frac{|n|}{\zeta})|f_n(s)|^2 \mathrm{d}s \lesssim \frac{\varepsilon^2}{6}, \ \forall \zeta \geqslant \zeta_4.\end{aligned} \tag{3.42}$$

Noticing that $z_\tau \in B_{E_\tau}[0,r]$, we infer that there is a $t_2 = t_2(t,r,\varepsilon) \leqslant t$ where $r > 0$, $\varepsilon > 0$ and $t \in \mathbb{R}$ are as given above, such that

$$\|z_\tau\|^2_{E_\tau} e^{-\sigma(t-\tau)} < \frac{\varepsilon^2}{6}, \ \forall \tau \leqslant t_2. \tag{3.43}$$

Substituting (3.42) and (3.43) into (3.41) gives

$$\sum_{|n|\geqslant 2\zeta_*} \xi(\frac{|n|}{\zeta})|z_n(t)|^2_{E_t} \leqslant \sum_{n\in\mathbb{Z}} \xi(\frac{|n|}{\zeta})[|z_n(t)|^2_{E_t} + (A(u(t)))^2_n] \lesssim \frac{\varepsilon^2}{2}, \ \forall \tau \leqslant t_*, \tag{3.44}$$

where $\zeta_* = \max\{\zeta_1,\zeta_2,\zeta_3,\zeta_4\}$, $t_* = \min\{t_0,t_1,t_2\}$. Therefore,

$$\begin{aligned}\sup_{z_\tau\in B_{E_\tau}[0,r]} \sum_{|n|\geqslant 2\zeta_*} |(V(t,\tau)z_\tau)_n|^2_{E_t} =& \sup_{z_\tau\in B_{E_\tau}[0,r]} \sum_{|n|\geqslant 2\zeta_*} |z_n(t)|^2_{E_t} \\ \leqslant& 2\sum_{|n|\geqslant 2\zeta_*} \xi(\frac{|n|}{\zeta})|z_n(t)|^2_{E_t} \lesssim \varepsilon^2, \ \forall \tau \leqslant t_*.\end{aligned} \tag{3.45}$$

This completes the proof. □

With Lemmas 3.1 and 3.2 in hand, it follows immediately from [1, Proposition 3.1] that

Theorem 3.1. *Suppose that* (A1)–(A2) *hold. Then the evolution process* $\{V(t,\tau) : E_\tau \to E_t, (t,\tau) \in \mathbb{R}^2_d\}$ *has a time-dependent pullback attractor* $\mathfrak{A} = \{\mathcal{A}_t\}_{t\in\mathbb{R}}$ *which fulfills the three conditions in the Definition 3.1(3).*

4. Invariant Borel probability measures and statistical solutions

In this section, we first prove a certain continuity of the evolution process $\{V(t,\tau) : E_\tau \to E_t, (t,\tau) \in \mathbb{R}^2_d\}$ with respect to the initial time τ, and then use the approaches of [1, 6] to establish a family of invariant Borel probability measures $\{\mathfrak{m}_t\}_{t\in\mathbb{R}}$. Lastly, we verify that $\{\mathfrak{m}_t\}_{t\in\mathbb{R}}$ satisfies Liouville's theorem and is a statistical solution of the addressed Zakharov equations.

Define

$$\Gamma_r = \Big\{\{\kappa(s)\}_{s\in\mathbb{R}} : \kappa(s) = (\varphi, \sqrt{\epsilon(s)}u, v)^T \in E_s, \kappa = (\varphi, u, v)^T \in B_E[0, r]\Big\}, \quad r \geqslant 0. \tag{4.1}$$

Note that for any $r \geqslant 0$ and any $z_* \in B_E[0, r]$, there holds

$$\|z_*(t)\|^2_{E_t} = \|z_*(\tau)\|^2_{E_\tau}, \quad \forall t, \tau \in \mathbb{R}. \tag{4.2}$$

We next prove an auxiliary lemma which helps us to verify that for any given $(t,\tau) \in \mathbb{R}^2_d$ and $\{z_*(s)\}_{s\in\mathbb{R}} \in \Gamma_r$ with $r > 0$, the map $\tau \mapsto V(t,\tau)z_*(\tau)$ is continuous.

Lemma 4.1. *Suppose that* (A1)–(A2) hold. Given $\tau_* \in \mathbb{R}$ and $\{z_*(s)\}_{s\in\mathbb{R}} \in \Gamma_r$ $(r > 0)$. *Then for every* $\varepsilon > 0$*, there is a positive constant* $\rho = \rho(\tau_*, r, \varepsilon)$ *such that*

$$\begin{cases} \|V(s,\tau_*)z_*(\tau_*) - z_*(s)\|^2_{E_s} \lesssim \varepsilon, \ \forall\, s \in [\tau_*, \tau_* + \rho), \\ \|V(\tau_*, s)z_*(s) - z_*(\tau_*)\|^2_{E_{\tau_*}} \lesssim \varepsilon, \ \forall\, s \in [\tau_* - \rho, \tau_*). \end{cases} \tag{4.3}$$

Proof. Given $r > 0$, $\tau_* \in \mathbb{R}$ and $\{z_*(\vartheta)\}_{\vartheta\in\mathbb{R}} \in \Gamma_r$ with $z_* = (\varphi_*, u_*, v_*)^T \in B_E[0, r]$. We consider any $s \in [\tau_*, \tau_* + 1]$, and let

$$V(s,\tau_*)z_*(\tau_*) = (\varphi(s), u(s), v(s))^T = (\varphi_m(s), u_m(s), v_m(s))^T_{m\in\mathbb{Z}} \in E_s$$

be the solution of problem (2.9)–(2.10) corresponding with the initial value

$$z_*(\tau_*) = (\varphi_{\tau_*}, u_{\tau_*}, v_{\tau_*})^T = (\varphi_*, \sqrt{\epsilon(\tau_*)}u_*, v_*)^T \in E_{\tau_*},$$

at initial time τ_*. Straightforward computations show that

$$\begin{aligned} &\|V(s,\tau_*)z_*(\tau_*) - z_*(s)\|^2_{E_s} \\ =&\|V(s,\tau_*)z_*(\tau_*)\|^2_{E_s} - \|z_*(\tau_*)\|^2_{E_{\tau_*}} - 2(V(s,\tau_*)z_*(\tau_*) - z_*(s), z_*(s))_{E_s}, \end{aligned} \tag{4.4}$$

where we have used (4.2). We next estimate separately each of the terms in the right-hand side of (4.4).

For the first two terms, by applying Eq (2.28) and utilizing the monotonicity of $\epsilon(t)$, we can deduce

$$\|V(s,\tau_*)z_*(\tau_*)\|^2_{E_s} - \|z_*(\tau_*)\|^2_{E_{\tau_*}} = \int_{\tau_*}^{s} \frac{\mathrm{d}\|V(\vartheta,\tau_*)z_*(\tau_*)\|^2_{E_\vartheta}}{\mathrm{d}\vartheta}\mathrm{d}\vartheta$$

$$\lesssim \int_{\tau_*}^{s} \|f(\vartheta)\|^2 \mathrm{d}\vartheta + \frac{1}{\epsilon(\tau_*+1)} \int_{\tau_*}^{s} \|g(\vartheta)\|^2 \mathrm{d}\vartheta + \frac{1}{\epsilon(\tau_*+1)} \int_{\tau_*}^{s} \|\varphi(\vartheta)\|^4 \mathrm{d}\vartheta. \tag{4.5}$$

Considering Eqs (2.39) and (3.7), we get

$$\begin{aligned}\int_{\tau_*}^{s} \|\varphi(\vartheta)\|^4 d\vartheta \lesssim & \int_{\tau_*}^{s} \|\varphi_{\tau_*}\|^4 \mathrm{d}\vartheta + \int_{\tau_*}^{s} (e^{-\gamma\vartheta} \int_{-\infty}^{\vartheta} e^{\gamma\eta} \|f(\eta)\|^2 \mathrm{d}\eta)^2 \mathrm{d}\vartheta \\ & + \|\varphi_{\tau_*}\|^2 \int_{\tau_*}^{s} e^{-r\vartheta} \int_{-\infty}^{\vartheta} e^{\gamma\eta} \|f(\eta)\|^2 \mathrm{d}\eta \mathrm{d}\vartheta \\ \lesssim & r^4 (s-\tau_*) + \int_{\tau_*}^{s} e^{(2\rho-\sigma)\vartheta} J^2(\vartheta) \mathrm{d}\vartheta + r^2 \int_{\tau_*}^{s} e^{(\rho-\frac{\sigma}{2})\vartheta} J(\vartheta) \mathrm{d}\vartheta. \end{aligned} \tag{4.6}$$

Since $g(\cdot) \in C(\mathbb{R}, l^2)$ is given, and $J(\cdot)$ and $J^2(\cdot) \in C(\mathbb{R}, \mathbb{R})$, we deduce that for any $\varepsilon > 0$, there exists a positive constant $\rho_1 = \rho_1(\tau_*, \varepsilon, r)$ yielding

$$\begin{aligned}\|V(s,\tau_*)z_*(\tau_*)\|^2_{E_s} - \|z_*(\tau_*)\|^2_{E_{\tau_*}} \lesssim & \int_{\tau_*}^{s} \|f(\vartheta)\|^2 \mathrm{d}\vartheta + \frac{1}{\epsilon(\tau_*+1)} \int_{\tau_*}^{s} \|g(\vartheta)\|^2 \mathrm{d}\vartheta \\ & + \frac{1}{\epsilon(\tau_*+1)} \int_{\tau_*}^{s} \|\varphi(\vartheta)\|^4 \mathrm{d}\vartheta < \frac{\varepsilon}{3}, \ \forall\, s \in (\tau_*, \tau_* + \rho_1). \end{aligned} \tag{4.7}$$

For the last term in the right-hand side of (4.4), it is not hard to check that

$$\begin{aligned} & 2(V(s,\tau_*)z_*(\tau_*) - z_*(s), z_*(s))_{E_s} \\ =& 2(V(s,\tau_*)z_*(\tau_*) - z_*(\tau_*), z_*(s))_{E_s} + 2(z_*(\tau_*) - z_*(s), z_*(s))_{E_s}. \end{aligned} \tag{4.8}$$

Seeing the definition of Γ_r, we have that

$$\begin{aligned}\|z_*(s)\|^2_{E_s} =& \|\varphi_*\|^2 + \|\sqrt{\epsilon(s)}u_*\|^2_{\epsilon(s)} + \|\nu_*\|^2 \\ =& \|\varphi_*\|^2 + \|u_*\|^2_{\mu} + \|\nu_*\|^2 \leqslant (6+\mu) r^2, \ \forall\, s \in \mathbb{R}, \end{aligned} \tag{4.9}$$

and that $z_*(s) \in B_{E_s}[0, \sqrt{6+\mu}r]$ for each $s \in \mathbb{R}$. Therefore,

$$\begin{aligned} & |(V(s,\tau_*)z_*(\tau_*) - z_*(\tau_*), z_*(s))_{E_s}| = |(\int_{\tau_*}^{s} \frac{\mathrm{d}V(\vartheta,\tau_*)z_*(\tau_*)}{\mathrm{d}\vartheta} \mathrm{d}\vartheta, z_*(s))_{E_s}| \\ \leqslant & \int_{\tau_*}^{s} \|\frac{\mathrm{d}V(\vartheta,\tau_*)z_*(\tau_*)}{\mathrm{d}\vartheta}\|_{E_s} \mathrm{d}\vartheta \|z_*(s)\|_{E_s} \leqslant \int_{\tau_*}^{s} \frac{\epsilon(\vartheta)}{\epsilon(s)} \|\frac{\mathrm{d}V(\vartheta,\tau_*)z_*(\tau_*)}{\mathrm{d}\vartheta}\|_{E_\vartheta} \mathrm{d}\vartheta \|z_*(s)\|_{E_s} \\ \lesssim & \frac{r\epsilon(\tau_*)}{\epsilon(\tau_*+1)} (\int_{\tau_*}^{s} \|\frac{\mathrm{d}V(\vartheta,\tau_*)z_*(\tau_*)}{\mathrm{d}\vartheta}\|^2_{E_\vartheta} \mathrm{d}\vartheta)^{\frac{1}{2}} (s-\tau_*)^{\frac{1}{2}}. \end{aligned} \tag{4.10}$$

We next verify that there exist constants $c_4 = c_4(\tau_*, r) > 0$ and $c_5 = c_5(\tau_*)$ yielding

$$\begin{aligned}\int_{\tau_*}^{s} \|\frac{\mathrm{d}V(\vartheta,\tau_*)z_*(\tau_*)}{\mathrm{d}\vartheta}\|^2_{E_\vartheta} \mathrm{d}\vartheta \lesssim & c_*(\tau_*, r) \\ := & c_4 c_5 + c_4^2 + \int_{\tau_*}^{\tau_*+1} \|g(\vartheta)\|^2 \mathrm{d}\vartheta + \int_{\tau_*}^{\tau_*+1} \|f(\vartheta)\|^2 \mathrm{d}\vartheta. \end{aligned} \tag{4.11}$$

In fact, we deduce from (2.9) and (2.15) that

$$\begin{aligned}
&\|\frac{\mathrm{d}V(\vartheta,\tau_*)z_*(\tau_*)}{\mathrm{d}\vartheta}\|^2_{E_\vartheta}\\
\lesssim&\|\Theta(\vartheta)V(\vartheta,\tau_*)z_*(\tau_*)\|^2_{E_\vartheta}+\|\Upsilon(V(\vartheta,\tau_*)z_*(\tau_*),\vartheta)\|^2_{E_\vartheta}\\
\lesssim&c_1(\vartheta)\|V(\vartheta,\tau_*)z_*(\tau_*)\|^2_{E_\vartheta}+\|\varphi(\vartheta)u(\vartheta)-f(\vartheta)\|^2+\frac{\||\varphi(\vartheta)|^2+g(\vartheta)\|^2}{\epsilon^2(\vartheta)}.
\end{aligned} \tag{4.12}$$

Referring to (2.24) and employing similar derivations as those used in (4.6), we get

$$\begin{aligned}
&\|V(\vartheta,\tau_*)z_*(\tau_*)\|^2_{E_\vartheta}\\
\leqslant&\|z_*(\tau_*)\|^2_{E_{\tau_*}}e^{-\sigma(\vartheta-\tau_*)}+c_2(\vartheta)e^{-\sigma\vartheta}\int_{\tau_*}^{\vartheta}e^{\sigma\eta}(\|f(\eta)\|^2+\|g(\eta)\|^2)\mathrm{d}\eta\\
&+c_2(\vartheta)e^{-\sigma\vartheta}\int_{\tau_*}^{\vartheta}e^{\sigma\eta}\|\varphi(\eta)\|^4\mathrm{d}\eta\\
\lesssim&r^2+c_2(\tau_*+1)\int_{\tau_*}^{\tau_*+1}(\|f(\eta)\|^2+\|g(\eta)\|^2)\mathrm{d}\eta\\
&+c_2(\tau_*+1)[\int_{\tau_*}^{\tau_*+1}r^4\mathrm{d}\vartheta+\int_{\tau_*}^{\tau_*+1}e^{(2\varrho-\sigma)\vartheta}J^2(\vartheta)\mathrm{d}\vartheta+r^2\int_{\tau_*}^{\tau_*+1}e^{(\varrho-\frac{\sigma}{2})\vartheta}J(\vartheta)\mathrm{d}\vartheta]\\
=:&\ c_4=c_4(\tau_*,r),\ \forall\,\vartheta\in[\tau_*,\tau_*+1],
\end{aligned} \tag{4.13}$$

which means

$$\|\varphi(\vartheta)\|^2\lesssim c_4\text{ and }\|u(\vartheta)\|^2\lesssim c_4. \tag{4.14}$$

At the same time, for any $t\in\mathbb{R}$, there holds $\mathfrak{s}(t)\in(0,\frac{\lambda}{4})$. By (2.16), we infer that

$$c_1(\vartheta)\ \leqslant c_5\ =c_5(\tau_*):=\max_{s\in[\tau_*,\tau_*+1]}c_1(s),\ \ \forall\vartheta\in[\tau_*,\tau_*+1].$$

Inserting estimates (4.13) and (4.14) into (4.12) yields

$$\|\frac{\mathrm{d}V(\vartheta,\tau_*)z_*(\tau_*)}{\mathrm{d}\vartheta}\|^2_{E_\vartheta}\lesssim c_4c_5+c_4^2+\|f(\vartheta)\|^2+\frac{c_4^2+\|g(\vartheta)\|^2}{\epsilon^2(\tau_*+1)},\ \forall\,\vartheta\in[\tau_*,\tau_*+1], \tag{4.15}$$

which proves (4.11). Combining (4.11) and (4.10), we have

$$\begin{aligned}
|(V(s,\tau_*)z_*(\tau_*)-z_*(\tau_*),z_*(s))_{E_s}|\lesssim&\frac{r\epsilon(\tau_*)}{\epsilon(\tau_*+1)}(\int_{\tau_*}^{s}\|\frac{\mathrm{d}V(\vartheta,\tau_*)z_*(\tau_*)}{\mathrm{d}\vartheta}\|^2_{E_\vartheta}\mathrm{d}\vartheta)^{\frac{1}{2}}(s-\tau_*)^{\frac{1}{2}}\\
\lesssim&\frac{rc_*(\tau_*,r)\epsilon(\tau_*)}{\epsilon(\tau_*+1)}(s-\tau_*)^{\frac{1}{2}},
\end{aligned}$$

which indicates that, for above ε, there is a positive constant $\rho_2=\rho_2(\tau_*,\varepsilon,r)$ yielding

$$2|(V(s,\tau_*)z_*(\tau_*)-z_*(\tau_*),z_*(s))_{E_s}|\lesssim\frac{\varepsilon}{6},\ \forall\,s\in[\tau_*,\tau_*+1]. \tag{4.16}$$

In addition, we have

$$|(z_*(\tau_*)-z_*(s),z_*(s))_{E_s}|=\Big|\Big((\sqrt{\epsilon(\tau_*)}-\sqrt{\epsilon(s)})u_*,\ \sqrt{\epsilon(s)}u_*\Big)_{\epsilon(s)}\Big|$$

$$=\Big|\frac{\sqrt{\epsilon(\tau_*)}-\sqrt{\epsilon(s)}}{\sqrt{\epsilon(s)}}\Big|\|u_*\|_\mu^2 \lesssim \frac{r^2|\sqrt{\epsilon(\tau_*)}-\sqrt{\epsilon(s)}|}{\sqrt{\epsilon(\tau_*+1)}}.$$

Due to the continuity of $\epsilon(t)$, we get that for above ε there is a positive constant $\rho_3=\rho_3(\tau_*,\varepsilon,r)$ yielding

$$|2(z_*(\tau_*)-z_*(s),z_*(s))_{E_s}|<\frac{\varepsilon}{6},\ \forall\, s\in[\tau_*,\tau_*+\rho_3). \tag{4.17}$$

Picking $\rho_4=\min\{\rho_2,\rho_3\}$ and using (4.8), (4.16) and (4.17), we get

$$|2(V(s,\tau_*)z_*(\tau_*)-z_*(s),z_*(s))_{E_s}|\lesssim\frac{\varepsilon}{3},\ \forall\, s\in[\tau_*,\tau_*+\rho_4). \tag{4.18}$$

Taking $\rho=\rho(\tau_*,r,\varepsilon)=\min\{\rho_1,\rho_4\}$, we get $(4.3)_1$ from (4.4), (4.7) and (4.18). One may repeat almost the same procedure to prove $(4.3)_2$. This completes the proof. □

Using Lemma 4.1 and applying the method in [6, Lemma 4.3], we get

Lemma 4.2. *Suppose that* (A1)–(A2) hold. *Then the map* $\tau\mapsto V(t,\tau)z_*(\tau)$ *is bounded and continuous on* $(-\infty,t]$ *for any given* $(t,\tau)\in\mathbb{R}_d^2$ *and any* $\{z_*(s)\}_{s\in\mathbb{R}}\in\Gamma_r$ *with any* $r>0$.

Proof. Consider any $t\in\mathbb{R}$ and let $\{z_*(s)\}_{s\in\mathbb{R}}$ with $z_*=(\varphi_*,u_*,\nu_*)^T\in B_E[0,r]$ $(r>0)$ be given. From (2.24) and Lemma 3.1, it follows readily that the map $\tau\mapsto V(t,\tau)z_*(\tau)$ is bounded.

Next, we proceed to prove for every $\tau_*\in(-\infty,t]$ and $\varepsilon>0$, there is a $\rho=\rho(\varepsilon,t,r)>0$ yielding

$$|s-\tau_*|<\rho\Longrightarrow\|V(t,s)z_*(s)-V(t,\tau_*)z_*(\tau_*)\|_{E_t}<\varepsilon. \tag{4.19}$$

We establish (4.19) by considering the cases that $\tau_*\leqslant s\leqslant\tau_*+1$ and $\tau_*-1\leqslant s\leqslant\tau_*$. For the first case, we employ the invariance of the evolution process and (2.32) to obtain

$$\begin{aligned}&\|V(t,s)z_*(s)-V(t,\tau_*)z_*(\tau_*)\|_{E_t}^2=\|V(t,s)z_*(s)-V(t,s)V(s,\tau_*)z_*(\tau_*)\|_{E_t}^2\\ \leqslant& c_3(r,t,s)\|z_*(s)-V(s,\tau_*)z_*(\tau_*)\|_{E_s}^2\leqslant\tilde{c}_3\|z_*(s)-V(s,\tau_*)z_*(\tau_*)\|_{E_s}^2,\end{aligned} \tag{4.20}$$

where $\tilde{c}_3=\tilde{c}_3(r,t,\tau_*)=\max_{s\in[\tau_*,\tau_*+1]}\{c_3(r,t,s)\}$. It is concluded from Lemma 4.1 that, for any $\varepsilon>0$ there is a $\rho'=\rho'(\tau_*,r,\varepsilon)>0$ yielding

$$\|V(t,s)z_*(s)-V(t,\tau_*)z_*(\tau_*)\|_{E_t}^2<\varepsilon,\ \forall\, s\in(\tau_*,\tau_*+\rho'], \tag{4.21}$$

indicating that the map $\tau\mapsto V(t,\tau)z_*(\tau)$ is right-continuous at $\tau=\tau_*$. We can prove the other case by the similar argument. Due to the arbitrariness of τ_*, we end the proof of Lemma 4.2. □

Combining Theorem 3.1, Lemma 4.2, [1, Theorem 4.1] and the concept of generalized Banach limits (see [14] for the definition), we can obtain the following result.

Theorem 4.1. *Suppose that* (A1)–(A2) *hold. Let* $\{z_*(s)\}_{s\in\mathbb{R}}\in\Gamma_r$ $(r>0)$ *with* $z_*=(\varphi_*,u_*,\nu_*)^T\in B_E[0,r]$ *be given. Then for any generalized Banach limit* $\mathrm{LIM}_{\cdot\to-\infty}$, *any* $t\in\mathbb{R}$ *and* $\varphi_t\in C(E_t)$, *there exists a unique family of Borel probability measures* $\{\mathfrak{m}_s\}_{s\in\mathbb{R}}$ *with support* $\operatorname{supp}\mathfrak{m}_s$ *contained in* $\mathcal{A}_s$ *for every* s *such that*

$$\mathrm{LIM}_{\tau\to-\infty}\frac{1}{t-\tau}\int_\tau^t\varphi_t(V(t,\vartheta)z_*(\vartheta))\mathrm{d}\vartheta=\int_{\mathcal{A}_t}\varphi_t(z)\mathrm{d}\mathfrak{m}_t(z)=\int_{E_t}\varphi_t(z)\mathrm{d}\mathfrak{m}_t(z)$$

$$=\mathrm{LIM}_{\tau\to-\infty}\frac{1}{t-\tau}\int_{\tau}^{t}\int_{E_\vartheta}\varphi_t(V(t,\vartheta)z)\mathrm{d}\mathfrak{m}_\vartheta(z)\mathrm{d}\vartheta. \tag{4.22}$$

What is more, $\{\mathfrak{m}_s\}_{s\in\mathbb{R}}$ *is invariant, that is,*

$$\int_{\mathcal{A}_t}\varphi_t(z)\mathrm{d}\mathfrak{m}_t(z)=\int_{\mathcal{A}_\tau}\varphi_t(V(t,\tau)z)\mathrm{d}\mathfrak{m}_\tau(z),\ \forall\, t\geqslant\tau. \tag{4.23}$$

In what follows, we investigate the statistical solutions for Eq (2.9). Firstly, rewrite Eq (2.9) as

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\mathcal{F}(z,t):=\Upsilon(z,t)-\Theta(t)z,\ t\in\mathbb{R}, \tag{4.24}$$

and define the family of class $\{\mathcal{T}_t\}_{t\in\mathbb{R}}$ (called test function class) for Eq (4.24) analogously to [6, Definition 4.5]. Also one can refer to [6] for the existences of the test functions. Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t(z)=(\mathcal{F}(z,t),\Phi_t'(z))_{E_t},\ t\in\mathbb{R}, \tag{4.25}$$

holds true for any test function $\Phi_t(\cdot)$ and any solution $z(\cdot)$ of Eq (4.24). We proceed to give the precise definition of statistical solutions for Eq (4.24) on the family $\{E_s,\|\cdot\|_{E_s}\}_{s\in\mathbb{R}}$.

Definition 4.1. *A family* $\{\rho_t\}_{t\in\mathbb{R}}\subset\mathcal{P}(E_t)$ *is referred to as a statistical solution of Eq* (4.24) *if it satisfies:*

(a) *The function* $z\mapsto(\mathcal{F}(z,t),\psi)_{E_t}$ *is* ρ_t*-integrable for almost* $t\in\mathbb{R}$ *and every* $\psi\in E_t$*. Furthermore, the mapping* $t\mapsto\int_{E_t}(\mathcal{F}(z,t),\psi)_{E_t}\mathrm{d}\rho_t(\psi)\in L^1_{\mathrm{loc}}(\mathbb{R})$ *for any* $\psi\in E_t$*;*

(b) *For every* $\{z(s)\}_{s\in\mathbb{R}}\in\bigcup_{r\geqslant0}\Gamma_r$ *and all* $t\geqslant\tau$*, the Liouville-type equation*

$$\int_{E_t}\Phi_t(z(t))\mathrm{d}\rho_t(z(t))-\int_{E_\tau}\Phi_\tau(z(\tau))\mathrm{d}\rho_\tau(z(\tau))$$
$$=\int_{\tau}^{t}\int_{E_\vartheta}(\mathcal{F}(z(\vartheta),\vartheta),\Phi_\vartheta'(z(\vartheta)))_{E_\vartheta}\mathrm{d}\rho_\vartheta(z(\vartheta))\mathrm{d}\vartheta, \tag{4.26}$$

holds for each $\{\Phi_t\}_{t\in\mathbb{R}}\in\{\mathcal{T}_t\}_{t\in\mathbb{R}}$

The main result of this section reads as follows.

Theorem 4.2. *Suppose that* (A1)–(A2) *hold. Then the family* $\{\mathfrak{m}_t\}_{t\in\mathbb{R}}$ *constructed in Theorem 4.1 is a statistical solution of Eq* (4.24)*.*

Proof. We will sketch the proof and describe the differences because that the detailed computations are analogous with those presented by [6] in the construction of the statistical solutions for the lattice Klein-Gordon-Schrödinger equations on the family of time-dependent phase spaces.

Step one. Let $\psi=(\psi_1,\psi_2,\psi_3)^T\in E_t$ and $t\in\mathbb{R}$ be given. Define $\varphi_t(\cdot):E_t\mapsto\mathbb{R}$ via

$$\varphi_t(z)=(\mathcal{F}(z,t),\psi)_{E_t},\ \forall\, z=(\varphi,u,v)^T\in E_t. \tag{4.27}$$

Then check that $\varphi_t(\cdot)\in C(E_t)$ and conclude from Theorem 4.1 that the function $z\mapsto(\mathcal{F}(z,t),\psi)_{E_t}:=\varphi_t(z)$ is $\mathfrak{m}_t$-integrable, and that the mapping

$$t\mapsto\int_{E_t}(\mathcal{F}(z,t),\psi)_{E_t}\mathrm{d}\mathfrak{m}_t(z)=\int_{E_t}\varphi_t(z)\mathrm{d}\mathfrak{m}_t(z)\in L^1_{\mathrm{loc}}(\mathbb{R}).$$

Step two. Prove that $\{\mathfrak{m}_t\}_{t\in\mathbb{R}}$ satisfies (4.26). This can be done as follows. Firstly, one can use (4.25) to obtain that for any $\{\phi(s)\}_{s\in\mathbb{R}} \in \bigcup_{r\geqslant 0}\Gamma_r$ there holds

$$\Phi_t(V(t,s)\phi(s)) - \Phi_\tau(V(\tau,s)\phi(s)) = \int_\tau^t (\mathcal{F}(V(\vartheta,s)\phi(s),\vartheta), \Phi'_\vartheta(V(\vartheta,s)\phi(s)))_{E_\vartheta}\mathrm{d}\vartheta. \tag{4.28}$$

Then by performing some calculations and utilizing (4.22) and (4.28), we can get

$$\begin{aligned}&\int_{E_t}\Phi_t(\phi(t))\mathrm{d}\mathfrak{m}_t(\phi(t)) - \int_{E_\tau}\Phi_\tau(\phi(\tau))\mathrm{d}\mathfrak{m}_\tau(\phi(\tau))\\ =&\mathrm{LIM}_{M\to-\infty}\frac{1}{\tau-M}\int_M^\tau\int_{E_s}(\mathcal{F}(V(\vartheta,s)\phi(s),\vartheta),\Phi'_\vartheta(V(\vartheta,s)\phi(s)))_{E_\vartheta}\mathrm{d}\mathfrak{m}_s(\phi(s))\mathrm{d}\vartheta\mathrm{d}s.\end{aligned} \tag{4.29}$$

Afterwards, by applying the invariance of the evolution process and (4.23), one has

$$\begin{aligned}&\int_{E_s}(\mathcal{F}(V(\vartheta,s)\phi(s),\vartheta),\Phi'_\vartheta(V(\vartheta,s)\phi(s)))_{E_\vartheta}\mathrm{d}\mathfrak{m}_s(\phi(s))\\ =&\int_{E_\tau}(\mathcal{F}(V(\vartheta,\tau)\phi(\tau),\vartheta),\Phi'_\vartheta(V(\vartheta,\tau)\phi(\tau)))_{E_\vartheta}\mathrm{d}\mathfrak{m}_\tau(\phi(\tau)).\end{aligned} \tag{4.30}$$

Lastly, by taking (4.28), (4.29), (4.30) and (4.23) into account, one can get

$$\begin{aligned}&\int_{E_t}\Phi_t(\phi(t))\mathrm{d}(\mathfrak{m}_t(\phi(t)) - \int_{E_\tau}\Phi_\tau(\phi(\tau))\mathrm{d}\mathfrak{m}_\tau(\phi(\tau))\\ =&\int_\tau^t\int_{E_\vartheta}(\mathcal{F}(\phi(\vartheta),\vartheta),\Phi'_\vartheta(\phi(\vartheta)))_{E_\vartheta}\mathrm{d}\mathfrak{m}_\vartheta(\phi(\vartheta))\mathrm{d}\vartheta.\end{aligned} \tag{4.31}$$

This ends the proof of Theorem 4.2. □

We want to point out that (4.31) is called Liouville's type equation satisfied by the statistical solution $\{\mathfrak{m}_t\}_{t\in\mathbb{R}}$ of Eq (4.24). In Statistical Mechanics, we say that evolution system attains its statistical equilibrium provided that $\Phi'_\vartheta(\cdot)\equiv 0$ for all $\vartheta\in\mathbb{R}$. In this case, Eq (4.31) turns to be

$$\int_{\mathcal{A}_t}\Phi_t(z(t))\mathrm{d}\mathfrak{m}_t(z(t)) = \int_{\mathcal{A}_\tau}\Phi_\tau(z(\tau))\mathrm{d}\mathfrak{m}_\tau(z(\tau)),\quad \forall\{z(s)\}_{s\in\mathbb{R}}\in\cup_{r\geqslant 0}\Gamma_r, \forall\, t,\tau\in\mathbb{R}, \tag{4.32}$$

which indicates that Liouville's theorem of Statistical Mechanics also holds true for the lattice Zakharov equations with varying coefficients.

5. Conclusions and discussions

In this paper, we follow the approaches of [1, 6] to prove the existences of the time-dependent pullback attractor and statistical solution for the lattice Zakharov equations with varying coefficients. The key difficulties come from the nonlinear term $A|\varphi|^2$, and the additional terms $D\varphi$ and Du corresponding to the higher-order derivative terms $|\varphi|^2_{xx}$, φ_{xxxx} and u_{xxxx}. These terms require us to construct subtle time-dependent phase spaces where to obtain a certain coercive property of the operator corresponding to the linear principle part extracted from the addressed equations. Our result indicates that the lattice Zakharov equations with varying coefficient satisfy Liouville's theorem.

There is an interesting issue. Consider the family of lattice Zakharov equations with varying coefficient

$$\begin{cases} i\dot{\varphi}_n + (A\varphi)_n - h^2(D\varphi)_n - u_n\varphi_n + i\gamma\varphi_n = f_n(t), \\ \epsilon_m(t)\ddot{u}_n + \lambda\dot{u}_n - (Au)_n + h^2(Du)_n - (A|\varphi|^2)_n + \mu u_n = g_n(t), \end{cases} \tag{5.1}$$

where $n \in \mathbb{Z}, \ \ m \in \mathbb{Z}_+$. We are curious about the upper semi-continuity of the time-dependent pullback attractors and statistical solutions for system (5.1) as the sequence of functions $\epsilon_m(\cdot)$ tends to zero uniformly on $\mathbb{R}$ as $m \to \infty$.

Author contributions

Anran Li participates in the theoretical analysis part, such as helping to prove the existence of pullback attractors, invariant measures. Caidi Zhao is involved in the key theoretical analysis and result verification. Tomás Caraballo works on the global well-posedness of solutions and helps to verify the relevant properties of the equations, like checking the continuity of solution mappings and related inequalities. All authors read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors of this paper hereby state that they do not have any competing interests that might compromise the objectivity, fairness, or honesty of the research findings presented in this paper.

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Appendix

Symbols index.

$\mathbb{C}$	Set of complex numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{Z}$	Set of integers
$\mathbb{Z}_+$	Set of positive integers
i	Imaginary unit
$\mathbb{R}^2_d$	$\mathbb{R}^2_d = \{(t,\tau) \in \mathbb{R}^2 : \tau \leqslant t\}$
$\lesssim$ (or $\gtrsim$)	$a \lesssim cb$ (or $a \gtrsim cb$) for some absolute constant c
$\bar{v}$	Conjugate of v
I	Identity operator
$(\cdot,\cdot,\cdot)^T$	Transposition of a vector
$B_\bullet[a,r]$	Closed balls in space $\bullet$ centered at $a \in \bullet$ with radius r
$C(\bullet)$	Set of continuous functions from $\bullet$ to $\mathbb{R}$
$\mathcal{P}(\bullet)$	Set of Borel probability measure on space $\bullet$