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Research article

Quasi-idempotent graphs of rings

Shifeng Luo*

School of Mathematics and Physics, Hechi University, Yizhou 546300, China

* **Correspondence:** Email: siephoon@163.com.

Abstract: Let R be a ring. An element $a \in R$ is called a quasi-idempotent if there exists a central unit k in R such that $a^2 = ka$. The quasi-idempotent graph of R, denoted by $G_{Qid}(R)$, is the simple undirected graph with vertex set R itself, where two distinct vertices a and b are adjacent if and only if $a + b$ is a quasi-idempotent. This paper presents a systematic study of the graph $G_{Qid}(R)$. We examine its basic structural properties, including connectivity and girth. We introduce a new invariant of the ring, termed the quasi-idempotent sum number, and establish the precise relationship between this invariant and the graph diameter. Furthermore, a complete classification is obtained for all finite commutative rings R according to the genus of $G_{Qid}(R)$, thereby characterizing the rings for which this graph has genus 0, 1, or 2.

Keywords: quasi-idempotent graph; girth; diameter; genus; finite commutative ring
Mathematics Subject Classification: 05C25, 13A99, 16L99

1. Introduction

Over the past three decades, the association of graph structures with algebraic systems has emerged as a vibrant and enduring research frontier. Many researchers have employed graph-theoretic concepts to investigate the properties of related algebraic systems. This line of inquiry was initiated by Beck in 1988 [1], who introduced the zero-divisor graph of a commutative ring. This graph, which is simple and undirected, has the entire ring as its vertex set, with two distinct vertices adjacent precisely when their product is zero. Beck used the chromatic number of this graph to characterize the structure of finite commutative rings whose zero-divisor graphs have a chromatic number not exceeding three. Since then, graph structures defined on various algebraic systems have gained widespread attention and have been extensively studied. Examples include the power graph of a group [2], the total graph of a ring [3], the unit graph of a ring [4], the enhanced power graph of a group [5], the commuting graph of a group [6], and the line graph of the comaximal graph of a ring [7].

Zero divisors, units, and idempotents are fundamental components for characterizing the structural properties of rings. In the interdisciplinary field of ring theory and graph theory, graph structures defined based on ring zero divisors include the zero-divisor graph [8] and the total graph of a ring. The study of graphs derived from ring units, including the unit graph and the unit Cayley graph, has grown into a fruitful interdisciplinary subfield.

The study of unit graphs was pioneered by Grimaldi in 1990, who defined the unit graph of a ring, taking all ring elements as vertices, where two vertices are adjacent if and only if their sum is a unit. Grimaldi investigated several graph-theoretic invariants for the unit graphs of rings of integers modulo n, including vertex degrees, Hamiltonian cycles, covering number, independence number, and chromatic polynomials. In 2010, Ashrafi et al. [9] analyzed the connectivity, diameter, girth, and planarity of unit graphs over finite commutative rings and derived relevant conclusions. Subsequently, Su et al. [10–13] further investigated properties of unit graphs concerning their girth, genus, domination number, and other characteristics. Research related to idempotent elements was initiated by Akbari et al. in 2013 [14]. They introduced the idempotent graph, whose vertices are the idempotent elements of the ring, with two vertices adjacent if and only if their product is zero. They explored graph properties, such as connectivity, diameter, girth, and genus, of such idempotent graphs. In 2021, Razzaghi et al. [15] defined a new type of idempotent graph, where the vertex set consists of all ring elements and two vertices are adjacent if and only if their sum is an idempotent element. They studied vertex degrees, connectivity, diameter, and girth for this class of graphs. Subsequently, a number of studies, including [16–18], have further investigated this idempotent graph. These research efforts on idempotent graphs have advanced interdisciplinary studies between algebraic systems and graph theory while providing new methodologies and tools for investigating the structure and properties of algebraic systems.

With the rapid advancement of information technology, interdisciplinary research in algebra and graph theory has garnered increasing importance in fields, such as cryptography, coding theory, and network security. Particularly in cryptography, the integration of algebraic and graph-theoretic methods provides not only a theoretical foundation for designing new cryptosystems but also powerful tools for cryptanalysis. The study of rings and their associated graphs establishes a crucial foundation for practical cryptographic applications. Against this backdrop, the study of ring-based graphs takes on added significance. Notably, due to their complex structure, idempotent-type graphs of rings exhibit graph-theoretic properties, such as connectivity, diameter, and clique number, that are deeply determined by the algebraic features of the underlying ring, including its direct product decomposition and ideal structure. This intricate algebra-graph interplay makes such graphs natural sources of hard computational problems, rendering them highly suitable for cryptographic applications.

Graph-theoretic invariants, including diameter, girth, clique number, chromatic number, domination number, energy, integrality, planarity, crossing number, thickness, and genus, along with properties, such as bipartiteness and completeness, serve as effective tools for characterizing graph structures derived from algebraic systems. Consequently, investigating these graph-theoretic properties of ring-based graphs offers a novel and insightful approach to studying the algebraic structure of rings.

Let R be a ring. Denote by $Z(R)$, $U(R)$, $C(R)$, and $C_U(R)$ the sets of zero divisors, units, central elements, and central units of R, respectively. If R has a unique maximal ideal, denoted by $\mathfrak{m}$, then R is called a local ring. In this case, for any $a \in R$, at least one of a and $1 + a$ is a unit. For further reading on algebra and graph theory, refer to [19–22]. In [23, 24], Tang and Su defined an element $a \in R$ to be quasi-idempotent if $a^2 = ka$ for some $k \in C_U(R)$, or equivalently, if $a = ke$ where k is a central unit and e is an idempotent in R. The ring R is called a quasi-Boolean ring if every element of R is a quasi-idempotent. As shown in [24], R is a quasi-Boolean ring if and only if it is a subdirect product of fields and all its prime ideals are maximal. Motivated by these concepts, we introduce the quasi-idempotent graph of a ring, thereby extending the research paradigm from idempotent to quasi-idempotent structures. We define the quasi-idempotent graph of an associative ring R with nonzero identity, denoted by $G_{Qid}(R)$, as a simple undirected graph with vertex set R. Two distinct vertices $a, b \in R$ are adjacent if and only if $a + b$ is a quasi-idempotent.

The paper is organized as follows. In Section 2, we formally investigate $G_{Qid}(R)$ and establish several of its elementary properties, such as connectivity and girth. Section 3 introduces the quasi-idempotent sum number of a ring and explores its relationship with the diameter of the quasi-idempotent graph. Finally, Section 4 focuses on determining the genus of the quasi-idempotent graphs of finite commutative rings.

2. Definitions, examples and basic properties

In this section, we explore some fundamental properties of the quasi-idempotent graph of a ring, starting with its formal definition and followed by several illustrative examples.

Definition 2.1. *Let R be a ring and $Qid(R)$ be the set of its quasi-idempotent elements. The quasi-idempotent graph of R, denoted by $G_{Qid}(R)$, is the simple undirected graph with vertex set R, where two distinct vertices $a, b \in R$ are adjacent if and only if $a + b \in Qid(R)$.*

Example 2.2. *It is clear that $Qid(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)) = \{(0,0), (0,1), (0,1+x), (1,0), (1,1), (1,1+x)\}$, $Qid(\mathbb{Z}_2 \times \mathbb{Z}_4) = \{(0,0), (0,1), (0,3), (1,0), (1,1), (1,3)\}$, and $Qid(\mathbb{Z}_3 \times \mathbb{Z}_3) = \mathbb{Z}_3 \times \mathbb{Z}_3$. The quasi-idempotent graphs of $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_3[x]/(x^2 + 1)$ are shown as Figures* 1–4.

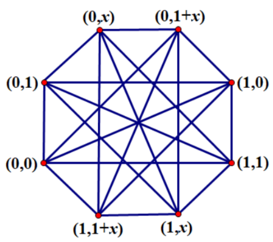

Figure 1. $G_{Qid}(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2))$.

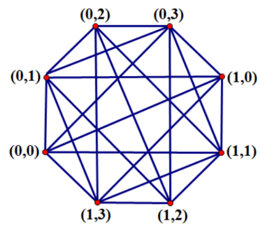

Figure 2. $G_{Qid}(\mathbb{Z}_2 \times \mathbb{Z}_4)$.

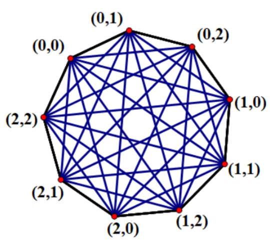

Figure 3. $G_{Qid}(\mathbb{Z}_3 \times \mathbb{Z}_3)$.

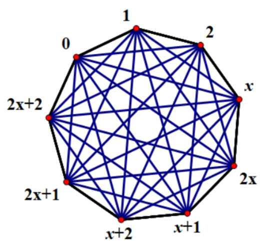

Figure 4. $G_{Qid}(\mathbb{Z}_3[x]/(x^2 + 1))$.

Remark 2.3. *Two graphs G_1 and G_2 are said to be isomorphic if there is a bijection f between the vertex set of G_1 and the vertex set of G_2 that preserves the adjacency relation. For any two rings R_1*

and R_2*, if* $R_1 \cong R_2$*, then* $G_{Qid}(R_1) \cong G_{Qid}(R_2)$*. However,* $G_{Qid}(R_1) \cong G_{Qid}(R_2)$ *does not imply* $R_1 \cong R_2$ *in general. For instance, from the above examples we can see that* $G_{Qid}(\mathbb{Z}_3 \times \mathbb{Z}_3) \cong G_{Qid}(\mathbb{Z}_3[x]/(x^2+1))$*, but* $\mathbb{Z}_3 \times \mathbb{Z}_3 \ncong \mathbb{Z}_3[x]/(x^2+1)$.

Let G be a graph. We denote the vertex set of G by $V(G)$ and the edge set as $E(G)$. For a vertex $a \in V(G)$, the degree of a, denoted by $deg(a)$, is the number of edges of G incident with a. For a given vertex $a \in V(G)$, the neighbor set of a is the set $N_G(a) = \{b \in V(G) \mid b \text{ is adjacent to } a\}$.

Lemma 2.4. *Let* R *be a ring and* $|Qid(R)|$ *be finite. For* $a \in R$*, then the following hold:*

(1) If $2a \in Qid(R)$*, then* $deg(a) = |Qid(R)| - 1$.
(2) If $2a \notin Qid(R)$*, then* $deg(a) = |Qid(R)|$.

Proof. Let $a \in R$. Consider the map $f : Qid(R) \to R$ defined by $f(e) = e - a$ for all $e \in Qid(R)$. The map f is injective: $f(e_1) = f(e_2)$ implies $e_1 - a = e_2 - a$, and hence $e_1 = e_2$. For any $b \in R$, we have $b \in Im(f)$ if and only if $a + b \in Qid(R)$, because $b = e - a$ for some $e \in Qid(R)$ is equivalent to $e = a + b$.

If $2a \in Qid(R)$, then $f(2a) = a$, so $a \in Im(f)$. In this case, $Im(f)$ coincides with the closed neighborhood $N_{G_{Qid}(R)}[a]$ of a, and therefore $\deg(a) = |Im(f)| - 1 = |Qid(R)| - 1$.

If $2a \notin Qid(R)$, then $a \notin Im(f)$, otherwise $a = f(e)$ would give $e = 2a \in Qid(R)$, a contradiction. Hence, $Im(f)$ equals the open neighborhood $N_{G_{Qid}(R)}(a)$, and we obtain $\deg(a) = |Im(f)| = |Qid(R)|$. □

A graph G is called a regular graph if every vertex has the same degree. That is, there exists a nonnegative integer k such that $\deg(a) = k$ for all vertices $a \in V$. In this case, the graph is also referred to as a k-regular graph. Lemma 2.4 implies that $G_{Qid}(R)$ is regular if and only if every element $a \in R$ satisfies $2a \in Qid(R)$

A graph G is a connected graph if there exists a path between any two distinct vertices of G. Otherwise, G is said to be disconnected.

Theorem 2.5. *Let* R *be a ring. Then,* $G_{Qid}(R)$ *is a connected graph if and only if* $(R, +) = \langle Qid(R) \rangle$*. That is,* R *is additively generated by* $Qid(R)$*.*

Proof. Assume that $G_{Qid}(R)$ is connected. Fix an arbitrary element $a \in R$ and choose a quasi-idempotent $e \in Qid(R)$. By connectivity, there exists a path from e to a in $G_{Qid}(R)$, say

$$e = v_0 \to v_1 \to v_2 \to \cdots \to v_n \to v_{n+1} = a,$$

where $v_i + v_{i+1} \in Qid(R)$ for each $0 \leq i \leq n$. Using a telescoping sum, we can express a as

$$a = e + \sum_{i=0}^{n} (-1)^{i+1} (v_i + v_{i+1}).$$

More explicitly,

$$a = \begin{cases} e - (e + v_1) + (v_1 + v_2) - \cdots + (v_n + a), & \text{if } n \text{ is odd,} \\ -e + (e + v_1) - (v_1 + v_2) + \cdots + (v_n + a), & \text{if } n \text{ is even.} \end{cases}$$

Since each term $v_i + v_{i+1}$ belongs to $Qid(R)$, the expression above shows that a is an integer linear combination of quasi-idempotents. Hence, $a \in \langle Qid(R) \rangle$, and consequently $(R, +) = \langle Qid(R) \rangle$.

Conversely, assuming that $(R, +) = \langle Qid(R) \rangle$. To prove that $G_{Qid}(R)$ is connected, it suffices to show that every nonzero element $b \in R$ is connected to 0. Since b lies in the additive subgroup generated by $Qid(R)$, it can be written as a finite sum of the form

$$b = \varepsilon_1 e_1 + \varepsilon_2 e_2 + \cdots + \varepsilon_k e_k,$$

where $e_i, -e_i \in Qid(R)$ and $\varepsilon_i \in \{1, -1\}$ for each i. By inserting zero terms if necessary, we may assume that the signs alternate; that is, we can express b as an alternating sum

$$b = e_1 - e_2 + e_3 - \cdots + (-1)^{m+1} e_m$$

for some $m \geq 1$ and $e_1, \ldots, e_m \in Qid(R)$. Now consider the following sequence of vertices:

$$0,\ e_1,\ -e_1 + e_2,\ e_1 - e_2 + e_3,\ \ldots,\ e_1 - e_2 + e_3 - \cdots + (-1)^{m+1} e_m = b.$$

We claim that consecutive vertices in this sequence are adjacent in $G_{Qid}(R)$. Indeed,

$$\begin{aligned} 0 + e_1 &= e_1 \in Qid(R), \\ e_1 + (-e_1 + e_2) &= e_2 \in Qid(R), \\ (-e_1 + e_2) + (e_1 - e_2 + e_3) &= e_3 \in Qid(R), \\ &\ \vdots \\ (e_1 - \cdots + (-1)^m e_{m-1}) + (e_1 - \cdots + (-1)^{m+1} e_m) &= e_m \in Qid(R). \end{aligned}$$

Thus, the sequence forms a path from 0 to b in $G_{Qid}(R)$. Since b was arbitrary, the graph is connected. □

The girth of a graph G, denoted by $gr(G)$, is the length of the shortest cycle in G. If G has no cycles, its girth is defined to be infinite. A graph without cycles is called a forest; a tree is a connected forest. In any tree or forest, there exists at least one vertex u with $\deg(u) \leq 1$.

Lemma 2.6. *Let R be a local ring with characteristic 2 and residue field $\mathbb{F} \cong R/\mathfrak{m}$. There exist three distinct central units $a, b, c \in C_U(R)$ such that $a + b = c$ if and only if $\mathbb{F} \not\cong \mathbb{F}_2$.*

Proof. Let $f : R \to \mathbb{F}$ be the natural quotient map. Since the image of the center under a homomorphism is central in the image ring and $\mathbb{F}$ is commutative, we have $f(C(R)) = \mathbb{F}$.

Since a, b, c are central units, their images $f(a), f(b), f(c)$ lie in $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$. From $a + b = c$ we obtain $f(a) + f(b) = f(c)$. If $f(a) = f(b)$, then $f(a) + f(b) = 0$ because $\mathrm{char}(\mathbb{F}) = 2$, which would imply $f(c) = 0$, contradicting $f(c) \in \mathbb{F}^*$. Hence, $f(a) \neq f(b)$. Similarly, $f(a) \neq f(c)$ and $f(b) \neq f(c)$. Thus, $\mathbb{F}^*$ contains at least three distinct elements, so $|\mathbb{F}| \geq 4$ and consequently $\mathbb{F} \not\cong \mathbb{F}_2$.

Conversely, suppose $\mathbb{F} \not\cong \mathbb{F}_2$; then $|\mathbb{F}| \geq 4$ and $|\mathbb{F}^*| \geq 3$. We construct three distinct elements $u, v, w \in \mathbb{F}^*$ with $u + v = w$. Take $u = 1$, choose $v \in \mathbb{F}^* \setminus \{0, 1\}$, and set $w = 1 + v$. Since $v \notin \{0, 1\}$, we have $w \notin \{0, 1, v\}$. Hence, u, v, and w are distinct.

Since $f(C(R)) = \mathbb{F}$, there exist $a, b \in C(R)$ such that $f(a) = u$ and $f(b) = v$. Because $u, v \neq 0$, we have $a, b \notin \mathfrak{m}$; in a local ring this means a and b are units, so $a, b \in C_U(R)$. Let $c = a + b$. Then,

$c \in C(R)$ because $C(R)$ is a subring. Moreover, $f(c) = u + v = w \neq 0$, so $c \notin \mathfrak{m}$, and thus c is a unit, that is, $c \in C_U(R)$. If $a = b$, then $u = v$, a contradiction. If $a = c$, then $a = a + b$ implies $b = 0$, contradicting $f(b) = v \neq 0$. Similarly, $b \neq c$. Therefore, a, b, and c are three distinct central units satisfying $a + b = c$. □

Theorem 2.7. *Let R be a local ring. Then, the following statements hold:*

(1) $gr(G_{Qid}(R)) = \infty$ *if and only if* $char(R) = 2$, $|Qid(R)| = 2$.
(2) $gr(G_{Qid}(R)) = 4$ *if and only if* $char(R) = 2$, $|Qid(R)| \geq 3$, $R/\mathfrak{m} \cong \mathbb{F}_2$.
(3) $gr(G_{Qid}(R)) = 3$ *if and only if either* $char(R) \geq 3$, *or* $char(R) = 2$ *and* $R/\mathfrak{m} \ncong \mathbb{F}_2$.

Proof. Suppose $gr(G_{Qid}(R)) = \infty$. Then, $G_{Qid}(R)$ contains no cycles; in particular, it is a forest. Hence, there exists a vertex v with $\deg(v) \leq 1$. If $\mathrm{char}(R) \geq 3$, then $-1 \in U(R) \subseteq Qid(R)$ and $-1 \neq 1$. Thus, the vertices 0, 1, -1 form a 3-cycle, contradicting $gr(G_{Qid}(R)) = \infty$. Therefore, $\mathrm{char}(R) = 2$.

When $\mathrm{char}(R) = 2$, for any $v \in R$ we have $2v = 0 \in Qid(R)$. By Lemma 2.4, $\deg(v) = |Qid(R)| - 1 \leq 1$. Since $\{0, 1\} \subseteq Qid(R)$, we have $|Qid(R)| \geq 2$. Hence, $|Qid(R)| = 2$ and $\deg(v) = 1$. In this case, because $Qid(R)$ maps onto $R/\mathfrak{m}$ and $|Qid(R)| = 2$, we obtain $R/\mathfrak{m} \cong \mathbb{F}_2$.

Conversely, if $\mathrm{char}(R) = 2$ and $|Qid(R)| = 2$, then every vertex has degree 1, so $G_{Qid}(R)$ is a disjoint union of edges and contains no cycles. Hence, $gr(G_{Qid}(R)) = \infty$.

Suppose $gr(G_{Qid}(R)) = 4$. Clearly $\mathrm{char}(R) = 2$ and $|Qid(R)| \geq 3$. Assume, to the contrary, that $R/\mathfrak{m} \ncong \mathbb{F}_2$. Then, $|R/\mathfrak{m}| \geq 4$. By Lemma 2.6, there exist distinct non-zero elements $a, b, c \in Qid(R)$ such that $a+b = c$, $b+c = a$, and $c+a = b$. These yield a 3-cycle $a - b - c - a$ in $G_{Qid}(R)$, contradicting $gr(G_{Qid}(R)) = 4$. Therefore, $R/\mathfrak{m} \cong \mathbb{F}_2$.

Conversely, assume $\mathrm{char}(R) = 2$, $|Qid(R)| \geq 3$, and $R/\mathfrak{m} \cong \mathbb{F}_2$. We first show that $G_{Qid}(R)$ contains no 3-cycle. Suppose for a contradiction that there exists a 3-cycle $a_1 \to a_2 \to a_3 \to a_1$, where $a_i + a_{i+1} = e_i \in Qid(R)$ and indices are taken modulo 3. Using that $\mathrm{char}(R) = 2$, we obtain the relations $e_1 = e_2 + e_3$, $e_2 = e_1 + e_3$, and $e_3 = e_1 + e_2$. Observe that 0 cannot belong to e_1, e_2, e_3; if $0 = e_i = e_j + e_k$ for some permutation, then $e_j = e_k$, which contradicts the fact that the edges of a cycle are distinct. Hence, $e_1, e_2, e_3 \in Qid(R) \setminus \{0\}$. Now 1 is also an element of $Qid(R)$. Under the given relations and because $R/\mathfrak{m} \cong \mathbb{F}_2$, one checks that 1 is different from $0, e_1, e_2, e_3$. Thus, $|Qid(R)| \geq 4$. But $Qid(R)$ maps onto $R/\mathfrak{m}$, and $|R/\mathfrak{m}| = 2$, a contradiction. Therefore, no 3-cycle exists.

Now, since $|Qid(R)| \geq 3$, we can choose an element $e \in Qid(R) \setminus \{0, 1\}$. Consider the cycle $0 \to 1 \to (1 + e) \to e \to 0$. The sums along its edges are $0 + 1 = 1$, $1 + (1 + e) = e$, $(1 + e) + e = 1$, and $e + 0 = e$, all of which lie in $Qid(R)$. Hence, this is a 4-cycle in $G_{Qid}(R)$, and consequently $gr(G_{Qid}(R)) = 4$.

The statement follows from the negation of the previous two cases. If $gr(G_{Qid}(R)) = 3$, then by the above we must have either $\mathrm{char}(R) \geq 3$, or $\mathrm{char}(R) = 2$ and $R/\mathfrak{m} \ncong \mathbb{F}_2$. Conversely, under either condition, $G_{Qid}(R)$ contains a 3-cycle: if $\mathrm{char}(R) \geq 3$, the cycle $0 - 1 - (-1) - 0$ exists; if $\mathrm{char}(R) = 2$ and $R/\mathfrak{m} \ncong \mathbb{F}_2$, then there exist three non-zero quasi-idempotents forming a 3-cycle. Hence, $gr(G_{Qid}(R)) = 3$. □

Proposition 2.8. *Let* $R \cong R_1 \times R_2 \times \cdots \times R_s$ *be a finite ring with* $s \geq 2$. *Then,* $gr(G_{Qid}(R)) = 3$.

Proof. Since the quasi-idempotent property is checked componentwise, we have $Qid(R) = Qid(R_1) \times Qid(R_2) \times \cdots \times Qid(R_s)$, that is, $Qid(R) = \{(a_1, a_2, \ldots, a_s) | a_i \in Qid(R_i), i = 1, 2, \ldots, s\}$. Consider the following three distinct vertices in $G_{Qid}(R)$: $a = (0, 0, \ldots, 0)$, $b = (1, 0, \ldots, 0)$, $c = (0, 0, \ldots, 0, 1)$. Their pairwise sums are: $a + b = (1, 0, \ldots, 0)$, $a + c = (0, 0, \ldots, 0, 1)$, $b + c = (1, 0, \ldots, 0, 1)$. Each of these sums belongs to $Qid(R)$ because every component is either 0 or 1, both of which are quasi-idempotents in the respective component rings. Therefore, a, b, and c are pairwise adjacent, forming a cycle of length 3. This shows that $gr(G_{Qid}(R)) \leq 3$.

On the other hand, since $G_{Qid}(R)$ is a simple undirected graph, any cycle must have length at least 3. Hence, $gr(G_{Qid}(R)) \geq 3$.

Combining both inequalities, we conclude that $gr(G_{Qid}(R)) = 3$. □

A graph G with n vertices is called a complete graph if every pair of distinct vertices in G is adjacent. Such a graph is denoted by K_n. Equivalently, G is complete if and only if each vertex of G has degree $n - 1$.

Theorem 2.9. *Let R be a ring. Then, $G_{Qid}(R)$ is a complete graph if and only if R is a quasi-Boolean ring.*

Proof. Suppose $G_{Qid}(R)$ is complete. Then, for any two distinct vertices $a, b \in V(G_{Qid}(R))$, we have $a + b \in Qid(R)$. This shows that every element of R is a quasi-idempotent, hence R is a quasi-Boolean ring. Consequently, every element of R is a quasi-idempotent, and thus R is a quasi-Boolean ring by definition.

Conversely, suppose R is a quasi-Boolean ring, that is, $r \in Qid(R)$ for every $r \in R$. Then, for any two distinct vertices $a, b \in V(G_{Qid}(R))$, their sum $a + b$ is an element of R; hence, $a + b \in Qid(R)$. This implies that a and b are adjacent in $G_{Qid}(R)$. Since a and b were arbitrary, every pair of distinct vertices is adjacent. Therefore, each vertex a satisfies $\deg(a) = |V(G_{Qid}(R))| - 1$, and so $G_{Qid}(R)$ is a complete graph. □

A graph G is called a bipartite graph if its vertex set $V(G)$ can be partitioned into two nonempty, disjoint subsets V_1 and V_2 such that every edge of G joins a vertex in V_1 with a vertex in V_2. Such a partition (V_1, V_2) is called a bipartition of G. If, in addition, every vertex in V_1 is adjacent to every vertex in V_2, then G is said to be a complete bipartite graph. When $|V_1| = s$ and $|V_2| = t$, where $s, t \geq 1$, this graph is denoted by $K_{s,t}$. A fundamental characterization of bipartite graphs is that a graph is bipartite if and only if it contains no odd cycles.

Theorem 2.10. *Let $R \cong R_1 \times R_2 \times \cdots \times R_s$ be a finite ring. Then, $G_{Qid}(R)$ is a bipartite graph if and only if R is a finite local ring with* $\mathrm{char}(R) = 2$ *and $R/\mathfrak{m} \cong \mathbb{F}_2$. Moreover, when $G_{Qid}(R)$ is bipartite, it is a complete bipartite graph if and only if R is commutative.*

Proof. Assume that $G_{\mathrm{Qid}}(R)$ is bipartite. Then, it contains no odd cycles. By Theorem 2.7 and Proposition 2.8, we conclude that R must be an indecomposable ring, $\mathrm{char}(R) = 2$, and $R/\mathfrak{m} \cong \mathbb{F}_2$. Thus, R is a local ring.

Conversely, suppose R is a finite local ring with $\mathrm{char}(R) = 2$ and $R/\mathfrak{m} \cong \mathbb{F}_2$, where $\mathfrak{m}$ is the maximal ideal of R. Then $R = U(R) \cup \mathfrak{m}$ and $U(R) \cap \mathfrak{m} = \emptyset$. We will show that $G_{\mathrm{Qid}}(R)$ is bipartite with the bipartition $V_1 = U(R)$ and $V_2 = \mathfrak{m}$.

Let $a, b \in R$, $a \neq b$, and suppose $a + b$ is a quasi-idempotent. Since $\mathrm{char}(R) = 2$, we have $a + b \neq 0$, otherwise $a = b$. Because in a local ring with residue field $\mathbb{F}_2$, the only quasi-idempotents are 0 and the central units. Hence, $a + b$ is a central unit. If both a and b belong to $U(R)$, since $a + b \equiv 0 \pmod{\mathfrak{m}}$, then $a + b \in \mathfrak{m}$, which contradicts that $a + b$ is a unit. Similarly, if both a and b belong to $\mathfrak{m}$, then $a + b \in \mathfrak{m}$, again a contradiction. Therefore, one of a and b must lie in $U(R)$ and the other in $\mathfrak{m}$. Consequently, every edge of $G_{\mathrm{Qid}}(R)$ connects a vertex in $U(R)$ to a vertex in $\mathfrak{m}$, so the graph is bipartite.

Now, assume further that $G_{\mathrm{Qid}}(R)$ is a complete bipartite graph. Then, for every $u \in U(R)$ and every $z \in \mathfrak{m}$, the sum $u + z$ is a quasi-idempotent. Since $u + z \equiv 1 \pmod{\mathfrak{m}}$, it is a unit, and hence a central unit. In particular, taking $z = 0$, we see that every $u \in U(R)$ is a central unit. Taking $u = 1$, we have $1 + z \in C_U(R)$ for all $z \in \mathfrak{m}$, which implies $z = (1 + z) - 1 \in C(R)$. Thus, every element of R is central, and R is commutative.

Conversely, if R is commutative, then every unit is central, so $U(R) \subseteq \mathrm{Qid}(R)$. For any $u \in U(R)$ and $z \in \mathfrak{m}$, since $u + z \equiv 1 \pmod{\mathfrak{m}}$, we have $u + z \in U(R)$, and hence $u + z \in C_U(R) \subseteq \mathrm{Qid}(R)$. Therefore, every vertex in $U(R)$ is adjacent to every vertex in $\mathfrak{m}$, making $G_{\mathrm{Qid}}(R)$ a complete bipartite graph. □

Let G be a graph. A mapping $f : E(G) \to \{1, 2, \ldots, k\}$ is called a k-edge-coloring if for any two adjacent edges $e_1, e_2 \in E(G)$, we have $f(e_1) \neq f(e_2)$. If such a coloring exists, G is said to be k-edge-colorable. The smallest integer k for which G is k-edge-colorable is called the chromatic index of G, denoted by $\chi'(G)$.

Let $\Delta(G)$ denote the maximum vertex degree in G. Vizing's theorem [19, Proposition 5.11, 5.13] states that

$$\Delta \leq \chi'(G) \leq \Delta + 1.$$

This leads to a classification of graphs: a graph G is of class 1 if $\chi'(G) = \Delta$, and of class 2 if $\chi'(G) = \Delta + 1$.

For complete graphs K_n on n vertices, the chromatic index is known explicitly [19]:

$$\chi'(K_n) = \begin{cases} n & \text{if } n \text{ is odd,} \\ n - 1 & \text{if } n \text{ is even.} \end{cases}$$

Thus, K_n is of class 1 when n is even, and of class 2 when n is odd.

A related classical result is Petersen's 2-factorization theorem [25, Corollary 8.4.5], which states that every regular graph of even degree is 2-factorable. This implies, in particular, that such graphs are of class 1.

Theorem 2.11. *Let R be a finite ring. The quasi-idempotent graph $G_{Qid}(R)$ is of class 2 if $|R|$ is odd and $2a \in Qid(R)$ for every $a \in R$. Otherwise, it is of class 1.*

Proof. Let $a, b, c \in V(G_{Qid}(R))$ with $ab, ac \in E(G_{Qid}(R))$. We color each edge ab with color $a+b$. Then, edges ab and ac receive the same color if and only if $a + b = a + c$, which implies $b = c$ and hence $ab = ac$. Therefore, adjacent edges receive distinct colors. Let $C = \{a + b \mid ab \text{ is an edge of } G_{Qid}(R)\}$. Then, C is the set of colors used, so $G_{Qid}(R)$ has a $|C|$-edge coloring, and hence $\chi'(G_{Qid}(R)) \leq |C|$. Clearly $|C| \leq |Qid(R)|$, so $\chi'(G_{Qid}(R)) \leq |Qid(R)|$.

Let Δ denote the maximum degree of $G_{Qid}(R)$. By Vizing's theorem, we have $\Delta \leq \chi'(G_{Qid}(R)) \leq \Delta + 1$.

If there exists an element $v \in R$ such that $2v \notin Qid(R)$, then by Lemma 2.4, $\deg(v) = |Qid(R)| = \Delta$. Thus, $|Qid(R)| \leq \chi'(G_{Qid}(R)) \leq |Qid(R)|$, so $\chi'(G_{Qid}(R)) = \Delta$. Hence, $G_{Qid}(R)$ is of class 1.

Now suppose $2u \in Qid(R)$ for every $u \in R$. If R is a quasi-Boolean ring, then $G_{Qid}(R)$ is a complete graph. It is well known that a complete graph K_n is of class 1 if n is even, and of class 2 if n is odd. Since $|V(G_{Qid}(R))| = |R|$, the statement follows in this case.

Assume now that R is not a quasi-Boolean ring. By Lemma 2.4, $G_{Qid}(R)$ is a $(|Qid(R)|-1)$-regular graph, so $\Delta = |Qid(R)| - 1$. We consider two cases.

Case 1. $|R|$ **is odd.** Assume, for contradiction, that $\chi'(G_{Qid}(R)) = \Delta = |Qid(R)| - 1$. In a proper edge-coloring with Δ colors, each vertex is incident to exactly one edge of each color. Fix a color c. Then, the edges of color c form a perfect matching in $G_{Qid}(R)$. However, a perfect matching exists only if the number of vertices is even, contradicting the oddness of $|R|$. Therefore, $\chi'(G_{Qid}(R)) \neq \Delta$, and so $\chi'(G_{Qid}(R)) = \Delta + 1 = |Qid(R)|$. Thus $G_{Qid}(R)$ is of class 2.

Case 2. $|R|$ **is even.** Then, $G_{Qid}(R)$ is a regular graph of even degree $\Delta = |Qid(R)|-1$. We distinguish two subcases.

Subcase 2.1. Assume that every element $a \in R$ satisfies $2a = 0$. We define an edge coloring of the graph $G_{\text{Qid}}(R)$ as follows: for an edge ab, if $a + b = e_i$ where e_i is a nonzero element in $Qid(R)$, we assign color i to this edge. The index i (where $1 \leq i \leq \Delta$) thus corresponds bijectively to the Δ distinct nonzero elements of $Qid(R)$.

Observe that for each fixed color i, the set of edges $\{ab : a + b = e_i\}$ forms a perfect matching. Indeed, for any vertex a, the unique element b satisfying $a + b = e_i$ is given by $b = e_i - a = e_i + a$ (since $2a = 0$). By definition, every vertex is incident to exactly one edge in this set. Therefore, the graph $G_{Qid}(R)$ decomposes into Δ disjoint perfect matchings, each corresponding to one color class.

By Lemma 2.4, the degree of each vertex a is Δ. Because each color class is a perfect matching, every vertex is incident with precisely one edge of each color. Consequently, all edges incident to a given vertex receive distinct colors among the Δ available.

Thus, the graph admits a proper edge coloring using exactly Δ colors, which equals its maximum degree. Hence, $G_{Qid}(R)$ is of class 1.

Subcase 2.2. Assume that there exists $b \in R$ such that $0 \neq 2b = e \in Qid(R)$. Then, $e \neq -e$, and consequently $|Qid(R)|$ is odd and $\Delta = |Qid(R)| - 1$ is even. By Petersen's 2-factorization theorem, $G_{Qid}(R)$ is 2-factorable. Then, $E(G_{Qid}(R))$ can be decomposed into several edge-disjoint 2-factors and each 2-factor itself can be properly colored with 2 colors. Since there are $\Delta/2$ such 2-factors, we can color the whole graph with Δ colors. Thus, $\chi'(G_{Qid}(R)) = \Delta$, and $G_{Qid}(R)$ is of class 1.

In all cases, the theorem is proved. □

3. The diameter of a quasi-idempotent graph

In this section, we focus on studying the diameter of the quasi-idempotent graph of a ring. To better understand the graph's structural properties, we introduce the concept of the quasi-idempotent sum number. After providing its definition and some illustrative examples, we then investigate the relationship between this number and the graph's diameter.

Definition 3.1. *Let R be a ring. An element $a \in R$ is called a k-fine element if k is the smallest positive integer such that $a = e_1 + \cdots + e_k$ for some $e_1, \ldots, e_k \in Qid(R)$. We denote this minimal k by $\ell(a)$ and call it the quasi-idempotent length of a. If no such decomposition exists, we set $\ell(a) = \infty$. If* $\sup\{\ell(a) \mid a \in R\} = k < \infty$, *then R is called a k-fine ring. In this case, every element of R can be expressed as a sum of at most k quasi-idempotents, and k is the smallest integer with this property.*

Definition 3.2. *Let R be a ring. The quasi-idempotent sum number of R, denoted by* **qisn**(R)*, is defined as follows.*

(1) If there exists a smallest positive integer k such that R is a k-fine ring, then $\mathbf{qisn}(R) = k$.
(2) If every element of R can be written as a sum of finitely many elements from $Qid(R)$*, but there is no upper bound on the number of summands required, then* $\mathbf{qisn}(R) = \omega$.
(3) If there exists an element $a \in R$ *that cannot be written as any finite sum of elements from* $Qid(R)$*, then* $\mathbf{qisn}(R) = \infty$.

Example 3.3. *Consider the following examples:*

- *For* $\mathbb{Z}_4$*, we have* $Qid(\mathbb{Z}_4) = \{0, 1, 3\}$ *and* $2 \notin Qid(\mathbb{Z}_4)$*. The elements can be decomposed as:* $0 = 0 + 0$, $1 = 1 + 0$, $3 = 3 + 0$*, and* $2 = 1 + 1 = 3 + 3$*. Hence,* $\mathbf{qisn}(\mathbb{Z}_4) = 2$.
- *For* $\mathbb{Z}$*, we have* $Qid(\mathbb{Z}) = \{0, 1, -1\}$*. In this case,* $\mathbf{qisn}(\mathbb{Z}) = \omega$.
- *For* $\mathbb{Z}_3[x]$*, we have* $Qid(\mathbb{Z}_3[x]) = \{0, 1, -1\}$*. Consequently,* $\mathbf{qisn}(\mathbb{Z}_3[x]) = \infty$.

Theorem 3.4. *Let R be a finite commutative local ring. Then, the following statements hold:*

(1) $\mathbf{qisn}(R) = 1$ *if and only if R is a field.*
(2) $\mathbf{qisn}(R) = 2$ *if and only if R is not a field.*

Proof. If $\mathbf{qisn}(R) = 1$, then every $a \in R$ is a 1-fine element, and hence a quasi-idempotent. Thus, R is a quasi-Boolean ring. Furthermore, a finite commutative local quasi-Boolean ring is necessarily a field. Conversely, if R is a field, then every element of R is a quasi-idempotent, whence $\mathbf{qisn}(R) = 1$.

If $\mathbf{qisn}(R) = 2$, suppose for contradiction that R is a field. Then, by the above, we would have $\mathbf{qisn}(R) = 1$, a contradiction. Therefore, R is not a field.

Conversely, assume R is not a field. Then, there exists a zero divisor $b \in R$ such that $b \notin Qid(R)$; in particular, b is not a 1-fine element. Since R is local, b lies in the maximal ideal. Consequently, $b - 1$ is a unit and hence a quasi-idempotent. Thus, $b = (b - 1) + 1$ is a sum of two quasi-idempotents, in other words, b is a 2-fine element. For any $e \in Qid(R)$, we also have $e = e + 0$. To see that every element of R is a sum of two quasi-idempotents, observe that any $x \in R$ can be written as $x = (x - 1) + 1$; if $x - 1$ is not a quasi-idempotent, it is either a unit or a zero divisor. If it is a unit, then it is a quasi-idempotent and we are done. If it is a zero divisor, we may apply a similar decomposition argument as for b. Therefore, R is a 2-fine ring. Since R contains elements that are not a 1-fine element, the smallest such integer is 2, and so $\mathbf{qisn}(R) = 2$. □

If R is a finite commutative ring, we know that $R \cong R_1 \times R_2 \times \cdots \times R_s$, where each R_i is a finite local ring, $1 \leq i \leq s$. It is clear that $U(R) = U(R_1) \times U(R_2) \times \cdots \times U(R_s)$ and $Id(R) = \{(e_1, e_2, \ldots, e_s) | e_i = 0, 1, i = 1, 2, \ldots, s\}$. So, $Qid(R) = \{(a_1, a_2, \ldots, a_s) | a_i \in U(R_i) \text{ or } a_i = 0, i = 1, 2, \ldots, s\}$.

Theorem 3.5. *Let R be a finite commutative ring. Then, the following statements hold:*

(1) $\mathbf{qisn}(R) = 1$ *if and only if each* R_i *is a field.*
(2) $\mathbf{qisn}(R) = 2$ *if and only if not all* R_i *are fields.*

Proof. If $\mathbf{qisn}(R) = 1$, then R is a 1-fine ring and every $v = (v_1, v_2, \ldots, v_s) \in R$, v is a quasi-idempotent, meaning each $v_i \in R_i$ is a quasi-idempotent of R_i. By Theorem 3.4, all finite commutative local rings R_i are fields. Conversely, if each R_i is a field, then clearly any $u = (u_1, u_2, \ldots, u_s) \in R$ has each component $u_i \in Qid(R_i)$. Hence, $u \in Qid(R)$, and thus $\mathbf{qisn}(R) = 1$.

If $\mathbf{qisn}(R) = 2$, then clearly not all R_i are fields; otherwise $\mathbf{qisn}(R) = 1$ by the above. Conversely, if not all R_i are fields, then there exists at least one index i such that R_i is not a field. Hence, there exists an element $e_i \in R_i$ that is not a quasi-idempotent. Consider the element $e = (e_1, e_2, \dots, e_s) \in R$ where we set $e_j = 0$ for $j \neq i$. Then, e is not a quasi-idempotent of R, meaning it is not a 1-fine element. Now, for any $a = (a_1, a_2, \dots, a_s) \in R$, we define $b = (b_1, b_2, \dots, b_s) \in R$ and $c = (c_1, c_2, \dots, c_s) \in R$ as follows:

$$b_i = \begin{cases} a_i, & a_i \in Qid(R_i), \\ a_i - 1, & a_i \notin Qid(R_i), \end{cases} \qquad c_i = \begin{cases} 0, & a_i \in Qid(R_i), \\ 1, & a_i \notin Qid(R_i). \end{cases}$$

We claim that b_i and c_i are quasi-idempotents in R_i. Indeed, if $a_i \in Qid(R_i)$, then $b_i = a_i$ and $c_i = 0$, both quasi-idempotents. If $a_i \notin Qid(R_i)$, since in a finite local ring, every unit is a quasi-idempotent, then a_i is a non-unit. Therefore a_i lies in the maximal ideal of R_i, and so $a_i - 1$ is a unit, hence a quasi-idempotent. Also, 1 is a quasi-idempotent. Thus, b_i and c_i are quasi-idempotents in both cases. Consequently, $b, c \in Qid(R)$. Clearly $a = b + c$, so every element of R is a sum of two quasi-idempotents. Since there exists an element that is not a 1-fine element, the smallest integer with this property is 2, and hence $\mathbf{qisn}(R) = 2$. □

Let G be a graph. The distance between two vertices $a, b \in V(G)$ is the length of a shortest path from a to b in G, denoted by $d(a, b)$. If there is no path from a to b in G, then $d(a, b) = \infty$. The diameter of G is the maximum distance between any two vertices in G, denoted by $\operatorname{diam}(G) = \max\{d(a, b) \mid a, b \in V(G)\}$.

Theorem 3.6. *Let R be a finite commutative local ring. Then, the following statements hold:*

(1) $\operatorname{diam}(G_{Qid}(R)) = 1$ *if and only if R is a field.*
(2) $\operatorname{diam}(G_{Qid}(R)) = 2$ *if and only if R is not a field.*

Proof. If $\operatorname{diam}(G_{Qid}(R)) = 1$, then every pair of distinct vertices in $G_{Qid}(R)$ is joined by an edge; hence $G_{Qid}(R)$ is a complete graph. By Theorem 2.9, R is a quasi-Boolean ring. Since R is a finite commutative local ring, it must be a field. Conversely, if R is a field, then R is a quasi-Boolean ring, and $G_{Qid}(R)$ is a complete graph, so $\operatorname{diam}(G_{Qid}(R)) = 1$.

Assume $\operatorname{diam}(G_{Qid}(R)) = 2$. If R were a field, then by the above we would have $\operatorname{diam}(G_{Qid}(R)) = 1$, a contradiction. Hence, R is not a field.

Conversely, suppose R is not a field. Then, R contains nonzero zero divisors, and these are not in $Qid(R)$. For any $a, b \in R$, we consider three cases.

Case 1. $a \in U(R)$ and $b \in \mathfrak{m}$. Then, $a + b \in U(R) \subseteq Qid(R)$, so $d(a, b) = 1$.

Case 2. $a, b \in \mathfrak{m}$. Then, $a + b \in \mathfrak{m}$. If $a + b = 0 \in Qid(R)$, then $d(a, b) = 1$. If $a + b \neq 0$, then $a + b \notin Qid(R)$; thus, a and b are not adjacent. Since $a, b \in \mathfrak{m}$, $1 + a$ and $1 + b$ are units and hence belong to $Qid(R)$. Therefore a is adjacent to 1 and 1 is adjacent to b. Hence, there exists a path $a \to 1 \to b$, giving $d(a, b) = 2$.

Case 3. $a, b \in U(R) \subseteq Qid(R)$. If $a + b \in Qid(R)$, then $d(a, b) = 1$. If $a + b \notin Qid(R)$, then a and b are not adjacent. Since $a, b \in Qid(R)$, both are adjacent to 0; thus, there exists a path $a \to 0 \to b$, and $d(a, b) = 2$.

In all cases, the distance between any two vertices is at most 2, and since R is not a field, the graph is not complete, so $\operatorname{diam}(G_{Qid}(R)) = 2$. □

Theorem 3.7. *Let R be a finite commutative ring. Then the following statements hold:*

(1) $\operatorname{diam}(G_{Qid}(R)) = 1$ *if and only if each R_i is a field.*

(2) $\mathrm{diam}(G_{Qid}(R)) = 2$ *if and only if not all R_i are fields.*

Proof. Suppose $\mathrm{diam}(G_{Qid}(R)) = 1$. Then, $G_{Qid}(R)$ is a complete graph. By Theorem 2.9, R is a quasi-Boolean ring, so every element $a = (a_1, a_2, \ldots, a_s) \in R$ is a quasi-idempotent. This implies that for each i, every element of R_i is a quasi-idempotent, namely, $G_{Qid}(R_i)$ is a complete graph. Hence, $\mathrm{diam}(G_{Qid}(R_i)) = 1$ for each i. By Theorem 3.6, each R_i is a field. Conversely, if each R_i is a field, then R is a direct product of fields, hence a quasi-Boolean ring. By Theorem 2.9, $G_{Qid}(R)$ is a complete graph, so $\mathrm{diam}(G_{Qid}(R)) = 1$.

Suppose $\mathrm{diam}(G_{Qid}(R)) = 2$. If all R_i were fields, then by the above we would have $\mathrm{diam}(G_{Qid}(R)) = 1$, a contradiction. Therefore, not all R_i are fields. Conversely, assume that not all R_i are fields. Then, there exists at least one index j such that R_j is not a field. By Theorem 3.6, $\mathrm{diam}(G_{Qid}(R_j)) = 2$, so there exist non-adjacent vertices in $G_{Qid}(R_j)$. This implies that $G_{Qid}(R)$ is not complete, hence $\mathrm{diam}(G_{Qid}(R)) \geq 2$.

It remains to show that $\mathrm{diam}(G_{Qid}(R)) \leq 2$. Let $a = (a_1, \ldots, a_s)$ and $b = (b_1, \ldots, b_s)$ be any two vertices in $G_{Qid}(R)$. If $a + b \in Qid(R)$, then $d(a, b) = 1$. Now suppose $a + b \notin Qid(R)$. For each i, consider the local ring R_i. By Theorem 3.6, $\mathrm{diam}(G_{Qid}(R_i)) \leq 2$. Therefore, for each i, there exists an element $c_i \in R_i$ such that $a_i + c_i \in Qid(R_i)$ and $b_i + c_i \in Qid(R_i)$. Now let $c = (c_1, \ldots, c_s)$ as follows:

$$c_i = \begin{cases} 0, & a_i \in U(R_i), B_i \in U(R_i) \\ -a_i, & a_i \in U(R_i), B_i \in Z(R_i) \\ 1, & a_i \in Z(R_i), B_i \in Z(R_i). \end{cases}$$

Then, $a + c = (a_1 + c_1, \ldots, a_s + c_s)$ and $b + c = (b_1 + c_1, \ldots, b_s + c_s)$ are both in $Qid(R)$ because each component is in $Qid(R_i)$. Hence, a and c are adjacent, and c and b are adjacent, so there is a path $a \to c \to b$ of length 2. Thus, $d(a, b) \leq 2$.

Since we have shown that the distance between any two vertices is at most 2, and that there exist vertices at distance exactly 2, we conclude that $\mathrm{diam}(G_{Qid}(R)) = 2$. □

Comparing the diameter of the quasi-idempotent graph of R with its quasi-idempotent sum number yields the following result.

Corollary 3.8. *Let R be a finite commutative ring. Then,* $\mathbf{qisn}(R) = \mathrm{diam}(G_{Qid}(R))$.

Proof. We consider two cases.

Case 1. R is a quasi-Boolean ring. Then, by Theorems 3.4 and 3.5, $\mathbf{qisn}(R) = 1$. Moreover, since R is a quasi-Boolean ring, $G_{Qid}(R)$ is a complete graph, and by Theorem 3.6 and 3.7, we have $\mathrm{diam}(G_{Qid}(R)) = 1$. Hence, $\mathbf{qisn}(R) = \mathrm{diam}(G_{Qid}(R)) = 1$.

Case 2. R is not a quasi-Boolean ring. Then, there exists an element $a \notin Qid(R)$. By Theorems 3.4 and 3.5, every element of R can be written as a sum of two quasi-idempotents, so in particular $a = b + c$ for some $b, c \in Qid(R)$. This shows that $\mathbf{qisn}(R) \leq 2$. Since $a \notin Qid(R)$, we have $\mathbf{qisn}(R) > 1$; hence, $\mathbf{qisn}(R) = 2$.

On the other hand, since R is not a quasi-Boolean ring, $G_{Qid}(R)$ is not a complete graph. Therefore, $\mathrm{diam}(G_{Qid}(R)) \geq 2$. By Theorems 3.6 and 3.7, we have $\mathrm{diam}(G_{Qid}(R)) = 2$. Hence, $\mathbf{qisn}(R) = \mathrm{diam}(G_{Qid}(R)) = 2$.

In both cases, we obtain $\mathbf{qisn}(R) = \mathrm{diam}(G_{Qid}(R))$. □

4. The genus of a quasi-idempotent graph

The symbol $\mathbb{S}_g$ denotes an orientable surface with genus g, which is a sphere with g handles,up to a topologically homeomorphism. A graph G that can be embedded on $\mathbb{S}_g$ but cannot be embedded on $\mathbb{S}_{g-1}$ without crossings is called a graph with genus g, and we write $\gamma(G) = g$. A planar graph is a graph with genus zero. Clearly, if H is a subgraph of a graph G, then $\gamma(H) \leq \gamma(G)$. Determining the genus of a graph is one of the fundamental problems in topological graph theory. In [26], Thomassen proved that the graph genus problem is indeed NP-complete. This implies that, even for most practical graphs, determining their genus is computationally infeasible. In [27], Duke identifies exactly three genus 2 irreducible subgraphs in the complete graph K_8 and uses this to determine the genus of every graph with fewer than nine vertices. It is evident that when R is a finite commutative ring, the unit graph of R is a subgraph of its quasi-idempotent graph. For related properties of the genus of the unit graph over finite commutative rings, one may refer to [28]. By contrast, the quasi-idempotent graph possesses more edges and exhibits a more complex structure, making it considerably more challenging to determine its genus. In this section, we study the genus of quasi-idempotent graphs of rings. We give a classification of finite commutative rings R whose $\gamma(G_{Qid}(R))$ is 0,1,2, respectively. The first two lemmas from graph theory will be frequently used.

Lemma 4.1. [20, Theorems 6.37, 6.38] Let $m \geq 2$, $n \geq 3$, $p \geq 2$ be integers. Then, $\gamma(K_n) = \left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil$, $\gamma\left(K_{m,p}\right) = \left\lceil \frac{1}{4}(m-2)(p-2) \right\rceil$, where $\lceil x \rceil$ is the least integer that is greater than or equal to x.

Lemma 4.2. [20, Corollary 6.14] Suppose a simple graph G is connected with $p \geq 3$ vertices and q edges. Then, $\gamma(G) \geq \frac{q}{6} - \frac{p}{2} + 1$.

The following lemma is a key result, which states the number of quasi-idempotents of a ring R is no more than 8 if $\gamma(G_{Qid}(R)) \leq 2$.

Lemma 4.3. *Let R be a finite ring with n elements, k quasi-idempotents. If $k \geq 9$, then $\gamma(G_{Qid}(R)) \geq 3$.*

Proof. For any element $a \in G_{Qid}(R)$, we know $deg(a) \geqslant k-1$ by Lemma 2.4, and thus $G_{Qid}(R)$ at least has $\frac{n(k-1)}{2}$ edges. Therefore, $\gamma(G_{Qid}(R)) \geq \lceil \frac{n(k-1)}{12} - \frac{n}{2} + 1 \rceil \geq \lceil \frac{n(k-7)}{12} + 1 \rceil$ by Lemma 4.2. When $k \geqslant 9$, then $\gamma(G_{Qid}(R)) \geq \lceil \frac{9\times 2}{12} + 1 \rceil = \lceil \frac{30}{12} \rceil = 3$. □

Theorem 4.4. *Let R be a finite commutative ring. Then, the following statements hold:*

(1) $\gamma(G_{Qid}(R)) = 0$ if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{F}_4$, $\mathbb{Z}_4$, $\mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(2) $\gamma(G_{Qid}(R)) = 1$ if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_5$, $\mathbb{Z}_7$, $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x, x^2)$, $\mathbb{Z}_4[x]/(2x, x^2-2)$, $\mathbb{Z}_2[x, y]/(x, y)^2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2 \times \mathbb{Z}_4$.

(3) $\gamma(G_{Qid}(R)) = 2$ if and only if R is isomorphic to one of the following rings: $\mathbb{F}_8$, $\mathbb{Z}_9$, $\mathbb{Z}_3[x]/(x^2)$, $\mathbb{Z}_2 \times \mathbb{F}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. As R is a finite commutative ring, we know that $R \cong R_1 \times R_2 \times \cdots \times R_s$, where each R_i is a finite local ring, $1 \leq i \leq s$. It is clear that $U(R) = U(R_1) \times U(R_2) \times \cdots \times U(R_s)$ and $Id(R) = \{(e_1, e_2, \ldots, e_s) | e_i = 0, 1, i = 1, 2, \ldots, s\}$. So, $Qid(R) = \{(a_1, a_2, \ldots, a_s) | a_i \in U(R_i)$ or $a_i = 0, i = 1, 2, \ldots, s\}$. Thus, $|Qid(R)| = \prod_{i=1}^{s} (|U(R_i)| + 1) \geq 2^s$.

If $s \geq 4$, then $|Qid(R)| \geq 16$. By Lemma 4.3, we have $\gamma(G_{Qid}(R)) \geq 3$, thus we need only consider the case of $s \leq 3$ when we explore $\gamma(G_{Qid}(R)) \leq 2$. We proceed with three cases.

Case 1. $s = 1$. In this case, R is a finite commutative local ring. If $\mathfrak{m} = \{0\}$, then R is a field. Thus, $G_{Qid}(R)$ is a complete graph. Therefore, when $R \cong \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{F}_4$, $\gamma(G_{Qid}(R)) = 0$; $R \cong \mathbb{Z}_5, \mathbb{Z}_7$, $\gamma(G_{Qid}(R)) = 1$; $R \cong \mathbb{F}_8$, $\gamma(G_{Qid}(R)) = 2$. When $|R| \geq 9$, $\gamma(G_{Qid}(R)) \geq 3$. If $\mathfrak{m} \neq \{0\}$, then we know $R/\mathfrak{m} \cong \mathbb{F}$ is a field and $|U(R)| = |\mathbb{F}^*| \cdot |\mathfrak{m}|$, where $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$. By Lemma 4.3, we have $|U(R)| + 1 = |\mathbb{F}^*| \cdot |\mathfrak{m}| + 1 \leq 8$ if $\gamma(G_{Qid}(R)) \leq 2$. So, $|\mathbb{F}^*| \cdot |\mathfrak{m}| \leq 7$, which deduces that $|\mathbb{F}^*| \leq 3$.

If $|\mathbb{F}^*| = 1$, then $\mathbb{F} \cong \mathbb{Z}_2$ and then $|\mathfrak{m}| = 2$ or 4. If $|\mathfrak{m}| = 2$, then R is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$. It is clear that $G_{Qid}(R)$ is a planar graph. If $|\mathfrak{m}| = 4$, then R is a local ring of order 8 but not a field. So R is isomorphic to one of the following rings: $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x, x^2)$, $\mathbb{Z}_4[x]/(2x, x^2-2)$, $\mathbb{Z}_2[x, y]/(x, y)^2$. It is easy to see that $K_{4,4}$ is a subgraph of $G_{Qid}(R)$. So, $\gamma(G_{Qid}(R)) \geq 1$ by Lemma 4.1. On the other hand, we can be embedded these graphs into $\mathbb{S}_1$ as shown in Figures 5–9. Hence, $\gamma(G_{Qid}(\mathbb{Z}_2[x]/(x^3))) = \gamma(G_{Qid}(\mathbb{Z}_2[x, y]/(x, y)^2)) = \gamma(G_{Qid}(\mathbb{Z}_4[x]/(2x, x^2))) = \gamma(G_{Qid}(\mathbb{Z}_4[x]/(2x, x^2 - 2))) = \gamma(G_{Qid}(\mathbb{Z}_8)) = 1$.

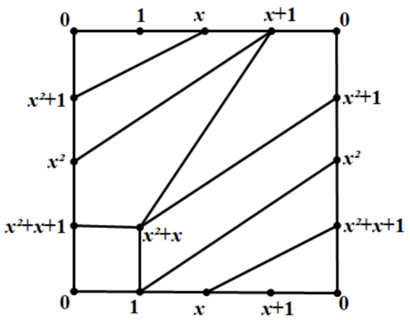

Figure 5. $G_{Qid}(\mathbb{Z}_2[x]/(x^3))$.

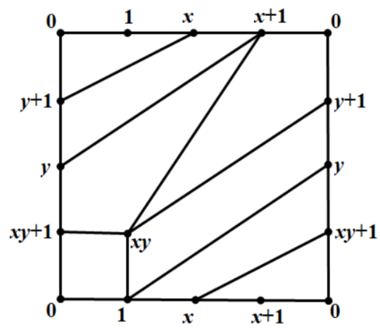

Figure 6. $G_{Qid}(\mathbb{Z}_2[x, y]/(x, y)^2)$.

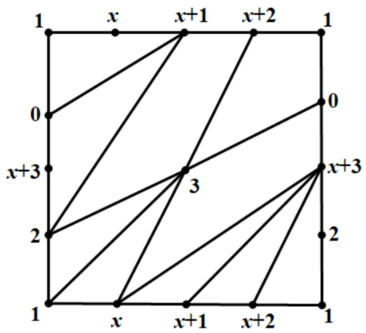

Figure 7. $G_{Qid}(\mathbb{Z}_4[x]/(2x, x^2))$.

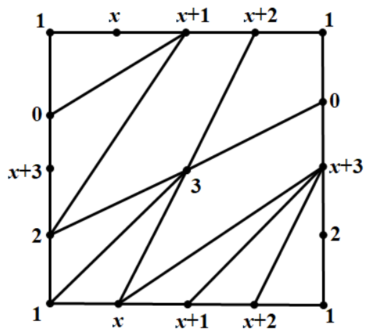

Figure 8. $G_{Qid}(\mathbb{Z}_4[x]/(2x, x^2 - 2))$.

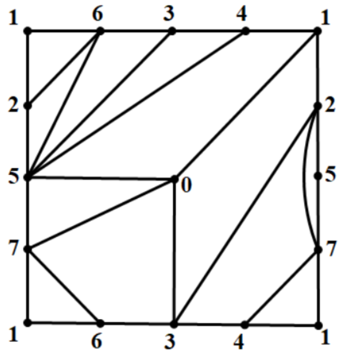

Figure 9. $G_{Qid}(\mathbb{Z}_8)$.

If $|\mathbb{F}^*| = 2$, then $\mathbb{F} \cong \mathbb{Z}_3$ and then $|\mathfrak{m}| = 3$. R is isomorphic to $\mathbb{Z}_9$ or $\mathbb{Z}_3[x]/(x^2)$. In this case, it is easy to verify that $|V(G_{Qid}(R))| = 9$ and $|E(G_{Qid}(R))| = 28$. Then, by Lemma 4.2, we know $\gamma(G_{Qid}(R)) \geq 2$. Figures 10 and 11 show that $G_{Qid}(R)$ can be embedded into $\mathbb{S}_2$. Therefore, $\gamma(G_{Qid}(\mathbb{Z}_9)) = \gamma(G_{Qid}(\mathbb{Z}_3[x]/(x^2))) = 2$.

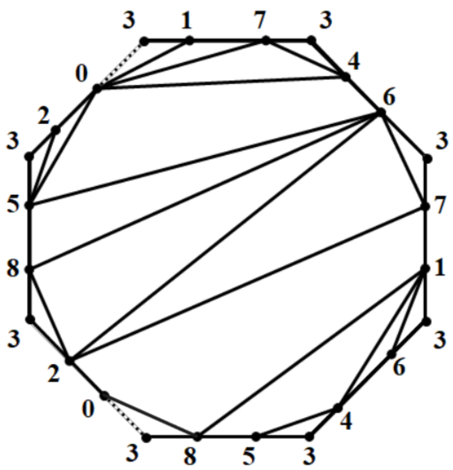

Figure 10. $G_{Qid}(\mathbb{Z}_9)$.

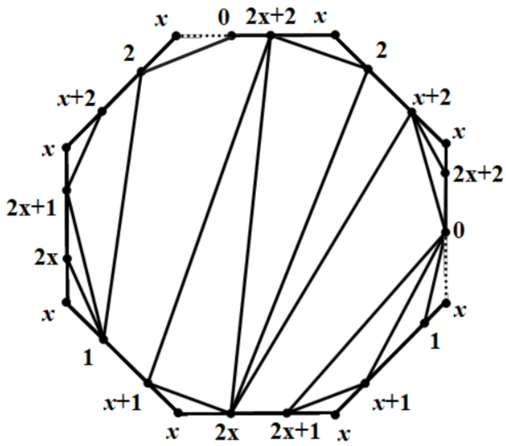

Figure 11. $G_{Qid}(\mathbb{Z}_3[x]/(x^2))$.

If $|\mathbb{F}^*| = 3$, then $\mathbb{F} \cong \mathbb{F}_4$ and then $|\mathfrak{m}| = 2$ by $|\mathbb{F}^*| \cdot |\mathfrak{m}| \leq 7$. There is no local rings R with $|\mathfrak{m}| = 2$ and $R/\mathfrak{m} \cong \mathbb{F}_4$.

Case 2. $s = 2$. In this case, $R = R_1 \times R_2$. By $|Qid(R)| = (|U(R_1)| + 1)(|U(R_2)| + 1) \leq 8$, we know that $|U(R_i)| = 1$ for some $i = 1, 2$. Without loss of generality. We let $|U(R_1)| = 1$. So, $|U(R_2)| = 1, 2, 3$.

When $|U(R_2)| = 1$, we know $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. In this case, $G_{Qid}(R)$ is clearly a planar graph.

When $|U(R_2)| = 2$, we know $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3$, or $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3$, $G_{Qid}(R) = K_6$, and thus its genus is one by Lemma 4.1. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$, then it is not difficult to verify that $G_{Qid}(R)$ has a subgraph $K_{3,3}$ (take the vertices: $(0,0)$, $(0,2)$, $(1,2)$, $(1,1)$, $(0,3)$, $(1,3)$ in $G_{Qid}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ or $(0,0)$, $(0,x)$, $(1,x)$, $(1,1)$, $(0,1+x)$, $(1,1+x)$ in $G_{Qid}(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2))$). So, $\gamma(G_{Qid}(R)) \geq 1$. Note that $G_{Qid}(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2))$ and $G_{Qid}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ can be embedded into $\mathbb{S}_1$ as shown in Figures 12 and 13, and thus $\gamma(G_{Qid}(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2))) = \gamma(G_{Qid}(\mathbb{Z}_2 \times \mathbb{Z}_4)) = 1$.

When $|U(R_2)| = 3$, we know $R \cong \mathbb{Z}_2 \times \mathbb{F}_3$. In this case, $G_{Qid}(R) = K_8$, and thus its genus is two.

Case 3. $s = 3$. In this case, by $|Qid(R)| = (|U(R_1)| + 1)(|U(R_2)| + 1)(|U(R_3)| + 1) \leq 8$, we know that $|U(R_i)| = 1$, $i = 1, 2, 3$. So, $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus, $G_{Qid}(R) = K_8$ and $\gamma(G_{Qid}(R)) = 2$.

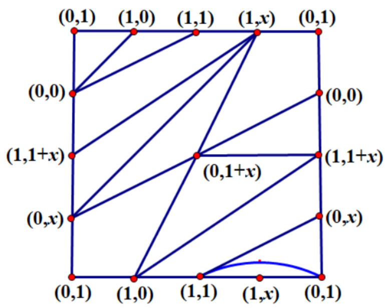

Figure 12. $G_{Qid}(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2))$.

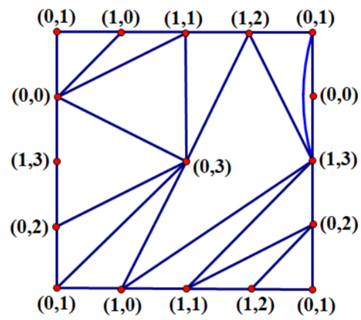

Figure 13. $G_{Qid}(\mathbb{Z}_2 \times \mathbb{Z}_4)$.

□

5. Conclusions

This paper systematically investigates the quasi-idempotent graph $G_{Qid}(R)$ of a ring R, where two distinct vertices $a, b \in R$ are adjacent if and only if $a + b$ is a quasi-idempotent element. We establish fundamental structural properties of the graph, including connectivity, regularity, completeness, bipartiteness, and a complete characterization of its girth, revealing deep connections with the algebraic structure of the ring.

We introduce the quasi-idempotent sum number **qisn**(R), a new ring-theoretic invariant, and for finite commutative rings, prove the equality **qisn**$(R) = diam(G_{Qid}(R))$, directly linking this algebraic invariant to the diameter of the graph.

Furthermore, we provide a complete classification of all finite commutative rings R according to the genus $\gamma(G_{Qid}(R))$ of their quasi-idempotent graphs, distinguishing those with genus 0, 1, or 2. This classification highlights the intricate relationship between the combinatorial complexity of the graph and the arithmetic structure of the ring.

In summary, the quasi-idempotent graph serves as a powerful tool for studying ring structures through graph-theoretic methods. The connections established between algebraic and graph invariants enrich the theory of ring-based graphs and suggest potential applications in areas such as algebraic cryptography. Future work may extend these results to non-commutative rings, investigate other graph invariants, or explore computational aspects of these graphs in cryptographic protocols.

Use of Generative-AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest in this paper.

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