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Research article

Oscillation of functional differential equations with a delayed damping term: Enhanced criteria and numerical simulation

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Abstract: This study aims to derive criteria for examining the asymptotic and oscillatory behavior of solutions to functional differential equations with delayed damping. By employing the Riccati technique together with an improved approach, we establish new criteria that complement the existing literature while distinguishing themselves by accounting for the delay effect in the damping term and by providing the well-known sharp criterion for Euler-type equations. Numerical examples illustrate the theoretical findings and clarify how the delay in the damping term affects the solution dynamics.

Keywords: differential equations; second-order equation; delayed damping term; asymptotic performance of nonoscillatory solutions; oscillatory properties
Mathematics Subject Classification: 34C10, 34K11

1. Introduction

This study seeks to analyze the asymptotic and oscillatory characteristics of solutions to the delay differential equation (DDE) with a delayed damping term

$$y''(t) + p(t)\, y'(h(t)) + q(t)\, y(g(t)) = 0, \tag{1.1}$$

where $t \geq t_0$, p and q are continuous functions on $[t_0, \infty)$, $p(t) \geq 0$, $q(t) > 0$, h and g are continuous delay functions on $[t_0, \infty)$, $g'(t) \geq 0$, $\lim_{t\to\infty} h(t) = \infty$, and $\lim_{t\to\infty} g(t) = \infty$.

A continuous real function y is considered a solution of (1.1) on $[t_y, \infty)$ for $t_y \geq t_0$, if it is differentiable twice, satisfies Eq (1.1), and satisfies the condition $\sup\{|y(t)| : t \geq t_*\} > 0$ for any $t_* \geq t_y$. This solution exhibits oscillatory behavior if it possesses arbitrarily large zeros; otherwise, it is termed nonoscillatory.

Damping is regarded as a fundamental and central concept in the study of differential equations (DE), particularly those that model physical systems exhibiting oscillatory behavior. The presence of damping terms in such equations typically arises from dissipative forces, such as friction or resistance, which inherently act to reduce the amplitude of oscillations over time. The impact of damping on system dynamics varies: A system may be underdamped, overdamped, or critically damped, with each regime corresponding to a distinct qualitative behavior. From a theoretical perspective, damping modifies both the stability and asymptotic behavior of solutions, effectively suppressing periodicity and guiding the system toward equilibrium. In practical applications, damping plays a crucial role in the design of engineering systems (e.g., vibration control in mechanical structures), signal processing, and control theory, where precise regulation of oscillatory responses is essential; see [1–3]. Consequently, understanding the interplay between damping and oscillatory dynamics provides a foundational framework for both qualitative analysis and applied modeling across a wide range of scientific and technological disciplines; see [4, 5].

Several studies have resorted to additional analytical techniques in conjunction with numerical approaches to capture subtle dynamical features and enhance the general understanding of various classes of delay differential equations (DDEs) as well as preserving important physical properties and structures with certain conditions; see, for example, [6–9].

Oscillation theory is an essential subfield of qualitative theory and a crucial element of the qualitative analysis of DDEs. This theory examines the oscillatory characteristics of solutions to differential equations together with their asymptotic and monotonic qualities. Recent investigations of the oscillatory characteristics of DDE solutions have demonstrated significant progress. Various studies on second-order equations have focused on establishing more accurate and efficient criteria than those now recognized (see [10–12]). Third-order neutral equations have also undergone significant research and examination (see [13–15]). Higher-order DDEs have also received considerable attention in investigating their oscillatory behavior; see, for example, [16–18].

The effect of ordinary damping on the oscillatory behavior of solutions to differential equations has been studied extensively using various analytical methods and techniques; see, for example, [19–21] for ordinary DEs and [22–24] for DDEs. In contrast, the impact of delayed damping has received significantly less attention. This is primarily due to the analytical difficulties that arise from incorporating delay into the damping term, which often leads to the failure of conventional techniques which analyze oscillatory behavior.

Delayed damping was addressed in the works of Grace [25] and Saker et al. [26], where the authors employed, respectively, the comparison technique and Riccati substitution to investigate the asymptotic behavior of solutions. However, the approach adopted in their analysis relies on neglecting the influence of one of the terms in the equation (either the second or the third). As a result, the criteria they obtained disregard the effect of one of the equation's coefficients as well as the associated delay functions.

Very recently, Moaaz and Ramos [27] employed an improved approach to derive oscillation criteria for Eq (1.1), thereby refining the results previously established in [25, 26].

In this work, we employ the Riccati substitution technique to investigate the asymptotic and oscillatory behavior of solutions to Eq (1.1), which incorporates a delayed damping term. We begin by deriving certain monotonicity properties of positive solutions and subsequently utilize them to establish the desired oscillation criteria. Our results are then compared with existing findings through their application to a special case of Euler-type equations. The second part of the study provides a numerical illustration of the impact of delayed damping, based on simulations of selected solution profiles for specific instances of the studied equation.

The main contributions of the paper can be summarized as follows:

– Investigating the oscillatory behavior of solutions to DDEs with delayed damping, a topic that has not been sufficiently explored.
– Establishing new oscillation criteria that explicitly incorporate the effect of the delay function h, providing a more accurate assessment of the system's dynamics.
– Conduct numerical simulations to support the theoretical results and illustrate the influence of delayed damping on solution behavior.

2. Main theoretical results

Remark 2.1. *For Eq (1.1), one can readily establish that any positive solution is monotonic. If this were not the case, the oscillation of the derivative y' would imply that $y'' < 0$ at each point where $y'(h(t)) = 0$, which necessarily restricts y' from attaining further zeros beyond the initial one. Consequently, any positive solution must eventually exhibit either increasing or decreasing monotonic behavior.*

By employing an approach identical to that used in [25, Theorem 1], we can establish the following result. Therefore, its proof is omitted.

Lemma 2.1. *Let*

$$\begin{cases} \varphi'(t) \geq 0, \ \left(\dfrac{\varphi(t)\,p(t)}{h'(t)}\right)' \leq 0, \\ \int_{t_0}^{\infty} \varphi(\upsilon)\,q(\upsilon)\,\mathrm{d}\upsilon = \infty, \\ \int_{t_0}^{\infty} \dfrac{1}{\varphi(\upsilon)} \int_{t_0}^{u} \varphi(\upsilon)\,q(\upsilon)\,\mathrm{d}\upsilon\mathrm{d}u = \infty, \end{cases} \tag{2.1}$$

for some $\varphi \in \mathbf{C}^1([t_0,\infty),\mathbb{R}^+)$. Then, every positive decreasing solution of (1.1) converges to zero.

Lemma 2.2. *Let*

$$\int_{t_1}^{\infty} \upsilon\,q(\upsilon)\,\mathrm{d}\upsilon = \infty. \tag{2.2}$$

Then, any positive increasing solution y of (1.1) eventually satisfies that both y' and y/t are decreasing.

Proof. Let $y(t) > 0$ and $y'(t) > 0$ for $t \geq t_1$. Then, from (1.1), we obtain

$$y''(t) = -p(t)\,y'(h(t)) - q(t)\,y(g(t)) \leq 0.$$

Equation (1.1) implies that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(t^2 \frac{\mathrm{d}}{\mathrm{d}t}\frac{y(t)}{t}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(t\,y'(t) - y(t)\right) \\ &= t\,y''(t) \\ &= t\left[-p(t)\,y'(h(t)) - q(t)\,y(g(t))\right]. \end{aligned}$$

As a result, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t^2\frac{\mathrm{d}}{\mathrm{d}t}\frac{y(t)}{t}\right)\leq -t\,q(t)\,y(g(t)),$$

which, upon integration, yields

$$t^2\frac{\mathrm{d}}{\mathrm{d}t}\frac{y(t)}{t}-t_1^2\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{y(t)}{t}\right]_{t=t_1}\leq -\int_{t_1}^{t}\upsilon\,q(\upsilon)\,y(g(\upsilon))\,\mathrm{d}\upsilon.$$

Because $g'(t)\geq 0$ and $y'(t)>0$, we obtain that $y(g(v))\geq y(g(t_1))$ for all $v\in[t_1,t]$, and so

$$\int_{t_1}^{t}\upsilon\,q(\upsilon)\,y(g(\upsilon))\,\mathrm{d}\upsilon\geq y(g(t_1))\int_{t_1}^{t}\upsilon\,q(\upsilon)\,\mathrm{d}\upsilon.$$

This implies that

$$t^2\frac{\mathrm{d}}{\mathrm{d}t}\frac{y(t)}{t}-t_1^2\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{y(t)}{t}\right]_{t=t_1}\leq -\int_{t_1}^{t}\upsilon\,q(\upsilon)\,y(g(\upsilon))\,\mathrm{d}\upsilon\leq -y(g(t_1))\int_{t_1}^{t}\upsilon\,q(\upsilon)\,\mathrm{d}\upsilon,$$

or

$$-t^2\frac{\mathrm{d}}{\mathrm{d}t}\frac{y(t)}{t}\geq -t_1^2\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{y(t)}{t}\right]_{t=t_1}+y(g(t_1))\int_{t_1}^{t}\upsilon\,q(\upsilon)\,\mathrm{d}\upsilon.$$

Accordingly, it follows from (2.2) that y/t is necessarily decreasing.

This concludes the proof. □

Theorem 2.1. *Let (2.1) hold for some* $\varphi\in\mathbf{C}^1([t_0,\infty),\mathbb{R}^+)$. *If (2.2) holds, and there is* $\theta\in\mathbf{C}^1([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left(\theta(\upsilon)\,\delta(\upsilon)\,q(\upsilon)\frac{g(\upsilon)}{\upsilon}-\frac{1}{4}\frac{\delta(\upsilon)\,(\theta'(\upsilon))^2}{\theta(\upsilon)}\right)\mathrm{d}\upsilon=\infty, \tag{2.3}$$

then every solution to (1.1) either exhibits oscillatory behavior or tends to zero, where

$$\delta(t)=\exp\left(\int_{t_1}^{t}p(\upsilon)\,\mathrm{d}\upsilon\right).$$

Proof. Based on the symmetry between positive negative solutions, assuming that Eq (1.1) has a positive solution does not affect the generality. Suppose that (1.1) has a positive solution y. Then, from Remark 2.1, there are two possibilities: Either y is decreasing or increasing.

Let $y'(t)>0$ for $t\geq t_1$. As a consequence of (1.1), we have that

$$\begin{aligned}-q(t)\,y(g(t)) &\geq y''(t)+p(t)\,y'(t)\\ &= \frac{1}{\delta(t)}\left(\delta(t)\,y'(t)\right)'.\end{aligned} \tag{2.4}$$

Because y/t is decreasing, we obtain

$$\frac{y(g(t))}{y(t)}\geq\frac{g(t)}{t}. \tag{2.5}$$

Now, we define

$$\Theta = \theta\,\delta\,\frac{y'}{y} > 0. \tag{2.6}$$

Then,

$$\Theta' = \frac{\theta'}{\theta}\Theta + \theta\left[\frac{(\delta\,y')'}{y} - \delta\frac{(y')^2}{y^2}\right],$$

which with (2.4)–(2.6) gives

$$\begin{aligned}\Theta' &\le \frac{\theta'}{\theta}\Theta - \theta\,\delta\,q\frac{y\,(g)}{y} - \theta\,\delta\frac{\Theta^2}{\theta^2\,\delta^2}\\ &\le \frac{\theta'}{\theta}\Theta - \theta\,\delta\,q\frac{g}{t} - \frac{1}{\theta\,\delta}\Theta^2.\end{aligned} \tag{2.7}$$

It is easy to verify that

$$\frac{\theta'}{\theta}\Theta - \frac{1}{\theta\,\delta}\Theta^2 \le \frac{1}{4}\frac{\delta\,(\theta')^2}{\theta},$$

which, together with (2.7), yields

$$\Theta' \le -\theta\,\delta\,q\frac{g}{t} + \frac{1}{4}\frac{\delta\,(\theta')^2}{\theta}.$$

Upon integrating this inequality, we deduce that

$$\int_{t_1}^{t}\left(\theta\,(\upsilon)\,\delta\,(\upsilon)\,q\,(\upsilon)\,\frac{g\,(\upsilon)}{\upsilon} - \frac{1}{4}\frac{\delta\,(\upsilon)\,(\theta'\,(\upsilon))^2}{\theta\,(\upsilon)}\right)\mathrm{d}\upsilon \le \Theta\,(t_1)\,.$$

This contradicts condition (2.3).

Next, let $y'\,(t) < 0$ for $t \ge t_1$. It follows from Lemma 2.1 that y converges to zero.

This concludes the proof. □

The previous theorem neglects the effect of delay within the damping term, whereas the upcoming result establishes a criterion that incorporates the role of the function $h\,(t)$.

Theorem 2.2. *Let (2.1) hold for some* $\varphi \in \mathbf{C}^1\,([t_0,\infty)\,,\mathbb{R}^+)$. *If (2.2) holds, and there is* $\theta \in \mathbf{C}^1\,([t_0,\infty)\,,(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left(\theta\,(\upsilon)\,\widehat{q}\,(\upsilon)\,\frac{g\,(h\,(\upsilon))}{\upsilon} - \frac{1}{4}\frac{(\theta'\,(\upsilon))^2}{\theta\,(\upsilon)}\right)\mathrm{d}\upsilon = \infty, \tag{2.8}$$

then every solution to (1.1) either exhibits oscillatory behavior or tends to zero, where

$$\widehat{q}\,(t) = p\,(t)\int_{h(t)}^{\infty} q\,(\upsilon)\,\mathrm{d}\upsilon + q\,(t)\,.$$

Proof. Based on the symmetry between positive and negative solutions, assuming that Eq (1.1) has a positive solution does not affect the generality. Suppose that (1.1) has a positive solution y. Then, from Remark 2.1, there are two possibilities: Either y is decreasing or increasing.

Let $y'(t) > 0$ for $t \geq t_1$. From (1.1), we get $y''(t) \leq -q(t)\, y(g(t))$. Upon integrating this inequality, we arrive at

$$\begin{aligned} y'(h(t)) &\geq \int_{h(t)}^{\infty} q(\upsilon)\, y(g(\upsilon))\, \mathrm{d}\upsilon \\ &\geq y(g(h(t))) \int_{h(t)}^{\infty} q(\upsilon)\, \mathrm{d}\upsilon, \end{aligned}$$

which with (1.1) gives

$$\begin{aligned} 0 &\geq y''(t) + p(t)\, y(g(h(t))) \int_{h(t)}^{\infty} q(\upsilon)\, \mathrm{d}\upsilon + q(t)\, y(g(t)) \\ &\geq y''(t) + \left(p(t) \int_{h(t)}^{\infty} q(\upsilon)\, \mathrm{d}\upsilon + q(t) \right) y(g(h(t))) \\ &= y''(t) + \widehat{q}(t)\, y(g(h(t))). \end{aligned} \tag{2.9}$$

Now, we define

$$\Theta = \theta \frac{y'}{y} > 0.$$

Hence,

$$\Theta' = \frac{\theta'}{\theta}\Theta + \theta \left[\frac{y''}{y} - \frac{(y')^2}{y^2} \right].$$

From (2.9), we find

$$\Theta' \leq \frac{\theta'}{\theta}\Theta - \theta \widehat{q} \frac{y(g(h))}{y} - \frac{1}{\theta}\Theta^2.$$

Because y/t is decreasing, we obtain

$$\Theta' \leq -\theta \widehat{q} \frac{g(h)}{t} + \frac{1}{4} \frac{(\theta')^2}{\theta}.$$

Upon integrating this inequality, we deduce that

$$\int_{t_1}^{t} \left(\theta(\upsilon)\, \widehat{q}(\upsilon) \frac{g(h(\upsilon))}{\upsilon} - \frac{1}{4} \frac{(\theta'(\upsilon))^2}{\theta(\upsilon)} \right) \mathrm{d}\upsilon \leq \Theta(t_1).$$

This contradicts condition (2.8).

Next, let $y'(t) < 0$ for $t \geq t_1$. It follows from Lemma 2.1 that y converges to zero.

This concludes the proof. □

Example 2.1. *Consider the DDE*

$$y''(t) + \frac{p_0}{t} y'(\mu t) + \frac{q_0}{t^2} y(\lambda t) = 0, \tag{2.10}$$

where $\mu, \lambda \in (0, 1]$, p_0, *and* q_0 *are positive. By choosing* $\varphi(t) = t$, *we note that* $\varphi'(t) = 1 \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\varphi(t)\, p(t)}{h'(t)} \right)' = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{p_0}{\mu} \right) = 0,$$

$$\int_{t_0}^{\infty} \varphi(\upsilon) q(\upsilon) \mathrm{d}\upsilon = \int_{t_0}^{\infty} \upsilon q(\upsilon) \mathrm{d}\upsilon = q_0 \int_{t_0}^{\infty} \frac{1}{\upsilon} \mathrm{d}\upsilon = \infty,$$

and

$$\int_{t_0}^{\infty} \frac{1}{\varphi(\upsilon)} \int_{t_0}^{u} \varphi(\upsilon) q(\upsilon) \mathrm{d}\upsilon \mathrm{d}u = q_0 \int_{t_0}^{\infty} \frac{\ln(u/t_0)}{u} \mathrm{d}u = \infty.$$

Accordingly, both conditions (2.1) and (2.2) are fulfilled. Moreover, we have $\delta(t) = t^{p_0}$ *and*

$$\widehat{q}(t) = \frac{q_0}{t^2}\left(\frac{p_0}{\mu} + 1\right).$$

By applying the previous theorems, we obtain the following:

– *By selecting* $\theta(t) = t^{1-p_0}$*, Theorem 2.1 asserts that the solutions of (2.10) oscillate or converge to zero when*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(\lambda q_0 - \frac{(1-p_0)^2}{4}\right) \frac{1}{\upsilon} \mathrm{d}\upsilon = \infty,$$

which is satisfied when

$$q_0 > \frac{1}{4\lambda}(1-p_0)^2. \tag{2.11}$$

– *Setting* $\theta(t) = t$ *and invoking Theorem 2.2 yields that the solutions of (2.10) exhibit oscillatory behavior or tend to zero whenever*

$$\lambda\mu q_0 \left(\frac{p_0}{\mu} + 1\right) > \frac{1}{4}. \tag{2.12}$$

Figure 1 provides a comparison between criteria (2.11) and (2.12). The results show that criterion (2.11) exhibits greater sharpness for small values of p_0 *(near 1), and criterion (2.12) is sharper for larger values of* p_0*. Moreover, we observe that criterion (2.12) has an advantage over (2.11) in that it takes into account the effect of the parameter* μ*.*

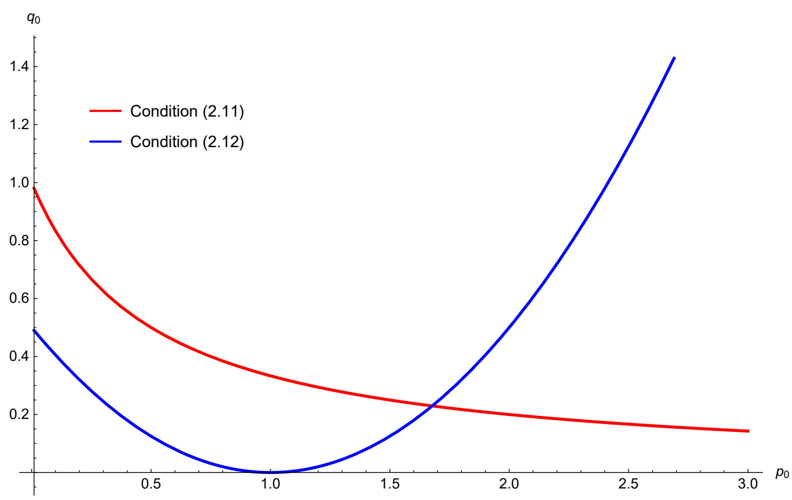

Figure 1. Comparison between conditions (2.11) and (2.12).

Remark 2.2. *Very recently, studies [27, 28] investigated the oscillatory and asymptotic behavior of solutions to DDEs involving delayed damping. According to [28, Example 2.2], the solutions of Eq (2.10) exhibit either oscillatory behavior or convergence to zero if*

$$q_0 \left[(2 + p_0) \lambda + (1 + p_0) \lambda \ln \frac{1}{\lambda} \right] > 1. \tag{2.13}$$

Figure 2 shows the lower bounds of the values of q_0 for conditions (2.12) and (2.13) across the values of λ in $(0, 1]$. It becomes clear that no absolute superiority can be ascribed to one condition over the other; condition (2.13) exhibits greater sharpness in $(0, 0.3679]$, whereas condition (2.12) demonstrates sharper behavior in the interval $[0.3679, 1]$. Consider the DDEs

$$y''(t) + \frac{1}{2t} y'(0.5t) + \frac{11}{10t^2} y(0.2t) = 0 \tag{2.14}$$

and

$$y''(t) + \frac{1}{2t} y'(0.5t) + \frac{4}{10t^2} y(0.8t) = 0. \tag{2.15}$$

It is not difficult to observe that condition (2.13) ensures that the solutions of (2.14) either oscillate or converge to zero, whereas (2.12) fails to apply. Conversely, the same property for (2.15) is established through condition (2.12), whereas (2.13) does not hold.

In addition, condition (2.12) is characterized by its consideration of the delayed effect μ within the damping term.

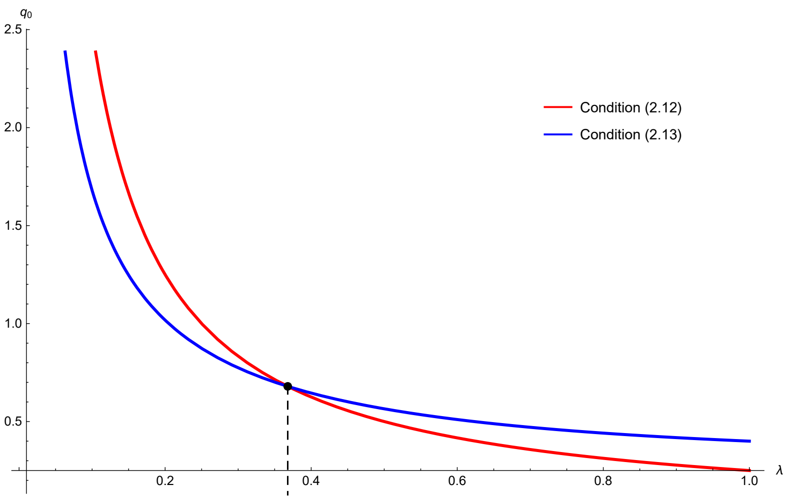

Figure 2. Comparison between conditions (2.12) and (2.13).

3. Numerical simulation of solutions

In this section, we present the numerical solutions of several examples of a delay differential equation with a delayed damping term. The aim of this discussion is to illustrate the effect of the damping term as well as the impact of the delay in the damping term.

We solve Eq (1.1) numerically by transforming it into a first-order system of delay differential equations and use Matlab 2024 to find the numerical solutions of the transformed system. In this discussion, we consider two different cases of delays as presented in the following subsections.

3.1. Numerical results with constant delays

In this case, we consider the configuration in which the delays are of the form $t - a$, $t - b$, and so forth. That is, we focus on the cases where $h(t) = t - a$ and $g(t) = t - b$. These types of constant delay are commonly used in applications such as population growth models.

Example 3.1. Consider the DDE

$$y''(t) + \frac{p_0}{t} y'(t - a) + \frac{q_0}{t^2} y(t - b) = 0, \tag{3.1}$$

where p_0, q_0, a and b are constants. The numerical solutions of this example are illustrated in Figures 3 and 4.

Figure 3 shows the solution of Eq (3.1) with $a = 0$, $q_0 = 3$, and $b = 3$ for all cases, where the solid line represents the case without a damping term ($p_0 = 0$). The other curves represent the case with a damping term but no delay ($a = 0$) with different values of p_0: The dotted line represents the case with $p_0 = 0.25$, the dashed line with $p_0 = 1$, and the dashed-dotted line with $p_0 = 2$. This figure illustrates the effect of the damping term. It is apparent that the damping term has a significant impact on the solution, as it leads to a reduction in the amplitude of the oscillations. We observe that the effect of damping is greater with larger values of p_0.

Figure 4 shows the solution of Eq (3.1) with $q_0 = 3$ and $b = 3$ for all cases, where the solid line represents the case without a damping term ($p_0 = 0$), the dashed line represents the case with a damping term but without a delay (with $p_0 = 1$), and the starred line represents the case with a delay in the damping term ($p_0 = 1$ and $a = 4$). Comparing the dashed and starred curves corresponding to the case with damping, we note that the impact of the damping is greater in the absence of a delay. The presence of a delay in the damping term affects the damping process.

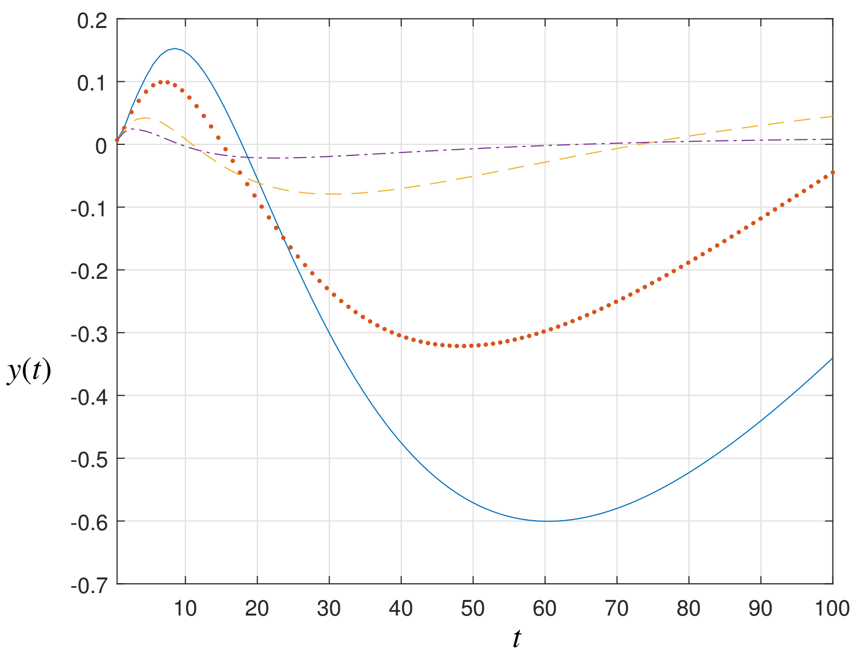

Figure 3. Numerical solution to Eq (3.1) when $a = 0$, $q_0 = 3$, and $b = 3$.

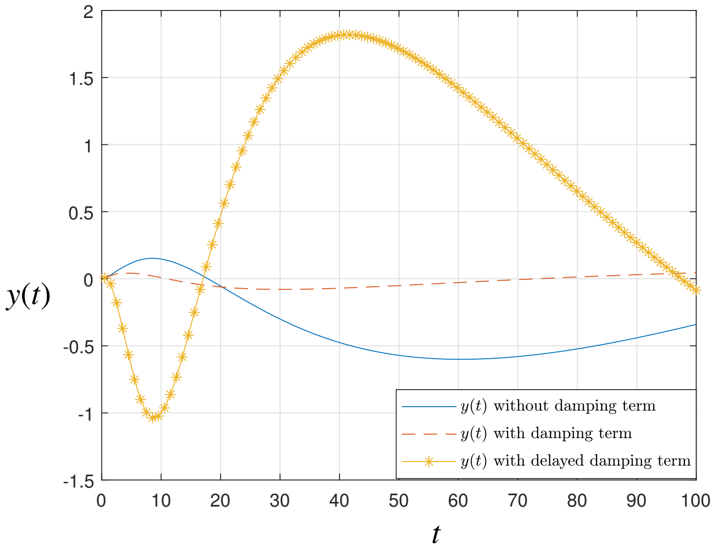

Figure 4. Numerical solution to Eq (3.1) when $q_0 = 3$ and $b = 3$.

Example 3.2. Consider the DDE

$$y''(t) + p_0 y'(t-a) + \frac{q_0}{t^2} y(t-b) = 0, \tag{3.2}$$

where p_0, q_0, a and b are constants.

Figure 5 shows the solution of Eq (3.2) with $q_0 = 3$ and $b = 3$ for all cases, where the solid line represents the case without a damping term ($p_0 = 0$), the dashed line represents the case with a damping term but no delay (with $p_0 = 1$), and the starred line represents the case with a delay in the damping term ($p_0 = 1$ and $a = 4$). In this figure, we notice that the effect of damping is present in the case without the delay, as before. In addition, we notice huge oscillations in the starred curve compared to Example 1. This is due to the form of $p(t)$ in Example 1, where there is a division by t. However, this was not the case in the absence of a delay, where we note the impact of damping in both examples.

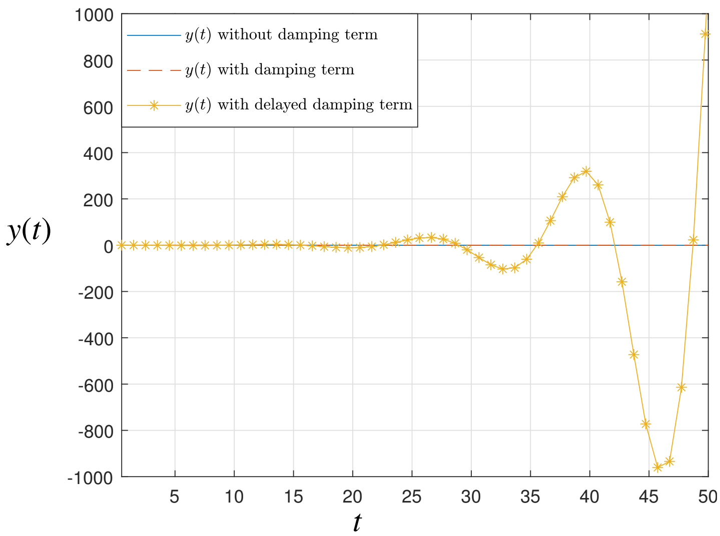

Figure 5. Numerical solution to Eq (3.2) when $q_0 = 3$ and $b = 3$.

Figure 6 shows the solution of Eq (3.2) with $q_0 = 0.25$ for all cases, where the solid line represents the case without a damping term ($p_0 = 0$ and $b = 0$), the dashed line represents the case with a damping term but without delay (with $p_0 = 1$, $a = 0$, $b = 3$), and the starred line represents the case with a delay in the damping term ($p_0 = 1$, $b = 3$, and $a = 4$). We note from this figure that this solid line solution is nonoscillatory. However, the addition of the damping term leads to an oscillatory solution as shown by the dashed and starred curves. This indicates that including a damping term can change the form of the solution to become an oscillatory solution.

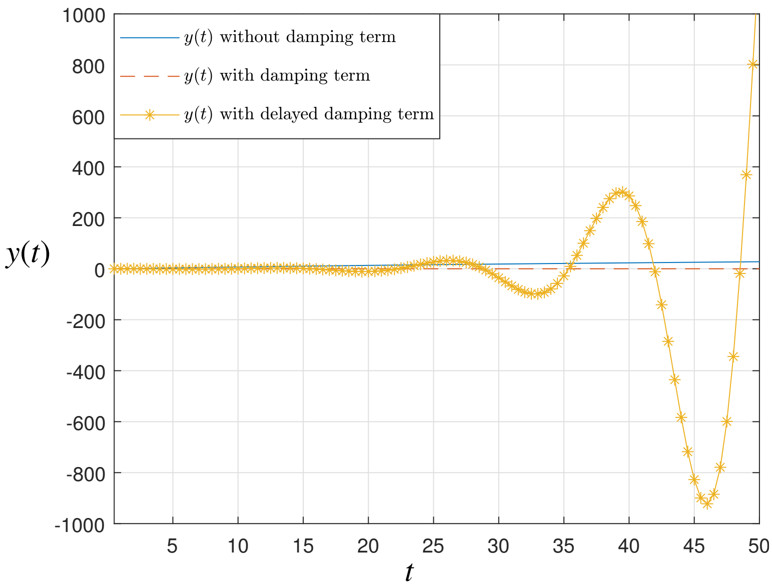

Figure 6. Numerical solution to Eq (3.2) when $q_0 = 0.25$.

3.2. Numerical results with time-dependent delays

In this subsection, we consider the situation where the delay changes with time; that is, the delay is represented as a function of time, such as $h(t) = at$ and $g(t) = bt$. This type of time-dependent delay is common in applications such as control systems and mechanical systems.

Example 3.3. Consider the DDE

$$y''(t) + \frac{p_0}{t} y'(at) + \frac{q_0}{t^2} y(bt) = 0, \tag{3.3}$$

where p_0, q_0, a, and b are constants.

Figure 7 shows the solution of Eq (3.3) with $q_0 = 3$ and $b = 0.3$ for all cases, where the solid line represents the case without a damping term ($p_0 = 0$), the dashed line represents the case with a damping term but without a delay (with $p_0 = 1$, $a = 0$, $b = 0.3$), and the starred line represents the case with a delay in the damping term ($p_0 = 1$, $b = 0.3$, and $a = 0.4$). We observe that the addition of the damping term has a significant impact on the solution, as illustrated by the dashed curve. The starred curve solution, which represents the case with a delayed damping, shows a damping behavior compared to the solid line curve. However, the damping effect is more apparent in the case of nondelayed damping. The damping behavior of the case with delayed damping is noticeable in this example compared to the constant delay configuration discussed in the previous subsection.

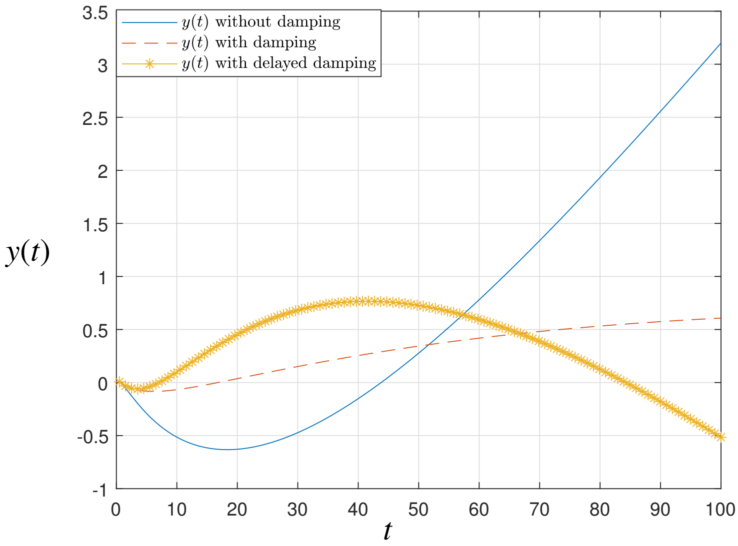

Figure 7. Numerical solution to Eq (3.3) when $q_0 = 3$ and $b = 0.3$.

4. Conclusions

In this work, we examined the oscillatory and asymptotic behavior of solutions of second-order functional differential equations with a delayed damping term. Applying the Riccati substitution method with advanced analytical techniques, we established novel oscillation criteria that expand and reinforce numerous previously reported results in the literature. In contrast to previous approaches, the criteria established here explicitly incorporate the influence of a delay in the damping term, thus providing a more precise assessment of the dynamics of delayed systems.

Two principal theorems were derived: the first provides refined conditions for oscillation without fully accounting for the delay effect in the damping coefficient, and the second presents an enhanced criterion that directly incorporates the impact of the delay. Comparison with the latest findings indicates that our conditions attain a more comprehensive understanding of the behavior of Euler-type equations.

To emphasize the theoretical results, numerical simulations were carried out for both constant and time-dependent delay configurations. These numerical solutions clearly illustrated the role of damping and delayed damping in solution profiles. In particular, they showed how including a delay may either weaken or influence the damping effect. In addition, it was found that in some cases, nonoscillatory solutions are transformed into oscillatory ones. This behavior indicates the practical importance of accurately modeling the delay within the damping term.

Author contributions

Ahmed S. Almohaimeed: Methodology, software, investigation, writing–original draft, writing–review and editing; Osama Moaaz: Conceptualization, methodology, investigation, writing–original draft, writing–review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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