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Research article

Certain novel generalized subclasses of uniformly starlike and convex functions: coefficient characterizations and inclusion properties through Bessel and Gaussian hypergeometric functions

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Abstract: In this scholarly article, we present novel generalized subclasses of ν uniformly starlike functions of order ρ, designated as $M(\tau, \rho, \nu)$, alongside ν uniformly convex functions of order ρ, referred to as $N(\tau, \rho, \nu)$. We provide comprehensive coefficient characterizations that delineate the conditions under which analytic functions are classified within the newly established subclasses of uniformly starlike and uniformly convex families, respectively. Furthermore, we conduct an analysis of the implications of the Bessel function and explore the consequences of the Gaussian hypergeometric function on these mathematical classes to substantiate an inclusion property for analytic functions that reside within these subclasses.

Keywords: analytic functions; starlike functions; convex functions; Bessel function of the first kind; Gaussian hypergeometric function
Mathematics Subject Classification: 30C45, 30C80

1. Introduction

Numerous subclasses of analytic functions associated with a complex variable that is well-defined within the open unit disk, known as

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\},$$

have received notable interest from a wide range of scholars in the field of geometric function theory. The development of subclasses pertaining to starlike and convex functions of analytic functions is fundamentally based on the open unit disk.

In this article, we carefully examine the subclasses of starlike and convex functions in the context of analytic functions, which helps to expand the particular subclasses of analytic functions defined in the existing literature.

The wide variety of generalized Bessel functions of the first kind has had a significant influence on geometric function theory in recent years. The generalized Bessel functions of the first kind are being studied in detail by a growing number of researchers, who have given them a definite categorization in the classes of starlike and convex analytic functions. Recent investigations have involved the Bessel function of the first kind in the study of geometric properties for various classes of analytic functions, like seen in [1–3]. The Bessel function of the first kind was also investigated regarding monotony conditions in recent studies like [4, 5] and was applied in order to introduce new operators in recent publications like [6–8]. Given the extensive use of this special function in recent developments, the present study was inspired to further obtain applications involving the new classes considered here.

The $F(i, b, d, z)$ hypergeometric function acted as an effective role in the analysis related to various topics of geometric function theory. Recently, interested researchers followed the Gaussian hypergeometric function to investigate the inclusion characteristics of particular subclasses of analytic functions. Numerous differential and integral operators are defined in the open unit disk and linked with the Gaussian hypergeometric function. Recent such studies can be seen in [9, 10]. Cho et al. in [11] applied the function $F(i, b, d, z)$ and studied its properties related to convexity. Recent studies on convexity properties of this function can be seen in [12]. At the beginning of the study of this function, Merkes and Scott in [13] first applied the function $F(i, b, d, z)$ and examined it for starlikeness properties. Later on, Carlson and Shaffer in [14] and Ruscheweyh and Singh in [15] additionally applied the Gaussian hypergeometric function and investigated the starlikeness, prestarlikeness, and the order of starlikeness of the hypergeometric functions. Recently, the Gaussian hypergeometric function has been adapted to fit the studies related to strong differential subordination theory, as seen in [16].

A wide variety of scholarly contributions constitute a substantial foundation of inspiration for this research's endeavors, particularly their investigation into the geometric characterizations associated with uniformly starlike and convex functions of various orders and their connections to conic sections. Sharma et al. [17] employ Gaussian hypergeometric functions to define the functions, deriving results pertinent due to their geometric characteristics. Murugusundaramoorthy, in [18], conducted an examination of diverse subclasses of starlike and convex functions. Porwal and Dixit proposed and scrutinized in [19] novel subclasses of analytic functions. Srinivas, Reddy and Niranjan investigate in [20] the properties of analytic functions through the applications of Bessel functions, while Silverman in [21] elucidated the criteria under which the Gaussian hypergeometric function maps the open unit disk onto specific subclasses of univalent functions. This exhaustive analysis has generated new ideas to investigate the potential introduction of new subclasses within $\mathbb{U}$. In the examination of simply connected starlike and convex domains, scholars could consult articles authored by Faisal and Darus [22–24].

In the subsequent section, we present the novel subclasses of analytic functions within the context of the open unit disk, specifically denoted as $M(\tau, \rho, \nu)$ and $N(\tau, \rho, \nu)$ as outlined in Definitions 2.1

and 2.2, respectively, along with the numerous classes generalized by them and with known results used as lemmas for the proofs of the outcome developed here. The new results obtained related to these classes are presented in Section 3 of the paper. A brief overview of the new results presented by this study is done in Section 4 and future research directions that include these new results are suggested.

2. Preliminaries

The context for obtaining the new results is formed of functions based on the equation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{2.1}$$

where z is in the open disk $\mathbb{U}$. The class consisting of those functions that are analytic and univalent within $\mathbb{U}$ will be referred to as $\mathbb{A}$.

For $f \in \mathbb{A}$ given by (2.1) and $g \in \mathbb{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{U}. \tag{2.2}$$

A function f, as defined in Eq (2.1), is classified as a starlike function of order ρ $(0 \leq \rho < 1)$, denoted $f \in S^*(\rho)$, if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \rho. \tag{2.3}$$

Furthermore, a function f, as defined in the formula (2.1), is said to be a convex function of order ρ $(0 \leq \rho < 1)$, represented as $f \in C(\rho)$, if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \rho. \tag{2.4}$$

The classes of starlike function of order ρ $(0 \leq \rho < 1)$ and convex function of order ρ $(0 \leq \rho < 1)$ were introduced by Robertson in 1936 [25].

In addition, the subclass T of $\mathbb{A}$ comprises functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad z \in \mathbb{U}.$$

Note that $T^*(\rho)$ and $K(\rho)$ represent the classes of starlike and convex functions of order ρ, as introduced and studied by Silverman [26].

Let the function f defined by (2.1) be categorized within the class $\mathfrak{R}^{\tau}(A, C)$ if the condition

$$\left|\frac{f'(z)-1}{(A-C)\tau-C[f'(z)-1]}\right| < 1$$

is satisfied, where τ is an element of $\mathbb{C}$, and the parameters satisfy

$$-1 \leq C \leq A \leq 1$$

with z residing in $\mathbb{U}$. The notation $\mathfrak{R}^{\tau}(A, C)$ was previously established by Dixit and Pal [27].

For the complex variables $p, a, b \in \mathbb{C}$, the generalized Bessel function

$$\omega(p, a, b) = \omega$$

(see [28]) is expressed and represented there by

$$\omega(z) = \omega(p, a, b)(z) = \sum_{0}^{\infty} \frac{(-1)^n b^n}{n!\Gamma(p + n + \frac{a+1}{2})} \left(\frac{z}{2}\right)^{2n+p}, \tag{2.5}$$

which serves as the specific solution to the equation

$$z^2\omega''(z) + az\omega'(z) + [bz^2 - p^2 + 1 - a]\omega(z) = 0. \tag{2.6}$$

Recently, numerous scholars have conducted investigations into the Bessel function; herein, we endorse references [29–31].

Employing the concept of the shifted factorial, Murugusundaramoorthy and Janani (see [32]), developed the function $\upsilon_{p,a,b}(z)$ in the following manner:

$$\upsilon_{p,a,b}(z) = 2^p\Gamma\left(p + \frac{a+1}{2}\right) z^{\frac{-p}{2}} \omega(p, a, b)(\sqrt{z}).$$

Subsequently, through the implementation of the shifted factorial in relation to the Gamma function, it is feasible to articulate $\upsilon_{p,a,b}(z)$ in the following manner:

$$\upsilon_{p,a,b}(z) = \sum_{n=0}^{\infty} \frac{\left(\frac{-b}{4}\right)^n}{\left(p + \frac{a+1}{2}\right)_n} \left(\frac{z^n}{n!}\right),$$

$$p + \frac{a+1}{2} \neq 0, -1, -2, \ldots,$$

and $\upsilon_{p,a,b}(z)$ signifies the analytic function that satisfies the second-order linear differential equation

$$4z^2\upsilon''(z) + 2(2p + a + 1)z\upsilon'(z) + bz\upsilon(z) = 0.$$

Using the Hadamard product (*), as defined in Eq (2.2), the linear operator $T(b, q)f(z)$ is defined as

$$T(b, q)f(z) = z\upsilon_{p,a,b} * f(z) = z + \sum_{n=2}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n-1}}{(q)_{n-1}(n-1)!} a_n z^n, \quad q = p + \frac{a+1}{2} \neq 0. \tag{2.7}$$

For the sake of clarity and ease of reference in the subsequent discussions, we shall employ the following symbols and notations:

$$\upsilon_{p,a,b}(z) = \upsilon_p(z) \text{ and } q = p + \frac{a+1}{2},$$

and defining for $b < 0$ and $q > 0$:

$$z\left(2 - \upsilon_p(z)\right) = z - \sum_{n=2}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n-1}}{(q)_{n-1}(n-1)!} z^n. \tag{2.8}$$

The Gauss hypergeometric function $F(i, b, d, z)$ denotes the function defined by

$$F(i, b, d, z) = \sum_{n=0}^{\infty} \frac{(i)_n (b)_n}{(d)_n (1)_n} z^n = F(z),$$

where i, b, d are complex numbers and d cannot take the value $0, -1, \ldots$, which is comprehensively discussed in [33]. The $F(i, b, d, z)$ hypergeometric function made important contributions in studies regarding several topics, including the quasiconformal theory and conformal mappings [21]. The function $F(i, b, d, z)$ satisfies numerous identities that were studied in [34]. Moreover, the behavior of the function $F(i, b, d, z)$ near $z = 1$ was presented in three different ways in [35]. The value of the function $F(i, b, d, z)$ is given by the formula (see [34])

$$F(i, b, d, 1) = \frac{\Gamma(d)\Gamma(d - i - b)}{\Gamma(d - i)\Gamma(d - b)} = F(1),$$

provided

$$\Re(d - i - b) > 0.$$

Remark 2.1. *It is important to note that the Euler integral representation of the hypergeometric function $F(i, b, d, z)$ for*

$$\Re(d) > \Re(b) > 0 \quad \text{and} \quad |z| < 1$$

is

$$F(i, b, d, z) = \frac{\Gamma(d)}{\Gamma(b)\Gamma(d - b)} \int_0^1 t^{b-1}(1 - t)^{d-b-1}(1 - zt)^{-i} dt.$$

Then,

$$\begin{aligned} F(i, b, d, 1) &= \frac{\Gamma(d)}{\Gamma(b)\Gamma(d - b)} \int_0^1 t^{b-1}(1 - t)^{d-b-1}(1 - t)^{-i} dt \\ &= \frac{\Gamma(d)}{\Gamma(b)\Gamma(d - b)} \int_0^1 t^{b-1}(1 - t)^{d-b-1-i}, \quad (for\ z = 1), \end{aligned}$$

is the Euler integral for the Beta function, denoted by $B(x, y)$, defined as

$$B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)},$$

and comparing the Euler integral with the definition of the Beta function, it follows that

$$\begin{aligned} F(i,b,d,1) &= \frac{\Gamma(d)}{\Gamma(b)\Gamma(d-b)} B(b,d-b-i) \\ &= \frac{\Gamma(d)}{\Gamma(b)\Gamma(d-b)} \frac{\Gamma(b)\Gamma(d-b-i)}{\Gamma(b+d-b-i)} \\ &= \frac{\Gamma(d)\Gamma(d-i-b)}{\Gamma(d-i)\Gamma(d-b)} = F(1). \end{aligned}$$

Definition 2.1. *For* $0 \leq \tau < 1$ *and* $\nu \geq 0$*, let* $M(\tau,\rho,\nu)$ *be the subset of* $\mathbb{A}$ *consisting of functions as given in* (2.1) *that meet the analytic criterion*

$$\begin{aligned} &\Re\left\{\frac{zf'(z)}{4(\tau-\tau^2)z+(2\tau^2-\tau)zf'(z)+(2\tau^2-3\tau+1)f(z)}-\rho\right\} \\ &> \nu\left|\frac{zf'(z)}{4(\tau-\tau^2)z+(2\tau^2-\tau)zf'(z)+(2\tau^2-3\tau+1)f(z)}-1\right|, \quad z\in\mathbb{U}. \end{aligned} \tag{2.9}$$

If we set

$$\tau = \nu = 0,$$

it follows that

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \rho,$$

which results in

$$M(\tau,\rho,\nu) = S^*(\rho)$$

as defined in (2.3) *(see [26]). If we assume* $\tau = 0$*, it follows that*

$$\Re\left\{\frac{zf'(z)}{f(z)}-\rho\right\} > \nu\left|\frac{zf'(z)}{f(z)}-1\right|,$$

leads to

$$M(\tau,\rho,\nu) = SD(\rho,\nu)$$

as specified in ([36]). Observe that if $f \in SD(\rho,\nu)$*, then* $f \in S^*(\frac{\rho-\nu}{1-\nu})$*, as illustrated in ([37]). Also, the previously mentioned class represents the generalized representation of various classes of starlike functions, which will be discussed immediately following Definition 2.2.*

Definition 2.2. *Let* $N(\tau,\rho,\nu)$ *be the subset of* $\mathbb{A}$ *consisting of functions as given in* (2.1) *that meet the analytic criterion*

$$\begin{aligned} &\Re\left\{\frac{f'(z)+zf''(z)}{4(\tau-\tau^2)+(2\tau^2-\tau)zf''(z)+(4\tau^2-4\tau+1)f'(z)}-\rho\right\} \\ &> \nu\left|\frac{f'(z)+zf''(z)}{4(\tau-\tau^2)+(2\tau^2-\tau)zf''(z)+(4\tau^2-4\tau+1)f'(z)}-1\right|. \end{aligned} \tag{2.10}$$

If one assumes

$$\tau = \nu = 0,$$

it follows that

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\rho,$$

resulting in the class $M(\tau,\rho,\nu)$ *being equivalent to* $C(\rho)$ *as defined in* (2.4) *([26]). If we examine the case where* $\tau=0$*, then*

$$\Re\left\{\frac{zf'(z)}{f(z)}-\rho\right\}>\nu\left|\frac{zf'(z)}{f(z)}-1\right|,$$

leading to

$$M(\tau,\rho,\nu)=KD(\rho,\nu)$$

as defined in ([36]). It is important to recognize that if $f\in KD(\rho,\nu)$*, then it follows that* $f\in K(\frac{\rho-\nu}{1-\nu})$ *as illustrated in ([37]). Similarly, the previously mentioned class serves as the generalized representation of diverse subclasses of convex functions, which are elaborated upon below.*

From (2.9) and (2.10), it is obvious that $f\in N(\tau,\rho,\nu)$ if and only if $zf'\in M(\tau,\rho,\nu)$. Moreover, let

$$TM(\tau,\rho,\nu)=T\cap M(\tau,\rho,\nu)$$

and

$$TN(\tau,\rho,\nu)=T\cap N(\tau,\rho,\nu).$$

Further, a straightforward computation shows that numerous subclasses of analytic functions introduced in different papers are particular cases of the subclasses $TM(\tau,\rho,\nu)$ and $TN(\tau,\rho,\nu)$:

In the case of $\tau=0$, we identify the classes $TS_P(\rho,\nu)$ and $TS_P(\rho,\nu)$ which were considered by Bharati et al. in [38]. For $\tau=0$ and $\rho=0$, we find the classes $TS_P(\nu)$ and $TK_P(\nu)$, which were previously examined by Subramanian et al. in [39,40]. When $\tau=0$ and $\nu=1$, we observe the classes $TS_P(\rho)$ and $TK_P(\rho)$, studied by Bharati et al. in [38]. In the case of $\nu=0$, we notice that $T^*(\tau,\rho)$ and $C(\tau,\rho)$ are the classes previously investigated by Altintas et al. in [41]. For $\tau=0$ and $\nu=0$, we uncover the classes $T^*(\rho)$ and $C(\rho)$, which were investigated earlier by Silverman in [26].

Lemma 2.1. *[42] If* $p,a,c\in\mathbb{C}$ *and* $m\neq 0,-1,-2,\ldots$*, then the function* υ_p *satisfies the recursive relation*

$$4m\upsilon_p'(z)=-c\upsilon_{p+1}(z),\quad\forall z\in\mathbb{U}.$$

Lemma 2.2. *[27] If* $f\in\Re^\tau(A,C)$ *has the functions of the form* (2.1)*, then*

$$|a_n|\leq(A-C)\frac{|\tau|}{n},\quad n\in\mathbb{N}\ \{0\}.$$

In the ensuing section, we initially derive the characterization properties, specifically related to the coefficient bounds, of the newly established subclasses of analytic functions $M(\tau,\rho,\nu)$ and $N(\tau,\rho,\nu)$ as expressed in Theorems 3.1 and 3.2. The necessary and sufficient conditions for the function given by Eq (2.8) to be part of the classes $M(\tau,\rho,\nu)$ and $N(\tau,\rho,\nu)$ are provided in Theorems 3.3 and 3.4. The impact of the Bessel function on the class $TN(\tau,\rho,\nu)$ is investigated in Theorems 3.5 and 3.6. The coefficient restriction proved in Theorem 3.2 is applied to investigate the impact of the hypergeometric function on the class $TN(\tau,\rho,\nu)$ in Theorems 3.7 and 3.8.

3. Results

Theorem 3.1. *Let* $0 \leq \rho < 1$, $0 \leq \tau < 1$, *and* $\nu \geq 0$. *A function* f *defined in* (2.1) *belongs to* $TM(\tau, \rho, \nu)$ *if and only if*

$$\sum_{n=2}^{\infty}[n(1+\nu) - (\rho+\nu)(1 + n(2\tau^2 - \tau) + (2\tau^2 - 3\tau))]|a_n| \leq 1 - \rho.$$

Proof. For proving sufficiency, we need to get

$$\nu\left|\frac{zf'(z)}{g(z)} - 1\right| - \Re\left\{\frac{zf'(z)}{g(z)} - \rho\right\} \leq 1 - \rho,$$

where

$$g(z) = 4(\tau - \tau^2)z + (2\tau^2 - \tau)zf'(z) + (2\tau^2 - 3\tau + 1)f(z).$$

Then

$$\frac{(1+\nu)\sum\limits_{n=2}^{\infty}[n - n(2\tau^2 - \tau) - (2\tau^2 - 3\tau + 1)]|a_n||z|^{n-1}}{1 - \sum\limits_{n=2}^{\infty}[n(2\tau^2 - \tau) + (2\tau^2 - 3\tau + 1)]|a_n||z|^{n-1}} \leq 1 - \rho,$$

which implies

$$\sum_{n=2}^{\infty}[n(1+\nu) - (\rho+\nu)(1 + n(2\tau^2 - \tau) + (2\tau^2 - 3\tau))]|a_n| \leq 1 - \rho.$$

For proving the necessity, let $f \in TM(\tau, \rho, \nu)$ and z be real. Then

$$\frac{1 - \sum\limits_{n=2}^{\infty} a_n z^n}{1 - \sum\limits_{n=2}^{\infty}[2\tau^2 - 3\tau + 1 + (2\tau^2 - \tau)n]} - \rho > \nu\left|\frac{1 - \sum\limits_{n=2}^{\infty} a_n z^n}{1 - \sum\limits_{n=2}^{\infty}[2\tau^2 - 3\tau + 1 + (2\tau^2 - \tau)n]} - 1\right|.$$

After several computations, supposing that z approaches to 1 on the real axes, the inequality

$$\sum_{n=2}^{\infty}[n(1+\nu) - (\rho+\nu)(1 + n(2\tau^2 - \tau) + (2\tau^2 - 3\tau))]|a_n| \leq 1 - \rho$$

is obtained. □

Corollary 3.1. *For the parametric value* $\tau = 0$ *in Theorem 3.1, we derive Theorem 2.1 in [36], which asserts*

$$\text{if} \sum_{n=2}^{\infty}[n(1+\nu) - (\rho+\nu)]|a_n| \leq 1 - \rho,$$

then $f \in SD(\rho, \nu)$.

Theorem 3.2. *Let* $0 \leq \rho < 1$, $0 \leq \tau < 1$, *and* $\nu \geq 0$. *A function* f *given in* (2.1) *belongs to* $TN(\tau, \rho, \nu)$ *if and only if*

$$\sum_{n=2}^{\infty} n[n(1+\nu) - (\rho+\nu)(1+n(2\tau^2-\tau)+(2\tau^2-3\tau))]|a_n| \leq 1-\rho.$$

Proof. The proof steps are similar to those for Theorem 3.1. □

Corollary 3.2. *For the parametric value* $\tau = 0$ *in Theorem 3.2, we derive Theorem 2.2 in [36], which asserts if*

$$\sum_{n=2}^{\infty} n[n(1+\nu) - (\rho+\nu)]|a_n| \leq 1-\rho,$$

then $f \in Sk(\rho, \nu)$.

In the ensuing theorems, Lemma 2.1 is employed, and both the necessary and sufficient conditions for the function $z(2 - \upsilon_p(z))$ given by (2.8) to be categorized within the classes $TN(\tau, \rho, \nu)$ and $TM(\tau, \rho, \nu)$, respectively, are given.

Theorem 3.3. *For* $0 \leq \rho < 1$, $0 \leq \tau < 1$, *and* $\nu \geq 0$, *given the conditions where* $b < 0$ *and* $q > 0$, *it follows that* $z(2 - \upsilon_p(z))$, *as described in (2.8), belongs to* $TN(\tau, \rho, \nu)$ *if and only if*

$$\begin{aligned}&[1-(2\tau^2-\tau)(\rho+\nu)+\nu]\upsilon_p''(1) + [2\nu-\rho-2\tau(\rho+\nu)(4\tau^2-3)+3]\upsilon_p'(1)\\ &+[4\tau(\rho+\nu)(1-\tau)+1-\rho][\upsilon_p(1)-1] \leq (1-\rho)\end{aligned} \tag{3.1}$$

holds true.

Proof. Let $z(2 - \upsilon_p(z)) \in TN(\tau, \rho, \nu)$. Then, using Theorem 3.2, we write

$$\sum_{n=2}^{\infty} \frac{n\left(\frac{-b}{4}\right)^{n-1}[(1+\nu)n - (2\tau^2-3\tau+1+(2\tau^2-\tau)n)(\rho+\nu)]}{(q)_{n-1}(n-1)!} \leq (1-\rho).$$

Since

$$\begin{aligned}&n[(1+\nu)n - (2\tau^2-3\tau+1+(2\tau^2-\tau)n)(\rho+\nu)]\\ &= (n+2)^2[1-(2\tau^2-\tau)(\rho+\nu)+\nu] - (n+2)(\rho+\nu)(2\tau^2-3\tau+1),\end{aligned}$$

it implies that

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{n\left(\frac{-b}{4}\right)^{n-1}[(1+\nu)n - (2\tau^2-3\tau+1+(2\tau^2-\tau)n)(\rho+\nu)]}{(q)_{n-1}(n-1)!} &= \sum_{n=0}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n+1}(n+2)^2[1-(2\tau^2-\tau)(\rho+\nu)+\nu]}{(q)_{n+1}(n+1)!}\\ -\sum_{n=0}^{\infty} \frac{(\rho+\nu)(n+2)\left(\frac{-b}{4}\right)^{n+1}(2\tau^2-3\tau+1)}{(q)_{n+1}(n+1)!}.\end{aligned}$$

Since

$$n+2 = n+1+1, \quad (n+2)^2 = (n+1)^2 + 2(n+1) + 1,$$

and

$$(n+1)! = (n+1)n!,$$

it follows that

$$\sum_{n=2}^{\infty} \frac{n\left(\frac{-b}{4}\right)^{n-1}[(1+\nu)n - (2\tau^2 - 3\tau + 1 + (2\tau^2 - \tau)n)(\rho + \nu)]}{(q)_{n-1}(n-1)!}$$
$$= \sum_{n=0}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n+1}(n+2)^2[1 - (2\tau^2 - \tau)(\rho + \nu) + \nu]}{(q)_{n+1}(n+1)!} - \sum_{n=0}^{\infty} \frac{(\rho+\nu)(n+2)\left(\frac{-b}{4}\right)^{n+1}(2\tau^2 - 3\tau + 1)}{(q)_{n+1}(n+1)!}$$
$$= [1 - (2\tau^2 - \tau)(\rho + \nu) + \nu]\left\{\sum_{n=0}^{\infty} \frac{(n+1)\left(\frac{-b}{4}\right)^{n+1}}{(q)_{n+1}(n)!} + 2\sum_{n=0}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n+1}}{(q)_{n+1}(n)!} + \sum_{n=0}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n+1}}{(q)_{n+1}(n+1)!}\right\},$$

and since

$$\sum_{n=0}^{\infty} \frac{(n+1)\left(\frac{-b}{4}\right)^{n+1}}{(q)_{n+1}(n)!} + 2\sum_{n=0}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n+1}}{(q)_{n+1}(n)!} = \sum_{n=0}^{\infty} \frac{n\left(\frac{-b}{4}\right)^{n+1}}{(q)_{n+1}(n)!} + 3\sum_{n=0}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n+1}}{(q)_{n+1}(n)!},$$

this implies that

$$\sum_{n=2}^{\infty} \frac{n\left(\frac{-b}{4}\right)^{n-1}[(1+\nu)n - (2\tau^2 - 3\tau + 1 + (2\tau^2 - \tau)n)(\rho + \nu)]}{(q)_{n-1}(n-1)!}$$
$$= \sum_{n=0}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n+1}(n+2)^2[1 - (2\tau^2 - \tau)(\rho + \nu) + \nu]}{(q)_{n+1}(n+1)!} - \sum_{n=0}^{\infty} \frac{(\rho+\nu)(n+2)\left(\frac{-b}{4}\right)^{n+1}(2\tau^2 - 3\tau + 1)}{(q)_{n+1}(n+1)!}$$
$$= [1 - (2\tau^2 - \tau)(\rho + \nu) + \nu]\left\{\sum_{n=0}^{\infty} \frac{(n+1)\left(\frac{-b}{4}\right)^{n+1}}{(q)_{n+1}(n)!} + 2\sum_{n=0}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n+1}}{(q)_{n+1}(n)!} + \sum_{n=0}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n+1}}{(q)_{n+1}(n+1)!}\right\}$$
$$= [1 - (2\tau^2 - \tau)(\rho + \nu) + \nu]\left\{\sum_{n=0}^{\infty} \frac{n\left(\frac{-b}{4}\right)^{n+1}}{(q)_{n+1}(n)!} + 3\sum_{n=0}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n+1}}{(q)_{n+1}(n)!} + \sum_{n=0}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n+1}}{(q)_{n+1}(n+1)!}\right\}$$
$$- (2\tau^2 - 3\tau + 1)(\rho + \nu)\left\{\sum_{n=0}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n+1}}{(q)_{n+1}(n)!} + \sum_{n=0}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n+1}}{(q)_{n+1}(n+1)!}\right\} \leq (1-\rho).$$

By employing the factorial identity

$$n! = n(n-1)!$$

and amalgamating analogous terms, we can subsequently reformulate as

$$\sum_{n=0}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n+1}[1 - (2\tau^2 - \tau)(\rho + \nu) + \nu]}{(q)_{n+1}(n-1)!} + \sum_{n=0}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n+1}[3 - 2\tau(4\tau^2 - 3)(\rho + \nu) + 2\nu - \rho]}{(q)_{n+1}n!}$$
$$+ [4\tau(1-\tau)(\rho + \nu) + 1 - \rho]\sum_{n=0}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n+1}}{(q)_{n+1}(n+1)!} \leq (1-\rho).$$

Applying

$$(q)_{n+1} = q(q+1)_n = q(q+1)(q+2)_{n-1},$$

we have

$$\sum_{n=0}^{\infty} \frac{[1-(2\tau^2-\tau)(\rho+\nu)+\nu]\left(\frac{-b}{4}\right)^2\left(\frac{-b}{4}\right)^{n-1}}{q(q+1)(q+2)_{n-1}(n-1)!} + \sum_{n=0}^{\infty} \frac{[3-2\tau(4\tau^2-3)(\rho+\nu)+2\nu-\rho]\left(\frac{-b}{4}\right)\left(\frac{-b}{4}\right)^n}{q(q+1)_n n!}$$
$$+ [4\tau(1-\tau)(\rho+\nu)+1-\rho]\sum_{n=0}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n+1}}{(q)_{n+1}(n+1)!} \le (1-\rho).$$

By using Lemma 2.1,

$$[(1+\nu)-(\rho+\nu)(2\tau^2-\tau)]\upsilon_p''(1) + [3+2\nu-\rho-2\tau(\rho+\nu)(4\tau^2-3)]\upsilon_p'(1)$$
$$+ [1-\rho+4\tau(\rho+\nu)(1-\tau)][\upsilon_p(1)-1] \le (1-\rho).$$

This completes the proof. □

Theorem 3.4. *For $0 \le \rho < 1$, $0 \le \tau < 1$, and $\nu \ge 0$, given the conditions where $b < 0$ and $m > 0$, it follows that $z(2-\upsilon_p(z))$, given by (2.8), belongs to $TM(\tau,\rho,\nu)$ if and only if*

$$[1-(2\tau^2-\tau)(\rho+\nu)+\nu]\upsilon_p'(1) + [1-(4\tau^2-4\tau)(\rho+\nu)-\rho][\upsilon_p(1)-1] \le (1-\rho) \tag{3.2}$$

holds true.

Proof. Let $z(2-\upsilon_p(z)) \in TM(\tau,\rho,\nu)$, and using Theorem 3.1, we write

$$\sum_{n=2}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n-1}[(1+\nu)n-(2\tau^2-3\tau+n(2\tau^2-\tau)+1)(\rho+\nu)]}{(q)_{n-1}(n-1)!} \le (1-\rho).$$

Since

$$[n(1+\nu)-(\rho+\nu)(1+n(2\tau^2-\tau)+(2\tau^2-3\tau)]$$
$$= [[1-(2\tau^2-\tau)(\rho+\nu)+\nu](n-1)+1-(4\tau^2-4\tau)(\rho+\nu)-\rho],$$

we have that

$$\sum_{n=2}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n-1}[[1-(2\tau^2-\tau)(\rho+\nu)+\nu](n-1)+1-(4\tau^2-4\tau)(\rho+\nu)-\rho]}{(q)_{n-1}(n-1)!}$$
$$= \sum_{n=2}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n-1}[1-(2\tau^2-\tau)(\rho+\nu)+\nu](n-1)}{(q)_{n-1}(n-1)!} + \sum_{n=2}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n-1}[1-(4\tau^2-4\tau)(\rho+\nu)-\rho]}{(q)_{n-1}(n-1)!} \le (1-\rho).$$

Then after computation, one can write it as

$$= \sum_{n=1}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n}[1-(2\tau^2-\tau)(\rho+\nu)+\nu]}{(q)_{n}(n-1)!} + \sum_{n=1}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n}[1-(4\tau^2-4\tau)(\rho+\nu)-\rho]}{(q)_{n}(n)!} \le (1-\rho),$$

which implies

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{-b}{4m}\right)\left(\frac{-b}{4}\right)^{n}[1-(2\tau^{2}-\tau)(\rho+\nu)+\nu]}{(m+1)_{n}(n)!} + \sum_{n=0}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n+1}[1-(4\tau^{2}-4\tau)(\rho+\nu)-\rho]}{(q)_{n+1}(n+1)!} \leq (1-\rho).$$

Employing Lemma 2.1, one can reformulate it as

$$= \left(\frac{-b}{4m}\right)[(1+\nu)-(\rho+\nu)(2\tau^{2}-\tau)]\upsilon_{p+1}(1) + [(1-\rho)-(\rho+\nu)(4\tau^{2}-4\tau)][\upsilon_{p}(1)-1]$$

or

$$= [(1+\nu)-(\rho+\nu)(2\tau^{2}-\tau)]\upsilon'_{p}(1) + [(1-\rho)-(\rho+\nu)(4\tau^{2}-4\tau)][\upsilon_{p}(1)-1].$$

It is bounded above by $(1-\rho)$ if and only if (3.2) holds. □

In the subsequent theorems, Lemmas 2.1 and 2.2 are applied to investigate the impact of the Bessel function on the class $TN(\tau, \rho, \nu)$ and to demonstrate an inclusion property for $T(b, q)f(z)$ and $\int_0^z (2 - \upsilon_p(t))dt$ to be included within the class $TN(\tau, \rho, \nu)$.

Theorem 3.5. *For $0 \leq \rho < 1$, $0 \leq \tau < 1$, and $\nu \geq 0$, consider $b < 0$, $q > 0$, and $f \in \mathfrak{R}^{\tau}(A, C)$. If*

$$\begin{aligned}&(A-C)|\tau|\{[(1+\nu)-(\rho+\nu)(2\tau^{2}-\tau)]\upsilon'_{p}(1)\\&+\left[(1-\rho)-(\rho+\nu)(4\tau^{2}-4\tau)\right][\upsilon_{p}(1)-1]\} \leq (1-\rho),\end{aligned} \tag{3.3}$$

then $T(b, q)f(z) \in TN(\tau, \rho, \nu)$.

Proof. If $T(b, q) \in TN(\tau, \rho, \nu)$, and using Theorem 3.2 then

$$\sum_{n=2}^{\infty} \frac{n[(2\tau^{2}-3\tau)-(\rho+\nu)(1+n(2\tau^{2}-\tau)+n(1+\nu))]\left(\frac{-b}{4}\right)^{n-1}}{(q)_{n-1}(n-1)!}|a_{n}| \leq (1-\rho).$$

Let the function f defined by (2.1) be an element of $\mathfrak{R}^{\tau}(A, C)$; consequently, by employing Lemma 2.2, it follows that

$$|a_{n}| \leq (A-C)\frac{|\tau|}{n},$$

which leads to the implication

$$\begin{aligned}&\sum_{n=2}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n-1} n[(2\tau^{2}-3\tau)-(\rho+\nu)(1+n(2\tau^{2}-\tau)+n(1+\nu))]}{(q)_{n-1}(n-1)!}|a_{n}|\\&\leq \sum_{n=2}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n-1} n[(2\tau^{2}-3\tau)-(\rho+\nu)(1+n(2\tau^{2}-\tau)+n(1+\nu))]}{n(q)_{n-1}(n-1)!}(A-C)|\tau| \leq (1-\rho)\\&= \sum_{n=2}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n-1}[(2\tau^{2}-3\tau)-(\rho+\nu)[1+n(2\tau^{2}-\tau)+n(1+\nu)]]}{(q)_{n-1}(n-1)!}(A-C)|\tau| \leq (1-\rho).\end{aligned}$$

Since

$$\begin{aligned}&[(2\tau^2-3\tau)-(\rho+\nu)[1+n(2\tau^2-\tau)+n(1+\nu)]]\\&=(1-\rho)-(\rho+\nu)(4\tau^2-4\tau)+(n-1)[(1+\nu)-(2\tau^2-\tau)(\rho+\nu)],\end{aligned}$$

using this equality, we rewrite as

$$\begin{aligned}&\sum_{n=2}^{\infty}\frac{\left(\frac{-b}{4}\right)^{n-1}[(2\tau^2-3\tau)-(\rho+\nu)[1+n(2\tau^2-\tau)+n(1+\nu)]]}{(q)_{n-1}(n-1)!}(A-C)|\tau|\\&=\sum_{n=2}^{\infty}\frac{\left(\frac{-b}{4}\right)^{n-1}[(1-\rho)-(\rho+\nu)(4\tau^2-4\tau)+(n-1)[(1+\nu)-(2\tau^2-\tau)(\rho+\nu)]]}{(q)_{n-1}(n-1)!}(A-C)|\tau|\\&=(A-C)|\tau|\left\{\sum_{n=1}^{\infty}\frac{n[(1+\nu)-(\rho+\nu)(2\tau^2-\tau)]\left(\frac{-b}{4}\right)^{n}}{(q)_n(n)!}+\sum_{n=1}^{\infty}\frac{(1-\rho)-(\rho+\nu)(4\tau^2-4\tau)\left(\frac{-b}{4}\right)^{n}}{(q)_n(n)!}\right\},\end{aligned}$$

and since

$$n!=n(n-1)!,$$

then it follows that

$$\begin{aligned}&=(A-C)|\tau|\left\{\sum_{n=1}^{\infty}\frac{[(1+\nu)-(\rho+\nu)(2\tau^2-\tau)]\left(\frac{-b}{4}\right)^{n}}{(q)_n(n-1)!}+\sum_{n=1}^{\infty}\frac{(1-\rho)-(\rho+\nu)(4\tau^2-4\tau)\left(\frac{-b}{4}\right)^{n}}{(q)_n(n)!}\right\}\\&=(A-C)|\tau|\left\{\sum_{n=0}^{\infty}\frac{[(1+\nu)-(\rho+\nu)(2\tau^2-\tau)]\left(\frac{-b}{4}\right)^{n+1}}{(q)_{n+1}(n)!}+\sum_{n=1}^{\infty}\frac{(1-\rho)-(\rho+\nu)(4\tau^2-4\tau)\left(\frac{-b}{4}\right)^{n}}{(q)_n(n)!}\right\},\end{aligned}$$

which, with

$$(q)_{n+1}=q(q+1)_n,$$

gives

$$=(A-C)|\tau|\left\{\sum_{n=0}^{\infty}\frac{[(1+\nu)-(\rho+\nu)(2\tau^2-\tau)]\left(\frac{-b}{4}\right)^{1}\left(\frac{-b}{4}\right)^{n}}{q(q+1)_n(n)!}+\sum_{n=1}^{\infty}\frac{(1-\rho)-(\rho+\nu)(4\tau^2-4\tau)\left(\frac{-b}{4}\right)^{n}}{(q)_n(n)!}\right\}.$$

Then, applying

$$\sum_{n=1}^{\infty}\frac{\left(\frac{-b}{4}\right)^{n}}{(q)_n(n)!}=\upsilon_p(1)-1$$

and

$$\sum_{n=0}^{\infty}\frac{\left(\frac{-b}{4}\right)^{n}}{(q+1)_n(n)!}=\upsilon_{p+1}(1),$$

we have

$$=(A-C)|\tau|\{\frac{[(1+\nu)-(\rho+\nu)(2\tau^2-\tau)]\left(\frac{-b}{4}\right)\upsilon_{p+1}(1)}{m}+\Big[(1-\rho)-(\rho+\nu)(4\tau^2-4\tau)\Big][\upsilon_p(1)-1]\}.$$

Employing Lemma 2.1, the expression can be reformulated as

$$\begin{aligned}&= (A - C)|\tau|\{[(1+\nu) - (\rho+\nu)(2\tau^2 - \tau)]\upsilon_p'(1) \\ &+ \left[(1-\rho) - (\rho+\nu)(4\tau^2 - 4\tau)\right][\upsilon_p(1) - 1]\}.\end{aligned}$$

It is bounded above by $(1-\rho)$ if and only if (3.3) holds. □

Theorem 3.6. *For* $0 \le \rho < 1$*,* $0 \le \tau < 1$*, and* $\nu \ge 0$*, given the conditions where* $b < 0$ *and* $m > 0$*, it follows that* $\int_0^z (2 - \upsilon_p(t))dt$ *belongs to* $TN(\tau, \rho, \nu)$ *if and only if*

$$[1 + \nu - (\rho+\nu)(2\tau^2 - \tau)]\upsilon_p'(1) + [1 - \rho - (\rho+\nu)(4\tau^2 - 4\tau)][\upsilon_p(1) - 1] \le (1-\rho). \tag{3.4}$$

Proof. Let

$$\int_0^z (2 - \upsilon_p(t))dt \in TN(\tau, \rho, \nu),$$

where

$$\int_0^z (2 - \upsilon_p(t))dt = z - \sum_{n=2}^{\infty} \frac{\left(\frac{-b}{4}\right)^{n-1} z^n}{(q)_{n-1}(n)!}.$$

Subsequently, by employing Theorem 3.2, this implies

$$\sum_{n=2}^{\infty} n[n(1+\nu) - (\rho+\nu)(1 + n(2\tau^2 - \tau) + (2\tau^2 - 3\tau))] \left[\frac{\left(\frac{-b}{4}\right)^{n-1}}{(q)_{n-1}(n)!}\right] \le 1 - \rho,$$

and applying

$$n! = n(n-1)!,$$

we can rewrite as

$$= \sum_{n=2}^{\infty} [n(1+\nu) - (\rho+\nu)(1 + n(2\tau^2 - \tau) + (2\tau^2 - 3\tau))] \left[\frac{\left(\frac{-b}{4}\right)^{n-1} z^n}{(q)_{n-1}(n-1)!}\right] \le 1 - \rho.$$

And since

$$\begin{aligned}&[n(1+\nu) - (\rho+\nu)(1 + n(2\tau^2 - \tau) + (2\tau^2 - 3\tau)] \\ &= [(n-1)[(1+\nu) - (\rho+\nu)(2\tau^2 - \tau)] + (1-\rho) - (\rho+\nu)(4\tau^2 - 4\tau)],\end{aligned}$$

therefore we have

$$\begin{aligned}&= \sum_{n=2}^{\infty} [(n-1)[(1+\nu) - (\rho+\nu)(2\tau^2 - \tau)] + (1-\rho) - (\rho+\nu)(4\tau^2 - 4\tau)] \left[\frac{\left(\frac{-b}{4}\right)^{n-1}}{(q)_{n-1}(n-1)!}\right] \le 1 - \rho \\ &= \sum_{n=2}^{\infty} \frac{[(n-1)[(1+\nu) - (\rho+\nu)(2\tau^2 - \tau)]](\frac{-b}{4})^{n-1}}{(q)_{n-1}(n-1)!} + \sum_{n=2}^{\infty} \frac{[(1-\rho) - (\rho+\nu)(4\tau^2 - 4\tau)](\frac{-b}{4})^{n-1}}{(q)_{n-1}(n-1)!} \le 1 - \rho\end{aligned}$$

$$= \sum_{n=2}^{\infty} \frac{[(1+\nu)-(\rho+\nu)(2\tau^2-\tau)](\frac{-b}{4})^{n-1}}{(q)_{n-1}(n-2)!} + \sum_{n=2}^{\infty} \frac{[(1-\rho)-(\rho+\nu)(4\tau^2-4\tau)](\frac{-b}{4})^{n-1}}{(q)_{n-1}(n-1)!} \leq 1-\rho$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{-b}{4q}\right)[(1+\nu)-(\rho+\nu)(2\tau^2-\tau)]\left(\frac{-b}{4}\right)^{n}}{(q+1)_n(n)!} + \sum_{n=0}^{\infty} \frac{[(1-\rho)-(\rho+\nu)(4\tau^2-4\tau)]\left(\frac{-b}{4}\right)^{n+1}}{(q)_{n+1}(n+1)!} \leq 1-\rho$$

$$= \left(\frac{-b}{4q}\right)[(1+\nu)-(\rho+\nu)(2\tau^2-\tau)]\upsilon_{p+1}(1) + [(1-\rho)-(\rho+\nu)(4\tau^2-4\tau)][\upsilon_p(1)-1].$$

Utilizing Lemma 2.1, this implies

$$[(1+\nu)-(\rho+\nu)(2\tau^2-\tau)]\upsilon_p'(1) + [(1-\rho)-(\rho+\nu)(4\tau^2-4\tau)][\upsilon_p(1)-1] \leq 1-\rho.$$

This completes the proof. □

In the subsequent theorems, the coefficient restriction proved in Theorem 3.2 is applied to investigate the impact of the hypergeometric function on the class $TN(\tau,\rho,\nu)$ and to substantiate an inclusion property for $zF(a,b;c;z)$ and $z(2-F(z))$ to be contained within the class $TN(\tau,\rho,\nu)$.

Theorem 3.7. *For* $0 \leq \rho < 1$, $0 \leq \tau < 1$, *and* $\nu \geq 0$, *if* $i, b > -1$, $ib < 0$, *and* $d > i+b+2$, *then* $zF(z)$ *belongs to* $TN(\tau,\rho,\nu)$ *if and only if*

$$\Big[(i)_2[(1+\nu)-(2\tau^2-\tau)(\rho+\nu)](b)_2 + (d-i-b-2)_2[1-\rho-(4\tau^2-4\tau)(\rho+\nu)] + ib(d-i-b-2)[-\rho-2(4\tau^2-3)\tau(\rho+\nu)+3+2\nu]\Big] \geq 0.$$

Proof. Let $zF(z) \in TN(\tau,\rho,\nu)$, where

$$zF(z) = z + \frac{ib}{d}\sum_{n=2}^{\infty}\frac{(i+1)_{n-2}(b+1)_{n-2}}{(d+1)_{n-2}(1)_{n-1}}z^n = z - \left|\frac{ib}{d}\right|\sum_{n=2}^{\infty}\frac{(i+1)_{n-2}(b+1)_{n-2}}{(d+1)_{n-2}(1)_{n-1}}z^n.$$

Then, by employing Theorem 3.2, we get

$$\sum_{n=2}^{\infty}\frac{n[n(1+\nu)-(\rho+\nu)(1+n(2\tau^2-\tau)+(2\tau^2-3\tau))](i+1)_{n-2}(b+1)_{n-2}}{(d+1)_{n-2}(1)_{n-1}} \leq \left|\frac{d}{ib}\right|(1-\rho).$$

Since

$$\begin{aligned} & n[n(1+\nu)-(\rho+\nu)(1+n(2\tau^2-\tau)+(2\tau^2-3\tau))] \\ &= (n+2)^2[1-(2\tau^2-\tau)(\rho+\nu)+\nu] - (n+2)(\rho+\nu)(2\tau^2-3\tau+1)', \end{aligned}$$

therefore we have

$$\sum_{n=2}^{\infty}\frac{(n+2)^2[1-(2\tau^2-\tau)(\rho+\nu)+\nu]-(n+2)(\rho+\nu)(2\tau^2-3\tau+1)(i+1)_{n-2}(b+1)_{n-2}}{(d+1)_{n-2}(1)_{n-1}} \leq \left|\frac{d}{ib}\right|(1-\rho)$$

$$= \sum_{0}^{\infty}\frac{(i+1)_n(n+2)^2(b+1)_n[-(2\tau^2-\tau)(\rho+\nu)+(1+\nu)]}{(d+1)_n(1)_{n+1}} - \frac{(n+2)(i+1)_n(b+1)_n(2\tau^2-3\tau+1)(\rho+\nu)}{(d+1)_n(1)_{n+1}} \leq \left|\frac{d}{ib}\right|(1-\rho).$$

And since

$$n+2=n+1+1, \quad (n+2)^2=(n+1)^2+2(n+1)+1,$$

and

$$(1)_{n+1}=(n+1)n!=(n+1)(1)_n,$$

therefore we obtain

$$= \Big[\sum_0^\infty \frac{(n+1)(i+1)_n(b+1)_n}{(d+1)_n(1)_n} + 2\sum_0^\infty \frac{(i+1)_n(b+1)_n}{(d+1)_n(1)_n} + \sum_0^\infty \frac{(i+1)_n(b+1)_n}{(d+1)_n(1)_{n+1}}\Big][1-(2\tau^2-\tau)(\rho+\nu)+\nu)]$$

$$-(2\tau^2-3\tau+1)(\rho+\nu)[\sum_0^\infty \frac{(i+1)_n(b+1)_n}{(d+1)_n(1)_n} + \sum_0^\infty \frac{(i+1)_n(b+1)_n}{(d+1)_n(1)_{n+1}}] \le \left|\frac{d}{ib}\right|(1-\rho)$$

$$= \Big[\sum_0^\infty \frac{(n)(i+1)_n(b+1)_n}{(d+1)_n(1)_n} + 3\sum_0^\infty \frac{(i+1)_n(b+1)_n}{(d+1)_n(1)_n} + \sum_0^\infty \frac{(i+1)_n(b+1)_n}{(d+1)_n(1)_{n+1}}\Big][1-(2\tau^2-\tau)(\rho+\nu)+\nu)]$$

$$-(2\tau^2-3\tau+1)(\rho+\nu)[\sum_0^\infty \frac{(i+1)_n(b+1)_n}{(d+1)_n(1)_n} + \sum_0^\infty \frac{(i+1)_n(b+1)_n}{(d+1)_n(1)_{n+1}}] \le \left|\frac{d}{ib}\right|(1-\rho),$$

$$= \sum_0^\infty \frac{[-(2\tau^2-\tau)(\rho+\nu)+(1+\nu)]n(b+1)_n(i+1)_n}{(d+1)_n(1)_n} + \sum_0^\infty \frac{[-2(4\tau^2-3)\tau(\rho+\nu)+3+2\nu-\rho](b+1)_n(i+1)_n}{(d+1)_n(1)_n}$$

$$+[4\tau(1-\tau)(\rho+\nu)+1-\rho]\sum_{n=1}^\infty \frac{(i+1)_{n-1}(b+1)_{n-1}}{(d+1)_{n-1}(1)_n} \le \left|\frac{d}{ib}\right|(1-\rho).$$

Using the properties

$$(1)_n = n(1)_{n-1}$$

and

$$(a+1)_n=(a+1)(a+2)_{n-1},$$

then we get

$$= \sum_0^\infty \frac{[-(2\tau^2-\tau)(\rho+\nu)+(1+\nu)](b+2)_{n-2}(i+2)_{n-2}(i+1)(b+1)}{(d+2)_{n-2}(1)_{n-1}(d+1)}$$

$$+\sum_0^\infty \frac{[-2(4\tau^2-3)\tau(\rho+\nu)+3+2\nu-\rho](b+1)_n(i+1)_n}{(d+1)_n(1)_n}$$

$$+c[-(4\tau^2-4\tau)(\rho+\nu)-1-\rho]\sum_1^\infty \frac{(b)_n(i)_n}{ib(d)_n(1)_n} \le \left|\frac{d}{ib}\right|(1-\rho).$$

Applying

$$\Gamma(i+2)=(i+1)\Gamma(i+1)$$

and

$$\Gamma(d-i-b-1)=(d-i-b-2)\Gamma(d-i-b-2),$$

we obtain

$$= \frac{\Gamma(d+1)\Gamma(d-i-b-2)}{\Gamma(d-i)\Gamma(d-b)}\Big[[(1+\nu)-(\rho+\nu)(2\tau^2-\tau)](i+1)(b+1)$$

$$+\frac{(d-i-b-2)[-2(4\tau^2-3)\tau(\rho+\nu)+3+2\nu-\rho]}{} + \frac{(d-i-b-2)_2[-(4\tau^2-4\tau)(\rho+\nu)+1-\rho]}{ib}\Bigg]$$

$$\leq (1-\rho)\Big[\left|\frac{d}{ib}\right| + \frac{d}{ib}\Big] - \frac{(4\tau^2-4\tau)(\rho+\nu)d}{ib} \leq 0.$$

It is imperative to observe that the condition $\tau < 1$, in conjunction with $ib < 0$, leads to the conclusion that

$$\frac{-(4\tau^2-4\tau)d(\rho+\nu)}{ib} < 0.$$

Ultimately, upon the multiplication by $ib < 0$, we arrive at

$$\Big[[(1+\nu)-(\rho+\nu)(2\tau^2-\tau)](i)_2(b)_2 + [1-\rho-(\rho+\nu)(4\tau^2-4\tau)](d-i-b-2)_2$$
$$+ ib[3+2\nu-\rho-2\tau(\rho+\nu)(4\tau^2-3)](d-i-b-2)\Big] \geq 0.$$

This completes the proof. □

Theorem 3.8. *For* $0 \leq \rho < 1$, $0 \leq \tau < 1$, *and* $\nu \geq 0$, *if* $d > i+b+1$ *and* $i, b > 0$, *then* $(2z - zF(z))$ *belongs to the class* $TN(\tau,\rho,\nu)$ *if and only if*

$$\frac{\Gamma(d-i-b)\Gamma(d)}{\Gamma(d-b)\Gamma(d-a)}\left[\frac{[3-2\tau(\rho+\nu)(4\tau^2-3)+2\nu-\rho]ib}{(1-\rho)(d-i-b-1)} + \frac{(i)_2[(1+\nu)-(2\tau^2-\tau)(\rho+\nu)](b)_2}{(1-\rho)(d-i-b-2)_2} + 1\right] \leq 2.$$

Proof. Let $z(2-F(z)) \in TN(\tau,\rho,\nu)$. Then, using Theorem 3.2, we write

$$\sum_{2}^{\infty} \frac{n(i)_{n-1}(b)_{n-1}[(1+\nu)n-(1+n(2\tau^2-\tau)+(2\tau^2-3\tau))(\rho+\nu)]}{(d)_{n-1}(1)_{n-1}} \leq (1-\rho).$$

Since

$$\begin{aligned}&[(1+\nu)n-((2\tau^2-3\tau)+1+(2\tau^2-\tau)n)(\rho+\nu)]n\\ &=[(1+\nu)-(2\tau^2-\tau)(\rho+\nu)](n+2)^2-(1+2\tau^2-3\tau)(n+2)(\rho+\nu),\end{aligned}$$

then we get

$$\begin{aligned}&\sum_{2}^{\infty} \frac{n(i)_{n-1}(b)_{n-1}[(1+\nu)n-(1+n(2\tau^2-\tau)+(2\tau^2-3\tau))(\rho+\nu)]}{(d)_{n-1}(1)_{n-1}}\\ &=\sum_{2}^{\infty} \frac{(i)_{n-1}(b)_{n-1}[[(1+\nu)-(2\tau^2-\tau)(\rho+\nu)](n+2)^2-(1+2\tau^2-3\tau)(n+2)(\rho+\nu)]}{(d)_{n-1}(1)_{n-1}},\end{aligned}$$

which implies

$$=\sum_{0}^{\infty} \frac{(i)_{n+1}(n+2)^2[(1+\nu)-(2\tau^2-\tau)(\rho+\nu)](b)_{n+1}}{(1)_{n+1}(d)_{n+1}} - \sum_{0}^{\infty} \frac{(i)_{n+1}(n+2)(1+2\tau^2-3\tau)(\rho+\nu)(b)_{n+1}}{(d)_{n+1}(1)_{n+1}} \leq (1-\rho).$$

And because

$$n+2 = n+1+1$$

and

$$(n+2)^2 = (n+1)^2 + 2(n+1) + 1,$$

hence we have

$$= [1 + \nu - (2\tau^2 - \tau)(\rho + \nu)]\Big[\sum_0^\infty \frac{(i)_{n+1}(b)_{n+1}(n+1)}{(1)_n(d)_{n+1}} + 2\sum_0^\infty \frac{(b)_{n+1}(i)_{n+1}}{(1)_n(d)_{n+1}} + \sum_0^\infty \frac{(b)_{n+1}(i)_{n+1}}{(1)_{n+1}(d)_{n+1}}\Big]$$
$$- (1 - 3\tau + 2\tau^2)(\rho + \nu)\Big[\sum_0^\infty \frac{(b)_{n+1}(i)_{n+1}}{(1)_n(d)_{n+1}} + \sum_0^\infty \frac{(b)_{n+1}(i)_{n+1}}{(1)_{n+1}(d)_{n+1}}\Big] \leq (1 - \rho)$$
$$= [1 + \nu - (2\tau^2 - \tau)(\rho + \nu)]\Big[\sum_0^\infty \frac{(i)_{n+1}(b)_{n+1}(n)}{(1)_n(d)_{n+1}} + 3\sum_0^\infty \frac{(b)_{n+1}(i)_{n+1}}{(1)_n(d)_{n+1}} + \sum_0^\infty \frac{(b)_{n+1}(i)_{n+1}}{(1)_{n+1}(d)_{n+1}}\Big]$$
$$- (1 - 3\tau + 2\tau^2)(\rho + \nu)\Big[\sum_0^\infty \frac{(b)_{n+1}(i)_{n+1}}{(1)_n(d)_{n+1}} + \sum_0^\infty \frac{(b)_{n+1}(i)_{n+1}}{(1)_{n+1}(d)_{n+1}}\Big] \leq (1 - \rho)$$
$$= \sum_0^\infty \frac{(i)_{n+2}[(1+\nu) - (2\tau^2 - \tau)(\rho + \nu)](b)_{n+2}}{(1)_n(d)_{n+2}} + \sum_0^\infty \frac{(i)_{n+1}[-\rho - 2\tau(\rho + \nu)(4\tau^2 - 3) + 3 + 2\nu](b)_{n+1}}{(1)_n(d)_{n+1}}$$
$$+ [1 - \rho + 4\tau(\rho + \nu)(1 - \tau)]\sum_{n=1}^\infty \frac{(i)_n(b)_n}{(d)_n(1)_n} \leq (1 - \rho).$$

Finally, using

$$(1)_n = n(1)_{n-1}$$

and

$$(b)_{n+1} = b(b+1)_n = b(b+1)(b+2)_{n-1},$$

and by simplifying, we get

$$\frac{\Gamma(d-i-b)\Gamma(d)}{\Gamma(d-b)\Gamma(d-a)}\left[\frac{[3 - 2\tau(\rho+\nu)(4\tau^2 - 3) + 2\nu - \rho]ib}{(1-\rho)(d-i-b-1)} + \frac{(i)_2[(1+\nu) - (2\tau^2 - \tau)(\rho+\nu)](b)_2}{(1-\rho)(d-i-b-2)_2} + 1\right] \leq 2,$$

hence proved. □

4. Conclusions

The article presents generalized subclasses of ν uniformly starlike and convex functions, characterized by the order ρ, and offers coefficient bounds along with implications derived from Bessel and hypergeometric functions. In Section 2, following the introduction of the newly defined classes given by Definitions 2.1 and 2.2, it is shown that various subclasses of analytic functions investigated in previous works represent particular instances of these generalized subclasses. Coefficient bounds are obtained such that certain analytic functions reside within the newly established subclasses of uniformly starlike and uniformly convex families. Furthermore, we have evaluated the impact of the Bessel function and analyzed the implications of the hypergeometric function on these classes to validate an inclusion property for analytic functions included within these subclasses.

We have stated that these innovative subclasses of starlike and convex functions represent a generalized form of the classes discussed in the research articles [26, 36, 37]. By specializing the parameters ρ, τ, and ν, we can highlight various intriguing results, as provided in the aforementioned references. The connection of the present results with those already established in [38–41] is also highlighted.

For future research, these subclasses can be extended to encompass multivalent and meromorphic analytic functions, allowing us to explore properties related to coefficient bounds, growth and distortion theorems, as well as subordination and superordination properties, among others.

Furthermore, the results presented here can be refined by deriving properties for the coefficients of the functions within the classes examined and by exploring additional coefficient-related issues concerning them.

Author contributions

Muhammad Imran Faisal: conceptualization, methodology, validation, formal analysis, investigation, writing–original draft preparation, writing–review and editing; Maslina Darus: conceptualization, methodology, validation, formal analysis, investigation, writing–original draft preparation, writing–review and editing, supervision, Georgia Irina Oros: conceptualization, methodology, validation, formal analysis, investigation, writing–review and editing, project administration, funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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