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Research article

Subalgebra lattices of algebras determined by two unary relations and an equivalence relation

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Abstract: In this paper, we show the subalgebra lattice of an algebra determined by two nontrivial unary relations and a nontrivial equivalence relation on a finite set. Moreover, we use an algorithm to indicate isomorphic covering graphs of subalgebra lattices. This gives us a lower bound of the number of categorically equivalent classes of clones on a given finite set.

Keywords: lattices; subalgebras; categorical equivalence of clones; isomorphic covering graphs
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1. Introduction

An algebra is a mathematical structure $\underline{A} := (A; F^A)$ consisting of a nonempty set A and a set F^A of operations on A. Many researchers study its properties and apply it in various areas, especially in computer science and engineering such as language theory, data structures and routing algorithms [1,6].

Given an algebra $\underline{A} := (A; F^A)$, the set A is called the *universe* and an element in F^A is called a *fundamental operation* of the algebra. For a nonempty subset S of A, we say that S is a *subuniverse* of an algebra $\underline{A}$ if S is closed under all fundamental operations of $\underline{A}$. Strictly speaking, a *subalgebra* is the algebra $\underline{S} = (S; F^{\underline{S}})$ formed by restricting all fundamental operations of $\underline{A}$ to S. Let $\mathrm{Sub}(\underline{A})$ be the set of all subuniverses of $\underline{A}$ together with the empty set. It is well known that $\mathrm{Sub}(\underline{A})$ is always closed under the intersection, but not always closed under the union. Consequently, $\mathrm{Sub}(\underline{A})$ forms a lattice under set inclusion, with $X \wedge Y = X \cap Y$ and $X \vee Y$ is the smallest subalgebra containing $X \cup Y$ for all $X, Y \in \mathrm{Sub}(\underline{A})$. This lattice is called the *subalgebra lattice* of $\underline{A}$ (see [3] for more details). Term operations of the algebra are operations defined inductively using fundamental operations of the algebra (see [3]). The set of all term operations of $\underline{A}$ is denoted by $T(\underline{A})$.

Let A be a nonempty set and O_A be the set of all operations on A. For each natural number n, let $O_A^{(n)}$ be the set of all n-ary operations on A. Then $O_A = \bigcup_{n\in\mathbb{N}} O_A^{(n)}$. A subset C_A of O_A is called a *clone* on A

if it contains all projections $pr_j^k : A^k \to A : (x_1, \ldots, x_k) \mapsto x_j$ and is closed under the superposition in the sense that, for $f \in O_A^{(n)} \cap C_A$ and $g_1, \ldots, g_n \in O_A^{(k)} \cap C_A$, the k-ary operation $f(g_1, \ldots, g_n) : A^k \to A$, defined by

$$f(g_1, \ldots, g_n)(x_1, \ldots, x_k) = f(g_1(x_1, \ldots, x_k), \ldots, g_n(x_1, \ldots, x_k)),$$

for all $(x_1, \ldots, x_k) \in A^k$, is also in C_A. It is well known that $T(\underline{A})$ is a clone on A and it is called a *clone of term operations* of $\underline{A}$.

In clone theory, characterizing clones on an arbitrary nonempty finite set A is an interesting problem. Researchers have used various methods to tackle the challenge of classifying these structures. When A contains only one element, there is only one possible clone. The case where A has exactly two elements was fully gathered by E. L. Post [13], who provided a complete structure of all clones on such a set. However, determining the structure of all clones when A has three or more elements remains an unsolved problem (see [10] for more details).

To further understand how clones are formed, it's crucial to consider the concept of preservation. For an n-ary operation f and an m-ary relation θ on A, we say that f *preserves* θ, written as $f \triangleright \theta$, if

$$(f(a_{11}, a_{21}, \ldots, a_{n1}), f(a_{12}, a_{22}, \ldots, a_{n2}), \ldots, f(a_{1m}, a_{2m}, \ldots, a_{nm})) \in \theta$$

for all $(a_{11}, a_{12}, \ldots, a_{1m}), (a_{21}, a_{22}, \ldots, a_{2m}), \ldots, (a_{n1}, a_{n2}, \ldots, a_{nm}) \in \theta$. For any set Q of relations on a set A, we denote $\mathrm{Pol}_A Q$ the set of all operations that preserve all relations in the set Q. A fundamental result in clone theory establishes that $\mathrm{Pol}_A Q$ always forms a clone for any set Q of relations on A [9].

This result enabled I. G. Rosenberg [14] to contribute a significant categorization in 1970. He classified all maximal clones on a finite set A into six classes based on six different types of relations defined on A. These types of relations are a partial order with least and greatest elements, the graph of a permutation of prime order, a non-trivial equivalence relation, a prime-affine relation, a central relation, and a k-regularly generated relation. Later in 2000, K. Denecke and O. Lüder [5] approached the study of clones from a categorical viewpoint. Their work focused on determining when two clones are equivalent as categories. This involves examining structure-preserving mappings, known as functors, between the categories of algebras associated with each clone, and the existence of natural isomorphisms between these functors. This categorical equivalence provides a distinct way to understand the fundamental similarities between clones, beyond simply looking at the operations they contain. Regarding Rosenberg's central relations, Denecke and Lüder focused on how different types of preserved relations impact the categorical structure of the resulting clones. They refined the categorization of central relations into three specific groups (singleton unary, non-singleton unary, and non-unary central).

Two clones C_A and C_B defined on the sets A and B, respectively, are considered categorically equivalent under two specific conditions. Firstly, C_A must represent the set of all term operations of some algebra $\underline{A}$, and similarly, C_B must be the set of all term operations of some algebra $\underline{B}$. Secondly, there exists an equivalence functor between the variety generated by $\underline{A}$ and the variety generated by $\underline{B}$, with this functor mapping $\underline{A}$ to $\underline{B}$. Furthermore, a key result by Davey and Werner [4] demonstrates that an equivalence functor between two varieties preserves the structure of subalgebra lattices; specifically, for any algebra $\underline{A}$ in a variety $\mathcal{V}$ and an equivalence functor F mapping to a variety $\mathcal{W}$, the subalgebra lattice of $\underline{A}$ is isomorphic to the subalgebra lattice of $F(\underline{A})$. Consequently, if the subalgebra lattices of the algebras $\underline{A}$ and $\underline{B}$ (whose clones of term operations are C_A and C_B, respectively) are not isomorphic, then C_A and C_B cannot be categorically equivalent. This shows the significant role that the study of subalgebra lattices can play in determining the categorical equivalence of clones. Ongoing research in the field leverages categorical equivalence as a principal methodology for the classification of clones, particularly in contexts involving [5, 11, 12, 15].

Building upon these fundamental ideas, W. Supaporn [15] in 2014 investigated and characterized the clones on a finite set A in the following forms:

- $\mathrm{Pol}_A\{B, C\}$ where B and C are nontrivial unary relations on A,
- $\mathrm{Pol}_A\{B, \theta\}$ where B is a nontrivial unary relation and θ is a nontrivial equivalence relation on A.

In his work, the subalgebra lattices play an important role in the characterization.

In this work, we are interested in determining the subalgebra lattices of the algebra $\underline{A}$ whose clone of term operations is $\mathrm{Pol}_A\{B, C, \theta\}$ where B and C are nontrivial unary relations and θ is a nontrivial equivalence relation on the set A. This result will be beneficial in characterizing some particular subclones of maximal clones on A by using the same technique found in [15].

2. Main results

In this section, we show all elements in $\mathrm{Sub}(\underline{A})$ for a given algebra $\underline{A}$. From now on, let A be a finite set and let B and C be nontrivial unary relations on A, i.e., they are nonempty proper subsets of A. Additionally, let θ be a nontrivial equivalence relation on A. For each $a \in A$, we denote $\{x \in A \mid (x, a) \in \theta\}$ by $[a]_\theta$ and call it an *equivalence class* of a. It is well known that the set of all equivalence classes determined by a given equivalence relation forms a partition of A. Furthermore, for any nonempty subset S of A, we denote $[S]_\theta := \bigcup_{x \in S}[x]_\theta$. Let $\underline{A}$ be the algebra with the universe A and the clone of term operations $\mathrm{Pol}_A\{B, C, \theta\}$.

Lemma 1. *Let f be an n-ary operation on A and θ be an equivalence relation on A. Then f preserves θ if and only if*

$$f([a_1]_\theta \times \cdots \times [a_n]_\theta) \subseteq [f(a_1, \ldots, a_n)]_\theta$$

for all $a_1, \ldots, a_n \in A$.

Proof. Assume that $f \triangleright \theta$. Let $a_1, \ldots, a_n \in A$ and let $x \in f([a_1]_\theta \times \cdots \times [a_n]_\theta)$. Then $x = f(b_1, \ldots, b_n)$ for some $(b_1, \ldots, b_n) \in [a_1]_\theta \times \cdots \times [a_n]_\theta$. This implies that $(a_i, b_i) \in \theta$ for all $1 \leq i \leq n$. Thus $x = f(b_1, \ldots, b_n) \in [f(a_1, \ldots, a_n)]_\theta$ because $f \triangleright \theta$.

Next, assume that, for all $a_1, \ldots, a_n \in A$,

$$f([a_1]_\theta \times \cdots \times [a_n]_\theta) \subseteq [f(a_1, \ldots, a_n)]_\theta.$$

Let $(b_i, c_i) \in \theta$ for all $1 \leq i \leq n$. Then $(c_1, \ldots, c_n) \in [b_1]_\theta \times \cdots \times [b_n]_\theta$. Hence $f(c_1, \ldots, c_n) \in f([b_1]_\theta \times \cdots \times [b_n]_\theta) \subseteq [f(b_1, \ldots, b_n)]_\theta$. It follows that $(f(b_1, \ldots, b_n), f(c_1, \ldots, c_n)) \in \theta$, and so $f \triangleright \theta$. □

Roughly speaking, an operation f preserves an equivalence relation θ if and only if f maps a product of equivalence classes into an equivalence class. The following lemma will show that

$$\left\{\bigcap \mathcal{S} | \mathcal{S} \subseteq \{\emptyset, A, B, C, [B]_\theta, [C]_\theta, [B \cap C]_\theta\}\right\} \subseteq \mathrm{Sub}(\underline{A}).$$

Lemma 2. *If $\mathcal{S} \subseteq \{\emptyset, A, B, C, [B]_\theta, [C]_\theta, [B \cap C]_\theta\}$, then $\bigcap \mathcal{S} \in \mathrm{Sub}(\underline{A})$.*

Proof. As Sub($\underline{A}$) is closed under intersection, it suffices to show that

$$\{\emptyset, A, B, C, [B]_\theta, [C]_\theta, [B \cap C]_\theta\} \subseteq \text{Sub}(\underline{A}).$$

Obviously, $\{\emptyset, A, B, C\} \subseteq \text{Sub}(\underline{A})$. First, we will show that if S is a nonempty proper subset of A and $f \in \text{Pol}_A\{B, C, \theta\}$ such that $f \triangleright S$, then $f \triangleright [S]_\theta$. Let S be a nonempty proper subset of A. For $n \in \mathbb{N}$, let $f \in \text{Pol}_A^{(n)}\{B, C, \theta\}$. Given $s_1, \dots, s_n \in [S]_\theta$ There exist $x_1, \dots, x_n \in S$ such that $(s_i, x_i) \in \theta$ for all $1 \leq i \leq n$. Then $f(x_1, \dots, x_n) \in S$ because $f \triangleright S$. Moreover, $(f(s_1, \dots, s_n), f(x_1, \dots, x_n)) \in \theta$ because $f \triangleright \theta$. Thus $f(s_1, \dots, s_n) \in [S]_\theta$, and so $f \triangleright [S]_\theta$. Then we can conclude that $f \triangleright [B]_\theta$, $f \triangleright [C]_\theta$ and $f \triangleright [B \cap C]_\theta$. Therefore

$$\{\bigcap \mathcal{S} | \mathcal{S} \subseteq \{\emptyset, A, B, C, [B]_\theta, [C]_\theta, [B \cap C]_\theta\}\} \subseteq \text{Sub}(\underline{A}).$$

□

Next, we will show that any element in Sub($\underline{A}$) will be in the form $\bigcap \mathcal{S}$ for some

$$\mathcal{S} \subseteq \{\emptyset, A, B, C, [B]_\theta, [C]_\theta, [B \cap C]_\theta\}.$$

Lemma 3. *For each $S \in \text{Sub}(\underline{A})$, if $S \subsetneq A$, then $S \subseteq [B]_\theta$ or $S \subseteq [C]_\theta$.*

Proof. Let $S \in \text{Sub}(\underline{A})$ be such that $S \subsetneq A$. Then there exists $a \in A \setminus S$. Suppose, on the contrary, that $S \nsubseteq [B]_\theta$ and $S \nsubseteq [C]_\theta$. Then there are $s_1 \in S \setminus [B]_\theta$ and $s_2 \in S \setminus [C]_\theta$. Let $f : A^2 \to A$ be defined by

$$f(x, y) = \begin{cases} a, & (x, y) \in [s_1]_\theta \times [s_2]_\theta, \\ x, & \text{otherwise.} \end{cases}$$

Since $s_1 \notin [B]_\theta$, we have $[s_1]_\theta \cap B = \emptyset$, and so $([s_1]_\theta \times [s_2]_\theta) \cap B^2 = \emptyset$. Similarly, $([s_1]_\theta \times [s_2]_\theta) \cap C^2 = \emptyset$. Thus $f(B^2) \subseteq B$ and $f(C^2) \subseteq C$, i.e., $f \triangleright B$ and $f \triangleright C$. Moreover, $f([s_1]_\theta \times [s_2]_\theta) = \{a\} \subseteq [a]_\theta = [f(s_1, s_2)]_\theta$ and $f([x]_\theta \times [y]_\theta) = [x]_\theta = [f(x, y)]_\theta$ for all $(x, y) \in A^2$ such that $(x, y) \notin [s_1]_\theta \times [s_2]_\theta$. Then Lemma 1 implies that $f \triangleright \theta$. Hence $f \in \text{Pol}_A\{B, C, \theta\}$. As $f(s_1, s_2) = a \notin S$, we have that $f(S \times S) \nsubseteq S$, and so $S \notin \text{Sub}(\underline{A})$ which is a contradiction. Therefore $S \subseteq [B]_\theta$ or $S \subseteq [C]_\theta$. □

Lemma 4. *For each $S \in \text{Sub}(\underline{A})$, if $S \subsetneq [B]_\theta$, then $S \subseteq B$ or $S \subseteq [B]_\theta \cap [C]_\theta$.*

Proof. Let $S \in \text{Sub}(\underline{A})$ be such that $S \subsetneq [B]_\theta$. Then there exists $d \in [B]_\theta \setminus S$. Suppose, on the contrary, that $S \nsubseteq B$ and $S \nsubseteq [B]_\theta \cap [C]_\theta$. Then there exists $s_1 \in S \setminus B$ and $s_2 \in S \setminus ([B]_\theta \cap [C]_\theta)$. Since $d \in [B]_\theta$, there exists $b \in B$ such that $d \in [b]_\theta$. Let $f : A^2 \to A$ be defined by

$$f(x, y) = \begin{cases} b, & x \in ([s_1]_\theta \times [s_2]_\theta) \cap B^2, \\ d, & x \in ([s_1]_\theta \times [s_2]_\theta) \setminus B^2, \\ x, & \text{otherwise.} \end{cases}$$

Lemma 1 implies that $f \triangleright \theta$ because $d \in [b]_\theta$. Since $f(x, y) \in \{x, b\} \subseteq B$ for all $(x, y) \in B^2$, we have that $f(B^2) \subseteq B$, and so $f \triangleright B$. Since $s_2 \in [B]_\theta$ and $s_2 \notin [B]_\theta \cap [C]_\theta$, we have $s_2 \notin [C]_\theta$. This implies that $[s_2]_\theta \cap C = \emptyset$. Thus $f(x, y) = x \in C$ for all $(x, y) \in C^2$, and so $f(C^2) = C$. It follows that $f \triangleright C$. Hence $f \in \text{Pol}_A\{B, C, \theta\}$. Thus $f(S) \nsubseteq S$ because $f(s_1, s_2) = d \notin S$. This implies that $S \notin \text{Sub}(\underline{A})$ contradicting to the assumption. Therefore $S \subseteq B$ or $S \subseteq [B]_\theta \cap [C]_\theta$. □

Lemma 5. *For each $S \in \mathrm{Sub}(\underline{A})$, if $S \subsetneq B$, then $S \subseteq B \cap [C]_\theta$.*

Proof. Let $S \in \mathrm{Sub}(\underline{A})$ be such that $S \subsetneq B$. Then there exists $b \in B \setminus S$. Suppose, on the contrary, that $S \nsubseteq B \cap [C]_\theta$. Then there exists $s \in S \setminus (B \cap [C]_\theta)$. Let $f : A \to A$ be defined by

$$f(x) = \begin{cases} b, & x \in [s]_\theta, \\ x, & \text{otherwise.} \end{cases}$$

Then Lemma 1 implies that $f \triangleright \theta$. Since $f(x) \in \{x, b\} \subseteq B$ for all $x \in B$, we have that $f(B) \subseteq B$, and so $f \triangleright B$. Since $s \notin [C]_\theta$, this implies that $[s]_\theta \cap [C]_\theta = \emptyset$, and so $f(x) = x$ for all $x \in C$. It follows that $f \triangleright C$. Hence $f \in \mathrm{Pol}_A\{B, C, \theta\}$. As $f(s) = b \notin S$, we have that $f(S) \nsubseteq S$. Then $S \notin \mathrm{Sub}(\underline{A})$ contradicting with the assumption. Therefore $S \subseteq B \cap [C]_\theta$. □

Lemma 6. *For each $S \in \mathrm{Sub}(\underline{A})$, if $S \subsetneq [B]_\theta \cap [C]_\theta$, then $S \subseteq B \cap [C]_\theta$, or $S \subseteq C \cap [B]_\theta$, or $S \subseteq [B \cap C]_\theta$.*

Proof. Let $S \in \mathrm{Sub}(\underline{A})$ be such that $S \subsetneq [B]_\theta \cap [C]_\theta$. Then there exists $a \in ([B]_\theta \cap [C]_\theta) \setminus S$. Suppose, on the contrary, that $S \nsubseteq B \cap [C]_\theta$, $S \nsubseteq C \cap [B]_\theta$ and $S \nsubseteq [B \cap C]_\theta$. Then there are $s_1 \in S \setminus (B \cap [C]_\theta)$, $s_2 \in S \setminus (C \cap [B]_\theta)$ and $s_3 \in S \setminus [B \cap C]_\theta$. Let $b \in B$ and $b' \in C$ be such that $b, b' \in [a]_\theta$. Let $f : A^3 \to A$ be defined by

$$f(x, y, z) = \begin{cases} a, & (x, y, z) \in ([s_1]_\theta \times [s_2]_\theta \times [s_3]_\theta) \setminus (B^3 \cup C^3), \\ b, & (x, y, z) \in ([s_1]_\theta \times [s_2]_\theta \times [s_3]_\theta) \cap B^3, \\ b', & (x, y, z) \in ([s_1]_\theta \times [s_2]_\theta \times [s_3]_\theta) \cap C^3\ , \\ x, & \text{otherwise.} \end{cases}$$

Since $s_3 \notin [B \cap C]_\theta$, we have that $[s_3]_\theta \cap B \cap C = \emptyset$. Then $([s_1]_\theta \times [s_2]_\theta \times [s_3]_\theta) \cap B^3 \cap C^3 = \emptyset$. Thus f is well-defined. Since $f(x, y, z) \in \{x, b\}$ for all $(x, y, z) \in B^3$, we have that $f(B^3) \subseteq B$, and so $f \triangleright B$. Similarly, $f \triangleright C$. Moreover, Lemma 1 implies that $f \triangleright \theta$ because $b, b' \in [a]_\theta$. These imply that $f \in \mathrm{Pol}_A\{B, C, \theta\}$. Since $s_1 \in S \subsetneq [B]_\theta \cap [C]_\theta \subseteq [C]_\theta$ and $s_1 \notin B \cap [C]_\theta$, we have $s_1 \notin B$. Since $s_2 \in S \subsetneq [B]_\theta \cap [C]_\theta \subseteq [B]_\theta$ and $s_2 \notin C \cap [B]_\theta$, we have $s_2 \notin C$. Thus $(s_1, s_2, s_3) \in ([s_1]_\theta \times [s_2]_\theta \times [s_3]_\theta) \setminus (B^3 \cup C^3)$. As $f(s_1, s_2, s_3) = a \notin S$, we have that $f(S) \nsubseteq S$, and so $S \notin \mathrm{Sub}(\underline{A})$ contradicting to the assumption. Therefore $S \subseteq B \cap [C]_\theta$, or $S \subseteq C \cap [B]_\theta$, or $S \subseteq [B \cap C]_\theta$. □

Lemma 7. *For each $S \in \mathrm{Sub}(\underline{A})$, if $S \subsetneq B \cap [C]_\theta$, then $S \subseteq B \cap [B \cap C]_\theta$.*

Proof. Let $S \in \mathrm{Sub}(\underline{A})$ be such that $S \subsetneq B \cap [C]_\theta$. Then there exists $b \in (B \cap [C]_\theta) \setminus S$. Suppose, on the contrary, that $S \nsubseteq B \cap [B \cap C]_\theta$. Then there exists $s \in S \setminus (B \cap [B \cap C]_\theta)$. Since $b \in [C]_\theta$, there exists $b' \in C$ such that $b' \in [b]_\theta$. Let $f : A \to A$ be defined by

$$f(x) = \begin{cases} b, & x \in [s]_\theta \setminus C, \\ b', & x \in [s]_\theta \cap C, \\ x, & \text{otherwise.} \end{cases}$$

Since $s \notin B \cap [B \cap C]_\theta$, we have $s \notin B$ or $s \notin [B \cap C]_\theta$. Then $s \notin [B \cap C]_\theta$ because $s \in S \subsetneq B \cap [C]_\theta$. It follows that $B \cap ([s]_\theta \cap C) = \emptyset$. This implies that $f(B) \subseteq B$ because $b \in B$. Then $f \triangleright B$. Since $b' \in C$, we have $f(C) \subseteq C$, and so $f \triangleright C$. Moreover, Lemma 1 implies that $f \triangleright \theta$ because $b' \in [b]_\theta$. Hence $f \in \mathrm{Pol}_A\{B, C, \theta\}$. As $s \in B$ and $s \notin [B \cap C]_\theta$, we can conclude that $s \notin C$. Thus $f(s) = b \notin S$. This implies that $S \notin \mathrm{Sub}(\underline{A})$ which is a contradiction. Therefore $S \subseteq B \cap [B \cap C]_\theta$. □

Lemma 8. *For each* $S \in \mathrm{Sub}(\underline{A})$*, if* $S \subsetneq [B \cap C]_\theta$*, then* $S \subseteq B \cap C$*, or* $S \subseteq B \cap [B \cap C]_\theta$*, or* $S \subseteq C \cap [B \cap C]_\theta$*.*

Proof. Let $S \in \mathrm{Sub}(\underline{A})$ be such that $S \subsetneq [B \cap C]_\theta$. Then there exists $a \in ([B \cap C]_\theta) \setminus S$. Suppose, on the contrary, that $S \nsubseteq B \cap C$, $S \nsubseteq B \cap [B \cap C]_\theta$ and $S \nsubseteq C \cap [B \cap C]_\theta$. Then there are $s_1 \in S \setminus (B \cap [B \cap C]_\theta)$, $s_2 \in S \setminus (C \cap [B \cap C]_\theta)$ and $s_3 \in S \setminus (B \cap C)$. Since $a \in [B \cap C]_\theta$, there is $b \in B \cap C$ such that $b \in [a]_\theta$. Let $f : A^3 \to A$ be defined by

$$f(x,y,z) = \begin{cases} a, & (x,y,z) \in ([s_1]_\theta \times [s_2]_\theta \times [s_3]_\theta) \setminus (B^3 \cup C^3), \\ b, & (x,y,z) \in ([s_1]_\theta \times [s_2]_\theta \times [s_3]_\theta) \cap (B^3 \cup C^3), \\ x, & \text{otherwise.} \end{cases}$$

Since $f(x,y,z) \in \{x, b\}$ for all $(x,y,z) \in B^3 \cup C^3$, we have that $f(B^3) \subseteq B$, and $f(C^3) \subseteq C$. Then $f \triangleright B$ and $f \triangleright C$. Lemma 1 implies that $f \triangleright \theta$ because $b \in [a]_\theta$. Thus $f \in \mathrm{Pol}_A\{B, C, \theta\}$. Since $s_1 \in S \subsetneq [B \cap C]_\theta$ and $s_1 \notin B \cap [B \cap C]_\theta$, we have $s_1 \notin B$. Since $s_2 \in S \subsetneq [B \cap C]_\theta$ and $s_2 \notin C \cap [B \cap C]_\theta$, we have $s_2 \notin C$. Thus $(s_1, s_2, s_3) \in ([s_1]_\theta \times [s_2]_\theta \times [s_3]_\theta) \setminus (B^3 \cup C^3)$. Then $f(s_1, s_2, s_3) = a \notin S$, and so $S \notin \mathrm{Sub}(\underline{A})$ which is a contradiction. Therefore $S \subseteq B \cap [B \cap C]_\theta$, or $S \subseteq C \cap [B \cap C]_\theta$, or $S \subseteq B \cap C$. □

Lemma 9. *For each* $S \in \mathrm{Sub}(\underline{A})$*, if* $S \subsetneq B \cap [B \cap C]_\theta$*, then* $S \subseteq B \cap C$*.*

Proof. Let $S \in \mathrm{Sub}(\underline{A})$ be such that $S \subsetneq B \cap [B \cap C]_\theta$. Then there exists $b \in (B \cap [B \cap C]_\theta) \setminus S$. Suppose on the contrary, that $S \nsubseteq B \cap C$. Then there exits $s \in S \setminus (B \cap C)$. Since $b \in [B \cap C]_\theta$, there exists $b' \in B \cap C$ such that $b' \in [b]_\theta$. Let $f : A \to A$ be defined by

$$f(x) = \begin{cases} b, & x = s, \\ b', & x \in [s]_\theta \setminus \{s\}, \\ x, & \text{otherwise.} \end{cases}$$

Obviously, $f(B) \subseteq B$ because both $b, b' \in B$. Then $f \triangleright B$. Since $s \notin B \cap C$ and $s \in B$, we have $s \notin C$. Since $b' \in C$, we have that $f(C) \subseteq \{x, b'\} \subseteq C$, and so $f \triangleright C$. Lemma 1 implies that $f \triangleright \theta$ because $b \in [b']_\theta$. Hence $f \in \mathrm{Pol}_A\{B, C, \theta\}$. Then $S \notin \mathrm{Sub}(\underline{A})$ because $f(s) = b \notin S$. This is a contradiction. Therefore $S \subseteq B \cap C$. □

Lemma 10. *For each* $S \in \mathrm{Sub}(\underline{A})$*, if* $S \subseteq B \cap C$*, then* $S = \emptyset$ *or* $S = B \cap C$*.*

Proof. Let $S \in \mathrm{Sub}(\underline{A})$ be such that $S \subseteq B \cap C$. If $B \cap C = \emptyset$, then we are done. So assume that $B \cap C \neq \emptyset$ and $S \subsetneq B \cap C$. Let $b \in (B \cap C) \setminus S$ and let $f : A \to A$ be defined by $f(x) = b$ for all $x \in A$. Then $f(B) = f(C) = \{b\} \subseteq B \cap C$. Thus $f \triangleright B$ and $f \triangleright C$. Moreover, $f \triangleright \theta$ by Lemma 1. Hence $f \in \mathrm{Pol}_A\{B, C, \theta\}$. This implies that $S = \emptyset$, otherwise $\{b\} = f(S) \subseteq S$, which is a contradiction. □

Theorem 1. *Let* $\underline{A}$ *be the algebra whose universe is a nonempty finite set A and the clone of term operations* $Pol_A\{B, C, \theta\}$*, where B and C are nontrivial unary relations and* θ *is a nontrivial equivalence relation on A. Then*

$$\mathrm{Sub}(\underline{A}) = \left\{ \bigcap \mathcal{S} | \mathcal{S} \subseteq \{\emptyset, A, B, C, [B]_\theta, [C]_\theta, [B \cap C]_\theta\} \right\}.$$

Proof. It follows from Lemma 2 to Lemma 10. Note that $\bigcap \emptyset = A$. □

The set $\mathrm{Sub}(\underline{A})$ together with the inclusion forms a lattice whose minimum and maximum elements are $\emptyset$ and A, respectively. The Hasse diagram of $(\mathrm{Sub}(\underline{A}), \subseteq)$ is depicted in Figure 1. Under the specific case where $B = C$, Theorem 1 indicates that $\mathrm{Sub}(\underline{A})$ is simplified to $\{\emptyset, A, B, [B]_\theta\}$, which aligns perfectly with Lemma 4.2 in [15].

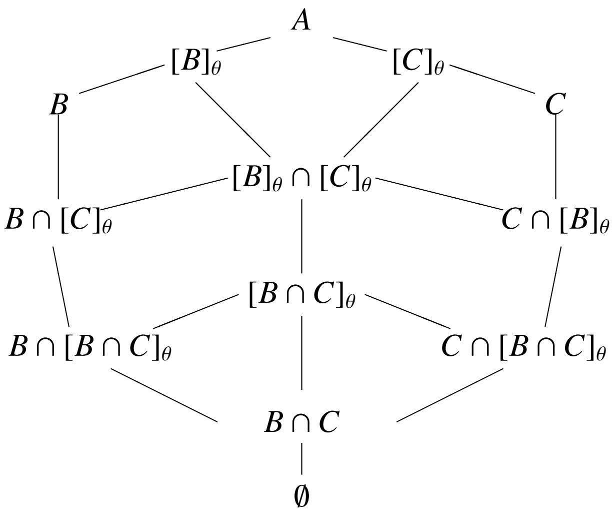

Figure 1. The Hasse diagram of the lattice $(\mathrm{Sub}(\underline{A}), \subseteq)$ where $\underline{A}$ is the algebra whose set of all term operations is $\mathrm{Pol}_A\{B, C, \theta\}$.

Example 1. *Let $A = \mathbb{Z}_7$ be the set of integers modulo 7. We define the algebra $\underline{A} = (A; F^A)$ such that F^A is the set of all operations on A preserving the subsets $B = \{\bar{0}\}$ and $C = \{\bar{1}, \bar{4}, \bar{5}\}$, as well as the equivalence relation θ on A defined by $x \equiv \pm y \pmod 7$. Note that the partition corresponding to θ is $\{\{\bar{0}\}, \{\bar{1}, \bar{6}\}, \{\bar{2}, \bar{5}\}, \{\bar{3}, \bar{4}\}\}$. Then $T(\underline{A}) = \mathrm{Pol}_A\{B, C, \theta\}$, and so Theorem 1 implies that*

$$\begin{aligned} \mathrm{Sub}(\underline{A}) &= \left\{\bigcap \mathcal{S} \mid \mathcal{S} \subseteq \{\emptyset, \{\bar{0}\}, \{\bar{1}, \bar{4}, \bar{5}\}, \{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}\}, \mathbb{Z}_7\}\right\} \\ &= \left\{\emptyset, \{\bar{0}\}, \{\bar{1}, \bar{4}, \bar{5}\}, \{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}\}, \mathbb{Z}_7\right\}. \end{aligned}$$

3. Isomorphisms of Subalgebra Lattices via Covering Graphs

Let $V_{\mathrm{Sub}(\underline{A})}$ represent the set of all elements in $\mathrm{Sub}(\underline{A})$ and let $E_{\mathrm{Sub}(\underline{A})}$ represent the set of all tuples (S_1, S_2) whenever $S_1 \prec S_2$ (meaning that $S_1 \subsetneq S_2$ and no S_3 exists such that $S_1 \subsetneq S_3 \subsetneq S_2$) or $S_2 \prec S_1$. Then Figure 1 can be visualized as the undirected graph $G_{\mathrm{Sub}(\underline{A})} := (V_{\mathrm{Sub}(\underline{A})}, E_{\mathrm{Sub}(\underline{A})})$. This graph is called the *covering graph* of $(\mathrm{Sub}(\underline{A}), \subseteq)$. It is shown that if two lattices are isomorphic, then their covering graphs are also isomorphic. A more comprehensive discussion can be found in [2] and [8]. Consequently, if the covering graphs of two associated subalgebra lattices are found to be non-isomorphic, it logically follows that the subalgebra lattices themselves are also non-isomorphic.

In this section, we will utilize the algorithms outlined by [16] to check for the necessary and sufficient conditions to determine if covering graphs of two subalgebra lattices are isomorphic. The algorithm proceeds by first constructing the incidence matrix of the covering graph. Specifically, we define this as a vertex-edge incidence matrix, where rows correspond to vertices and columns correspond to edges. The algorithm is shown in Algorithm 1.

Algorithm 1 Isomorphism Test for Covering Graphs of Subalgebra Lattices

Require: Two subalgebra lattices Sub_1 and Sub_2
Ensure: **True** if $G_{\text{Sub}_1} \cong G_{\text{Sub}_2}$, otherwise **False**
Compute incidence matrices I_1 and I_2 of the graphs associated with Sub_1 and Sub_2
Compute initial matrices: $M_1^{(0)} = I_1 I_1^\top$ and $M_2^{(0)} = I_2 I_2^\top$
— Necessity Check Phase (cf. Algorithm 1 in [16]) —
if $\det(M_1^{(0)}) \neq \det(M_2^{(0)})$ **then**
    return False
end if
$i \leftarrow 0$
Compute row-sum sequences $SM_1^{(0)}$ and $SM_2^{(0)}$ from $M_1^{(0)}$ and $M_2^{(0)}$
Set $NM_1^{(0)} \leftarrow$ number of distinct values in $SM_1^{(0)}$
Set $NM_2^{(0)} \leftarrow$ number of distinct values in $SM_2^{(0)}$
loop
    $i \leftarrow i + 1$
    $M_1^{(i)} \leftarrow M_1^{(i-1)}(M_1^{(i-1)})^\top$
    $M_2^{(i)} \leftarrow M_2^{(i-1)}(M_2^{(i-1)})^\top$
    Compute row-sum sequences $SM_1^{(i)}$ and $SM_2^{(i)}$
    Set $NM_1^{(i)} \leftarrow$ number of distinct values in $SM_1^{(i)}$
    Set $NM_2^{(i)} \leftarrow$ number of distinct values in $SM_2^{(i)}$
    if $NM_1^{(i)} = NM_1^{(i-1)}$ **and** $NM_2^{(i)} = NM_2^{(i-1)}$ **then**
        break
    end if
end loop
Let $SM_{1f} \leftarrow SM_1^{(i-1)}$, $SM_{2f} \leftarrow SM_2^{(i-1)}$ be the last stabilized sequences
if $\text{multiset}(SM_{1f}) \neq \text{multiset}(SM_{2f})$ **then**
    return False
end if
— Sufficiency Check Phase (cf. Algorithm 2 in [16]) —
Attempt to construct vertex mapping $\phi : V(\text{Sub}_1) \to V(\text{Sub}_2)$ using SM_{1f}, SM_{2f}, and adjacency structure
if no consistent bijection ϕ exists **then**
    return False
end if
Construct adjacency matrices A_1 for Sub_1 and A_2 for Sub_2 using the vertex order induced by ϕ
if $A_1 = A_2$ **then**
    return True
else
    return False
end if

This method efficiently verifies the isomorphism of covering graphs through the analysis of incident matrices and adjacency details. It significantly reduces the computational burden compared to traditional isomorphism tests by minimizing permutation checks, making it more feasible for larger structures. Following this algorithm, we developed a Python tool for the systematic structural comparison of two covering graphs. The number of non-isomorphic subalgebra lattices for an algebra yields a lower bound for the number of non-categorically equivalent classes of clones it contains.

Let A be a finite set with at least 3 elements. According to the algorithm and Lemma 2.2 presented in Duffus and Rival's work [7], there are exactly 2 non-isomorphic subalgebra lattices when the clone of term operations is $\mathrm{Pol}_A\{B,\theta\}$ where B is a nonempty proper subset of A and θ is a nontrivial equivalence relation on A. Based on the results of Davey and Werner [4] concerning categorical equivalence, this implies that there are at least 2 categorically equivalent classes of clones expressible in the form $\mathrm{Pol}_A\{B,\theta\}$. Expanding on this, Supaporn's research [15] demonstrated through the application of categorical equivalence of clones that

- If $|A| = 3$, then there are exactly 4 categorically equivalent classes of clones,
- If $|A| = 4$, then there are exactly 9 categorically equivalent classes of clones, and
- If $|A| \geq 5$, then there are exactly 10 categorically equivalent classes of clones.

In the case that the clone of term operations is $\mathrm{Pol}_A\{B,C,\theta\}$ where B and C are nonempty proper subsets of A and θ is a nontrivial equivalence relation on A, we employed the algorithm to find the number of non-isomorphic covering graphs of subalgebra lattices for some cardinalities of A. The results are shown in Table 1. The number indicates a lower bound on the number of categorically equivalent classes of clones in the form $\mathrm{Pol}_A\{B,C,\theta\}$.

Table 1. The number of non-isomorphic covering graphs of subalgebra lattices for different cardinalities of A.

$\lvert A\rvert$	The number of non-isomorphic covering graphs of subalgebra lattices
3	7
4	23
5	38
6	48

4. Conclusions and future work

In this work, we consider the algebra $\underline{A}$ defined on a nonempty finite set A with the clone of term operations $\mathrm{Pol}_A\{B,C,\theta\}$, where B and C are nontrivial unary relations and θ is a nontrivial equivalence relation. We show that

$$\mathrm{Sub}(\underline{A}) = \left\{\bigcap \mathcal{S} \middle| \mathcal{S} \subseteq \{\emptyset, A, B, C, [B]_\theta, [C]_\theta, [B\cap C]_\theta\}\right\}.$$

Moreover, by examining the covering graph of the lattice $(\mathrm{Sub}(\underline{A}), \subseteq)$, we determine a lower bound on the number of categorical equivalence classes for the clone $\mathrm{Pol}_A\{B,C,\theta\}$. This is achieved by applying the algorithm proposed in [16] with respect to the size of A. Some selected examples are presented in Table 1.

In future work, we aim to utilize the structural isomorphism of subalgebra lattices to classify clones, determined by two unary relations and an equivalence relation, on a finite set up to categorical equivalence.

Author contributions

All authors contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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