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Research article

Analyzing the existence, uniqueness, and stability of solutions to boundary value problems involving a generalized fractional derivative

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Abstract: This paper examines tempered fractional derivatives with respect to a kernel function, extending classical operators like Caputo and tempered derivatives. We investigate fractional differential equations (FDEs) that incorporate these generalized derivatives, focusing on the existence and uniqueness of solutions for boundary value problems. Using fixed-point theorems, we establish conditions for the existence and uniqueness of solutions. Additionally, we analyze the stability of these equations under different criteria. Our approach addresses inaccuracies in previous studies and contributes to the broader theory of fractional equations with generalized derivatives.

Keywords: fractional calculus; fractional differential equations; boundary value problems; stability analysis
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1. Introduction

Fractional calculus is an extension of classical calculus that allows for derivatives of non-integer order. Various definitions exist, but they all share the property that, when the fractional derivative's order is an integer, some form of an integer-order derivative is recovered. Due to the existence of multiple definitions, it becomes essential to consider generalized forms of fractional derivatives, where specific choices of the generalized derivative recover well-known cases. In this work, we adopt the concept of a fractional derivative with respect to another function, as presented in [1, 12]. These derivatives depend on a kernel ψ, and specific choices of this function yield, for example, the classical Caputo or Caputo–Hadamard derivatives. This derivative has already proven effective in modeling complex systems, offering advantages over integer-order derivatives and even the usual fractional derivatives [3, 7, 26]. Another possible generalization of classical fractional derivatives is the tempered fractional derivative, as presented in e.g. [8, 13, 14, 21]. The tempered fractional derivative introduces an exponential tempering parameter that moderates the long-range memory effects typical of classical fractional operators, allowing a more realistic description of phenomena in which heavy-tailed dynamics gradually weaken over time. This added flexibility enables more accurate modeling of systems with long-range memory and decay, capturing behaviours that classical fractional operators cannot fully describe, such as those observed in viscoelastic materials, anomalous diffusion, geophysics, or systems where past influences gradually diminish over long periods. In addition, the tempered formulation facilitates well-posedness, enhances the flexibility of boundary-value problems, and often yields more robust discretization schemes for practical computations.

Following the same reasoning, a fractional differential equation (FDE) extends the classical theory of differential equations by replacing integer-order with fractional-order derivatives. This generalization allows for a more flexible and accurate modeling of various complex systems, particularly those exhibiting memory effects. The literature provides a rich theoretical foundation for FDEs, covering topics such as the existence and uniqueness of solutions (see, for example, [9, 19, 23]), stability analysis ([11, 20]), and qualitative properties of solutions ([16]). Additionally, significant progress has been made in the development of numerical methods for solving these equations, addressing the challenges posed by the complexity of fractional operators ([15, 17, 24, 25]). A wide range of FDEs are studied in different contexts as researchers can explore various aspects, including the choice of fractional derivative definitions (e.g., Riemann–Liouville, Caputo, or Grünwald–Letnikov), different orders of differentiation, and distinct initial and boundary conditions. To develop a more unified theory and reduce redundancy across studies, we propose studying FDEs by focusing on the generalized tempered derivative, which encompasses derivatives such as Caputo-type derivatives and tempered derivatives.

In this paper, we investigate the existence and uniqueness of solutions for fractional boundary value problems incorporating tempered Caputo fractional derivatives with respect to a smooth kernel, applying fixed-point theorems. Additionally, we conduct the Ulam–Hyers and Ulam–Hyers–Rassias stability analyses of these FDEs under certain assumptions. The stability analysis of FDEs plays a crucial role in understanding the long-term behavior of systems with memory and hereditary effects. By establishing conditions for asymptotic stability, one can ensure that perturbations in initial data or parameters do not lead to unbounded responses, thereby supporting the design of robust numerical schemes and physically consistent models in complex dynamical settings. Ulam–Hyers stability in dynamical systems concerns whether an approximate solution to an equation remains close to an exact solution, even when the original problem is difficult to solve precisely. It plays a key role in mathematical modeling, ensuring that small errors in approximations do not amplify uncontrollably but instead remain bounded near a true solution, which is essential in real-world settings where exact solutions are rarely available.

The study in [18] examined FDEs involving a fractional derivative defined with respect to another function. However, the work appears to contain some inaccuracies; in particular, the statements of Proposition 3.1 and Theorem 3.2 seem to require correction, and several calculations throughout the proofs should be carefully reconsidered. We remark that the corrections of these results can be obtained from our results by considering $\lambda = 0$.

The paper is organized as follows. In Section 2, we present some necessary concepts about fractional operators, along with key results used in our analysis. In Section 3, we introduce the problem and rewrite it in integral equation form. Then, in Section 4, we prove the existence and uniqueness of solutions by applying well-known fixed-point theorems. In Section 5, we analyze the stability of the equation, considering three different types of stability. In Section 6, we provide an example to illustrate the applicability of our main results. Finally, in Section 7, we provide some concluding remarks.

2. Preliminaries

Next, we introduce the necessary definitions of the tempered fractional integral and Caputo derivative, depending on an arbitrary kernel, as presented in [4]. In what follows, $\lambda \geq 0$ is a parameter, $n \in \mathbb{N}$, and $\alpha \in (n-1, n]$. The function $\psi \in C^n([a,b],\mathbb{R})$ serves as the kernel, satisfying $\psi'(t) > 0$ for all $t \in [a,b]$.

Definition 2.1. Let $u : [a,b] \to \mathbb{R}$ be an integrable function. The tempered fractional integral of order α with parameter λ, defined with respect to the function ψ, is given by

$$\mathbb{I}_{a+}^{\alpha,\lambda,\psi} u(t) = \frac{\mathrm{e}^{-\lambda t}}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1} \mathrm{e}^{\lambda\tau} u(\tau)\, d\tau.$$

We note that when $\lambda = 0$, the definition reduces to the fractional integral with respect to another function (see [12])

$$\mathbb{I}_{a+}^{\alpha,\psi} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1} u(\tau)\, d\tau.$$

Definition 2.2. Let $u \in C^{n-1}([a,b],\mathbb{R})$. The Caputo tempered fractional derivative of order α with parameter λ, defined with respect to the function ψ, is defined as

$${}^C\mathbb{D}_{a+}^{\alpha,\lambda,\psi} u(t) = \mathrm{e}^{-\lambda t}\, {}^C\mathbb{D}_{a+}^{\alpha,\psi} (\mathrm{e}^{\lambda t} u(t)),$$

where ${}^C\mathbb{D}_{a+}^{\alpha,\psi}$ stands for the Caputo fractional derivative operator with respect to the kernel ψ, of order α (see [2]).

Remark 2.1. We make the following observations: Given $u \in C^n([a,b],\mathbb{R})$,

(1) if $\alpha \in \mathbb{N}$, then

$${}^C\mathbb{D}_{a+}^{\alpha,\lambda,\psi} u(t) = \mathrm{e}^{-\lambda t} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^{\alpha} (\mathrm{e}^{\lambda t} u(t)),$$

and if $\alpha \notin \mathbb{N}$, then

$${}^C\mathbb{D}_{a+}^{\alpha,\lambda,\psi} u(t) = \frac{\mathrm{e}^{-\lambda t}}{\Gamma(n-\alpha)} \int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{n-\alpha-1} \left(\frac{1}{\psi'(\tau)} \frac{d}{d\tau} \right)^{n} (\mathrm{e}^{\lambda\tau} u(\tau))\, d\tau,$$

where $n = [\alpha] + 1$;

(2) if $\lambda = 0$, Definition 2.2 reduces to the Caputo fractional derivative with respect to another function, as introduced in [2].

According to Lemmas 1 and 2 in [1], for $\beta > n-1$,

$$ {}^{C}\mathbb{D}_{a+}^{\alpha,\lambda,\psi}(\mathrm{e}^{-\lambda t}(\psi(t)-\psi(a))^{\beta}) = \mathrm{e}^{-\lambda t}\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(\psi(t)-\psi(a))^{\beta-\alpha}, $$

and, for $\Lambda \in \mathbb{R}$,

$$ {}^{C}\mathbb{D}_{a+}^{\alpha,\lambda,\psi}(\mathrm{e}^{-\lambda t}E_{\alpha}(\Lambda(\psi(t)-\psi(a))^{\alpha})) = \Lambda\mathrm{e}^{-\lambda t}E_{\alpha}(\Lambda(\psi(t)-\psi(a))^{\alpha}), $$

where E_α denotes the one-parameter Mittag–Leffler function

$$ E_{\alpha}(t) = \sum_{k=0}^{\infty}\frac{t^k}{\Gamma(k\alpha+1)}. $$

The following result is very useful for proving Theorem 3.1.

Theorem 2.1. For any $u \in C([a,b],\mathbb{R})$, the following holds:

$$ {}^{C}\mathbb{D}_{a+}^{\alpha,\lambda,\psi}\,\mathbb{I}_{a+}^{\alpha,\lambda,\psi}u(t) = u(t). $$

Moreover, for $u \in C^{n-1}([a,b],\mathbb{R})$, we have the identity

$$ \mathbb{I}_{a+}^{\alpha,\lambda,\psi}\,{}^{C}\mathbb{D}_{a+}^{\alpha,\lambda,\psi}u(t) = u(t) - \mathrm{e}^{-\lambda t}\sum_{k=0}^{n-1}\frac{(\psi(t)-\psi(a))^k}{k!}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^k(\mathrm{e}^{\lambda t}u(t))\Big|_{t=a}. $$

Proof. The formulae are a direct consequence of Theorem 1 from [2]. □

Next, we present the fixed-point theorems that are essential to prove our main results (see, for example, [10, 22]). We begin by presenting Krasnoselskii's fixed-point theorem, which guarantees the existence of a solution to our fractional boundary value problem (FBVP).

Theorem 2.2. (Krasnoselskii's fixed-point theorem) Let M be a closed, bounded, convex, and nonempty subset of a Banach space $\mathbb{B}$, and let $\mathcal{P}$ and Q be operators on M satisfying the following conditions:

(1) $\mathcal{P}(M) + Q(M) \subseteq M$;
(2) $\mathcal{P}$ is a contraction mapping; and
(3) Q is continuous, and $Q(M)$ is a relatively compact subset of $\mathbb{B}$.

Then, there exists $z \in M$ such that $z = \mathcal{P}z + Qz$.

Additionally, we present the Arzelà–Ascoli theorem, a fundamental result in analysis that establishes necessary and sufficient conditions for the compactness of a family of continuous functions.

Theorem 2.3. (Arzelà–Ascoli theorem) Let (X,d) be a compact metric space. Then, $M \subseteq C(X)$ is relatively compact if and only if M is uniformly bounded and uniformly equicontinuous.

We conclude this section with the well-known Banach fixed-point theorem.

Theorem 2.4. (Banach's fixed point theorem) Let (X,d) be a complete metric space, and let $\mathcal{T} : X \to X$ be a contraction mapping, that is,

$$ d(\mathcal{T}u,\mathcal{T}v) \le Ld(u,v), \quad \forall u,v \in X, $$

for some constant $0 \le L < 1$. Then,

(1) $\mathcal{T}$ admits a unique fixed point $u^\star$; and
(2) for any $u \in X$, the fixed point $u^\star$ satisfies the following inequality:

$$d(u, u^\star) \leq \frac{1}{1-L} d(u, \mathcal{T}u).$$

3. Tempered FDEs

In this paper, we study an FDE of order between 2 and 3, incorporating the generalized tempered fractional derivative along with initial and terminal conditions. The problem is formulated as follows: Let $\alpha \in (2, 3)$ be the fractional order, and let $\psi \in C^3([a, b], \mathbb{R})$ be the kernel that satisfies $\psi'(t) > 0$ for all $t \in [a, b]$. Consider the functions $p, q, r \in C([a, b], \mathbb{R})$. The FBVP we are studying is as follows:

$$\begin{cases} {}^C\mathbb{D}_{a+}^{\alpha,\lambda,\psi} u(t) + p(t)u'(t) + q(t)u(t) = r(t), \quad t \in [a, b], \\ u(a) = u'(a) = u(b) = 0, \end{cases} \tag{3.1}$$

where $u \in C^2([a, b], \mathbb{R})$.

Remark 3.1. For $\lambda = 0$, i.e., when we are dealing with the fractional Caputo derivative with respect to another function, our results coincide with those of [18], except for minor corrections. Furthermore, when $\psi(t) = t$, meaning that the equation follows the classical Caputo fractional derivative, we recover as special cases the results of [6].

Remark 3.2. We restrict our analysis to homogeneous boundary conditions for simplicity, but the same techniques developed in the following sections allow the results to be extended to general non-homogeneous boundary conditions.

The equivalence between the FBVP (3.1) and a fractional integral equation is shown by the following result.

Theorem 3.1. A function $u \in C^2([a, b], \mathbb{R})$ is a solution of the FBVP (3.1) if and only if u satisfies the integral equation:

$$u(t) = \frac{e^{-\lambda t}(\psi(t) - \psi(a))^2}{\Gamma(\alpha)(\psi(b) - \psi(a))^2} \int_a^b \psi'(\tau)(\psi(b) - \psi(\tau))^{\alpha-1} e^{\lambda\tau} f(\tau)\, d\tau - \frac{e^{-\lambda t}}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} e^{\lambda\tau} f(\tau)\, d\tau, \tag{3.2}$$

where

$$f(t) = p(t)u'(t) + q(t)u(t) - r(t).$$

Proof. Let $u \in C^2([a, b], \mathbb{R})$ be a solution of (3.1). Taking the tempered fractional integral operator of order α, with respect to ψ, $\mathbb{I}_{a+}^{\alpha,\lambda,\psi}$, to both sides of the FDE given in (3.1), we obtain from Theorem 2.1

$$u(t) = e^{-\lambda t}\left[(e^{\lambda t}u(t))\Big|_{t=a} + (\psi(t) - \psi(a))\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)(e^{\lambda t}u(t))\Big|_{t=a} + \frac{(\psi(t) - \psi(a))^2}{2}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^2 (e^{\lambda t}u(t))\Big|_{t=a} \right] - \frac{e^{-\lambda t}}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} e^{\lambda\tau} f(\tau)\, d\tau.$$

Since

$$\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{k}(\mathrm{e}^{\lambda t}u(t))\Big|_{t=a}=0,\quad k\in\{0,1\},$$

we get

$$u(t)=\mathrm{e}^{-\lambda t}\frac{(\psi(t)-\psi(a))^2}{2}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{2}(\mathrm{e}^{\lambda t}u(t))\Big|_{t=a}-\frac{\mathrm{e}^{-\lambda t}}{\Gamma(\alpha)}\int_a^t\psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}\mathrm{e}^{\lambda\tau}f(\tau)\,d\tau.$$

Using the condition $u(b)=0$, one gets

$$\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{2}(\mathrm{e}^{\lambda t}u(t))\Big|_{t=a}=\frac{2}{\Gamma(\alpha)(\psi(b)-\psi(a))^2}\int_a^b\psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}\mathrm{e}^{\lambda\tau}f(\tau)\,d\tau.$$

Hence, we can conclude that

$$\begin{aligned}u(t)=\frac{\mathrm{e}^{-\lambda t}(\psi(t)-\psi(a))^2}{\Gamma(\alpha)(\psi(b)-\psi(a))^2}&\int_a^b\psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}\mathrm{e}^{\lambda\tau}f(\tau)\,d\tau\\&-\frac{\mathrm{e}^{-\lambda t}}{\Gamma(\alpha)}\int_a^t\psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}\mathrm{e}^{\lambda\tau}f(\tau)\,d\tau.\end{aligned}$$

For the converse, we assume that $u\in C^2([a,b],\mathbb{R})$ satisfies the fractional integral equation (3.2). From Theorem 2.1, by applying the differential operator ${}^{C}\mathbb{D}_{a+}^{\alpha,\lambda,\psi}$ on both sides of (3.2), we obtain that

$${}^{C}\mathbb{D}_{a+}^{\alpha,\lambda,\psi}u(t)=\frac{\int_a^b\psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}\mathrm{e}^{\lambda\tau}f(\tau)\,d\tau}{\Gamma(\alpha)(\psi(b)-\psi(a))^2}\cdot{}^{C}\mathbb{D}_{a+}^{\alpha,\lambda,\psi}\left(e^{-\lambda t}(\psi(t)-\psi(a))^2\right)-f(t).$$

Since

$${}^{C}\mathbb{D}_{a+}^{\alpha,\lambda,\psi}\left(\mathrm{e}^{-\lambda t}(\psi(t)-\psi(a))^2\right)=\mathrm{e}^{-\lambda t}\cdot{}^{C}\mathbb{D}_{a+}^{\alpha,\psi}(\psi(t)-\psi(a))^2=0,$$

one gets that

$${}^{C}\mathbb{D}_{a+}^{\alpha,\lambda,\psi}u(t)=-f(t),$$

proving that

$${}^{C}\mathbb{D}_{a+}^{\alpha,\lambda,\psi}u(t)+p(t)u'(t)+q(t)u(t)=r(t),$$

as desired. □

4. Existence and uniqueness results

In this section, we establish results on the existence and uniqueness of solutions to problem (3.1). To this end, we apply Krasnoselskii's and Banach's fixed-point theorems. Although Theorems 4.1 and 4.2 share the same hypotheses, the former establishes existence, while the latter guarantees existence and uniqueness, avoiding repetition of calculations by relying on the arguments of the former.

In what follows, $C^2([a,b],\mathbb{R})$ is endowed with the norm defined by

$$||u||_{C^2}=\sup_{t\in[a,b]}|u(t)|+\sup_{t\in[a,b]}|u'(t)|+\sup_{t\in[a,b]}|u''(t)|.$$

Equipped with this norm, $C^2([a,b],\mathbb{R})$ is a Banach space.

4.1. Existence result via Krasnoselskii's fixed-point theorem

Theorem 4.1. Let

$$M := \max_{t\in[a,b]}\{|p(t)|, |q(t)|\} \quad \text{and} \quad W := \max_{t\in[a,b]}\{|\psi'(t)|, |\psi''(t)|\}.$$

Define

$$A := \frac{e^{\lambda(b-a)}}{\Gamma(\alpha+1)}(\psi(b)-\psi(a))^{\alpha-2}\Big(2(1+\lambda+\lambda^2)(\psi(b)-\psi(a))^2 + 2W(2+\alpha)(1+\lambda)(\psi(b)-\psi(a)) + W^2(\alpha^2-\alpha+2)\Big).$$

If

$$A\cdot M < 1,$$

then the FBVP (3.1) has a least one solution in $C^2([a,b],\mathbb{R})$.

Proof. From Theorem 3.1, we know that $u \in C^2([a,b],\mathbb{R})$ is a solution of (3.1) if and only if u satisfies (3.2). Let

$$N := \max_{t\in[a,b]}\{|r(t)|\},$$

and choose R such that

$$R \geq \frac{AN}{1-AM}.$$

Define

$$B_R := \{u \in C^2([a,b],\mathbb{R}) : \|u\|_{C^2} \leq R\}.$$

Clearly, $B_R \neq \emptyset$, B_R is bounded, closed, and convex. We now define the operators $\mathcal{P}$ and $\mathcal{Q}$ on B_R as follows:

$$(\mathcal{P}u)(t) := -\frac{e^{-\lambda t}}{\Gamma(\alpha)}\int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}e^{\lambda\tau}f(\tau)\,d\tau, \tag{4.1}$$

and

$$(\mathcal{Q}u)(t) := \frac{e^{-\lambda t}(\psi(t)-\psi(a))^2}{\Gamma(\alpha)(\psi(b)-\psi(a))^2}\int_a^b \psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}e^{\lambda\tau}f(\tau)\,d\tau, \tag{4.2}$$

where

$$f(t) = p(t)u'(t) + q(t)u(t) - r(t).$$

Let

$$A_1 := \frac{e^{\lambda(b-a)}}{\Gamma(\alpha+1)}(\psi(b)-\psi(a))^{\alpha-2}\Big((1+\lambda+\lambda^2)(\psi(b)-\psi(a))^2 + 2\alpha W(1+\lambda)(\psi(b)-\psi(a)) + \alpha(\alpha-1)W^2\Big),$$

and

$$A_2 := \frac{e^{\lambda(b-a)}}{\Gamma(\alpha+1)}(\psi(b)-\psi(a))^{\alpha-2}\Big((1+\lambda+\lambda^2)(\psi(b)-\psi(a))^2 + 4W(1+\lambda)(\psi(b)-\psi(a)) + 2W^2\Big).$$

The proof will be carried out in several steps.

Step 1: Let us prove that, for any $u, v \in B_R$, $\mathcal{P}u + \mathcal{Q}v \in B_R$. First, we will prove that

$$\|\mathcal{P}u\|_{C^2} \leq A_1(MR + N).$$

Note that, for any $t \in [a, b]$, we have

$$|f(t)| \leq |p(t)| \cdot |u'(t)| + |q(t)| \cdot |u(t)| + |r(t)| \leq M\|u\|_{C^2} + N \leq MR + N,$$

and

$$\begin{aligned} |(\mathcal{P}u)(t)| &\leq \frac{\mathrm{e}^{-\lambda a}}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} \mathrm{e}^{\lambda\tau} |f(\tau)|\, d\tau \\ &\leq \frac{\mathrm{e}^{\lambda(b-a)}}{\Gamma(\alpha)}(MR + N) \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1}\, d\tau \\ &\leq \frac{\mathrm{e}^{\lambda(b-a)}}{\Gamma(\alpha+1)}(MR + N) \cdot (\psi(b) - \psi(a))^{\alpha}. \end{aligned}$$

Observe that

$$\begin{aligned} (\mathcal{P}u)'(t) &= \frac{\lambda \mathrm{e}^{-\lambda t}}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} \mathrm{e}^{\lambda\tau} f(\tau)\, d\tau \\ &- \frac{\mathrm{e}^{-\lambda t}}{\Gamma(\alpha)}(\alpha - 1)\psi'(t) \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-2} \mathrm{e}^{\lambda\tau} f(\tau)\, d\tau. \end{aligned}$$

Thus,

$$\begin{aligned} |(\mathcal{P}u)'(t)| &\leq \frac{\lambda \mathrm{e}^{\lambda(b-a)}}{\Gamma(\alpha)}(MR + N) \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1}\, d\tau \\ &+ \frac{\mathrm{e}^{\lambda(b-a)}}{\Gamma(\alpha)}(\alpha - 1)W(MR + N) \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-2}\, d\tau \\ &\leq \frac{\mathrm{e}^{\lambda(b-a)}}{\Gamma(\alpha)}(MR + N)\left(\lambda \frac{(\psi(b) - \psi(a))^{\alpha}}{\alpha} + W(\psi(b) - \psi(a))^{\alpha-1}\right). \end{aligned}$$

In a similar manner, we get

$$\begin{aligned} (\mathcal{P}u)''(t) &= -\frac{\lambda^2 \mathrm{e}^{-\lambda t}}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} \mathrm{e}^{\lambda\tau} f(\tau)\, d\tau \\ &+2\frac{\lambda \mathrm{e}^{-\lambda t}}{\Gamma(\alpha)}(\alpha - 1)\psi'(t) \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-2} \mathrm{e}^{\lambda\tau} f(\tau)\, d\tau \\ &- \frac{\mathrm{e}^{-\lambda t}}{\Gamma(\alpha)}(\alpha - 1)\Bigg(\psi''(t) \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-2} \mathrm{e}^{\lambda\tau} f(\tau)\, d\tau \\ &+(\psi'(t))^2(\alpha - 2) \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-3} \mathrm{e}^{\lambda\tau} f(\tau)\, d\tau\Bigg), \end{aligned}$$

and so

$$\begin{aligned} |(\mathcal{P}u)''(t)| &\leq \frac{\lambda^2 \mathrm{e}^{\lambda(b-a)}}{\Gamma(\alpha)}(MR + N)\frac{(\psi(b) - \psi(a))^{\alpha}}{\alpha} + \frac{2\lambda \mathrm{e}^{\lambda(b-a)}}{\Gamma(\alpha)}(MR + N)W(\psi(b) - \psi(a))^{\alpha-1} \\ &+ \frac{\mathrm{e}^{\lambda(b-a)}}{\Gamma(\alpha)}(MR + N)W(\psi(b) - \psi(a))^{\alpha-1} + \frac{\mathrm{e}^{\lambda(b-a)}}{\Gamma(\alpha)}(MR + N)(\alpha - 1)W^2(\psi(b) - \psi(a))^{\alpha-2} \\ &\leq \frac{\mathrm{e}^{\lambda(b-a)}}{\Gamma(\alpha)}\Bigg(\lambda^2 \frac{(\psi(b) - \psi(a))^{\alpha}}{\alpha} + (1 + 2\lambda)W(\psi(b) - \psi(a))^{\alpha-1} \\ &+(\alpha - 1)W^2(\psi(b) - \psi(a))^{\alpha-2}\Bigg)(MR + N). \end{aligned}$$

Then, we can conclude that

$$\begin{aligned}\|\mathcal{P}u\|_{C^2} &= \sup_{t\in[a,b]}|(\mathcal{P}u)(t)| + \sup_{t\in[a,b]}|(\mathcal{P}u)'(t)| + \sup_{t\in[a,b]}|(\mathcal{P}u)''(t)|\\ &\le \frac{e^{\lambda(b-a)}}{\Gamma(\alpha)}\Bigg((1+\lambda+\lambda^2)\frac{(\psi(b)-\psi(a))^{\alpha}}{\alpha} + 2W(1+\lambda)(\psi(b)-\psi(a))^{\alpha-1}\\ &+(\alpha-1)W^2(\psi(b)-\psi(a))^{\alpha-2}\Bigg)(MR+N)\\ &= \frac{e^{\lambda(b-a)}}{\Gamma(\alpha+1)}(\psi(b)-\psi(a))^{\alpha-2}\Bigg((1+\lambda+\lambda^2)(\psi(b)-\psi(a))^2\\ &+2\alpha W(1+\lambda)(\psi(b)-\psi(a)) + \alpha(\alpha-1)W^2\Bigg)(MR+N),\end{aligned}$$

proving that

$$\|\mathcal{P}u\|_{C^2} \le A_1(MR+N).$$

Next, we prove that

$$\|\mathcal{Q}u\|_{C^2} \le A_2(MR+N).$$

Since

$$\begin{aligned}|(\mathcal{Q}u)(t)| &\le \frac{e^{-\lambda a}}{\Gamma(\alpha)}\int_a^b \psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}e^{\lambda\tau}|f(\tau)|\,d\tau\\ &\le \frac{e^{\lambda(b-a)}}{\Gamma(\alpha)}(MR+N)\int_a^b \psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}\,d\tau\\ &\le \frac{e^{\lambda(b-a)}}{\Gamma(\alpha+1)}(\psi(b)-\psi(a))^{\alpha}(MR+N),\end{aligned}$$

$$\begin{aligned}|(\mathcal{Q}u)'(t)| &\le \Big(e^{-\lambda a}\lambda(\psi(b)-\psi(a))^2 + 2e^{-\lambda a}W(\psi(b)-\psi(a))\Big)\\ &\times\frac{e^{\lambda b}}{\Gamma(\alpha)(\psi(b)-\psi(a))^2}(MR+N)\int_a^b \psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}\,d\tau\\ &\le \frac{e^{\lambda(b-a)}}{\Gamma(\alpha+1)}(\psi(b)-\psi(a))^{\alpha-2}\Big(\lambda(\psi(b)-\psi(a))^2 + 2W(\psi(b)-\psi(a))\Big)(MR+N)\\ &= \frac{e^{\lambda(b-a)}}{\Gamma(\alpha+1)}(\psi(b)-\psi(a))^{\alpha-1}\Big(\lambda(\psi(b)-\psi(a)) + 2W\Big)(MR+N),\end{aligned}$$

and

$$\begin{aligned}|(\mathcal{Q}u)''(t)| &\le e^{-\lambda a}\Big(\lambda^2(\psi(b)-\psi(a))^2 + 4\lambda W(\psi(b)-\psi(a)) + 2W^2 + 2W(\psi(b)-\psi(a))\Big)\\ &\times\frac{e^{\lambda b}(\psi(b)-\psi(a))^{\alpha-2}}{\Gamma(\alpha+1)}(MR+N)\\ &\le \frac{e^{\lambda(b-a)}}{\Gamma(\alpha+1)}(\psi(b)-\psi(a))^{\alpha-2}\\ &\times\Big(\lambda^2(\psi(b)-\psi(a))^2 + (4\lambda W+2W)(\psi(b)-\psi(a)) + 2W^2\Big)(MR+N),\end{aligned}$$

we may conclude that

$$\|\mathcal{Q}u\|_{C^2} \leq A_2(MR + N).$$

Hence,

$$\|\mathcal{P}u + \mathcal{Q}u\|_{C^2} \leq \|\mathcal{P}u\|_{C^2} + \|\mathcal{Q}u\|_{C^2} \leq (A_1 + A_2)(MR + N).$$

Since $A = A_1 + A_2$, we get

$$\|\mathcal{P}u + \mathcal{Q}u\|_{C^2} \leq A(MR + N) \leq R,$$

and therefore, $\mathcal{P}u + \mathcal{Q}v \in B_R$.

Step 2: Now, we show that $\mathcal{P}$ is a contraction. Let $u, v \in B_R$. Since

$$\begin{aligned} \|\mathcal{P}u - \mathcal{P}v\|_{C^2} &= \sup_{t\in[a,b]} |(\mathcal{P}u)(t) - (\mathcal{P}v)(t)| + \sup_{t\in[a,b]} |(\mathcal{P}u)'(t) - (\mathcal{P}v)'(t)| + \sup_{t\in[a,b]} |(\mathcal{P}u)''(t) - (\mathcal{P}v)''(t)| \\ &\leq MA_1\|u - v\|_{C^2} \end{aligned}$$

and $MA_1 < 1$, we conclude that $\mathcal{P}$ is a contraction.

Step 3: In the following, we prove that $\mathcal{Q}$ is continuous and $\mathcal{Q}(B_R)$ is a relatively compact subset of $C^2([a, b], \mathbb{R})$.

Clearly, $\mathcal{Q}$ is continuous. To prove that $\mathcal{Q}(B_R)$ is relatively compact, we apply the Arzelà–Ascoli theorem. Since, for any $u \in B_R$,

$$\|\mathcal{Q}u\|_{C^2} \leq A_2(MR + N),$$

we may conclude that $\mathcal{Q}(B_R)$ is uniformly bounded.

Let us prove that $\mathcal{Q}(B_R)$ is uniformly equicontinuous. Let $u \in B_R$ and $t_1, t_2 \in [a, b]$ such that $t_2 > t_1$. Observe that

$$|(\mathcal{Q}u)(t_2) - (\mathcal{Q}u)(t_1)| \leq \frac{\left|e^{-\lambda t_2}(\psi(t_2) - \psi(a))^2 - e^{-\lambda t_1}(\psi(t_1) - \psi(a))^2\right|}{\Gamma(\alpha + 1)}(\psi(b) - \psi(a))^{\alpha-2}e^{\lambda b}(MR + N),$$

which goes to 0 as $t_2 \to t_1$.

Since

$$\begin{aligned} |(\mathcal{Q}u)'(t_2) - (\mathcal{Q}u)'(t_1)| \leq& \Big| -\lambda e^{-\lambda t_2}(\psi(t_2) - \psi(a))^2 + 2e^{-\lambda t_2}(\psi(t_2) - \psi(a))\psi'(t_2) \\ &+\lambda e^{-\lambda t_1}(\psi(t_1) - \psi(a))^2 - 2e^{-\lambda t_1}(\psi(t_1) - \psi(a))\psi'(t_1)\Big| \\ &\times\frac{e^{\lambda b}(\psi(b) - \psi(a))^{\alpha-2}}{\Gamma(\alpha + 1)}(MR + N), \end{aligned}$$

we conclude that $(\mathcal{Q}u)'(t_2) - (\mathcal{Q}u)'(t_1) \to 0$ as $t_2 \to t_1$.

Similarly, it can be proved that

$$|(\mathcal{Q}u)''(t_2) - (\mathcal{Q}u)''(t_1)| \leq \Big|G(t_2) - G(t_1)\Big| \cdot \frac{e^{\lambda b}(\psi(b) - \psi(a))^{\alpha-2}}{\Gamma(\alpha + 1)}(MR + N),$$

where

$$G(t) = e^{-\lambda t}\Big(\lambda^2(\psi(t) - \psi(a))^2 - 4\lambda(\psi(t) - \psi(a))\psi'(t) + 2(\psi'(t))^2 + 2(\psi(t) - \psi(a))\psi''(t)\Big).$$

Clearly, when $t_2 \to t_1$, we get $(\mathcal{Q}u)''(t_2)-(\mathcal{Q}u)''(t_1) \to 0$. Hence, $\mathcal{Q}(B_R)$ is uniformly equicontinuous.

Step 4: Applying Krasnoselskii's fixed-point theorem, we may conclude that there exists at least one $z \in B_R$ such that $z = \mathcal{P}z + \mathcal{Q}z$, proving that the FBVP (3.1) has a least one solution. □

4.2. Existence and uniqueness result via Banach's fixed-point theorem

Here, we establish the existence and uniqueness of a solution to problem (3.1), by applying Banach's fixed-point theorem.

Theorem 4.2. Under the assumptions of Theorem 4.1, the FBVP (3.1) admits a unique solution in the set

$$B_R := \{u \in C^2([a,b],\mathbb{R}) : \|u\|_{C^2} \le R\}$$

for some constant $R > 0$.

Proof. Define again

$$N := \max_{t\in[a,b]}\{|r(t)|\},$$

and consider the set

$$B_R := \{u \in C^2([a,b],\mathbb{R}) : \|u\|_{C^2} \le R\},$$

where

$$R \ge \frac{AN}{1-AM}.$$

Define the operator $\mathcal{T} : B_R \to \mathbb{C}^2([a,b],\mathbb{R})$ by

$$(\mathcal{T}u)(t) := \frac{e^{-\lambda t}(\psi(t)-\psi(a))^2}{\Gamma(\alpha)(\psi(b)-\psi(a))^2}\int_a^b \psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}e^{\lambda\tau}f(\tau)\,d\tau - \frac{e^{-\lambda t}}{\Gamma(\alpha)}\int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}e^{\lambda\tau}f(\tau)\,d\tau, \quad (4.3)$$

where

$$f(t) = p(t)u'(t) + q(t)u(t) - r(t).$$

Note that

$$(\mathcal{T}u)(t) = (\mathcal{P}u)(t) + (\mathcal{Q}u)(t),$$

where $\mathcal{P}$ and $\mathcal{Q}$ are defined in (4.1) and (4.2), respectively. We now show that $\mathcal{T}(B_R) \subseteq B_R$. Given $u \in B_R$, by using the calculations from the proof of Theorem 4.1, we obtain

$$\|\mathcal{T}u\|_{C^2} \le \|\mathcal{P}u\|_{C^2} + \|\mathcal{Q}u\|_{C^2} \le A(MR+N).$$

Hence,

$$\|\mathcal{T}u\|_{C^2} \le R$$

by the definition of R. Therefore, the operator $\mathcal{T} : B_R \to B_R$ is well-defined.

Next, we prove that $\mathcal{T}$ is a contraction.

Similarly, using the calculations from the proof of Theorem 4.1, we obtain for $u, v \in B_R$

$$\|\mathcal{T}u - \mathcal{T}v\|_{C^2} \le AM\|u-v\|_{C^2}.$$

Since $AM < 1$, this proves that $\mathcal{T}$ is a contraction. By Banach's fixed-point theorem, we conclude the proof. □

5. Stability analysis

This section is dedicated to establishing sufficient conditions for Ulam–Hyers stability, generalized Ulam–Hyers stability, and Ulam–Hyers–Rassias stability of the FBVP (3.1). The following definitions are adapted from [5].

Definition 5.1. (Ulam–Hyers stability) FBVP (3.1) is said to exhibit Ulam–Hyers stability if there exists a constant $C > 0$ such that for any $\epsilon > 0$, whenever a function $v \in C^2([a, b], \mathbb{R})$ satisfies

$$\begin{cases} \left|{}^{C}\mathbb{D}_{a+}^{\alpha,\lambda,\psi} v(t) + p(t)v'(t) + q(t)v(t) - r(t)\right| \leq \epsilon, \quad t \in [a, b], \\ v(a) = v'(a) = v(b) = 0, \end{cases} \tag{5.1}$$

there exists a solution $u \in C^2([a, b], \mathbb{R})$ of Eq (3.1) fulfilling the condition

$$\|u - v\|_{C^2} \leq C\epsilon.$$

Definition 5.2. (Generalized Ulam–Hyers stability) FBVP (3.1) is generalized Ulam–Hyers stable if there exists a continuous function $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ that satisfies $\phi(0) = 0$, such that for any $\epsilon > 0$ and for every function $v \in C^2([a, b], \mathbb{R})$ meeting the condition (5.1), there exists a solution $u \in C^2([a, b], \mathbb{R})$ of Eq (3.1) for which

$$\|u - v\|_{C^2} \leq \phi(\epsilon).$$

Definition 5.3. (Ulam–Hyers–Rassias stability) FBVP (3.1) is Ulam–Hyers–Rassias stable with respect to a function $\phi : [a, b] \to \mathbb{R}^+$ if there exists a constant $C_\phi > 0$ such that for every $\epsilon > 0$, if a function $v \in C^2([a, b], \mathbb{R})$ satisfies

$$\begin{cases} \left|{}^{C}\mathbb{D}_{a+}^{\alpha,\lambda,\psi} v(t) + p(t)v'(t) + q(t)v(t) - r(t)\right| \leq \epsilon\phi(t), \quad t \in [a, b], \\ v(a) = v'(a) = v(b) = 0, \end{cases} \tag{5.2}$$

then there exists a solution $u \in C^2([a, b], \mathbb{R})$ of Eq (3.1) ensuring that

$$\|u - v\|_{C^2} \leq C_\phi \epsilon \phi(t), \quad t \in [a, b].$$

For what follows, we use the symbols M and A as defined in Theorem 4.2.

Theorem 5.1. If $A \cdot M < 1$, then the FBVP (3.1) is Ulam–Hyers stable.

Proof. By Theorem 4.2, the problem (3.1) has a unique solution, which we denote by u. Also, observe that $v \in C^2([a, b], \mathbb{R})$ is a solution of (5.1) if and only if, for any $\epsilon > 0$, there exists $h \in C^2([a, b], \mathbb{R})$ satisfying $|h(t)| \leq \epsilon$ for all $t \in [a, b]$, such that

$$\begin{cases} h(t) = {}^{C}\mathbb{D}_{a+}^{\alpha,\lambda,\psi} v(t) + p(t)v'(t) + q(t)v(t) - r(t), \quad t \in [a, b], \\ v(a) = v'(a) = v(b) = 0. \end{cases}$$

Recalling the operator $\mathcal{T} : B_R \to B_R$, defined in (4.3), and proceeding as in Theorem 3.1, we obtain

$$v(t) = \frac{e^{-\lambda t}(\psi(t)-\psi(a))^2}{\Gamma(\alpha)(\psi(b)-\psi(a))^2}\int_a^b \psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}e^{\lambda\tau}\Big(p(\tau)v'(\tau)+q(\tau)v(\tau)-r(\tau)-h(\tau)\Big)\,d\tau$$
$$-\frac{e^{-\lambda t}}{\Gamma(\alpha)}\int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}e^{\lambda\tau}\Big(p(\tau)v'(\tau)+q(\tau)v(\tau)-r(\tau)-h(\tau)\Big)\,d\tau. \quad (5.3)$$

By Theorem 2.4, since $\mathcal{T}: B_R \to B_R$ is a contraction mapping with contraction constant AM, it follows that

$$\|u-v\|_{C^2} \le \frac{1}{1-AM}\|\mathcal{T}v - v\|_{C^2}. \quad (5.4)$$

Moreover, since v satisfies Eq (5.3), we obtain

$$\begin{aligned}
\|\mathcal{T}v - v\|_{C^2} &= \sup_{t\in[a,b]}|(\mathcal{T}v)(t)-v(t)| + \sup_{t\in[a,b]}|(\mathcal{T}v)'(t)-v'(t)| + \sup_{t\in[a,b]}|(\mathcal{T}v)''(t)-v''(t)| \\
&\le \sup_{t\in[a,b]}\Bigg|\frac{e^{-\lambda t}(\psi(t)-\psi(a))^2}{\Gamma(\alpha)(\psi(b)-\psi(a))^2}\int_a^b \psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}e^{\lambda\tau}h(\tau)\,d\tau \\
&- \frac{e^{-\lambda t}}{\Gamma(\alpha)}\int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}e^{\lambda\tau}h(\tau)\,d\tau\Bigg| \\
&+ \sup_{t\in[a,b]}\Bigg|\frac{e^{-\lambda t}}{\Gamma(\alpha)(\psi(b)-\psi(a))^2}\Big[-\lambda(\psi(t)-\psi(a))^2 + 2\psi'(t)(\psi(t)-\psi(a))\Big] \\
&\times \int_a^b \psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}e^{\lambda\tau}h(\tau)\,d\tau + \frac{\lambda e^{-\lambda t}}{\Gamma(\alpha)}\int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}e^{\lambda\tau}h(\tau)\,d\tau \\
&- \frac{e^{-\lambda t}(\alpha-1)\psi'(t)}{\Gamma(\alpha)}\int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-2}e^{\lambda\tau}h(\tau)\,d\tau\Bigg| \\
&+ \sup_{t\in[a,b]}\Bigg|\frac{e^{-\lambda t}}{\Gamma(\alpha)(\psi(b)-\psi(a))^2}\Big[\lambda^2(\psi(t)-\psi(a))^2 - 4\lambda\psi'(t)(\psi(t)-\psi(a)) + 2(\psi'(t))^2 \\
&+ 2\psi''(t)(\psi(t)-\psi(a))\Big]\int_a^b \psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}e^{\lambda\tau}h(\tau)\,d\tau \\
&- \frac{\lambda^2 e^{-\lambda t}}{\Gamma(\alpha)}\int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}e^{\lambda\tau}h(\tau)\,d\tau \\
&+ \frac{2\lambda e^{-\lambda t}(\alpha-1)\psi'(t)}{\Gamma(\alpha)}\int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-2}e^{\lambda\tau}h(\tau)\,d\tau \\
&- \frac{e^{-\lambda t}(\alpha-1)\psi''(t)}{\Gamma(\alpha)}\int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-2}e^{\lambda\tau}h(\tau)\,d\tau \\
&- \frac{e^{-\lambda t}(\alpha-1)(\alpha-2)(\psi'(t))^2}{\Gamma(\alpha)}\int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-3}e^{\lambda\tau}h(\tau)\,d\tau\Bigg| \\
&\le A\epsilon,
\end{aligned}$$

where we use the fact that $|h(t)| \le \epsilon$, for all $t \in [a,b]$ and follow similar calculations as in the proof of Theorem 4.2.

In conclusion, we have

$$\|u - v\|_{C^2} \leq \frac{A}{1 - AM}\epsilon,$$

which completes the proof. □

Corollary 5.1. If $A \cdot M < 1$, then the problem (3.1) is generalized Ulam–Hyers Stable.

Proof. This is an immediate consequence of Theorem 5.1, by choosing $\phi(\epsilon) = \frac{A}{1-AM}\epsilon$. □

Theorem 5.2. Assume that $A \cdot M < 1$. Let u be the unique solution of the problem (3.1), and let v satisfy Eq (5.2). Suppose that the function ϕ, as given in Definition 5.3, satisfies the following three conditions:

(1) $\mathbb{I}_{a+}^{\alpha,\lambda,\psi}\phi(b) \leq \phi(t)$, for all $t \in [a, b]$;
(2) $\mathbb{I}_{a+}^{\alpha-1,\lambda,\psi}\phi(b) \leq \phi(t)$, for all $t \in [a, b]$; and
(3) $\mathbb{I}_{a+}^{\alpha-2,\lambda,\psi}\phi(b) \leq \phi(t)$, for all $t \in [a, b]$.

Then, the FBVP (3.1) is Ulam–Hyers–Rassias stable with respect to the function ϕ.

Proof. Since v satisfies (5.2), for any $\epsilon > 0$, there exists $h \in C^2([a, b], \mathbb{R})$ such that:

(1) $|h(t)| \leq \epsilon\phi(t)$ for all $t \in [a, b]$;
(2) $h(t) = {}^{C}\mathbb{D}_{a+}^{\alpha,\lambda,\psi}v(t) + p(t)v'(t) + q(t)v(t) - r(t)$ for all $t \in [a, b]$; and
(3) $v(a) = v'(a) = v(b) = 0$.

Thus,

$$\begin{aligned}
\|\mathcal{T}v - v\|_{C^2} \leq\ & 2\epsilon\mathbb{I}_{a+}^{\alpha,\lambda,\psi}\phi(b) + \epsilon\frac{\lambda(\psi(b) - \psi(a))^2 + 2W(\psi(b) - \psi(a))}{(\psi(b) - \psi(a))^2}\mathbb{I}_{a+}^{\alpha,\lambda,\psi}\phi(b) \\
& + \epsilon\lambda\mathbb{I}_{a+}^{\alpha,\lambda,\psi}\phi(b) + \epsilon W\mathbb{I}_{a+}^{\alpha-1,\lambda,\psi}\phi(b) + \epsilon\frac{\lambda^2(\psi(b) - \psi(a))^2 + W(4\lambda + 2)(\psi(b) - \psi(a)) + 2W^2}{(\psi(b) - \psi(a))^2}\mathbb{I}_{a+}^{\alpha,\lambda,\psi}\phi(b) \\
& + \epsilon\lambda^2\mathbb{I}_{a+}^{\alpha,\lambda,\psi}\phi(b) + \epsilon W(2\lambda + 1)\mathbb{I}_{a+}^{\alpha-1,\lambda,\psi}\phi(b) + \epsilon W^2\mathbb{I}_{a+}^{\alpha-2,\lambda,\psi}\phi(b) \\
\leq\ & \frac{(2\lambda^2 + 2\lambda + W^2 + 2W + 2\lambda W + 2)(\psi(b) - \psi(a))^2 + 4W(1 + \lambda)(\psi(b) - \psi(a)) + 2W^2}{(\psi(b) - \psi(a))^2}\epsilon\phi(t).
\end{aligned}$$

From (5.4), we conclude that

$$\|u - v\|_{C^2} \leq C_\phi\epsilon\phi(t), \quad t \in [a, b],$$

where

$$C_\phi = \frac{(2\lambda^2 + 2\lambda + W^2 + 2W + 2\lambda W + 2)(\psi(b) - \psi(a))^2 + 4W(1 + \lambda)(\psi(b) - \psi(a)) + 2W^2}{(1 - AM)(\psi(b) - \psi(a))^2}.$$

This concludes the proof. □

6. Example

Consider the following FBVP:

$$\begin{cases} {}^{C}\mathbb{D}_{0+}^{2.5,0.1,\psi}u(t) + \frac{\ln(t+1)}{5}u'(t) + \frac{\cos(t)}{5}u(t) = t+3, \quad t \in [0,1], \\ u(0) = u'(1) = u(0) = 0. \end{cases}$$

The fractional order is $\alpha = 2.5$, the parameter is $\lambda = 0.1$, and the kernel ψ is the function $\psi : [0,1] \to \mathbb{R}$ given by $\psi(t) = \arctan(t)$. For this problem, the values of M and W, as defined in Theorem 4.2, are $M = 1/5$ and $W = 1$. Consequently, the respective value of A is $A \approx 4.3897$, and since the condition $AM < 1$ holds, the existence and uniqueness of the solution is guaranteed.

Furthermore, the conditions of Theorem 5.1 and Corollary 5.1 are satisfied, ensuring both Ulam–Hyers stability and generalized Ulam–Hyers stability of the FBVP. Regarding Theorem 5.2, consider the function $\phi(t) = 1 + 1/(t+1)$ for $t \in [0,1]$. Since

$$\mathbb{I}_{0+}^{2.5,0.1,\psi}\phi(1) \approx 0.2940, \quad \mathbb{I}_{0+}^{1.5,0.1,\psi}\phi(1) \approx 0.8952, \quad \text{and} \quad \mathbb{I}_{0+}^{0.5,0.1,\psi}\phi(1) \approx 1.4385,$$

the remaining assumptions of the theorem are satisfied. Consequently, the FBVP is Ulam–Hyers–Rassias stable with respect to the function ϕ.

Remark 6.1. The values of the parameters α and λ, as well as the choice of the kernel ψ, may determine whether the obtained results are applicable. For instance, in the previous example, taking $\lambda = 1$ yields $A \approx 17.0985$, and therefore the condition $AM < 1$ fails. As another illustration, keeping $\alpha = 2.5$ and $\lambda = 0.1$ but choosing the kernel $\psi(t) = (t+1)^2$ gives $W = 4$ and $A \approx 132.9266$. In this case as well, the hypothesis of Theorem 4.2 is not satisfied.

7. Conclusions

In this study, we extend previous work on FBVPs by introducing a generalized form of the derivative. We investigate an FBVP involving a tempered fractional derivative of order between 2 and 3, with respect to a smooth kernel. We analyze the existence, uniqueness, and stability of solutions to this problem. By utilizing Krasnoselskii's and Banach's fixed-point theorems, we derive conditions that guarantee the existence and uniqueness of solutions. Furthermore, we establish various forms of stability for the FBVP, including Ulam–Hyers stability, generalized Ulam–Hyers stability, and Ulam–Hyers–Rassias stability.

Author contributions

All authors contributed equally to this work. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Prof. Ricardo Almeida is an editorial board member for AIMS Mathematics and was not involved in the editorial review and/or the decision to publish this article.

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