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Research article

Certain subfamily of multivalently Bazilevič and non-Bazilevič functions involving the bounded boundary rotation

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Abstract: In the present paper, we introduce a certain subfamily of multivalently Bazilevič and non-Bazilevič functions associated with bounded boundary rotation to study some interesting properties, inclusion results, and the upper bounds of the initial Taylor-Maclaurin coefficients for functions in this subfamily. In addition, some special cases of this subfamily are also discussed.

Keywords: analytic functions; Bazilevič function; non-Bazilevič function; bounded boundary rotation
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1. Introduction

Let $\mathcal{H}_p$ denote the family of analytic functions of the following form:

$$\varphi(x) = x^p \left(1 + \sum_{j=1}^{\infty} b_{p+j} x^j\right), \qquad (p \in \mathbb{N} = \{1, 2, 3, ...\}), \tag{1.1}$$

which are p-valent (multivalent of order p) in $\Delta = \{x \in \mathbb{C} : |x| < 1\}$ with $\mathcal{H}_1 = \mathcal{H}$ and also, the subfamily of $\mathcal{H}$ consisting of univalent (one-to-one) functions in Δ is denoted by $\mathcal{U}$. We denote by $\mathcal{K}$ and $\mathcal{S}^*$ the usual subclasses of $\mathcal{U}$ consisting of functions that are, respectively, bounded turning and starlike in Δ, and have the following geometric inequalities: $\Re\{\varphi'(x)\} > 0$ and $\Re\{x\varphi'(x)/\varphi(x)\} > 0$. Singh [21] introduced an important subfamily of $\mathcal{U}$ denoted by $\mathcal{B}^\alpha$ that consists of Bazilevič functions with the next inequality:

$$\Re\left\{\frac{x\varphi'(x)}{\varphi(x)}\left[\frac{\varphi(x)}{x}\right]^\alpha\right\} > 0,$$

for a non-negative real number α. He noted in his work, that the cases $\alpha = 0$ and $\alpha = 1$ correspond to $\mathcal{S}^*$ and $\mathcal{K}$, respectively. In [14], Obradovic introduced and studied the well-known subfamily of non-Bazilevič functions, that is,

$$\mathcal{N}^{\beta} = \left\{ \varphi \in \mathcal{H} : \Re \left\{ \frac{x\varphi'(x)}{\varphi(x)} \left[\frac{x}{\varphi(x)} \right]^{\beta} \right\} > 0 \right\}, 0 < \beta < 1.$$

Recently, several research papers have appeared on subfamilies related to Bazilevič functions, non-Bazilevič functions that are sometimes defined by linear operators, and their generalizations (see, for example, [7, 20, 22–24, 27, 28]).

Let $Q_k(\sigma)$ denote the family of analytic functions $g(x)$ of the form:

$$g(x) = 1 + \sum_{j=1}^{\infty} d_j x^j \quad (x \in \Delta), \tag{1.2}$$

satisfying the following inequality

$$\int_0^{2\pi} \left| \frac{\Re\{g(x)\} - \sigma}{1 - \sigma} \right| d\theta \leq k\pi, \tag{1.3}$$

where $k \geq 2$, $0 \leq \sigma < 1$, and $x = re^{i\theta} \in \Delta$. The family $Q_k(\sigma)$ was introduced and studied by Padmanabhan and Parvatham [17]. For $\sigma = 0$, we obtain the family $Q_k(0) = Q_k$ that was introduced by Pinchuk [18].

Remark 1. For $g(x) \in Q_k(\sigma)$, we can write

$$g(x) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + (1 - 2\sigma) xe^{-is}}{1 - xe^{-is}} d\mu(s) \quad (x \in \Delta), \tag{1.4}$$

where $\mu(s)$ is a function with bounded variation on $[0, 2\pi]$ such that

$$\int_0^{2\pi} d\mu(s) = 2\pi, \qquad \int_0^{2\pi} |d\mu(s)| < k\pi. \tag{1.5}$$

Since $\mu(s)$ has a bounded variation on $[0, 2\pi]$, we may put $\mu(s) = A(s) - B(s)$, where $A(s)$ and $B(s)$ are two non-negative increasing functions on $[0, 2\pi]$ satisfying (1.5). Thus, if we set $A(s) = \frac{k+2}{4}\mu_1(s)$ and $B(s) = \frac{k-2}{4}\mu_2(s)$, then (1.4) becomes

$$\begin{aligned} g(x) = &\left(\frac{k+2}{4}\right) \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + (1 - 2\sigma) xe^{-is}}{1 - xe^{-is}} d\mu_1(s) \\ &- \left(\frac{k-2}{4}\right) \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + (1 - 2\sigma) xe^{-is}}{1 - xe^{-is}} d\mu_2(s). \end{aligned} \tag{1.6}$$

Now, using Herglotz-Stieltjes formula for the family $Q(\sigma)$ of analytic functions with positive real part greater than σ and (1.6), we obtain

$$g(x) = \left(\frac{k+2}{4}\right) g_1(x) - \left(\frac{k-2}{4}\right) g_2(x), \tag{1.7}$$

where $g_1(x), g_2(x) \in Q(\sigma)$. Also, we have here $Q(0) = Q$, where Q is the family of analytic functions $g(x)$ in Δ with $\Re\{g(x)\} > 0$.

Remark 2. It is well-known from [13] that the family $Q_k(\sigma)$ is a convex set.

Remark 3. For $0 \leq \sigma_1 < \sigma_2 < 1$, we have $Q_k(\sigma_2) \subset Q_k(\sigma_1)$ (see [6]).

In recent years, researchers have been using the family $Q_k(\sigma)$ of analytic functions associated with bounded boundary rotation in various branches of mathematics very effectively, especially in geometric function theory (GFT). For further developments and discussion about this family, we can obtain selected articles produced by some mathematicians like [1,4,5,8,12,25] and many more.

Now, using the family $Q_k(\sigma)$, we introduce the subfamily $\mathcal{BN}_{p,k}^{\alpha,\beta}(\lambda,\sigma)$ of p-valent Bazilevič and non-Bazilevič functions of $\mathcal{H}_p$ as the following definition:

Definition 1. A function $\varphi \in \mathcal{H}_p$ is said to be in the subfamily $\mathcal{BN}_{p,k}^{\alpha,\beta}(\lambda,\sigma)$ if it satisfies the following condition:

$$\left(1 - \frac{\alpha-\beta}{\alpha+\beta}\lambda\right)\left[\frac{\varphi(x)}{x^p}\right]^{\alpha-\beta} + \frac{\alpha-\beta}{\alpha+\beta}\lambda\frac{x\varphi'(x)}{p\varphi(x)}\left[\frac{\varphi(x)}{x^p}\right]^{\alpha-\beta} \in Q_k(\sigma),$$

which is equivalent to

$$\int_0^{2\pi}\left|\frac{\Re\left\{\left(1 - \frac{\alpha-\beta}{\alpha+\beta}\lambda\right)\left[\frac{\varphi(x)}{x^p}\right]^{\alpha-\beta} + \frac{\alpha-\beta}{\alpha+\beta}\lambda\frac{x\varphi'(x)}{p\varphi(x)}\left[\frac{\varphi(x)}{x^p}\right]^{\alpha-\beta}\right\} - \sigma}{1-\sigma}\right| d\theta \leq k\pi,$$

where $\alpha, \beta \geq 0$; $\alpha+\beta \geq 0$; $\lambda > 0$; $k \geq 2$; $0 \leq \sigma < 1$; $x \in \Delta$; and all powers are principal ones.

Example 1. Let $\varphi(x) : \Delta \to \mathbb{C}$ be an analytic function given by

$$\varphi(x) = x^p\left(1 + \frac{p(\alpha+\beta)(1-\sigma)k}{[p(\alpha+\beta)+\lambda]}x\right)^{\frac{1}{\alpha-\beta}} \in \mathcal{H}_p \quad (\alpha \neq \beta).$$

Clearly $\varphi(x) \in \mathcal{H}_p$ (with all powers being principal ones). After some calculations, we find that

$$\left(1 - \frac{\alpha-\beta}{\alpha+\beta}\lambda\right)\left[\frac{\varphi(x)}{x^p}\right]^{\alpha-\beta} + \frac{\alpha-\beta}{\alpha+\beta}\lambda\left[\frac{\varphi(x)}{x^p}\right]^{\alpha-\beta}\frac{x\varphi'(x)}{p\varphi(x)} = 1 + (1-\sigma)kx.$$

Now, if $x = re^{i\theta}$ $(0 \leq r < 1)$, then

$$\frac{\Re\left\{\left(1 - \frac{\alpha-\beta}{\alpha+\beta}\lambda\right)\left[\frac{\varphi(x)}{x^p}\right]^{\alpha-\beta} + \frac{\alpha-\beta}{\alpha+\beta}\lambda\frac{x\varphi'(x)}{p\varphi(x)}\left[\frac{\varphi(x)}{x^p}\right]^{\alpha-\beta}\right\} - \sigma}{1-\sigma} = 1 + kr\cos\theta,$$

and

$$\begin{aligned}\int_0^{2\pi}\left|\frac{\Re\left\{\left(1 - \frac{\alpha-\beta}{\alpha+\beta}\lambda\right)\left[\frac{\varphi(x)}{x^p}\right]^{\alpha-\beta} + \frac{\alpha-\beta}{\alpha+\beta}\lambda\frac{x\varphi'(x)}{p\varphi(x)}\left[\frac{\varphi(x)}{x^p}\right]^{\alpha-\beta}\right\} - \sigma}{1-\sigma}\right| d\theta &= \int_0^{2\pi}(1 + kr\cos\theta)\,d\theta \\ &= 2\pi \leq k\pi \quad (k \geq 2).\end{aligned}$$

Hence, $\varphi(x)$ belongs to the subfamily $\mathcal{BN}_{p,k}^{\alpha,\beta}(\lambda,\sigma)$, and it is not empty.

By specializing the parameters α, β, λ, p, k, and σ involved in Definition 1, we get the following subfamilies, which were studied in many earlier works:
(i) For $k = 2, \beta = 0$ and $\sigma = \frac{1-L}{1-M}$ $(-1 \leq M < L \leq 1)$, we have $\mathcal{BN}_{2,p}^{\alpha,0}\left(\lambda, \frac{1-L}{1-M}\right) = \mathcal{B}_p^{\alpha}(\lambda, L, M)$

$$= \left\{\varphi \in \mathcal{H}_p : (1-\lambda)\left[\frac{\varphi(x)}{x^p}\right]^{\alpha} + \lambda\frac{x\varphi'(x)}{p\varphi(x)}\left[\frac{\varphi(x)}{x^p}\right]^{\alpha} \prec \frac{1+Lx}{1+Mx}\right\},$$

where $\prec$ denotes the usual meaning of subordination, $\mathcal{B}_p^{\alpha}(\lambda, L, M)$ is a subfamily of multivalently Bazilevič functions introduced by Liu [10] and $\mathcal{B}_p^{\alpha}(1, L, M) = \mathcal{B}_p^{\alpha}(L, M)$

$$= \left\{\varphi \in \mathcal{H}_p : \frac{x\varphi'(x)}{p\varphi(x)}\left[\frac{\varphi(x)}{x^p}\right]^{\alpha} \prec \frac{1+Lx}{1+Mx}\right\},$$

where the subfamily $\mathcal{B}_p^{\alpha}(L, M)$ was introduced by Yang [30];
(ii) $\mathcal{BN}_{2,1}^{\alpha,0}\left(1, \frac{1-L}{1-M}\right) = \mathcal{B}^{\alpha}(L, M)$

$$= \left\{\varphi \in \mathcal{H} : \frac{x\varphi'(x)}{\varphi(x)}\left[\frac{\varphi(x)}{x}\right]^{\alpha} \prec \frac{1+Lx}{1+Mx}\right\},$$

where the subfamily $\mathcal{B}^{\alpha}(L, M)$ was studied by Singh [21] (see also Owa and Obradovic [15]);
(iii) $\mathcal{BN}_{2,1}^{\alpha,0}(1, \sigma) = \mathcal{B}^{\alpha}(\sigma)$

$$= \left\{\varphi \in \mathcal{H} : \Re\left\{\frac{x\varphi'(x)}{\varphi(x)}\left[\frac{\varphi(x)}{x}\right]^{\alpha}\right\} > \sigma\right\},$$

where the family $\mathcal{B}^{\alpha}(\sigma)$ was considered by Owa [16];
(iv) For $k = 2$, $\alpha = 0$ and $\sigma = \frac{1-L}{1-M}$ $(-1 \leq M < L \leq 1)$, we have $\mathcal{BN}_{2,p}^{0,\beta}\left(\lambda, \frac{1-L}{1-M}\right) = \mathcal{N}_p^{\beta}(\lambda, L, M)$

$$= \left\{\varphi \in \mathcal{H}_p : (1+\lambda)\left[\frac{x^p}{\varphi(x)}\right]^{\beta} - \lambda\frac{x\varphi'(x)}{p\varphi(x)}\left[\frac{x^p}{\varphi(x)}\right]^{\beta} \prec \frac{1+Lx}{1+Mx}\right\},$$

where $\mathcal{N}_p^{\beta}(\lambda, L, M)$ is the family of non-Bazilevič multivalent functions introduced by Aouf and Seoudy [3], and $\mathcal{N}_1^{\beta}(\lambda, L, M) = \mathcal{N}^{\beta}(\lambda, L, M)$

$$= \left\{\varphi \in \mathcal{H} : (1+\lambda)\left[\frac{x}{\varphi(x)}\right]^{\beta} - \lambda\frac{x\varphi'(x)}{\varphi(x)}\left[\frac{x}{\varphi(x)}\right]^{\beta} \prec \frac{1+Lx}{1+Mx}\right\},$$

where $\mathcal{N}^{\beta}(\lambda, L, M)$ is the subclass of non-Bazilevič univalent functions defined by Wang et al. [29];
(v) $\mathcal{BN}_{2,1}^{0,\beta}(-1, \sigma) = \mathcal{N}^{\beta}(\sigma)$

$$= \left\{\varphi \in \mathcal{H} : \Re\left\{\frac{x\varphi'(x)}{\varphi(x)}\left[\frac{x}{\varphi(x)}\right]^{\beta}\right\} > \sigma\right\},$$

where $\mathcal{N}^{\beta}(\sigma)$ is the family of non-Bazilevič functions of order σ (see Tuneski and Daus [26]) and $\mathcal{N}^{\beta}(0) = \mathcal{N}^{\beta}$

$$= \left\{\varphi \in \mathcal{H} : \Re\left\{\frac{x\varphi'(x)}{\varphi(x)}\left[\frac{x}{\varphi(x)}\right]^{\beta}\right\} > 0\right\},$$

where $\mathcal{N}^{\beta}$ is the family of non-Bazilevič functions (see Obradovic [14]).

Also, we note that

(i) $\mathcal{BN}_{k,p}^{\alpha,0}(\lambda,\sigma) = \mathcal{B}_{k,p}^{\alpha}(\lambda,\sigma)$

$$= \left\{\varphi \in \mathcal{H}_p : (1-\lambda)\left[\frac{\varphi(x)}{x^p}\right]^{\alpha} + \lambda \frac{x\varphi'(x)}{p\varphi(x)}\left[\frac{\varphi(x)}{x^p}\right]^{\alpha} \in \mathcal{Q}_k(\sigma)\right\},$$

and $\mathcal{B}_{k,1}^{\alpha}(\lambda,\sigma) = \mathcal{B}_{k}^{\alpha}(\lambda,\sigma)$

$$= \left\{\varphi \in \mathcal{H} : (1-\lambda)\left[\frac{\varphi(x)}{x}\right]^{\alpha} + \lambda \frac{x\varphi'(x)}{\varphi(x)}\left[\frac{\varphi(x)}{x}\right]^{\alpha} \in \mathcal{Q}_k(\sigma)\right\};$$

(ii) $\mathcal{BN}_{k,p}^{0,\beta}(\lambda,\sigma) = \mathcal{N}_{k,p}^{\beta}(\lambda,\sigma)$

$$= \left\{\varphi \in \mathcal{H}_p : (1+\lambda)\left[\frac{x^p}{\varphi(x)}\right]^{\beta} - \lambda \frac{x\varphi'(x)}{p\varphi(x)}\left[\frac{x^p}{\varphi(x)}\right]^{\beta} \in \mathcal{Q}_k(\sigma)\right\},$$

and $\mathcal{N}_{k,1}^{\beta}(\lambda,\sigma) = \mathcal{N}_{k}^{\beta}(\lambda,\sigma)$

$$= \left\{\varphi \in \mathcal{H} : (1+\lambda)\left[\frac{x}{\varphi(x)}\right]^{\beta} - \lambda \frac{x\varphi'(x)}{\varphi(x)}\left[\frac{x}{\varphi(x)}\right]^{\beta} \in \mathcal{Q}_k(\sigma)\right\};$$

(iii) $\mathcal{BN}_{k,p}^{1,0}(\lambda,\sigma) = \mathcal{B}_{k,p}(\lambda,\sigma)$

$$= \left\{\varphi \in \mathcal{H}_p : (1-\lambda)\frac{\varphi(x)}{x^p} + \lambda \frac{\varphi'(x)}{px^{p-1}} \in \mathcal{Q}_k(\sigma)\right\},$$

and $\mathcal{B}_{k,1}(\lambda,\sigma) = \mathcal{B}_{k}(\lambda,\sigma)$

$$= \left\{\varphi \in \mathcal{H} : (1-\lambda)\frac{\varphi(x)}{x} + \lambda\varphi'(x) \in \mathcal{Q}_k(\sigma)\right\};$$

(iv) $\mathcal{BN}_{k,p}^{0,1}(\lambda,\sigma) = \mathcal{N}_{k,p}(\lambda,\sigma)$

$$= \left\{\varphi \in \mathcal{H}_p : (1+\lambda)\frac{x^p}{\varphi(x)} - \lambda \frac{x^{p+1}\varphi'(x)}{p\varphi^2(x)} \in \mathcal{Q}_k(\sigma)\right\},$$

and $\mathcal{N}_{k,1}(\lambda,\sigma) = \mathcal{N}_{k}(\lambda,\sigma)$

$$= \left\{\varphi \in \mathcal{H} : (1+\lambda)\frac{x}{\varphi(x)} - \lambda \frac{x^{2}\varphi'(x)}{\varphi^2(x)} \in \mathcal{Q}_k(\sigma)\right\};$$

(v) $\mathcal{BN}_{k,p}^{\alpha,0}(1,\sigma) = \mathcal{B}_{k,p}^{\alpha}(\sigma)$

$$= \left\{\varphi \in \mathcal{H}_p : \frac{x\varphi'(x)}{p\varphi(x)}\left[\frac{\varphi(x)}{x^p}\right]^{\alpha} \in \mathcal{Q}_k(\sigma)\right\},$$

and $\mathcal{B}_{k,1}^{\alpha}(\sigma) = \mathcal{B}_{k}^{\alpha}(\sigma)$

$$= \left\{\varphi \in \mathcal{H} : \frac{x\varphi'(x)}{\varphi(x)} \left[\frac{\varphi(x)}{x}\right]^{\alpha} \in \mathcal{Q}_k(\sigma)\right\};$$

(vi) $\mathcal{BN}_{k,p}^{0,\beta}(-1,\sigma) = \mathcal{N}_{k,p}^{\beta}(\sigma)$

$$= \left\{\varphi \in \mathcal{H}_p : \frac{x\varphi'(x)}{p\varphi(x)} \left[\frac{x^p}{\varphi(x)}\right]^{\beta} \in \mathcal{Q}_k(\sigma)\right\},$$

and $\mathcal{N}_{k,1}^{\beta}(\sigma) = \mathcal{N}_{k}^{\beta}(\sigma)$

$$= \left\{\varphi \in \mathcal{H} : \frac{x\varphi'(x)}{\varphi(x)} \left[\frac{x}{\varphi(x)}\right]^{\beta} \in \mathcal{Q}_k(\sigma)\right\};$$

(vii) $\mathcal{B}_{k,p}^{0}(\sigma) = \mathcal{S}_{k,p}(\sigma)$

$$= \left\{\varphi \in \mathcal{H}_p : \frac{x\varphi'(x)}{p\varphi(x)} \in \mathcal{Q}_k(\sigma)\right\},$$

and $\mathcal{S}_{k,1}(\sigma) = \mathcal{S}_k(\sigma)$

$$= \left\{\varphi \in \mathcal{H} : \frac{x\varphi'(x)}{\varphi(x)} \in \mathcal{Q}_k(\sigma)\right\}.$$

To prove our main results, the next lemmas will be required in our investigation.

Lemma 1. [11] Let $\gamma = \gamma_1 + i\gamma_2$ and $\delta = \delta_1 + i\delta_2$ and $\Theta(\gamma, \delta)$ be a complex-valued function satisfying the next conditions:

(i) $\Theta(\gamma, \delta)$ is continuous in a domain $D \in \mathbb{C}^2$.

(ii) $(0, 1) \in D$ and $\Theta(1, 0) > 0$.

(iii) $\Re\{\Theta(i\gamma_2, \delta_1)\} > 0$ whenever $(i\gamma_2, \delta_1) \in D$ and $\delta_1 \leq -\frac{1+\gamma_2^2}{2}$.

If $g(x)$ given by (1.2) is analytic in Δ such that $(g(x), xg'(x)) \in D$ and $\Re\{\Theta(g(x), xg'(x))\} > 0$ for $x \in \Delta$, then $\Re\{g(x)\} > 0$ in Δ.

Lemma 2. [2, Theorem 5 with $p = 1$] If $g(x) \in \mathcal{Q}_k(\sigma)$ is given by (1.2), then

$$\left|d_j\right| \leq (1-\sigma) k \quad (j \in \mathbb{N}). \tag{1.8}$$

This result is sharp.

Remark 4. For $\sigma = 0$ in Lemma 2, we get the result for the family $\mathcal{Q}_k$ obtained by Goswami et al. [9].

In the present article, we have combined Bazilevič and non-Bazilevič analytic functions into a new family $\mathcal{BN}_{k,p}^{\alpha,\beta}(\lambda,\sigma)$ associated with a bounded boundary rotation. In the next section, several properties like inclusion results, some connections with the generalized Bernardi-Libera-Livingston integral operator, and the upper bounds for $\left|b_{p+1}\right|$ and $\left|b_{p+2} + \frac{\alpha-\beta-1}{2}b_{p+1}^2\right|$ for this family $\mathcal{BN}_{k,p}^{\alpha,\beta}(\lambda,\sigma)$ and its special subfamilies are investigated. The motivation of this article is to generalize and improve previously known works.

2. Main results

Theorem 1. If $\varphi \in \mathcal{BN}_{k,p}^{\alpha,\beta}(\lambda, \sigma)$, then

$$\left[\frac{\varphi(x)}{x^p}\right]^{\alpha-\beta} \in \mathcal{Q}_k(\sigma_1), \tag{2.1}$$

where σ_1 is given by

$$\sigma_1 = \frac{2p(\alpha+\beta)\sigma + \lambda}{2p(\alpha+\beta)+\lambda}. \tag{2.2}$$

Proof. Let $\varphi \in \mathcal{BN}_{k,p}^{\alpha,\beta}(\lambda, \sigma)$ and set

$$\left[\frac{\varphi(x)}{x^p}\right]^{\alpha-\beta} = (1-\sigma_1)g(x) + \sigma_1 \qquad (x \in \Delta) \tag{2.3}$$

$$= \left(\frac{k+2}{4}\right)\{(1-\sigma_1)g_1(x)+\sigma_1\} - \left(\frac{k-2}{4}\right)\{(1-\sigma_1)g_2(x)+\sigma_1\},$$

where $g_i(x)$ is analytic in Δ with $g_i(0) = 1$, $i = 1, 2$. Differentiating (2.3) with respect to x, we obtain

$$\left(1 - \frac{\alpha-\beta}{\alpha+\beta}\lambda\right)\left[\frac{\varphi(x)}{x^p}\right]^{\alpha-\beta} + \frac{\alpha-\beta}{\alpha+\beta}\lambda\frac{x\varphi'(x)}{p\varphi(x)}\left[\frac{\varphi(x)}{x^p}\right]^{\alpha-\beta}$$

$$= (1-\sigma_1)g(x) + \sigma_1 + \frac{\lambda(1-\sigma_1)xg'(x)}{p(\alpha+\beta)} \in \mathcal{Q}_k(\sigma),$$

this implies that

$$\frac{1}{1-\sigma}\left\{(1-\sigma_1)g_i(x) + \sigma_1 - \sigma + \frac{\lambda(1-\sigma_1)xg_i'(x)}{p(\alpha+\beta)}\right\} \in \mathcal{Q} \quad (x \in \Delta; i = 1, 2).$$

We form the functional $\Theta(\gamma, \delta)$ by choosing $\gamma = g_i(x)$, $\delta = xg_i'(x)$,

$$\Theta(\gamma, \delta) = (1-\sigma_1)\gamma + \sigma_1 - \sigma + \frac{\lambda(1-\sigma_1)\delta}{p(\alpha+\beta)}.$$

Clearly, the first two conditions of Lemma 1 are satisfied. Now, we verify the condition (iii) of Lemma 1 as follows:

$$\begin{aligned}
\Re\{\Theta(i\gamma_2, \delta_1)\} &= \sigma_1 - \sigma + \Re\left\{\frac{\lambda(1-\sigma_1)\delta_1}{p(\alpha+\beta)}\right\} \\
&\leq \sigma_1 - \sigma - \frac{\lambda(1-\sigma_1)\left(1+\gamma_2^2\right)}{2p(\alpha+\beta)} \\
&= \frac{2p(\alpha+\beta)(\sigma_1-\sigma) - \lambda(1-\sigma_1) - \lambda(1-\sigma_1)\gamma_2^2}{2p(\alpha+\beta)} \\
&= \frac{A + B\gamma_2^2}{2C},
\end{aligned}$$

where

$$\begin{aligned} A &= 2p(\alpha+\beta)(\sigma_1-\sigma)-\lambda(1-\sigma_1), \\ B &= -\lambda(1-\sigma_1)<0, \\ C &= 2p(\alpha+\beta)>0. \end{aligned}$$

We note that $\Re\{\Theta(i\gamma_2,\delta_1)\}<0$ if and only if $A=0$, $B<0$, and $C>0$, and this gives us

$$\sigma_1=\frac{2p(\alpha+\beta)\sigma+\lambda}{2p(\alpha+\beta)+\lambda}.$$

Since $B=-\lambda(1-\sigma_1)<0$ gives us $0\le\sigma_1<1$. Therefore, applying Lemma 1, $g_i(x)\in\mathcal{Q}(i=1,2)$ and consequently $g(x)\in\mathcal{Q}_k(\sigma_1)$ for $x\in\Delta$. This completes the proof of Theorem 1. □

Putting $\beta=0$ in Theorem 1, we obtain the next result.

Corollary 1. If $\varphi\in\mathcal{B}^{\alpha}_{k,p}(\lambda,\sigma)$, then

$$\left[\frac{\varphi(x)}{x^p}\right]^{\alpha}\in\mathcal{Q}_k(\sigma_2),$$

where σ_2 is given by

$$\sigma_2=\frac{2p\alpha\rho+\lambda}{2p\alpha+\lambda}.$$

Putting $\alpha=0$ in Theorem 1, we get the following result.

Corollary 2. If $\varphi\in\mathcal{N}^{\beta}_{k,p}(\lambda,\sigma)$, then

$$\left[\frac{x^p}{\varphi(x)}\right]^{\beta}\in\mathcal{Q}_k(\sigma_3),$$

where σ_3 is given by

$$\sigma_3=\frac{2p\beta\rho+\lambda}{2p\beta+\lambda}.$$

Theorem 2. If $\varphi\in\mathcal{BN}^{\alpha,\beta}_{k,p}(\lambda,\sigma)$, then

$$\left[\frac{\varphi(x)}{x^p}\right]^{\frac{\alpha-\beta}{2}}\in\mathcal{Q}_k(\sigma_4), \tag{2.4}$$

where σ_4 is given by

$$\sigma_4=\frac{\lambda+\sqrt{\lambda^2+4[p(\alpha+\beta)+\lambda]\,p(\alpha+\beta)\sigma}}{2[p(\alpha+\beta)+\lambda]}. \tag{2.5}$$

Proof. Let $\varphi\in\mathcal{BN}^{\alpha,\beta}_{k,p}(\lambda,\sigma)$ and let

$$\begin{aligned} \left[\frac{\varphi(x)}{x^p}\right]^{\alpha-\beta} &= \left(\frac{k}{4}+\frac{1}{2}\right)[(1-\sigma_4)g_1(x)+\sigma_4]^2-\left(\frac{k}{4}+\frac{1}{2}\right)[(1-\sigma_4)g_1(x)+\sigma_4]^2 \\ &= [(1-\sigma_4)g(x)+\sigma_4]^2, \end{aligned} \tag{2.6}$$

where $g_i(x)$ is analytic in Δ with $g_i(0) = 1$, $i = 1, 2$. Differentiating both sides of (2.6) with respect to x, we obtain

$$\left(1 - \frac{\alpha-\beta}{\alpha+\beta}\lambda\right)\left[\frac{\varphi(x)}{x^p}\right]^{\alpha-\beta} + \frac{\alpha-\beta}{\alpha+\beta}\lambda\frac{x\varphi'(x)}{p\varphi(x)}\left[\frac{\varphi(x)}{x^p}\right]^{\alpha-\beta}$$
$$= \left\{\left[(1-\sigma_4)g(x)+\sigma_4\right]^2 + \left[(1-\sigma_4)g(x)+\sigma_4\right]\frac{2\lambda(1-\sigma_4)xg'(x)}{p(\alpha+\beta)}\right\} \in \mathcal{Q}_k(\sigma),$$

this implies that

$$\frac{1}{1-\sigma}\left\{\left[(1-\sigma_4)g_i(x)+\sigma_4\right]^2 + \left[(1-\sigma_4)g_i(x)+\sigma_4\right]\frac{2\lambda(1-\sigma_4)xg_i'(x)}{p(\alpha+\beta)} - \sigma\right\} \in \mathcal{Q} \quad (i = 1, 2).$$

We form the functional $\Theta(\gamma, \delta)$ by choosing $\gamma = g_i(x)$, $\delta = xg_i'(x)$,

$$\Theta(\gamma,\delta) = \left[(1-\sigma_4)\gamma+\sigma_4\right]^2 + \left[(1-\sigma_4)\gamma+\sigma_4\right]\frac{2\lambda(1-\sigma_4)\delta}{p(\alpha+\beta)} - \sigma.$$

Clearly, the conditions (i) and (ii) of Lemma 1 are satisfied. Now, we verify the condition (iii) of Lemma 1 as follows:

$$\begin{aligned}\Re\{\Theta(i\gamma_2,\delta_1)\} &= \sigma_4^2 - (1-\sigma_4)^2\gamma_2^2 + \frac{2\lambda\sigma_4(1-\sigma_4)\delta_1}{p(\alpha+\beta)} - \sigma \\ &\leq \sigma_4^2 - \sigma - (1-\sigma_4)^2\gamma_2^2 - \frac{\lambda\sigma_4(1-\sigma_4)\left(1+\gamma_2^2\right)}{p(\alpha+\beta)} \\ &= \frac{A+B\gamma_2^2}{2C},\end{aligned}$$

where

$$\begin{aligned}A &= p(\alpha+\beta)\left(\sigma_4^2-\sigma\right) - \lambda\sigma_4(1-\sigma_4), \\ B &= -(1-\sigma_4)(1-\sigma_4+\lambda\sigma_4) < 0, \\ C &= \frac{p(\alpha+\beta)}{2} > 0.\end{aligned}$$

We note that $\Re\{\Theta(i\gamma_2,\delta_1)\} < 0$ if and only if $A = 0$, $B < 0$, and $C > 0$, and this gives us σ_4 as given by (2.5), and $B < 0$ gives us $0 \leq \sigma_4 < 1$. Therefore, applying Lemma 1, $g_i(x) \in \mathcal{Q}$ ($i = 1, 2$), and consequently $g(x) \in \mathcal{Q}_k(\sigma_4)$ for $x \in \Delta$. This completes the proof of Theorem 2. □

Putting $\beta = 0$ in Theorem 2, we obtain the following.

Corollary 3. If $\varphi \in \mathcal{B}^{\alpha}_{k,p}(\lambda, \sigma)$, then

$$\left[\frac{\varphi(x)}{x^p}\right]^{\frac{\alpha}{2}} \in \mathcal{Q}_k(\sigma_5),$$

where σ_5 is given by

$$\sigma_5 = \frac{\lambda + \sqrt{\lambda^2 + 4(p\alpha+\lambda)\rho p\alpha}}{2(p\alpha+\lambda)}.$$

Putting $\alpha = 0$ in Theorem 2, we obtain the following.

Corollary 4. If $\varphi \in \mathcal{N}^{\beta}_{k,p}(\lambda,\sigma)$, then

$$\left[\frac{x^p}{\varphi(x)}\right]^{\frac{\beta}{2}} \in \mathcal{Q}_k(\sigma_6),$$

where σ_6 is given by

$$\sigma_6 = \frac{\lambda + \sqrt{\lambda^2 + 4(p\beta + \lambda)\rho p\beta}}{2(p\beta + \lambda)}.$$

For a function $\varphi \in \mathcal{H}_p$, the generalized Bernardi-Libera-Livingston integral operator $\Phi_{p,\mu} : \mathcal{H}_p \to \mathcal{H}_p$, with $\mu > -p$, is given by (see [19])

$$\Phi_{p,\mu}(\varphi(x)) = \frac{\mu + p}{x^{\mu}} \int_0^x \omega^{\mu-1}\varphi(\omega)\, d\omega \quad (\mu > -p). \tag{2.7}$$

It is easy to verify that for all $\varphi \in \mathcal{H}_p$ given by (1.2), we have

$$x\left(\Phi_{p,\mu}(\varphi(x))\right)' = (\mu + p)\varphi(x) - \mu\Phi_{p,\mu}(\varphi(x)). \tag{2.8}$$

Theorem 3. If the function $\varphi \in \mathcal{H}_p$ satisfies the next condition

$$\left(1 - \frac{\alpha-\beta}{\alpha+\beta}\lambda\right)\left[\frac{\Phi_{p,\mu}(\varphi(x))}{x^p}\right]^{\alpha-\beta} + \frac{\alpha-\beta}{\alpha+\beta}\lambda\frac{\varphi(x)}{\Phi_{p,\mu}(\varphi(x))}\left[\frac{\Phi_{p,\mu}(\varphi(x))}{x^p}\right]^{\alpha-\beta} \in \mathcal{Q}_k(\sigma), \tag{2.9}$$

with $\Phi_{p,\mu}$ is the integral operator defined by (2.7), then

$$\left[\frac{\Phi_{p,\mu}(\varphi(x))}{x^p}\right]^{\alpha-\beta} \in \mathcal{Q}_k(\sigma_7),$$

where σ_7 is given by

$$\sigma_7 = \frac{2(p+\mu)(\alpha+\beta)\sigma + \lambda}{2(p+\mu)(\alpha+\beta) + \lambda}. \tag{2.10}$$

Proof. Let

$$\begin{aligned}\left[\frac{\Phi_{p,\mu}(\varphi(x))}{x^p}\right]^{\alpha-\beta} &= \left(\frac{k}{4} + \frac{1}{2}\right)\{(1-\sigma_7)g_1(x) + \sigma_7\} - \left(\frac{k}{4} + \frac{1}{2}\right)\{(1-\sigma_7)g_2(x) + \sigma_7\} \\ &= (1-\sigma_7)g(x) + \sigma_7 \quad (x \in \Delta),\end{aligned} \tag{2.11}$$

then where $g_i(x)$ is analytic in Δ with $g_i(0) = 1$, $i = 1, 2$. Differentiating (2.11) with respect to x and using (2.8) in the resulting relation, we get

$$\begin{aligned}&\left(1 - \frac{\alpha-\beta}{\alpha+\beta}\lambda\right)\left[\frac{\Phi_{p,\mu}(\varphi(x))}{x^p}\right]^{\alpha-\beta} + \frac{\alpha-\beta}{\alpha+\beta}\lambda\frac{\varphi(x)}{\Phi_{p,\mu}(\varphi(x))}\left[\frac{\Phi_{p,\mu}(\varphi(x))}{x^p}\right]^{\alpha-\beta} \\ &= (1-\sigma_7)g(x) + \frac{\lambda(1-\sigma_7)xg'(x)}{(p+\mu)(\alpha+\beta)} \in \mathcal{Q}_k(\sigma) \quad (x \in \Delta).\end{aligned}$$

Using the same method we used to prove Theorem 1, the remaining part of this theorem can be derived in a similar way. □

Putting $\beta = 0$ in Theorem 3, we obtain the following.

Corollary 5. If the function $\varphi \in \mathcal{H}_p$ satisfies the next condition

$$(1-\lambda)\left[\frac{\Phi_{p,\mu}(\varphi(x))}{x^p}\right]^{\alpha} + \lambda\frac{\varphi(x)}{\Phi_{p,\mu}(\varphi(x))}\left[\frac{\Phi_{p,\mu}(\varphi(x))}{x^p}\right]^{\alpha} \in \mathcal{Q}_k(\sigma),$$

with $\Phi_{p,\mu}$ is defined by (2.7), then

$$\left[\frac{\Phi_{p,\mu}(\varphi(x))}{x^p}\right]^{\alpha} \in \mathcal{Q}_k(\sigma_8),$$

where σ_8 is given by

$$\sigma_8 = \frac{2(p+\mu)\alpha\rho + \lambda}{2(p+\mu)\alpha + \lambda}.$$

Putting $\alpha = 0$ in Theorem 3, we obtain the following.

Corollary 6. If the function $\varphi \in \mathcal{H}_p$ satisfies the next condition

$$(1+\lambda)\left[\frac{x^p}{\Phi_{p,\mu}(\varphi(x))}\right]^{\beta} - \lambda\frac{\varphi(x)}{\Phi_{p,\mu}(\varphi(x))}\left[\frac{x^p}{\Phi_{p,\mu}(\varphi(x))}\right]^{\beta} \in \mathcal{Q}_k(\sigma),$$

with $\Phi_{p,\mu}$ is defined by (2.7), then

$$\left[\frac{x^p}{\Phi_{p,\mu}(\varphi(x))}\right]^{\beta} \in \mathcal{Q}_k(\sigma_9),$$

where σ_9 is given by

$$\sigma_9 = \frac{2(p+\mu)(\alpha+\beta)\sigma + \lambda}{2(p+\mu)(\alpha+\beta) + \lambda}.$$

Theorem 4. If $0 \leq \lambda_1 < \lambda_2$, then

$$\mathcal{BN}_{k,p}^{\alpha,\beta}(\lambda_2, \sigma) \subset \mathcal{BN}_{k,p}^{\alpha,\beta}(\lambda_1, \sigma).$$

Proof. If we consider an arbitrary function $\varphi \in \mathcal{BN}_{k,p}^{\alpha,\beta}(\lambda_2, \sigma)$, then

$$\varphi_2(x) = \left(1 - \frac{\alpha-\beta}{\alpha+\beta}\lambda_2\right)\left[\frac{\varphi(x)}{x^p}\right]^{\alpha-\beta} + \frac{\alpha-\beta}{\alpha+\beta}\lambda_2\frac{x\varphi'(x)}{p\varphi(x)}\left[\frac{\varphi(x)}{x^p}\right]^{\alpha-\beta} \in \mathcal{Q}_k(\sigma).$$

According to Theorem 1, we have

$$\varphi_1(x) = \left[\frac{\varphi(x)}{x^p}\right]^{\alpha-\beta} \in \mathcal{Q}_k(\sigma_1),$$

where σ_1 is given by (2.2). From (2.2), it follows that $\sigma_1 \geq \sigma$, and from Remark 3, we conclude that $\mathcal{Q}_k(\sigma_1) \subset \mathcal{Q}_k(\sigma)$; hence, $\varphi_1(x) \in \mathcal{Q}_k(\sigma)$.

A simple computation shows that

$$\left(1 - \frac{\alpha-\beta}{\alpha+\beta}\lambda_1\right)\left[\frac{\varphi(x)}{x^p}\right]^{\alpha-\beta} + \frac{\alpha-\beta}{\alpha+\beta}\lambda_1\frac{x\varphi'(x)}{p\varphi(x)}\left[\frac{\varphi(x)}{x^p}\right]^{\alpha-\beta}$$

$$= \left(1 - \frac{\lambda_1}{\lambda_2}\right)\varphi_1(x) + \frac{\lambda_1}{\lambda_2}\varphi_2(x). \tag{2.12}$$

Since the class $\mathcal{Q}_k(\sigma)$ is a convex set (see Remark 2), it follows that the right-hand side of (2.12) belongs to $\mathcal{Q}_k(\sigma)$ for $0 \le \lambda_1 < \lambda_2$, which implies that $\varphi \in \mathcal{BN}_{k,p}^{\alpha,\beta}(\lambda_1, \sigma)$. □

Putting $\beta = 0$ in Theorem 4, we obtain the following.

Corollary 7. If $0 \le \lambda_1 < \lambda_2$, then

$$\mathcal{B}_{k,p}^{\alpha}(\lambda_2, \sigma) \subset \mathcal{B}_{k,p}^{\alpha}(\lambda_1, \sigma).$$

Putting $\alpha = 0$ in Theorem 4, we get the following.

Corollary 8. If $0 \le \lambda_1 < \lambda_2$, then

$$\mathcal{N}_{k,p}^{\beta}(\lambda_2, \sigma) \subset \mathcal{N}_{k,p}^{\beta}(\lambda_1, \sigma).$$

Theorem 5. If $\varphi \in \mathcal{BN}_{k,p}^{\alpha,\beta}(\lambda, \sigma)$ given by (1.1) with $\alpha \ne \beta$, $p(\alpha+\beta)+\lambda \ne 0$ and $p(\alpha+\beta)+2\lambda \ne 0$, then

$$\left|b_{p+1}\right| \le \frac{p\,|\alpha+\beta|\,(1-\sigma)\,k}{|\alpha-\beta|\,|p(\alpha+\beta)+\lambda|}, \tag{2.13}$$

and

$$\left|b_{p+2} + \frac{\alpha-\beta-1}{2}b_{p+1}^2\right| \le \frac{p\,|\alpha+\beta|\,(1-\sigma)\,k}{|\alpha-\beta|\,|p(\alpha+\beta)+2\lambda|}. \tag{2.14}$$

Proof. If $\varphi \in \mathcal{BN}_{k,p}^{\alpha,\beta}(\lambda, \sigma)$, from Definition 1, we have

$$\left(1 - \frac{\alpha-\beta}{\alpha+\beta}\lambda\right)\left[\frac{\varphi(x)}{x^p}\right]^{\alpha-\beta} + \frac{\alpha-\beta}{\alpha+\beta}\lambda\frac{x\varphi'(x)}{p\varphi(x)}\left[\frac{\varphi(x)}{x^p}\right]^{\alpha-\beta} = G(x), \tag{2.15}$$

where $G(x) \in \mathcal{Q}_k(\sigma)$ is given by

$$G(x) = 1 + d_1 x + d_2 x^2 + d_3 x^3 + \dots. \tag{2.16}$$

Since

$$\left(1 - \frac{\alpha-\beta}{\alpha+\beta}\lambda\right)\left[\frac{\varphi(x)}{x^p}\right]^{\alpha-\beta} + \frac{\alpha-\beta}{\alpha+\beta}\lambda\frac{x\varphi'(x)}{p\varphi(x)}\left[\frac{\varphi(x)}{x^p}\right]^{\alpha-\beta}$$
$$= 1 + \frac{(\alpha-\beta)\left[p(\alpha+\beta)+\lambda\right]}{p(\alpha+\beta)}b_{p+1}x + \frac{(\alpha-\beta)\left[p(\alpha+\beta)+2\lambda\right]}{p(\alpha+\beta)}\left(b_{p+2} + \frac{\alpha-\beta-1}{2}b_{p+1}^2\right)x^2 + \dots, \tag{2.17}$$

Comparing the coefficients in (2.15) by using (2.16) and (2.17), we obtain

$$\frac{(\alpha-\beta)\left[p(\alpha+\beta)+\lambda\right]}{p(\alpha+\beta)}b_{p+1} = d_1, \tag{2.18}$$

$$\frac{(\alpha-\beta)\left[p(\alpha+\beta)+2\lambda\right]}{p(\alpha+\beta)}\left(b_{p+2} + \frac{\alpha-\beta-1}{2}b_{p+1}^2\right) = d_2. \tag{2.19}$$

Therefore,

$$b_{p+1} = \frac{p(\alpha+\beta)}{(\alpha-\beta)\left[p(\alpha+\beta)+\lambda\right]}d_1,$$

and

$$b_{p+2}+\frac{\alpha-\beta-1}{2}b_{p+1}^{2}=\frac{p\left(\alpha+\beta\right)}{\left(\alpha-\beta\right)\left[p\left(\alpha+\beta\right)+2\lambda\right]}d_{2}.$$

Our result now follows by an application of Lemma 2. This completes the proof of Theorem 5. □

Putting $\beta=0$ in Theorem 5, we obtain the following.

Corollary 9. If $\varphi\in\mathcal{B}_{k,p}^{\alpha}\left(\lambda,\sigma\right)$ is given by (1.1) with $p\alpha+\lambda\neq0$ and $p\alpha+2\lambda\neq0$, then

$$\left|b_{p+1}\right|\leq\frac{p\left(1-\sigma\right)k}{\left|p\alpha+\lambda\right|},$$

and

$$\left|b_{p+2}+\frac{\alpha-1}{2}b_{p+1}^{2}\right|\leq\frac{p\left(1-\sigma\right)k}{\left|p\alpha+2\lambda\right|}.$$

Putting $k=2$ and $\sigma=\frac{1-L}{1-M}\left(-1\leq M<L\leq1\right)$ in Corollary 9, we obtain the following corollary, which improves the result of Liu [10, Theorem 4 with $n=1$].

Corollary 10. If $\varphi\in\mathcal{B}_{p}^{\alpha}\left(\lambda,L,M\right)$ is given by (1.1) with $p\alpha+\lambda\neq0$ and $p\alpha+2\lambda\neq0$, then

$$\left|b_{p+1}\right|\leq\frac{2p\left(L-M\right)}{\left|p\alpha+\lambda\right|\left(1-M\right)},$$

and

$$\left|b_{p+2}+\frac{\alpha-1}{2}b_{p+1}^{2}\right|\leq\frac{2p\left(L-M\right)}{\left|p\alpha+2\lambda\right|\left(1-M\right)}.$$

Putting $\alpha=0$ in Theorem 5, we get the following.

Corollary 11. If $\varphi\in\mathcal{N}_{k}^{\beta}\left(\lambda,\sigma\right)$ given by (1.1) with $p\beta+\lambda\neq0$ and $p\beta+2\lambda\neq0$, then

$$\left|b_{p+1}\right|\leq\frac{p\left(1-\sigma\right)k}{\left|p\beta+\lambda\right|},$$

and

$$\left|b_{p+2}-\frac{\beta+1}{2}b_{p+1}^{2}\right|\leq\frac{p\left(1-\sigma\right)k}{\left|p\beta+2\lambda\right|}.$$

Putting $k=2$ and $\sigma=\frac{1-L}{1-M}\left(-1\leq M<L\leq1\right)$ in Corollary 11, we obtain the following corollary, which improves the result of Aouf and Seoudy [3, Theorem 8 with $n=1$].

Corollary 12. If $\varphi\in\mathcal{N}_{k}^{\beta}\left(\lambda,L,M\right)$ given by (1.1) with $p\beta+\lambda\neq0$ and $p\beta+2\lambda\neq0$, then

$$\left|b_{p+1}\right|\leq\frac{2p\left(L-M\right)}{\left|p\beta+\lambda\right|\left(1-M\right)},$$

and

$$\left|b_{p+2}-\frac{\beta+1}{2}b_{p+1}^{2}\right|\leq\frac{2p\left(L-M\right)}{\left|p\beta+2\lambda\right|\left(1-M\right)}.$$

3. Conclusions

In this investigation, we have presented the subfamily $\mathcal{BN}_{k,p}^{\alpha,\beta}(\lambda,\sigma)$ of multivalent Bazilevič and non-Bazilevič functions related to bounded boundary rotation. Also, we have computed a number of important properties, including the inclusion results and the upper bounds for the first two Taylor-Maclaurin coefficients for this function subfamily. For different choices of the parameters α, β, λ, p, k, and σ in the above results, we can get the corresponding results for each of the next subfamilies: $\mathcal{B}_p^{\alpha}(\lambda,L,M)$, $\mathcal{B}_p^{\alpha}(L,M)$, $\mathcal{B}^{\alpha}(L,M)$, $\mathcal{B}^{\alpha}(L,M)$, $\mathcal{B}^{\alpha}(\sigma)$, $\mathcal{N}_p^{\beta}(\lambda,L,M)$, $\mathcal{N}^{\beta}(\lambda,L,M)$, $\mathcal{N}^{\beta}(\sigma)$, $\mathcal{N}^{\beta}$, $\mathcal{B}_{k,p}^{\alpha}(\lambda,\sigma)$, $\mathcal{B}_k^{\alpha}(\lambda,\sigma)$, $\mathcal{N}_{k,p}^{\beta}(\lambda,\sigma)$, $\mathcal{N}_k^{\beta}(\lambda,\sigma)$, $\mathcal{B}_{k,p}(\lambda,\sigma)$, $\mathcal{B}_k(\lambda,\sigma)$, $\mathcal{N}_{k,p}(\lambda,\sigma)$, $\mathcal{N}_k(\lambda,\sigma)$, $\mathcal{B}_{k,p}^{\alpha}(\sigma)$, $\mathcal{B}_k^{\alpha}(\sigma)$, $\mathcal{N}_{k,p}^{\beta}(\sigma)$, $\mathcal{N}_k^{\beta}(\sigma)$, $\mathcal{S}_{k,p}(\sigma)$, and $\mathcal{S}_k(\sigma)$, which are defined in an introduction section. In addition, this work lays the foundation for future research and encourages researchers to explore more Bazilevič and non-Bazilevič functions involving some linear operators in geometric function theory and related fields.

Author contributions

The authors contributed equally to this work. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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