Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(5): 12726–12744.
DOI: 10.3934/math.2025573
Received: 19 February 2025
Revised: 06 May 2025
Accepted: 21 May 2025
Published: 30 May 2025

Research article

Optimal development problem for a nonlinear population model with size structure in a periodic environment

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Abstract: This paper investigated the optimal development problem of a size-structured population model under a periodic environmental setting. The boundary condition of the novel model consists of a nonlinear recruitment process and a bounded input, which endow the model with more realistic and complex characteristics compared to traditional ones. First, we established the existence of a unique non-negative bounded solution and demonstrated the continuous dependence of the solutions on the control variable. Next, we showed that the adjoint system is also well-posed. Then, the Euler-Lagrange equations describing the exact structure of the optimal strategies were derived and the existence of a unique optimal policy was proved. Finally, some numerical results were presented. The obtained research results will contribute to the development of some renewable resources, such as fish resources.

Keywords: environmental periodicity; size structure; optimal harvesting; nonlinear recruitment process
Mathematics Subject Classification: 35F50, 49K15, 49K20, 92D25

1. Description of the problem

As is well-known, there are many differences among individuals, including age, body size, gender, and so on. The structured population model is used to distinguish different individuals based on these structural differences, so as to determine the birth rate, growth rate, and death rate, as well as the interactions among individuals and between individuals and the environment [1]. In the last century, population systems with individual differences have attracted the interest of many researchers and yielded numerous beneficial results. Especially, age-structured first-order partial differential equations provide a main tool for modeling population systems. To name a few research examples, see [2–10] and the references therein. Here, [3–5] discussed the dynamics of age-dependent population models, including global behavior, oscillations, extinction, and blow-up phenomena. [6–8] investigated the optimal control problems for age-structured population models. Moreover, [9] investigated the optimal harvesting problem for a hierarchical age-structured population system. Abia et al. [10] discussed the numerical solution of age-structured population models. The paper reviews different numerical methods as well as the stability and convergence results of these methods. Din et al. [11] investigated the ergodic stationary distribution of an age-structured Hepatitis B virus (HBV) epidemic model with standard incidence rate. Blasio et al. [12] studied the stability of equilibrium of an age-structured fish population model with a non-decreasing recruitment process.

However, compared with age, the body size of fish is a more important parameter, because the body size of fish is easier to determine than its age. Here by *body size*, we mean some indices that can reflect the physiological or statistical characteristics of the individuals in the population, such as mass, length, diameter, volume, and so on [13]. In addition, long-term ecological studies have shown that the body size of an individual has a significant impact on its dynamic processes such as feeding, growth, and reproduction [14], and these dynamic processes will, in turn, affect the dynamics of the entire population. Hence, modeling population dynamics with a size structure has become an active and effective topic in mathematical biology. To name a few examples, see [15–26]. Here, [15–17] investigated the stability and regularity of size-structured population models, while [18, 19] studied the dynamical behaviors of two-species models with size structure. [20–22] investigated the optimal harvesting problems for population models with size structure, while [23] focused on optimal birth control problem and [24] discussed optimal contraception control problem. Moreover, [25] investigated the optimal harvesting problem in a unidirectional consumer-resource mutualisms system with size structure in the consumer. Abia et al. [26] presented a review of the numerical methods for solving the size-structured population balance models, and made a comparison of these methods in terms of accuracy, efficiency, generality, and mathematical methodology.

However, few works directly target the optimal development of fish resources, please refer to [27, 28], and they do not take into account the migration of individuals. In addition, due to factors such as seasons, the habitats where fish live often change periodically. What impact will this external environment have on the development of fish? Motivated by the above discussion, the present paper aims at the harvesting problem for the following fish dynamics model with a size-structure in a periodical environment:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} + \dfrac{\partial(V(x,t)u(x,t))}{\partial x} = f(x,t) - \mu(x,t)u(x,t) - \alpha(x,t)u(x,t), & (x,t) \in D, \\ V(0,t)u(0,t) = \psi(t) + \phi\Big(\displaystyle\int_0^m \beta(x,t)u(x,t)\,\mathrm{d}x\Big)\displaystyle\int_0^m \beta(x,t)u(x,t)\,\mathrm{d}x, & t \in R_+, \\ u(x,t) = u(x,t+T), & (x,t) \in D, \end{cases} \tag{1.1}$$

where $D = (0,m) \times (0,+\infty)$, $R_+ = (0,+\infty)$. $m \in R_+$ is the maximum size of the individuals and $T \in R_+$ is the environmental evolution cycle. $u(x,t)$ represents the number of individuals in the population with a size of x at time t; $V(x,t)$ means the rate of change of the individual size over time, namely, $\mathrm{d}x/\mathrm{d}t = V(x,t)$; $\mu(x,t)$ and $\beta(x,t)$ are, respectively, the mortality and egg-laying rate; $E(t) \doteq \int_0^m \beta(x,t)u(x,t)\,\mathrm{d}x$ represents the number of eggs laid at time t; $\phi(\cdot)$ is the conversion rate of fish eggs into fry; $\psi(t)$ is the number of fry released by the fishermen t; and $f(x,t)$ are the individuals of the same species that flow into the living environment of this population at time t. The control variable $\alpha(x,t)$ is the harvesting efforts, and it belongs to

$$\mathcal{U} = \Big\{\alpha \in L_T^\infty(D) : 0 \leq \underline{\alpha}(x,t) \leq \alpha(x,t) \leq \overline{\alpha}(x,t) \ \text{ a.e. } \ (x,t) \in D\Big\},$$

where $\underline{\alpha}$, $\overline{\alpha} \in L_T^\infty(D) = \{h \in L^\infty(D) : h(x,t) = h(x,t+T)\}$ are given bounded functions. If $u^\alpha(x,t)$ is the solution of model (1.1) with $\alpha \in \mathcal{U}$, it can be seen from the third equation of (1.1) that $u^\alpha(x,t)$ is a T-periodic function with respect to time t. In the present paper, we will discuss the following optimization problem:

$$\max_{\alpha \in \mathcal{U}} J(\alpha), \tag{1.2}$$

where

$$J(\alpha) = \int_0^T \int_0^m \Big[\omega(x,t)\alpha(x,t)u^\alpha(x,t) - \frac{1}{2}c\alpha^2(x,t)\Big]\,\mathrm{d}x\,\mathrm{d}t.$$

Here $\omega(x,t) \geq 0$ represents the economic value of an individual with a size of x at time t. $c > 0$ is the cost coefficient of implementing control measures. Thus the objective functional represents the total net economic benefit generated from fishing within a time period of duration T.

Compared with the existing related works, our model has the following features. Aniţa et al. [6] discussed a linear periodic age-dependent harvesting model, while our model extends it in two aspects: from age-dependent to a size-structure, and from linearity to nonlinearity. If $\psi(t) \equiv 0$ and $\phi(s) \equiv 1$, then our model can be transformed into the size-structured harvesting model in [21]. Moreover, if $\alpha(x,t) \equiv 0$ as well, our model simplifies to the size-structured model discussed in [22]. As a periodic version of the models in [27, 28], our model introduces a size-dependent immigration rate, which is a key extension in this paper. Similarly, as a periodic version of the model in [23], our model introduces a size-dependent immigration rate and a nonlinear recruitment process. In conclusion, compared with the existing models, our model has some novel characteristics.

In the coming discussion, we make the following assumptions:

($\mathbf{A}_1$) $V : D \to R_+$ is a bounded continuous function, $V(x,t) = V(x,t+T)$ for $(x,t) \in D$, and $\lim_{x\uparrow m} V(x,t) = 0$. This indicates that the growth rate of the individual's size is a periodic function of time t, and the individual will no longer grow after it reaches the maximum size m. Moreover, there is a positive constant L_V such that

$$|V(x_1,t) - V(x_2,t)| \leq L_V|x_1 - x_2|, \quad \text{for } x_1, x_2 \in (0,m).$$

The Lipschitz condition guarantees that the initial-valued problem $x'(t) = V(x,t)$, $x(t_0) = x_0$ has a unique solution.

($\mathbf{A}_2$) There exists $\bar{\beta}$ such that $0 \leq \beta(x,t) = \beta(x,t+T) \leq \bar{\beta}$ for $(x,t) \in D$. This is reasonable because the birth rate of any species is a bounded function. Moreover, $\mu(x,t) = \mu(x,t+T) \geq 0$ and $\mu(x,t) + V_x(x,t) \geq 0$ for $(x,t) \in D$. This assumption guarantees that the survival rate of the individuals mentioned below satisfies $0 < S(x,t) < 1$.

($\mathbf{A}_3$) $\psi \in L^1(R_+)$ and there exists a constant $\bar{\psi} > 0$ such that $0 \leq \psi(t) = \psi(t+T) \leq \bar{\psi}$ for $t \in R_+$. This means that the amount of fry put in artificially is a bounded function. $\phi : R_+ \to R_+$ is a bounded continuous function and there is a constant $\bar{\phi}$ such that $\phi(s) \leq \bar{\phi}$ for $s \in R_+$. This ensures that the conversion rate of eggs into fry is a bounded function. Moreover, there is an increasing function $c_\phi : R_+ \to R_+$ such that $|\phi(s_1) - \phi(s_2)| \leq c_\phi(r)|s_1 - s_2|$ for $0 \leq s_1, s_2 \leq r$.

(A_4) $f, \omega \in L^\infty(D)$, $f(x,t) = f(x,t+T) \geq 0$, and $0 \leq \omega(x,t) = \omega(x,t+T) \leq \bar{\omega}$ for $(x,t) \in D$, where $\bar{\omega}$ is a positive constant.

2. Well-posedness of the system model

The purpose of this section is to discuss the well-posedness of model (1.1). For convenience, the following definitions are introduced.

Definition 2.1 ([27]). *The unique solution $x = \varphi(t; t_0, x_0)$ of the initial-valued problem $x'(t) = V(x,t)$, $x(t_0) = x_0$ is called the characteristic curve passing through the point (t_0, x_0). Denote $z(t) = \varphi(t; 0, 0)$ as the characteristic curve passing through the point $(0,0)$ in the x-t plane.*

Definition 2.2 ([27]). *The derivative of the function $u(x,t)$ at the point (x,t) along the characteristic curve φ is given by*

$$D_\varphi u(x,t) = \lim_{h\to 0} \frac{u(\varphi(t+h; t, x), t+h) - u(x,t)}{h}.$$

Noting that $u(x,t)$ is periodic, we only consider the case where $t \in [\bar{t}, \bar{t} + T]$. Here $\bar{t} = z^{-1}(m)$. First, we consider the case where $\alpha(x,t) \equiv 0$. Let $v(x,t) \in L_T^\infty(D)$ be arbitrary but fixed, as is the function

$$\phi\Big(\int_0^m \beta(x,t) v(x,t)\,\mathrm{d}x\Big) \doteq \phi(v)(t).$$

Consider the following linear system:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} + \dfrac{\partial (V(x,t)u(x,t))}{\partial x} = f(x,t) - \mu(x,t)u(x,t), & (x,t) \in D, \\ V(0,t)u(0,t) = \psi(t) + \phi(v)(t) \displaystyle\int_0^m \beta(x,t)u(x,t)\,\mathrm{d}x, & t \in R_+, \\ u(x,t) = u(x,t+T), & (x,t) \in D. \end{cases} \tag{2.1}$$

Utilizing the characteristic curve technique, the solution of (2.1) can be given as

$$u(x,t;v) = u(0,\tau;v)\Pi(t;x,t) + \int_\tau^t f(\varphi(s;t,x),s)\frac{\Pi(t;x,t)}{\Pi(s;x,t)}\,\mathrm{d}s, \tag{2.2}$$

where $\tau = \varphi^{-1}(0;t,x) = t - z^{-1}(x)$ and $\Pi(r;x,t) = \exp\left\{-\int_\tau^r [\mu(\varphi(\sigma;t,x),\sigma) + V_x(\varphi(\sigma;t,x),\sigma)]\,\mathrm{d}\sigma\right\}$. Let $s = \varphi(\sigma;t,x)$. By Definition 2.1, $s = 0$ when $\sigma = \tau$, while $s = x$ when $\sigma = t$. It is easy to show that $\mathrm{d}s = V(s, \varphi^{-1}(s;t,x))\,\mathrm{d}\sigma$. Thus,

$$\begin{aligned} \Pi(t;x,t) &= \exp\left\{-\int_0^x \frac{\mu(s,\varphi^{-1}(s;t,x)) + V_x(s,\varphi^{-1}(s;t,x))}{V(s,\varphi^{-1}(s;t,x))}\,\mathrm{d}s\right\} \\ &= \frac{V(0,\tau)}{V(x,t)}\exp\left\{-\int_0^x \frac{\mu(s,\varphi^{-1}(s;t,x))}{V(s,\varphi^{-1}(s;t,x))}\,\mathrm{d}s\right\} \doteq \frac{V(0,\tau)}{V(x,t)}S(x,t). \end{aligned}$$

Hence, (2.2) can be rewritten as

$$u(x,t;v) = V(0,\tau)u(0,\tau;v)\frac{S(x,t)}{V(x,t)} + \int_\tau^t f(\varphi(s;t,x),s)\frac{\Pi(t;x,t)}{\Pi(s;x,t)}\,\mathrm{d}s.$$

According to [16], the basic reproduction number of the population described in model (2.1) can be defined as

$$R_0(t) = \phi(v)(t) \int_0^m \beta(x,t) \frac{S(x,t)}{V(x,t)}\,\mathrm{d}x,$$

which represents the average number of newborns produced by an individual during its lifespan. Here

$$S(x,t) = \exp\left\{ -\int_0^x \frac{\mu(w, \varphi^{-1}(w;t,x))}{V(w, \varphi^{-1}(w;t,x))}\,\mathrm{d}w \right\}$$

is the survival rate of the individual, that is, the probability of an individual surviving from birth to size x. Clearly, $0 \le S(x,t) \le 1$. Let

$$R_0 \doteq \bar{\phi} \int_0^m \sup_{t \in R_+} \left\{ \beta(x,t) \frac{S(x,t)}{V(x,t)} \right\} \mathrm{d}x.$$

Similar to [21, Theorem 1], using the spectral radius theory of a linear operator, for the linear system (2.1), we have the following result.

Lemma 2.1. *If* (A_1)–(A_4) *hold and* $R_0 < 1$*, then* (2.1) *has a unique non-negative solution* $u(x,t;v)$.

To discuss the well-posedness of (1.1), let $X = L^\infty_T(R_+, L^1(0,m))$ and define the equivalent norm on the space X by

$$\|u\|_* = \operatorname{Ess} \sup_{t \in [\bar{t}, \bar{t}+T]} \left\{ e^{-\lambda t} \int_0^m |u(x,t)|\,\mathrm{d}x \right\},$$

for some $\lambda > 0$. Thus, $(X, \|\cdot\|_*)$ is a Banach space.

Theorem 2.1. *Assume that* (A_1)–(A_4) *hold and* $\alpha(x,t) \equiv 0$. *If* $R_0 < 1$*, then model* (1.1) *has a unique solution* $u(x,t) \in \mathcal{X}$*. Here*

$$\mathcal{X} = \left\{ v \in X \,\middle|\, 0 \le v(x,t) = v(x,t+T) \text{ for } (x,t) \in D \text{ and } \int_0^m v(x,t)\,\mathrm{d}x \le M \right\},$$

where $M = [\bar{\psi}\bar{t} + \|f(\cdot,\cdot)\|_{L^1(D)}] \exp\{\bar{\phi}\bar{\beta}(\bar{t}+T)\}$.

Proof. Clearly, $(\mathcal{X}, \|\cdot\|_*)$ is a Banach space. Define $\mathcal{A} : \mathcal{X} \to X$ by $(\mathcal{A}v)(x,t) = u(x,t;v)$, $v \in \mathcal{X}$, where $u(x,t;v)$ is the solution of (2.1) and has the form (2.2).

First, for any $v \in \mathcal{X}$, we have

$$\begin{aligned}
\int_0^m |\mathcal{A}v|(x,t)\,\mathrm{d}x &= \int_0^m u(0,\tau;v)\Pi(t;x,t)\,\mathrm{d}x + \int_0^m \int_\tau^t f(\varphi(s;t,x),s) \frac{\Pi(t;x,t)}{\Pi(s;x,t)}\,\mathrm{d}s\,\mathrm{d}x \\
&\le \int_0^m u(0,\tau;v) \exp\left\{ -\int_\tau^t V_x(\varphi(\sigma;t,x),\sigma)\,\mathrm{d}\sigma \right\} \mathrm{d}x \\
&\quad + \int_0^m \int_\tau^t f(\varphi(s;t,x),s) \exp\left\{ -\int_s^t V_x(\varphi(\sigma;t,x),\sigma)\,\mathrm{d}\sigma \right\} \mathrm{d}s\,\mathrm{d}x \\
&\doteq I_1 + I_2.
\end{aligned}$$

For I_1, let $s = \tau = \varphi^{-1}(0; t, x)$. From Definition 2.1, $s = t$ when $x = 0$, while $s = t - \bar{t}$ when $x = m$. It is clear that $\varphi(\sigma; s, 0) = \varphi(\sigma; t, x)$. Thus, from [25, Lemma 2], it follows that

$$\frac{\mathrm{d}x}{\mathrm{d}s} = -V(0, s)\exp\left\{\int_s^t V_x(\varphi(\sigma; s, 0), \sigma)\,\mathrm{d}\sigma\right\} = -V(0, s)\exp\left\{\int_s^t V_x(\varphi(\sigma; t, x), \sigma)\,\mathrm{d}\sigma\right\}.$$

Hence,

$$\begin{aligned} I_1 &= \int_{t-\bar{t}}^t V(0, s)u(0, s; v)\,\mathrm{d}s \\ &= \int_{t-\bar{t}}^t \left[\psi(s) + \phi(v)(s)\int_0^m \beta(x, s)u(x, s; v)\,\mathrm{d}x\right]\mathrm{d}s \\ &\le \bar{\psi}\bar{t} + \bar{\phi}\bar{\beta}\int_0^t \left[\int_0^m u(x, s; v)\,\mathrm{d}x\right]\mathrm{d}s. \end{aligned}$$

For I_2, just as in [15], by making use of Fubini's theorem and assumptions, we can obtain

$$I_2 \le \int_0^t \int_0^m f(\varphi(s; t, x), s)\exp\left\{-\int_s^t V_x(\varphi(\sigma; t, x), \sigma)\,\mathrm{d}\sigma\right\}\mathrm{d}x\,\mathrm{d}s.$$

Further, let $r = \varphi(s; t, x)$. From Definition 2.1, $r = \varphi(s; t, 0) = 0$ when $x = 0$, while $r = \varphi(s; t, m) < m$ when $x = m$. Moreover, we have $\varphi(\sigma; s, r) = \varphi(\sigma; t, x)$. Thus, from [25, Lemma 2], we have

$$\frac{\mathrm{d}x}{\mathrm{d}r} = \exp\left\{\int_s^t V_x(\varphi(\sigma; s, r), \sigma)\,\mathrm{d}\sigma\right\} = \exp\left\{\int_s^t V_x(\varphi(\sigma; t, x), \sigma)\,\mathrm{d}\sigma\right\}.$$

Hence,

$$I_2 \le \int_0^t \int_0^m f(r, s)\,\mathrm{d}r\,\mathrm{d}s \le \|f(\cdot, \cdot)\|_{L^1(D)}.$$

Thus, we can obtain

$$\int_0^m u(x, t; v)\,\mathrm{d}x \le \bar{\psi}\bar{t} + \|f(\cdot, \cdot)\|_{L^1(D)} + \bar{\phi}\bar{\beta}\int_0^t \left[\int_0^m u(x, s; v)\,\mathrm{d}x\right]\mathrm{d}s. \tag{2.3}$$

From Gronwall's inequality, it follows that

$$\int_0^m u(x, t; v)\,\mathrm{d}x \le \left[\bar{\psi}\bar{t} + \|f(\cdot, \cdot)\|_{L^1(D)}\right]\exp\left\{\bar{\phi}\bar{\beta}(\bar{t} + T)\right\} = M.$$

Thus, $\mathcal{A}$ is a mapping from $\mathcal{X}$ to itself.

Second, for any $v_1, v_2 \in \mathcal{X}$, from (2.2), it follows that

$$\begin{aligned} \int_0^m |(\mathcal{A}v_1) - (\mathcal{A}v_2)|(x, t)\,\mathrm{d}x &\le \int_0^m |u(0, \tau; v_1) - u(0, \tau; v_2)|\exp\left\{-\int_\tau^t V_x(\varphi(\sigma; t, x), \sigma)\,\mathrm{d}\sigma\right\}\mathrm{d}x \\ &= \int_{t-\bar{t}}^t \left|\phi(v_1)(s)\int_0^m \beta(x, s)u(x, s; v_1) - \phi(v_2)(s)\int_0^m \beta(x, s)u(x, s; v_2)\,\mathrm{d}x\right|\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^t |\phi(v_1)(s) - \phi(v_2)(s)| \int_0^m \beta(x,s)u(x,s;v_1)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^t \phi(v_2)(s) \int_0^m \beta(x,s)|u(x,s;v_1) - u(x,s;v_2)|\,\mathrm{d}x\,\mathrm{d}s \\
&\leq \bar{\beta}M \int_0^t \left|\phi\Big(\int_0^m \beta(x,s)v_1(x,s)\,\mathrm{d}x\Big) - \phi\Big(\int_0^m \beta(x,s)v_2(x,s)\,\mathrm{d}x\Big)\right| \mathrm{d}s \\
&\quad + \bar{\phi}\bar{\beta} \int_0^t \int_0^m |u(x,s;v_1) - u(x,s;v_2)|\,\mathrm{d}x\,\mathrm{d}s \\
&\leq c_\phi(r_0)\bar{\beta}^2 M \int_0^t \|v_1(\cdot,s) - v_2(\cdot,s)\|_{L^1(0,m)}\,\mathrm{d}s \\
&\quad + \bar{\phi}\bar{\beta} \int_0^t \int_0^m |u(x,s;v_1) - u(x,s;v_2)|\,\mathrm{d}x\,\mathrm{d}s,
\end{aligned}
$$

where $r_0 = \bar{\beta}M$. From Gronwall's inequality, it follows that

$$
\int_0^m |u(x,t;v_1) - u(x,t;v_2)|\,\mathrm{d}x \leq M_1 \int_0^t \|v_1(\cdot,s) - v_2(\cdot,s)\|_{L^1(0,m)}\,\mathrm{d}s,
$$

where $M_1 = c_\phi(r_0)\bar{\beta}^2 M[1 + \exp\{\bar{\phi}\bar{\beta}(\bar{t} + T)\}]$. Then

$$
\begin{aligned}
\|\mathcal{A}v_1 - \mathcal{A}v_2\|_* &= \operatorname{Ess}\sup_{t\in[\bar{t},\bar{t}+T]} e^{-\lambda t} \int_0^m |(\mathcal{A}v_1)(x,t) - (\mathcal{A}v_2)(x,t)|\,\mathrm{d}x \\
&\leq \operatorname{Ess}\sup_{t\in[\bar{t},\bar{t}+T]} \left\{(M_1 e^{-\lambda t} \int_0^t e^{\lambda s} e^{-\lambda s} \|v_1(\cdot,s) - v_2(\cdot,s)\|_{L^1(0,m)}\,\mathrm{d}s\right\} \\
&\leq \frac{M_1}{\lambda}\|v_1 - v_2\|_*.
\end{aligned}
$$

Choosing $\lambda > M_1$, then $\mathcal{A}$ is a contraction mapping on the Banach space $(\mathcal{X}, \|\cdot\|_*)$. Thus, $\mathcal{A}$ has a unique fixed point in $\mathcal{X}$, which is the solution of (1.1). The proof is complete. □

Finally, we present the following well-posedness result of model (1.1).

Theorem 2.2. *Assume that* (A_1)–(A_4) *hold and* $R_0 < 1$. *Then, for any* $\alpha \in \mathcal{U}$, *model* (1.1) *has a unique solution* $u^\alpha \in \mathcal{X}$. *Moreover, for any* $\alpha_1, \alpha_2 \in \mathcal{U}$, *there are positive constants* B_1 *and* B_2 (*independent of* α_i $(i = 1, 2)$) *such that*

$$
\|u_1 - u_2\|_{L_T^\infty(R_+;L^1(0,m))} \leq B_1 T \|\alpha_1 - \alpha_2\|_{L_T^\infty(R_+;L^1(0,m))},
$$

$$
\|u_1 - u_2\|_{L_T^1(D)} \leq B_2 T \|\alpha_1 - \alpha_2\|_{L_T^1(D)},
$$

where u_i *is the solution of* (1.1) *with* $\alpha_i \in \mathcal{U}$ $(i = 1, 2)$.

Proof. Using $\mu + \alpha$ instead of α in Theorem 2.1, we can obtain

$$
\exp\left\{-\int_0^x \frac{\mu(w,\varphi^{-1}(w;t,x)) + \alpha(w,\varphi^{-1}(w;t,x))}{V(w,\varphi^{-1}(w;t,x))}\,\mathrm{d}w\right\} \leq S(x,t).
$$

Thus, from Theorem 2.1, if (A_1)–(A_4) hold and $R_0 < 1$, then for any $\alpha \in \mathcal{U}$, model (1.1) has a unique solution $u^\alpha \in \mathcal{X}$.

Note that u_i is the solution of (1.1) with $\alpha_i \in \mathcal{U}$. Thus,

$$u_i(x,t) = u_i(0,\tau)\Pi(t,x;\alpha_i) + \int_\tau^t f(\varphi(s;t,x),s)\frac{\Pi(t,x;\alpha_i)}{\Pi(s,x;\alpha_i)}\,\mathrm{d}s,$$

where

$$\Pi(r,x;\alpha_i) = \exp\left\{-\int_\tau^r [\mu(\varphi(w;t,x),w) + V_x(\varphi(w;t,x),w) + \alpha_i(\varphi(w;t,x),w)]\,\mathrm{d}w\right\}.$$

Similar to the proof of Theorem 2.1, we can obtain

$$\begin{aligned}
\|u_1(\cdot,t) - u_2(\cdot,t)\|_{L^1(0,m)} \leq & \int_0^m |u_1 - u_2|(0,\tau)\Pi(t,x;\alpha_1)\,\mathrm{d}x + \int_0^m u_2(0,\tau)|\Pi(t,x;\alpha_1) - \Pi(t,x;\alpha_2)|\,\mathrm{d}x \\
& + \int_0^m \int_\tau^t f(\varphi(s;t,x),s)\left|\frac{\Pi(t,x;\alpha_1)}{\Pi(s,x;\alpha_1)} - \frac{\Pi(t,x;\alpha_2)}{\Pi(s,x;\alpha_2)}\right|\mathrm{d}s\,\mathrm{d}x \\
\leq & \int_0^t V(0,s)|u_1(0,s) - u_2(0,s)|\,\mathrm{d}s \\
& + (\bar{\psi} + \bar{\phi}\bar{\beta}M + \|f(\cdot,\cdot)\|_{L^1(D)}) \int_0^t \int_0^m |\alpha_1(x,s) - \alpha_2(x,s)|\,\mathrm{d}x\,\mathrm{d}s \\
\leq & \bar{\beta}M \int_0^t \left|\phi\Big(\int_0^m \beta(x,s)u_1(x,s)\,\mathrm{d}x\Big) - \phi\Big(\int_0^m \beta(x,s)u_2(x,s)\,\mathrm{d}x\Big)\right|\mathrm{d}s \\
& + \bar{\phi}\bar{\beta}\int_0^t \int_0^m |u_1(x,s) - u_2(x,s)|\,\mathrm{d}x \\
& + (\bar{\psi} + \bar{\phi}\bar{\beta}M + \|f(\cdot,\cdot)\|_{L^1(D)}) \int_0^t \int_0^m |\alpha_1(x,s) - \alpha_2(x,s)|\,\mathrm{d}x\,\mathrm{d}s \\
\leq & c_\phi(r_0)\bar{\beta}^2 M \int_0^t \int_0^m |u_1(x,s) - u_2(x,s)|\,\mathrm{d}x\,\mathrm{d}s \\
& + \bar{\phi}\bar{\beta}\int_0^t \int_0^m |u_1(x,s) - u_2(x,s)|\,\mathrm{d}x \\
& + (\bar{\psi} + \bar{\phi}\bar{\beta}M + \|f(\cdot,\cdot)\|_{L^1(D)}) \int_0^t \int_0^m |\alpha_1(x,s) - \alpha_2(x,s)|\,\mathrm{d}x\,\mathrm{d}s \\
\leq & M_2 \int_0^t \|u_1(\cdot,s) - u_2(\cdot,s)\|_{L^1(0,m)}\,\mathrm{d}s + M_3 \int_0^t \|\alpha_1(\cdot,s) - \alpha_2(\cdot,s)\|_{L^1(0,m)}\,\mathrm{d}s,
\end{aligned}$$

where $M_2 = \bar{\phi}\bar{\beta} + c_\phi(r_0)\bar{\beta}^2 M$ and $M_3 = \bar{\psi} + \bar{\phi}\bar{\beta}M + \|f(\cdot,\cdot)\|_{L^1(D)}$. The result follows immediately from Gronwall's inequality. □

3. Optimal harvesting problem

In this part, we will discuss the optimization problem (1.2). Let $\mathcal{T}_{\mathcal{U}}(\alpha)$ and $\mathcal{N}_{\mathcal{U}}(\alpha)$ be the tangent cone and the normal cone of $\mathcal{U}$ at element α, respectively.

3.1. Well-posedness of the adjoint system

First, consider the following system:

$$\begin{cases} D_\varphi \xi(x,t) - \mu(x,t)\xi(x,t) + \beta(x,t)\xi(0,t) = m(x,t), & (x,t) \in D, \\ \xi(m,t) = 0, & t \in R_+, \\ \xi(x,t) = \xi(x,t+T), & (x,t) \in D. \end{cases} \tag{3.1}$$

Define $\mathcal{B} : L_T^\infty(R_+) \to L_T^\infty(R_+)$ by

$$(\mathcal{B}h)(t) = \int_0^l K(t,s)h(\varphi^{-1}(s;t,0))\,\mathrm{d}s,$$

where $h \in L_T^\infty(R_+) = \{h \in L^\infty(R_+) : h(t) = h(t+T),\ \text{a.e. } t \in R_+\}$ and

$$K(t,s) = \begin{cases} \dfrac{\beta(s,\varphi^{-1}(s;t,0))}{V(s,\varphi^{-1}(s;t,0))} \exp\Big\{ -\displaystyle\int_0^s \frac{\mu(\sigma,\varphi^{-1}(\sigma;t,0))}{V(\sigma,\varphi^{-1}(\sigma;t,0))}\,\mathrm{d}\sigma \Big\}, & 0 \le s \le \min\{z(t), l\}, \\ 0, & \text{otherwise.} \end{cases}$$

Let $r(\mathcal{B})$ be the spectral radius of the linear operator $\mathcal{B}$. For system (3.1), by [22, Lemma 4.2], we can obtain the following result.

Lemma 3.1. *If $r(\mathcal{B}) < 1$, then system (3.1) has a unique solution $\xi(x,t) \in L_T^\infty(D)$. Moreover, $m(x,t) \ge 0$ implies that $\xi(x,t) \le 0$ a.e. $(x,t) \in D$.*

Now, we consider the following adjoint system:

$$\begin{cases} D_\varphi \xi(x,t) - [\mu(x,t) + \alpha(x,t)]\xi(x,t) + \beta(x,t)\eta(t)\xi(0,t) = \omega(x,t)\alpha(x,t), \\ \xi(m,t) = 0, \\ \xi(x,t) = \xi(x,t+T), \quad (x,t) \in D, \end{cases} \tag{3.2}$$

where

$$\eta(t) = \phi'\Big(\int_0^m \beta(x,t)u(x,t)\,\mathrm{d}x\Big)\int_0^m \beta(x,t)u(x,t)\,\mathrm{d}x + \phi\Big(\int_0^m \beta(x,t)u(x,t)\,\mathrm{d}x\Big).$$

Let

$$\bar{R}_0 \doteq (c_\phi(r_0)\bar{\beta}M + \bar{\phi})\int_0^m \sup_{t\in R_+}\Big\{\beta(x,t)\frac{S(x,t)}{V(x,t)}\Big\}\,\mathrm{d}x.$$

From a similar discussion as that in Theorem 2.2 and Lemma 3.1, we have the following result.

Theorem 3.1. *Assume that (A_1)–(A_4) hold and $\bar{R}_0 < 1$. Then, for any $\alpha \in \mathcal{U}$, system (3.2) has a unique solution $\xi^\alpha \in L_T^\infty(D)$ and $\xi^\alpha(x,t) \le 0$ for $(x,t) \in D$. Moreover, for any $\alpha_1, \alpha_2 \in \mathcal{U}$, there is a positive constant B_3 (independent of α_i $(i = 1,2)$) such that*

$$\|\xi_1 - \xi_2\|_{L_T^\infty(D)} \le B_3 T\|\alpha_1 - \alpha_2\|_{L_T^\infty(D)},$$

where ξ_i is the solution of system (3.2) corresponding to $\alpha_i \in \mathcal{U}$ $(i = 1,2)$.

3.2. Optimality conditions

Theorem 3.2. *Let $\alpha^*(x,t)$ be an optimal harvesting strategy for the optimization problem* (1.2)*. Under the conditions of Theorem 3.1, we have*

$$\alpha^*(x,t) = \mathcal{F}\left\{\frac{[\omega(x,t)+\xi(x,t)]\,u^*(x,t)}{c}\right\}, \tag{3.3}$$

in which the mapping $\mathcal{F}$ is given by

$$(\mathcal{F}\eta)(x,t) = \begin{cases} \underline{\alpha}(x,t), & \eta(x,t) < \underline{\alpha}(x,t), \\ \eta(x,t), & \underline{\alpha}(x,t) \le \eta(x,t) \le \overline{\alpha}(x,t), \\ \overline{\alpha}(x,t), & \eta(x,t) > \overline{\alpha}(x,t), \end{cases} \tag{3.4}$$

where $u^*(x,t)$ is the solution of (1.1) *with α^* and $\xi(x,t)$ satisfies*

$$\begin{cases} D_\varphi\xi = [\mu(x,t)+\alpha^*(x,t)]\xi(x,t) - \beta(x,t)\Big[\phi\Big(\int_0^m \beta(x,t)u^*(x,t)\,\mathrm{d}x\Big) \\ \qquad +\phi'\Big(\int_0^m \beta(x,t)u^*(x,t)\,\mathrm{d}x\Big)\int_0^m \beta(x,t)u^*(x,t)\,\mathrm{d}x\Big]\xi(0,t) \\ \qquad +\omega(x,t)\alpha^*(x,t), \\ \xi(m,t) = 0, \quad \xi(x,t) = \xi(x,t+T), \qquad (x,t)\in D. \end{cases} \tag{3.5}$$

Proof. According to Theorem 3.1, system (3.5) has a unique solution. For any $v \in \mathcal{T}_{\mathcal{U}}(\alpha^*)$ and for a sufficiently small $\varepsilon > 0$, we have $\alpha^\varepsilon \doteq \alpha^* + \varepsilon v \in \mathcal{U}$. Let $u^\varepsilon(x,t)$ be solution of (1.1) with α^ε. It follows from the optimality of α^* that

$$\int_0^T\int_0^m \omega(x,t)\Big[\alpha^*(x,t)[u^\varepsilon(x,t)-u^*(x,t)] + \varepsilon v(x,t)u^\varepsilon(x,t)\Big]\,\mathrm{d}x\,\mathrm{d}t$$
$$-\frac{c}{2}\int_0^T\int_0^m \Big[2\varepsilon\alpha^*(x,t)v(x,t)+\varepsilon^2 v^2(x,t)\Big]\,\mathrm{d}x\,\mathrm{d}t \le 0.$$

Thus,

$$\int_0^T\int_0^m \omega(x,t)[\alpha^*(x,t)z(x,t)+v(x,t)u^*(x,t)]\,\mathrm{d}x\,\mathrm{d}t - c\int_0^T\int_0^m \alpha^*(x,t)v(x,t)\,\mathrm{d}x\,\mathrm{d}t \le 0, \tag{3.6}$$

where

$$z(x,t) = \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}[u^\varepsilon(x,t)-u^*(x,t)].$$

From [21, Lemma 2], $z(x,t)$ makes sense. By a simple computation, $z(x,t)$ satisfies

$$\begin{cases} D_\varphi z(x,t) = -[\mu(x,t)+V_x(x,t)+\alpha^*(x,t)]z(x,t) - v(x,t)u^*(x,t), \\ V(0,t)z(0,t) = \Big[\phi'\Big(\int_0^m \beta(x,t)u^*(x,t)\,\mathrm{d}x\Big)\cdot\int_0^m \beta(x,t)u^*(x,t)\,\mathrm{d}x \\ \qquad +\phi\Big(\int_0^m \beta(x,t)u^*(x,t)\,\mathrm{d}x\Big)\Big]\int_0^m \beta(x,t)z(x,t)\,\mathrm{d}x, \\ z(x,t) = z(x,t+T), \qquad (x,t)\in D. \end{cases} \tag{3.7}$$

Let

$$E^*(t) \doteq \int_0^m \beta(x,t)u^*(x,t)\,\mathrm{d}x.$$

Multiplying the first equation of (3.7) by $\xi(x,t)$ and integrating on $D_T = (0,m)\times[0,T]$, we can obtain that

$$\begin{aligned}\int_0^T\int_0^m (D_\varphi\xi)z\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_0^m [\mu(x,t)+\alpha^*(x,t)]\xi(x,t)z(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &-\int_0^T\int_0^m \beta\Big[\phi'(E^*(t))E^*(t)+\phi(E^*(t))\Big]\xi(0,t)z\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_0^T\int_0^m v(x,t)u^*(x,t)\xi(x,t)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Multiplying the first equation of (3.5) by $z(x,t)$ and integrating on D_T, we yield

$$\begin{aligned}\int_0^T\int_0^m (D_\varphi\xi)z\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_0^m [\mu(x,t)+\alpha^*(x,t)]\xi(x,t)z(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &-\int_0^T\int_0^m \beta\Big[\phi'(E^*(t))E^*(t)+\phi(E^*(t))\Big]\xi(0,t)z\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_0^T\int_0^m \omega(x,t)\alpha^*(x,t)z(x,t)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Thus, we get

$$\int_0^T\int_0^m \omega(x,t)\alpha^*(x,t)z(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_0^m \xi(x,t)v(x,t)u^*(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

Substituting this into (3.6) yields

$$\int_0^T\int_0^m \left[(\omega(x,t)+\xi(x,t))u^*(x,t)-c\alpha^*(x,t)\right]v(x,t)\,\mathrm{d}x\,\mathrm{d}t \le 0,$$

for any $v\in\mathcal{T}_{\mathcal{U}}(\alpha^*)$. Thus, $(\omega+\xi)u^*-c\alpha^*\in\mathcal{N}_{\mathcal{U}}(\alpha^*)$ (see [29]). Hence, we get the conclusion of the theorem. □

3.3. Existence of a unique optimal policy

Definition 3.1. *The embedding mapping $\tilde{J}$ is given by*

$$\tilde{J}(\alpha) = \begin{cases} J(\alpha), & \alpha\in\mathcal{U},\\ -\infty, & \alpha\notin\mathcal{U}.\end{cases} \tag{3.8}$$

From a similar discussion as that in [23], we know that $\tilde{J}(\alpha)$ is upper semi-continuous. According to Ekeland's principle, for every $\varepsilon>0$, there exists $\alpha_\varepsilon\in\mathcal{U}$ such that

$$\tilde{J}(\alpha_\varepsilon) \ge \sup_{\alpha\in\mathcal{U}}\tilde{J}(\alpha)-\varepsilon, \tag{3.9}$$

$$\tilde{J}(\alpha_\varepsilon) \geq \sup_{\alpha \in \mathcal{U}} \left\{ \tilde{J}(\alpha) - \sqrt{\varepsilon} \|\alpha_\varepsilon - \alpha\|_{L^1_T(D)} \right\}. \tag{3.10}$$

Thus, $\tilde{J}_\varepsilon(\alpha) = \tilde{J}(\alpha) - \sqrt{\varepsilon}\|\alpha_\varepsilon - \alpha\|_{L^1_T(D)}$ achieves its supremum at α_ε. Then, similar to the discussion in Theorem 3.2, for any $v \in \mathcal{T}_{\mathcal{U}}(\alpha_\varepsilon)$, we have

$$\int_0^T \int_0^m \left[c\alpha_\varepsilon(x,t) - (\omega(x,t) + \xi^\varepsilon(x,t))u^\varepsilon(x,t)\right] v(x,t)\,\mathrm{d}x\,\mathrm{d}t + \sqrt{\varepsilon} \int_0^T \int_0^m |v(x,t)|\,\mathrm{d}x\,\mathrm{d}t \geq 0.$$

Among them, u^ε is the solution of (1.1) when $\alpha = \alpha_\varepsilon$, and ξ^ε is the solution of (3.2) when $\alpha = \alpha_\varepsilon$ and $u = u^\varepsilon$. Thus, based on the structure of normal cones, there exists $\theta \in L^\infty_T(D)$, $|\theta(x,t)| \leq 1$, such that $\sqrt{\varepsilon}\theta + [\omega + \xi^\varepsilon]u^\varepsilon - c\alpha_\varepsilon \in \mathcal{N}_{\mathcal{U}}(\alpha_\varepsilon)$. Hence,

$$\alpha_\varepsilon(x,t) = \mathcal{F}\left\{ \frac{\left[\omega(x,t) + \xi^\varepsilon(x,t)\right] u^\varepsilon(x,t)}{c} + \frac{\sqrt{\varepsilon}\theta(x,t)}{c} \right\}. \tag{3.11}$$

Theorem 3.3. *Under the conditions of Theorem 3.1, if $c^{-1}T$ is sufficiently small, then optimization problem* (1.2) *has a unique solution.*

Proof. Define the mapping $C : \mathcal{U} \to L^\infty_T(D)$ by

$$(C\alpha)(x,t) = \mathcal{F}\left\{ \frac{\left[\omega(x,t) + \xi^\alpha(x,t)\right] u^\alpha(x,t)}{c} \right\}. \tag{3.12}$$

It is obvious that $(\mathcal{U}, \|\cdot\|_{L^\infty(D)})$ is a Banach space and C maps $\mathcal{U}$ to itself. In addition, for any $(x,t) \in D$, it can be derived from (3.12) that

$$\begin{aligned}
&\left|(C\alpha_1)(x,t) - (C\alpha_2)(x,t)\right| \\
=&\left|\mathcal{F}\left\{ \frac{\left[\omega(x,t) + \xi^{\alpha_1}(x,t)\right] u^{\alpha_1}(x,t)}{c} \right\} - \mathcal{F}\left\{ \frac{\left[\omega(x,t) + \xi^{\alpha_2}(x,t)\right] u^{\alpha_2}(x,t)}{c} \right\}\right| \\
\leq& c^{-1}\Big[|\omega(x,t)||u^{\alpha_1}(x,t) - u^{\alpha_2}(x,t)| + |u^{\alpha_2}(x,t)||\xi^{\alpha_1}(x,t) - \xi^{\alpha_2}(x,t)| + |\xi^{\alpha_2}(x,t)||u^{\alpha_1}(x,t) - u^{\alpha_2}(x,t)|\Big].
\end{aligned}$$

From Theorems 2.2 and 3.1, u^α and ξ^α are continuous with respect to α. Thus, there exists a constant $K > 0$ such that

$$\|C\alpha_1 - C\alpha_2\|_{L^\infty(D)} \leq \frac{TK}{c} \|\alpha_1 - \alpha_2\|_{L^\infty(D)}. \tag{3.13}$$

Clearly, if $c^{-1}TK < 1$, then C has a unique fixed point $\hat{\alpha} \in \mathcal{U}$. In addition, as can be seen from Theorem 3.2, any optimal controller, if it exists, must be a fixed point of C.

Now, we prove that the control $\hat{\alpha} \in \mathcal{U}$ is actually optimal. That is to say, we need to show $\tilde{J}(\hat{\alpha}) = \sup\{\tilde{J}(\alpha) : \alpha \in \mathcal{U}\}$. From (3.11) and (3.12), it follows that

$$\begin{aligned}
\|C\alpha_\varepsilon - \alpha_\varepsilon\|_{L^\infty(D)} =& \left\| \mathcal{F}\left[\frac{[(\omega + \xi^\varepsilon)u^\varepsilon](x,t)}{c} + \frac{\sqrt{\varepsilon}\theta(x,t)}{c} \right] - \mathcal{F}\left[\frac{[(\omega + \xi^\varepsilon)u^\varepsilon](x,t)}{c} \right] \right\|_{L^\infty(D)} \\
\leq& c^{-1}\sqrt{\varepsilon}\|\theta(x,t)\|_{L^\infty(D)} \leq c^{-1}\sqrt{\varepsilon}.
\end{aligned}$$

Note that $C\hat{\alpha} = \hat{\alpha}$. Thus,

$$\|\hat{\alpha} - \alpha_\varepsilon\|_{L^\infty(D)} = \|C\hat{\alpha} - C\alpha_\varepsilon + C\alpha_\varepsilon - \alpha_\varepsilon\|_{L^\infty(D)} \leq \frac{TK}{c}\|\hat{\alpha} - \alpha_\varepsilon\|_{L^\infty(D)} + \frac{\sqrt{\varepsilon}}{c}.$$

If $c^{-1}TK < 1$, then $\|\hat{\alpha} - \alpha_\varepsilon\|_{L^\infty(D)} \leq (1 - c^{-1}TK)c^{-1}\sqrt{\varepsilon}$. Thus, $\alpha_\varepsilon \to \hat{\alpha}$ in $L^\infty_T(D)$ as $\varepsilon \to 0$. Then, the upper semi-continuity of $\tilde{J}(\alpha)$ implies that $\hat{\alpha} \in \mathcal{U}$ is the optimal strategy. The proof is complete. □

4. Numerical test

The numerical simulations in this section are only carried out for academic tests. According to [3], in order to obtain the numerical simulation of our model (1.1), it is carried out in the following three steps.

The first step is to present the numerical simulation of the following model with periodic parameters:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} + \dfrac{\partial (V(x,t)u(x,t))}{\partial x} = f(x,t) - \mu(x,t)u(x,t) - \alpha(x,t)u(x,t), & (x,t) \in D, \\ V(0,t)u(0,t) = \psi(t) + \phi\left(\displaystyle\int_0^m \beta(x,t)u(x,t)\,\mathrm{d}x\right)\displaystyle\int_0^m \beta(x,t)u(x,t)\,\mathrm{d}x, & t \in R_+, \\ u(x,0) = u_0(x), & x \in (0,m). \end{cases} \tag{4.1}$$

The second step is to select the numerical solution of model (4.1) on a certain interval $[kT, (k+1)T]$ for a sufficiently large k.

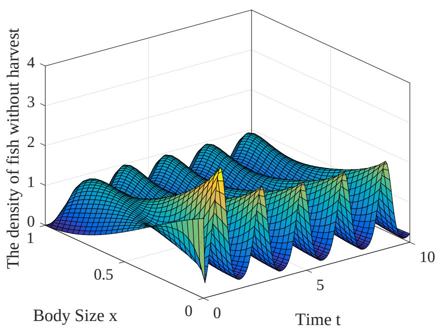
The third step is that by extending the numerical solution on this interval, we can obtain the numerical solution of model (1.1).

Next, we use the extrapolated upwind scheme in [10] to present the numerical simulation of model (4.1). In this simulation, the vital rates and other parameters take arbitrary values, and these values do not correspond to any specific biological population. Here, we take $c = 5$, $m = 1$, $T = 2$, and

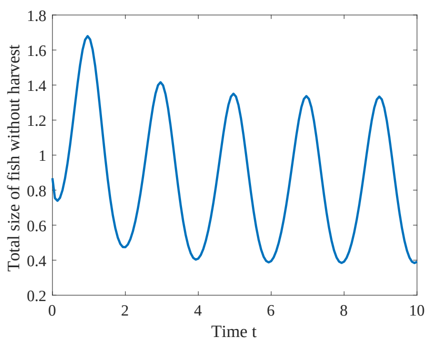
$$f(x,t) = 1 + \sin(\pi t), \quad \beta(x,t) = 20x^2(1-x)(2+\sin(\pi t)), \quad \mu(x,t) = e^{-4x}(1-x)^{-1.4}(2+\cos(\pi t)),$$
$$V(x,t) = 1 - x, \qquad \phi(s) = 0.1s, \qquad \omega(x,t) = \frac{1}{30}(0.8\pi x + \sin(\pi t) + 1).$$

Example 4.1. *Take $u_0(x) = 2(1+x)(1-x)^2$. Without taking fishing into account, Figure 1(a,b) shows the density function and the total number of fish when $\psi(t) = 0.04(1+\sin(\pi t))$; while Figure 2(a,b) shows the density function and the total size of fish when $\psi(t) = 0.8(1+\sin(\pi t))$.*

Example 4.2. *Take $\psi(t) = 0.04(1+\sin(\pi t))$ and $u_0(x) = 2(1+x)(1-x)^2$. Figure 3 displays the optimal densities $u^*(x,t)$, optimal harvesting strategy $\alpha^*(x,t)$, and the total size of fish when $\underline{\alpha}(x,t) = 0$ and $\overline{\alpha}(x,t) = 1$. Figure 4 shows the optimal densities $u^*(x,t)$, optimal harvesting strategy $\alpha^*(x,t)$, and the total size of fish when $\underline{\alpha}(x,t) = 0.8(1+0.2\sin(\pi t))$ and $\overline{\alpha}(x,t) = 1.6(1+0.2\sin(\pi t))$.*

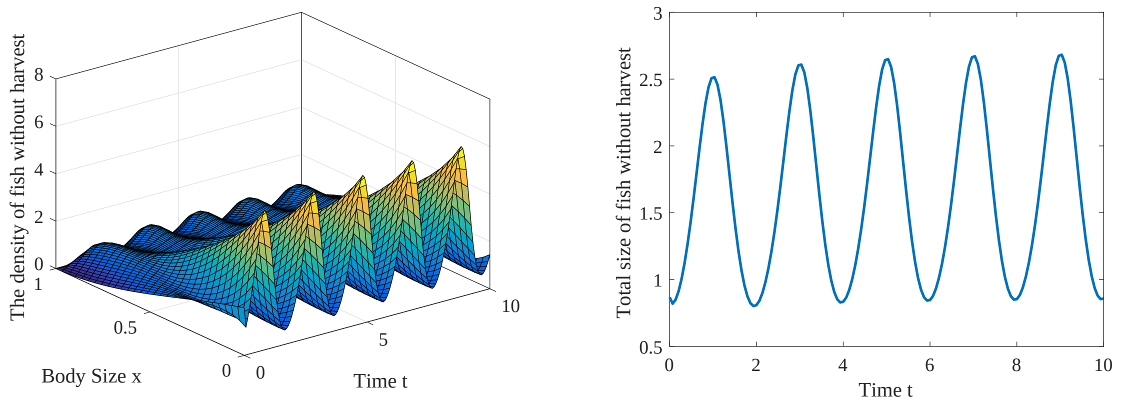


(a) The density of fish.



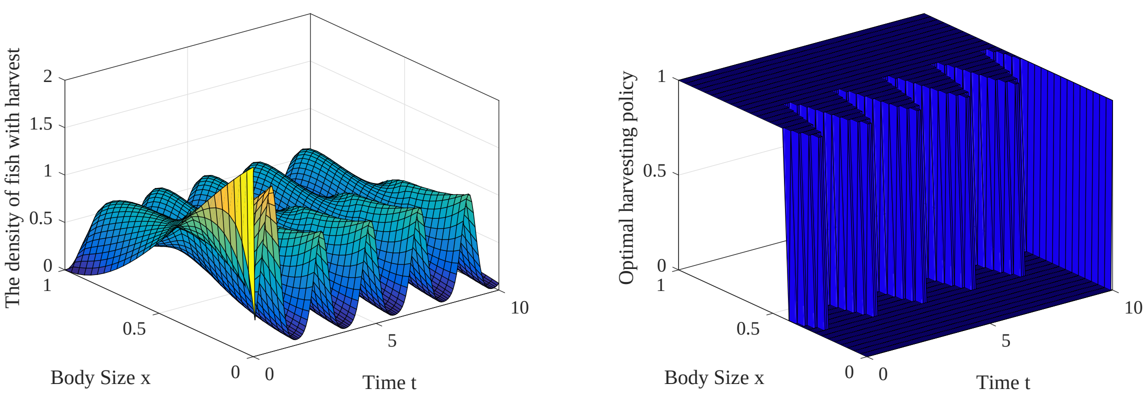
(b) The size of fish.

Figure 1. The density function and the size of fish with $\psi(t) = 0.04(1+\sin(\pi t))$ and $\alpha \equiv 0$.

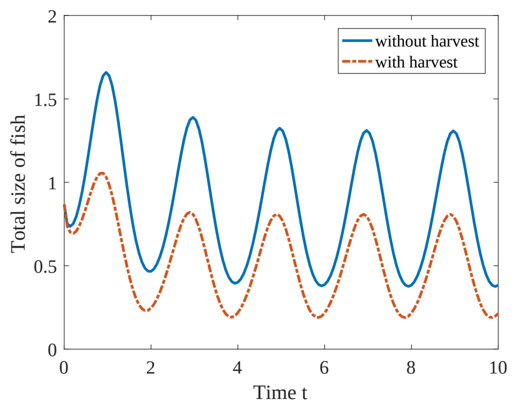


(a) The density of fish. (b) The size of fish.

Figure 2. The density function and the size of fish with $\psi(t) = 0.8(1 + \sin(\pi t))$ and $\alpha \equiv 0$.

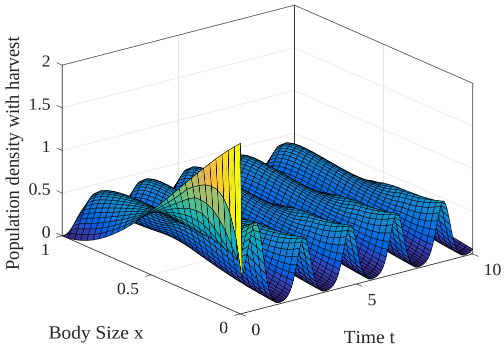


(a) The optimal densities $u^*(x, t)$. (b) The optimal strategies $\alpha^*(x, t)$.

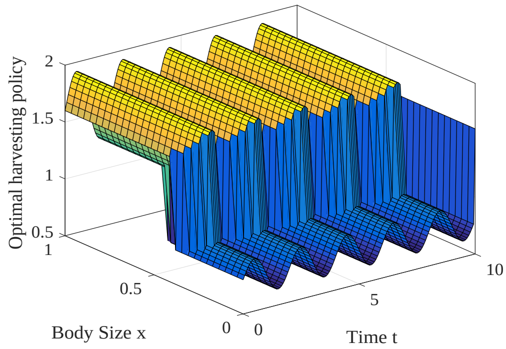


(c) The total size of fish.

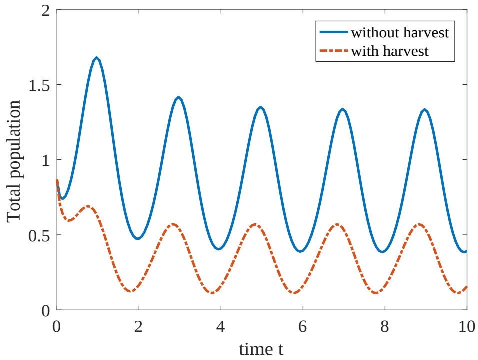
Figure 3. Optimal densities $u^*(x, t)$, optimal strategy $\alpha^*(x, t)$, and total size of fish.



(a) The optimal densities $u^*(x, t)$.



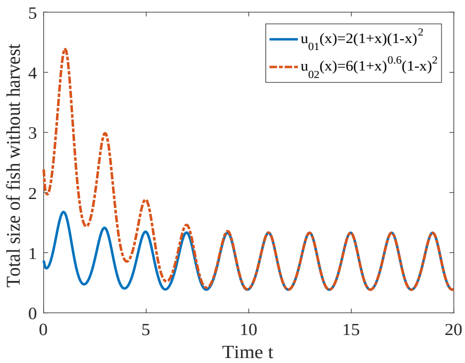
(b) The optimal strategies $\alpha^*(x, t)$.



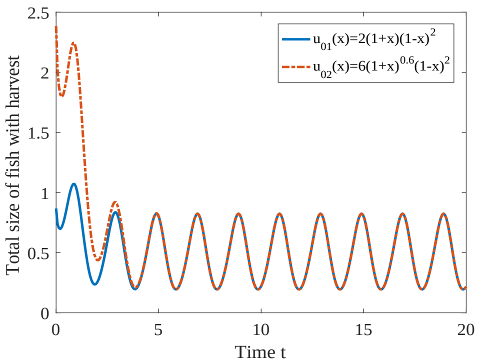
(c) The total size of fish.

Figure 4. Optimal densities $u^*(x, t)$, optimal strategy $\alpha^*(x, t)$, and total size of fish.

Example 4.3. *Take $\psi(t) = 0.04(1 + \sin(\pi t))$, $u_{10}(x) = 2(1 + x)(1 - x)^2$, and $u_{20}(x) = 6(1 + x)^{0.6}(1 - x)^2$. Assume that $\underline{\alpha}(x, t) = 0$ and $\overline{\alpha}(x, t) = 1$. Figure 5(a,b) shows the total size of fish without fishing and with optimal fishing.*



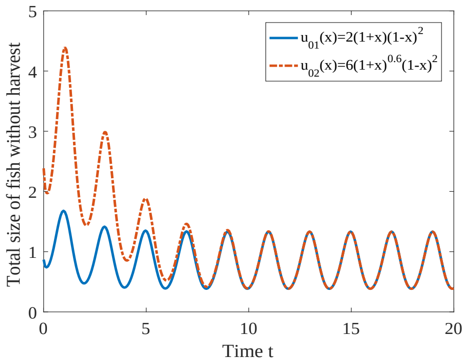
(a) Total size of fish without harvest.



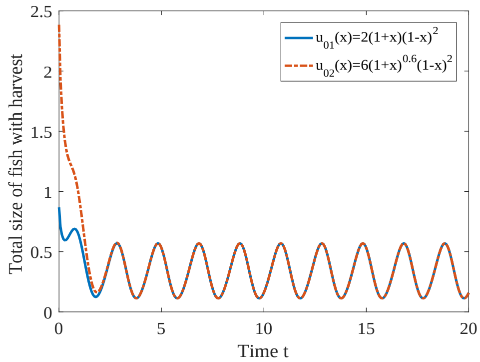
(b) Total size of fish with harvest.

Figure 5. The total size of fish with different initial values.

Example 4.4. *Take* $\psi(t) = 0.04(1 + \sin(\pi t))$, $u_{10}(x) = 2(1 + x)(1 - x)^2$, *and* $u_{20}(x) = 6(1 + x)^{0.6}(1 - x)^2$. *Assume that* $\underline{\alpha}(x, t) = 0.8(1 + 0.2 \sin(\pi t))$ *and* $\overline{\alpha}(x, t) = 1.6(1 + 0.2 \sin(\pi t))$. *Figure 6(a,b) shows the total size of fish without fishing and with optimal fishing.*



(a) Total size of fish without harvest.



(b) Total size of fish with harvest.

Figure 6. The total size of fish with different initial values.

As can be seen from Figures 1 and 2, when there is no fishing, the more fry are released, and the larger the final number of fish will be. Obviously, this is consistent with the basic facts. By comparing Figures 3(c) and 4(c), we can obtain the basic fact that a larger harvesting effort can reduce the size of fish. It can be seen from Figures 3(b) and 4(b) that the optimal fishing strategy has a bang-bang structure, which is consistent with the optimal control form of most practical problems. In addition, by comparing Figures 5 and 6, it can be observed that for different initial values, the total number of fish will tend toward the same periodic solution, which is in line with the theoretical results of the model. At the same time, from Figures 5(a) and 5(b), we can find that fishing can reduce the time it takes for the solution of the model to tend toward the periodic solution.

5. Conclusions and discussion

This paper studies an optimal development model of fish with a size structure and a non-decreasing renewal process in a periodic environment. As mentioned before, our model includes some existing models as special cases. First, we establish the existence of a unique non-negative bounded solution to the state system and the continuous dependence of the solution on the control variable. Then, by constructing an appropriate adjoint system and applying the normal cone technique, we derive the Euler-Lagrange equations that describe the exact structure of the optimal strategy. Finally, with the help of Ekeland's variational principle and the fixed-point method, we prove the existence of a unique optimal strategy.

Now, we give a brief introduction to the differences between our research results and methods and those in related works. The model established in our paper is an extension of models in [6, 12, 21, 22]. Moreover, the results of our paper extend the corresponding conclusions in [21, 22]. It is worth pointing out that [27, 28] assumes that $F(s) = s\phi(s)$ satisfies the local Lipschitz condition, while our paper assumes that $\phi(s)$ satisfies the local Lipschitz condition. This makes the well-posed analysis of our model more difficult. In [6], the existence of the optimal strategy is proved by means of a maximizing sequence, and its uniqueness is established by excluding the singular cases. Nevertheless, the corresponding problems we study are handled by means of Ekeland's principle and fixed-point methods. In addition, [20] only considered the existence of the optimal harvesting strategy without taking into account its uniqueness and the structure of the optimal strategy. [28] proved that there is at least one solution to the optimal harvesting problem, but did not pay attention to its uniqueness.

To conclude our paper, we propose several directions for further research. First of all, in this paper, it is assumed that the amount of fry artificially stocked is a bounded function. To reflect the actual situation more realistically, it should be assumed that the amount of fry stocked is dependent on the total population of fish. Second, similar to what was done in [30], the optimal harvesting problem of a fractional-order model with a size structure can be studied.

Author contributions

Rong Liu: Responsible for the preparation of the original draft of the manuscript; Xin Yi: Responsible for the numerical simulation; Yanmei Wang: Responsible for the review and editing of the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (No. 12001341), the Natural Science Foundation of Shanxi (No. 202403021221214), and Scientific and Technological Innovation Programs of Higher Education Institutions in Shanxi (No. 2024L298).

Conflict of interest

The authors declare there are no conflicts of interest.

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