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Research article

Bivariate Epanechnikov-Weibull distribution based on Sarmanov copula: properties, simulation, and uncertainty measures with applications

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Abstract: The modeling of bivariate data in statistics often requires constructing families of bivariate distributions with predefined marginals. In this study, we introduced a novel bivariate distribution, denoted as EP-WD-SAR, which combines the Sarmanov (SAR) copula with the Epanechnikov-Weibull marginal distribution (EP-WD). We analyzed its statistical properties, including product moments, correlation coefficient, moment-generating function, conditional distribution, and concomitants of order statistics. Additionally, we evaluated key reliability and information measures such as the hazard function, reversed hazard function, bivariate extropy, bivariate weighted extropy, and bivariate cumulative residual extropy. Parameter estimation was performed using maximum likelihood, asymptotic confidence intervals, and Bayesian methods. Finally, we demonstrated the advantages of the EP-WD-SAR model over existing alternatives, including the bivariate Weibull-SAR, bivariate Epanechnikov-exponential-SAR, bivariate exponential-SAR, and bivariate Chen-SAR distributions through applications to real data sets.

Keywords: Sarmanov family; Epanechnikov-Weibull distribution; maximum likelihood estimation; Bayesian estimation; confidence intervals; bootstrap; simulation
Mathematics Subject Classification: 60B12, 62G30

1. Introduction

One of the most widely used probability distributions is the Weibull distribution, which is used in many scientific fields, including biology, hydrology, engineering, reliability theory, and more; see McCool [1] and Weibull [2]. However, it does not fit well with data sets with bathtub-shaped or upside-down bathtub-shaped failure rates, which are frequently seen in reliability and engineering.

Numerous Weibull distribution models, including inverse Weibull, exponentiated Weibull, modified Weibull, transmuted inverse Weibull, and others, have been proposed in the literature to model such datasets. The Epanechnikov-Weibull distribution (EP-WD) was suggested by Alzoubi et al. [3] by combining the kernel function with the Weibull distribution. The distribution function (DF) and probability density function (PDF) of this model are, respectively, given by

$$\mathsf{F}_X(x;\gamma,\omega) = 1 + \frac{1}{2}e^{-3\left(\frac{x}{\omega}\right)^\gamma} - \frac{3}{2}e^{-2\left(\frac{x}{\omega}\right)^\gamma}, \quad \gamma,\omega > 0, \quad x > 0 \tag{1.1}$$

and

$$\mathsf{f}_X(x;\gamma,\omega) = \left(\frac{3\gamma}{2\omega^\gamma}\right)x^{\gamma-1}e^{-2\left(\frac{x}{\omega}\right)^\gamma}\left(2 - e^{-\left(\frac{x}{\omega}\right)^\gamma}\right), \quad \gamma,\omega > 0, \quad x > 0, \tag{1.2}$$

where γ and ω are the shape and scale parameters, respectively. The survival function of the EP-WD, also known as the reliability function and characterizing the system's life behavior, is

$$\mathsf{R}_X(x;\gamma,\omega) = 1 - \mathsf{F}_X(x;\gamma,\omega) = \frac{3}{2}e^{-2\left(\frac{x}{\omega}\right)^\gamma} - \frac{1}{2}e^{-3\left(\frac{x}{\omega}\right)^\gamma}.$$

The EP-WD, defined by (1.1) or (1.2), and denoted by, EP-WD(γ,ω), offers several advantages over the standard Weibull distribution, depending on the context in which it is used. Notably, it provides a smoother and more accurate representation of underlying data, particularly in small sample sizes or irregular distributions. Additionally, it minimizes the mean integrated squared error, enhancing the precision of PDF estimations.

The EP-WD provides improved control over tail behavior by combining the characteristics of two distributions, each of which addresses tail behavior in distinct ways. Specifically, the Epanechnikov distribution, commonly used as a kernel, has finite support, meaning its values are non-zero only within a specific range. This feature helps control the spread and ensures that the distribution's density does not extend excessively beyond certain limits, thereby influencing tail behavior in a controlled manner. In contrast, the Weibull distribution is highly versatile and widely used for modeling tail behavior, particularly in reliability analysis and extreme value theory.

By combining these two distributions, the EP-WD benefits from both the Epanechnikov distribution's ability to constrain spread and the Weibull distribution's strength in modeling tails. This makes the EP-WD a flexible and effective tool in applications where precise control over tail behavior is crucial. It also features a more adaptable hazard function, improving real-world predictive performance. Moreover, its computational efficiency can lead to faster and more stable calculations in data analysis applications.

Several techniques have been developed to construct new bivariate statistical distributions, enhancing modeling efficiency. One widely documented method in statistical literature involves the utilization of copulas (see Nelsen, [4]). Copulas are crucial in characterizing bivariate distributions, particularly when an explicit dependency structure exists between the variables. These functions link bivariate DFs with uniform $[0,1]$ marginals, making them valuable tools for analyzing bivariate relationships. The choice of a copula function depends on the type of dependence structure between the two random variables (RVs). In high-dimensional statistical applications, copulas are particularly useful because they simplify the modeling and estimation of random vectors by enabling the estimation of the marginals and the copula separately. Given two marginal distributions, $\mathsf{F}_{X_1}(x_1) = P(X_1 \leq x_1)$ and $\mathsf{F}_{X_2}(x_2) = P(X_2 \leq x_2)$, along with a copula function, $C(u,v)$, Sklar [5] introduced the corresponding joint DF (JDF) as follows:

$$\mathsf{G}_{X_1,X_2}(x_1,x_2) = C(\mathsf{F}_{X_1}(x_1), \mathsf{F}_{X_2}(x_2)). \tag{1.3}$$

When the JDF is absolutely continuous, then we get

$$\mathsf{g}_{X_1,X_2}(x_1,x_2) = \mathsf{f}_{X_1}(x_1)\mathsf{f}_{X_2}(x_2)c(\mathsf{F}_{X_1}(x_1), \mathsf{F}_{X_2}(x_2)), \tag{1.4}$$

where $\mathsf{g}_{X_1,X_2}(x_1,x_2)$ is the joint PDF (JPDF) of $\mathsf{G}_{X_1,X_2}(x_1,x_2)$ and $c(u,v) = \frac{\partial^2 C(u,v)}{\partial u \partial v}$ is the PDF of the copula $C(u,v)$ (for more details about copulas and the Sklar theorem, see Nelsen [4], Barakat et al. [6] and Iordanov and Chervenov [7]).

The Sarmanov family, indicated by SAR(.), is a highly adaptive and robust extension of the conventional Farlie-Gumbel-Morgenstern (FGM) family of bivariate DFs, proposed by Sarmanov [8] for the characterization of hydrological events. Recent studies, specifically those by Abd Elgawad et al. [9], Alawady et al. [10], Barakat et al. [11,12], and Husseiny et al. [13], have demonstrated the superiority of this family over all other expansions of the FGM family. The DF and PDF of SAR(α) are given, respectively, by

$$\begin{aligned} \mathsf{G}_{X_1,X_2}(x_1,x_2) &= \mathsf{F}_{X_1}(x_1)\mathsf{F}_{X_2}(x_2)\Big[1 + 3\alpha\overline{\mathsf{F}}_{X_1}(x_1)\overline{\mathsf{F}}_{X_2}(x_2) + 5\alpha^2(2\mathsf{F}_{X_1}(x_1) - 1) \\ &\times (2\mathsf{F}_{X_2}(x_2) - 1)\overline{\mathsf{F}}_{X_1}(x_1)\overline{\mathsf{F}}_{X_2}(x_2)\Big] \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} \mathsf{g}_{X_1,X_2}(x_1,x_2) &= \mathsf{f}_{X_1}(x_1)\mathsf{f}_{X_2}(x_2)\Big[1 + 3\alpha(2\mathsf{F}_{X_1}(x_1) - 1)(2\mathsf{F}_{X_2}(x_2) - 1) \\ &+ \frac{5}{4}\alpha^2(3(2\mathsf{F}_{X_1}(x_1) - 1)^2 - 1)(3(2\mathsf{F}_{X_2}(x_2) - 1)^2 - 1)\Big], \ |\alpha| \leq \frac{\sqrt{7}}{5}, \end{aligned} \tag{1.6}$$

where $\overline{\mathsf{F}}_{X_i}(x_i)$ is the survival function (or the reliability function $\mathsf{R}(x_i) = \overline{\mathsf{F}}_{X_i}(x_i) = P(X_i > x_i)$) of $\mathsf{F}_{X_i}(x_i)$, $i = 1,2$. Moreover, when the marginals are uniform, then the correlation coefficient is α. Thus, in this case, the minimal and maximal correlation coefficients ρ of this copula are -0.529 and 0.529, respectively (cf. Balakrishnan and Lin, [14], page 74). For more details about this family, see Abd Elgawad et al. [9], Alawady et al. [10], Barakat et al. [11,12], and Husseiny et al. [13].

Remark 1.1. *It is worth noting that the Sarmanov copula is a special case of the copulas discussed in Nelsen [4], specifically in Theorem 3.2.10. Moreover, the restriction* $|\alpha| \leq \frac{\sqrt{7}}{5}$ *can be directly deduced from this theorem. For a detailed explanation, refer to Example 3.15 in Nelsen [4].*

Here, we build upon the distribution defined in (1.1) and (1.2), in combination with the copula specified in (1.5) and (1.6), to introduce a novel bivariate distribution: The Epanechnikov-Weibull distribution Sarmanov family (EP-WD-SAR). This proposed distribution offers greater flexibility and sophistication in modeling complex datasets, enabling more precise predictions and enhanced decision-making across applications.

Notably, the EP-WD-SAR family significantly extends the bivariate Epanechnikov-exponential distribution, based on the FGM copula, denoted as EP-EX-FGM, as introduced by Barakat et al. [15]. The EP-WD-SAR family generalizes the marginal distributions and the copula structure of the EP-EX-FGM family, making it a more versatile and adaptive framework for statistical modeling.

Key advantages of the EP-WD-SAR family

(1) Motivation for choosing the Sarmanov copula: The dependence parameter α in the Sarmanov copula has a direct and interpretable effect on the correlation structure; for instance, $\alpha = 0$ implies independence. This clarity is particularly appealing in applied modeling. Furthermore, the Sarmanov copula strikes an effective balance between analytical tractability and modeling flexibility. Unlike Archimedean, copulas such as Clayton or Gumbel, are designed to capture strong lower or upper tail dependence, respectively. The Sarmanov copula provides a more general framework for modeling dependence, including positive and negative correlations, using a single, easily interpretable parameter.
Another key advantage is its ability to accommodate a broad class of marginal distributions without imposing restrictive conditions, making it especially suitable for integrating complex forms like the EP-WD. In contrast to more rigid bivariate models such as the Marshall-Olkin or Clayton-Weibull distributions, the Sarmanov approach supports moderate asymmetry and retains closed-form expressions for the JPDFs and DFs. These features make it a practical and versatile tool for capturing realistic dependence structures, particularly in applications such as reliability analysis, environmental studies, and biomedical research, where tail behavior may be moderate, but the flexibility of the marginal model is critical.

(2) The EP-WD-SAR family accommodates a broader spectrum of dependency structures, including positive and negative dependencies.

(3) The flexibility of the Weibull distribution enables robust modeling in fields such as:

   (a) Engineering failure analysis (e.g., reliability studies).
   (b) Financial risk assessment (e.g., modeling heavy-tailed distributions).
   (c) Medical survival analysis (e.g., modeling varying hazard rates).

(4) The EP-WD-SAR distribution provides better tail properties, making it more suitable for analyzing extreme events and long-term trends.

Overall, the EP-WD-SAR family represents a significant advancement in bivariate modeling, offering improved adaptability, greater accuracy, and broader applicability in diverse statistical and real-world domains.

The paper is structured as follows: In Section 2, we introduce the DF and PDF of the EP-WD-SAR distribution. In Section 3, we explore the distributional characteristics of the EP-WD-SAR family, including moments, the moment-generating function (MGF), the conditional distribution, concomitants of order statistics (OSs), mean residual life (MRL), and the vitality function. In Section 4, we evaluate recent information measures, such as the bivariate hazard function, bivariate reversed hazard (RH) function, bivariate extropy, bivariate weighted extropy, and bivariate cumulative residual extropy (CREX). In Section 5, model parameters are estimated using maximum likelihood (ML) and Bayesian techniques, with asymptotic confidence intervals (CIs) computed accordingly. In Section 6, we assess ML and Bayesian estimators through Monte Carlo simulations. In Section 7, we present an application of the proposed methodology to two bivariate real-world datasets, demonstrating its effectiveness. We conclude the paper in Section 8.

2. The Epanechnikov-Weibull distribution based on SAR copula

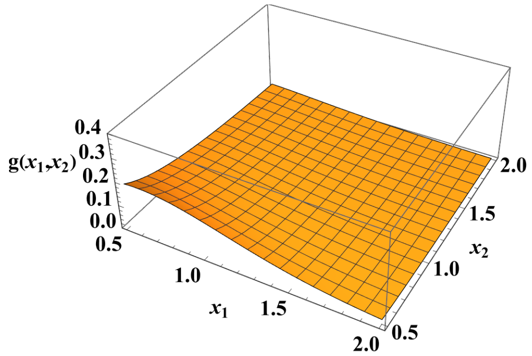
Let $X_1 \sim$ EP-WD(γ_1, ω_1) and $X_2 \sim$ EP-WD(γ_2, ω_2). Thus, according to (1.3), the DF of bivariate EP-WD based on the SAR copula, denoted by EP-WD-SAR$(\gamma_1, \omega_1, \gamma_2, \omega_2)$, is given by

$$
\begin{aligned}
\mathsf{G}_{X_1,X_2}(x_1,x_2) &= \left(1+\frac{1}{2}e^{-3\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}-\frac{3}{2}e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right)\left(1+\frac{1}{2}e^{-3\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}-\frac{3}{2}e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right)\Bigg[1+3\alpha\left(\frac{3}{2}e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right.\\
&\left.-\frac{1}{2}e^{-3\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right)\left(\frac{3}{2}e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}-\frac{1}{2}e^{-3\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right)+5\alpha^2\left(1+e^{-3\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}-3e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right)\\
&\times\left(1+e^{-3\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}-3e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right)\left(\frac{3}{2}e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}-\frac{1}{2}e^{-3\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right)\left(\frac{3}{2}e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}-\frac{1}{2}e^{-3\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right)\Bigg].
\end{aligned}
\tag{2.1}
$$

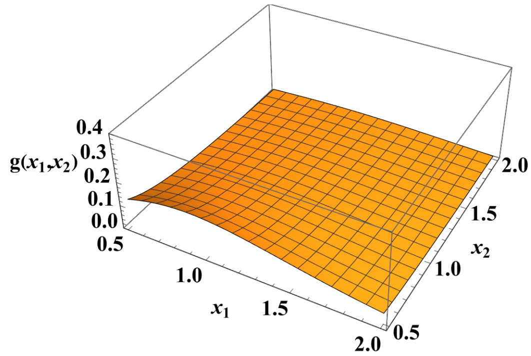
Moreover, according to (1.4), the corresponding PDF of (2.1) is given by

$$
\begin{aligned}
\mathsf{g}_{X_1,X_2}(x_1,x_2) &= \left(\frac{3\gamma_1}{2\omega_1^{\gamma_1}}\right)\left(\frac{3\gamma_2}{2\omega_2^{\gamma_2}}\right)x_1^{\gamma_1-1}e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}x_2^{\gamma_2-1}e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\left(2-e^{-\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right)\left(2-e^{-\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right)\\
&\times\Bigg[1+3\alpha\left(1+e^{-3\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}-3e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right)\left(1+e^{-3\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}-3e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right)+\frac{5\alpha^2}{4}\\
&\times\left(3\left(1+e^{-3\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}-3e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right)^2-1\right)\left(3\left(1+e^{-3\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}-3e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right)^2-1\right)\Bigg].
\end{aligned}
\tag{2.2}
$$

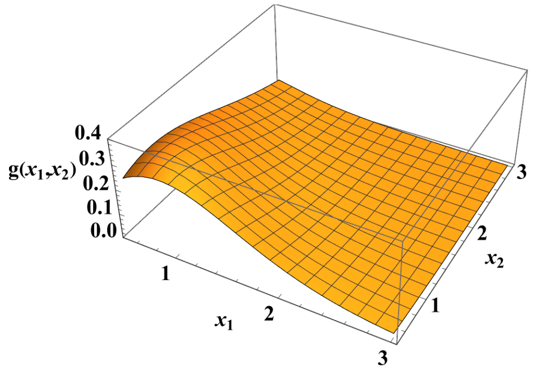
Figure 1 depicts the behavior of the JPDF of the EP-WD-SAR distribution for some selected values for the shape parameters $\gamma_1, \gamma_2, \alpha$, and the scale parameters ω_1, ω_2.



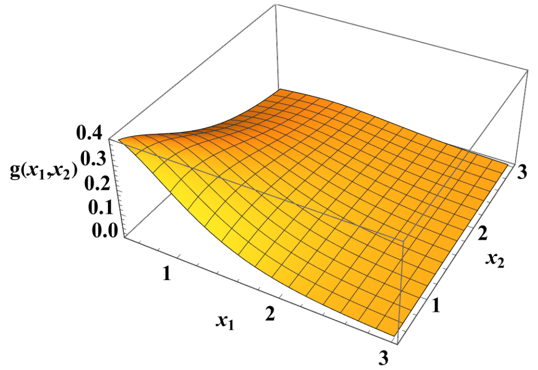
(a) $\gamma_1 = 0.5$, $\omega_1 = 1.6$, $\gamma_2 = 1.2$, $\omega_2 = 1.9$, $\alpha = -0.4$



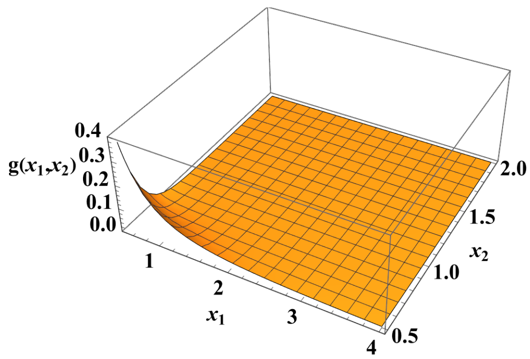
(b) $\gamma_1 = 0.5$, $\omega_1 = 1.6$, $\gamma_2 = 1.2$, $\omega_2 = 1.9$, $\alpha = 0.4$



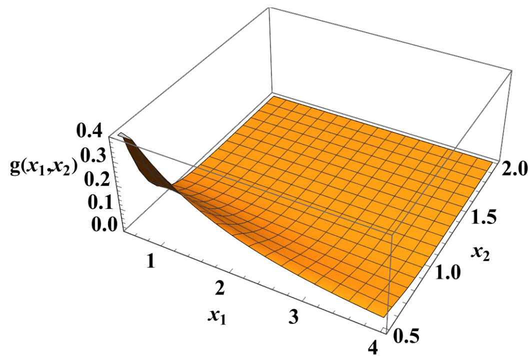
(c) $\gamma_1 = 1.4$, $\omega_1 = 1.6$, $\gamma_2 = 1.2$, $\omega_2 = 2.3$, $\alpha = -0.3$



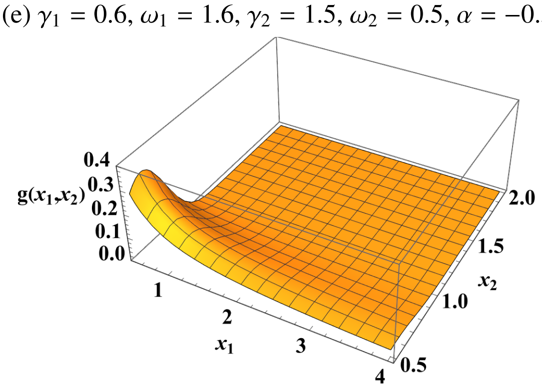
(d) $\gamma_1 = 1.4$, $\omega_1 = 1.6$, $\gamma_2 = 1.2$, $\omega_2 = 2.3$, $\alpha = 0.3$



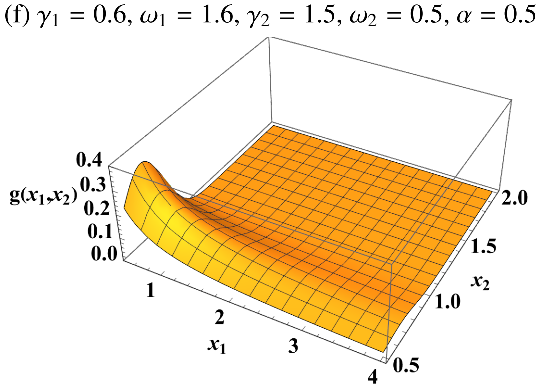
(e) $\gamma_1 = 0.6$, $\omega_1 = 1.6$, $\gamma_2 = 1.5$, $\omega_2 = 0.5$, $\alpha = -0.5$



(f) $\gamma_1 = 0.6$, $\omega_1 = 1.6$, $\gamma_2 = 1.5$, $\omega_2 = 0.5$, $\alpha = 0.5$



(g) $\gamma_1 = 0.2$, $\omega_1 = 2.6$, $\gamma_2 = 3.5$, $\omega_2 = 0.8$, $\alpha = -0.2$



(h) $\gamma_1 = 0.2$, $\omega_1 = 2.6$, $\gamma_2 = 3.5$, $\omega_2 = 0.8$, $\alpha = 0.2$

Figure 1. Some graphs of the JPDF of the EP-WD-SAR distribution.

3. Properties of the EP-WD-SAR distribution

In this section, we explore several key properties of the EP-WD-SAR distribution, including joint moments, the MGF, conditional distributions, concomitants of OSs, the MRL, and the vitality function. The joint moments reflect both the compact support of the Epanechnikov kernel and the tail characteristics introduced by the Weibull component. The MGF is a comprehensive tool for characterizing the joint distribution, enabling the derivation of moments, risk-related measures, analysis of sum distributions, and the application of limit theorems such as the central limit theorem. The conditional distributions, shaped by the Sarmanov copula structure, provide insight into the localized dependence between variables illustrating how the behavior of one variable influences the distributional shape of the other. These conditional distributions also serve as a foundation for predictive modeling, bayesian inference, and regression based analysis. Altogether, the properties discussed in this section underscore the flexibility and applicability of the proposed model, particularly in contexts that demand interpretable dependence structures, robust marginal behavior, and predictive capacity, such as reliability analysis and survival modeling.

3.1. Moments

The (l_1, l_2)th, $l_1, l_2 = 1, 2, ...,$ product moments of the EP-WD-SAR$(\gamma_1, \omega_1, \gamma_2, \omega_2)$ is given by

$$\begin{aligned}
\mathsf{E}(X_1^{l_1} X_2^{l_2}) &= \left(\frac{3\gamma_1}{2\omega_1^{\gamma_1}}\right)\left(\frac{3\gamma_2}{2\omega_2^{\gamma_2}}\right)\int_0^\infty \int_0^\infty x_1^{l_1+\gamma_1-1} e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}} x_2^{l_2+\gamma_2-1} e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}} \left(2 - e^{-\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right)\left(2 - e^{-\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right) \\
&\times \left[1 + 3\alpha\left(1 + e^{-3\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}} - 3e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right)\left(1 + e^{-3\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}} - 3e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right) + \frac{5\alpha^2}{4}\right. \\
&\times \left.\left(3\left(1 + e^{-3\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}} - 3e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right)^2 - 1\right)\left(3\left(1 + e^{-3\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}} - 3e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right)^2 - 1\right)\right] dx_1 dx_2 \\
&= \frac{\omega_1^{l_1}\omega_2^{l_2}}{4}\Gamma\left(1 + \frac{l_1}{\gamma_1}\right)\Gamma\left(1 + \frac{l_2}{\gamma_2}\right)\left(3 \times 2^{\frac{-l_1}{\gamma_1}} - 3^{\frac{-l_1}{\gamma_1}}\right)\left(3 \times 2^{\frac{-l_2}{\gamma_2}} - 3^{\frac{-l_2}{\gamma_2}}\right)\left[1 + 3\alpha\left(2^{\frac{-l_1}{\gamma_1}}\right.\right. \\
&- \left. 3^{\frac{-l_1}{\gamma_1}-1}\right)^{-1}\left(2^{\frac{-l_2}{\gamma_2}} - 3^{\frac{-l_2}{\gamma_2}-1}\right)^{-1}\left(5^{\frac{-l_1}{\gamma_1}} + 2^{\frac{-l_1}{\gamma_1}} - 3 \times 2^{\frac{-2l_1}{\gamma_1}-1} - 3^{\frac{-l_1}{\gamma_1}-1} - 2^{-1-\frac{l_1}{\gamma_1}} \times 3^{\frac{-l_1}{\gamma_1}-1}\right) \\
&\times \left(5^{\frac{-l_2}{\gamma_2}} + 2^{\frac{-l_2}{\gamma_2}} - 3 \times 2^{\frac{-2l_2}{\gamma_2}-1} - 3^{\frac{-l_2}{\gamma_2}-1} - 2^{-1-\frac{l_2}{\gamma_2}} \times 3^{\frac{-l_2}{\gamma_2}-1}\right) + 5\alpha^2\left(2^{\frac{-l_1}{\gamma_1}} - 3^{\frac{-l_1}{\gamma_1}-1}\right)^{-1} \\
&\times \left(2^{\frac{-l_2}{\gamma_2}} - 3^{\frac{-l_2}{\gamma_2}-1}\right)^{-1}\left(3 \times 5^{\frac{-l_1}{\gamma_1}} + 7 \times 2^{-1-\frac{l_1}{\gamma_1}} \times 3^{\frac{-l_1}{\gamma_1}} - 9 \times 2^{-1-\frac{2l_1}{\gamma_1}} + 3 \times 2^{-1-\frac{3l_1}{\gamma_1}} - \frac{9}{2} \times 7^{\frac{-l_1}{\gamma_1}}\right. \\
&- \left.\frac{1}{2} \times 3^{-1-\frac{2l_1}{\gamma_1}} + 2^{-1-\frac{l_1}{\gamma_1}} \times 3^{\frac{-l_1}{\gamma_1}} + 2^{\frac{-l_1}{\gamma_1}} - 3^{\frac{-l_1}{\gamma_1}-1}\right)\left(3 \times 5^{\frac{-l_2}{\gamma_2}} + 7 \times 2^{-1-\frac{l_2}{\gamma_2}} \times 3^{\frac{-l_2}{\gamma_2}} - 9\right. \\
&\times \left.\left. 2^{-1-\frac{2l_2}{\gamma_2}} + 3 \times 2^{-1-\frac{3l_2}{\gamma_2}} - \frac{9}{2} \times 7^{\frac{-l_2}{\gamma_2}} - \frac{1}{2} \times 3^{-1-\frac{2l_2}{\gamma_2}} + 2^{-1-\frac{l_2}{\gamma_2}} \times 3^{\frac{-l_2}{\gamma_2}} + 2^{\frac{-l_2}{\gamma_2}} - 3^{\frac{-l_2}{\gamma_2}-1}\right)\right]. \quad (3.1)
\end{aligned}$$

Thus, using (3.1) at $l_1 = l_2 = 1$, we get

$$
\begin{aligned}
\mathsf{E}(X_1X_2) \;=\;& \frac{\omega_1\omega_2}{4}\Gamma\left(1+\frac{1}{\gamma_1}\right)\Gamma\left(1+\frac{1}{\gamma_2}\right)\left(3\times 2^{\frac{-1}{\gamma_1}}-3^{\frac{-1}{\gamma_1}}\right)\left(3\times 2^{\frac{-1}{\gamma_2}}-3^{\frac{-1}{\gamma_2}}\right)\left[1+3\alpha\left(2^{\frac{-1}{\gamma_1}}-3^{\frac{-1}{\gamma_1}-1}\right)^{-1}\right.\\
\times\;& \left(2^{\frac{-1}{\gamma_2}}-3^{\frac{-1}{\gamma_2}-1}\right)^{-1}\left(5^{\frac{-1}{\gamma_1}}+2^{\frac{-1}{\gamma_1}}-3\times 2^{\frac{-2}{\gamma_1}-1}-3^{\frac{-1}{\gamma_1}-1}-2^{-1-\frac{1}{\gamma_1}}\times 3^{\frac{-1}{\gamma_1}-1}\right)\left(5^{\frac{-1}{\gamma_2}}+2^{\frac{-1}{\gamma_2}}\right.\\
-\;& \left.3\times 2^{\frac{-2}{\gamma_2}-1}-3^{\frac{-1}{\gamma_2}-1}-2^{-1-\frac{1}{\gamma_2}}\times 3^{\frac{-1}{\gamma_2}-1}\right)+5\alpha^2\left(2^{\frac{-1}{\gamma_1}}-3^{\frac{-1}{\gamma_1}-1}\right)^{-1}\left(2^{\frac{-1}{\gamma_2}}-3^{\frac{-1}{\gamma_2}-1}\right)^{-1}\\
\times\;& \left(3\times 5^{\frac{-1}{\gamma_1}}+7\times 2^{-1-\frac{1}{\gamma_1}}\times 3^{\frac{-1}{\gamma_1}}-9\times 2^{-1-\frac{2}{\gamma_1}}+3\times 2^{-1-\frac{3}{\gamma_1}}-\frac{9}{2}\times 7^{\frac{-1}{\gamma_1}}-\frac{1}{2}\times 3^{-1-\frac{2}{\gamma_1}}+2^{-1-\frac{1}{\gamma_1}}\right.\\
\times\;& \left.3^{\frac{-1}{\gamma_1}}+2^{\frac{-1}{\gamma_1}}-3^{\frac{-1}{\gamma_1}-1}\right)\left(3\times 5^{\frac{-1}{\gamma_2}}+7\times 2^{-1-\frac{1}{\gamma_2}}\times 3^{\frac{-1}{\gamma_2}}-9\times 2^{-1-\frac{2}{\gamma_2}}+3\times 2^{-1-\frac{3}{\gamma_2}}-\frac{9}{2}\times 7^{\frac{-1}{\gamma_2}}\right.\\
-\;& \left.\left.\frac{1}{2}\times 3^{-1-\frac{2}{\gamma_2}}+2^{-1-\frac{1}{\gamma_2}}\times 3^{\frac{-1}{\gamma_2}}+2^{\frac{-1}{\gamma_2}}-3^{\frac{-1}{\gamma_2}-1}\right)\right].
\end{aligned}
$$

Thus, the conditional expectation is non-linear with respect to x_1. Therefore, the coefficient of correlation between X_1 and X_2 is

$$
\begin{aligned}
\rho_{X_1,X_2}=\;& \frac{\left(2^{\frac{-1}{\gamma_1}}-3^{\frac{-1}{\gamma_1}-1}\right)^{-1}\left(2^{\frac{-1}{\gamma_2}}-3^{\frac{-1}{\gamma_2}-1}\right)^{-1}}{\sqrt{\frac{2\Gamma\left(1+\frac{2}{\gamma_1}\right)\left(3\times 2^{\frac{-2}{\gamma_1}}-3^{\frac{-2}{\gamma_1}}\right)}{\left(\Gamma\left(1+\frac{1}{\gamma_1}\right)\left(3\times 2^{\frac{-1}{\gamma_1}}-3^{\frac{-1}{\gamma_1}}\right)\right)^2}-1}\sqrt{\frac{2\Gamma\left(1+\frac{2}{\gamma_2}\right)\left(3\times 2^{\frac{-2}{\gamma_2}}-3^{\frac{-2}{\gamma_2}}\right)}{\left(\Gamma\left(1+\frac{1}{\gamma_2}\right)\left(3\times 2^{\frac{-1}{\gamma_2}}-3^{\frac{-1}{\gamma_2}}\right)\right)^2}-1}}\left[3\alpha\left(5^{\frac{-1}{\gamma_1}}+2^{\frac{-1}{\gamma_1}}-3\times 2^{\frac{-2}{\gamma_1}-1}\right.\right.\\
&-3^{\frac{-1}{\gamma_1}-1}-2^{-1-\frac{1}{\gamma_1}}\times 3^{\frac{-1}{\gamma_1}-1}\Big)\left(5^{\frac{-1}{\gamma_2}}+2^{\frac{-1}{\gamma_2}}-3\times 2^{\frac{-2}{\gamma_2}-1}-3^{\frac{-1}{\gamma_2}-1}-2^{-1-\frac{1}{\gamma_2}}\times 3^{\frac{-1}{\gamma_2}-1}\right)+5\alpha^2\left(3\times 5^{\frac{-1}{\gamma_1}}\right.\\
&+7\times 2^{-1-\frac{1}{\gamma_1}}\times 3^{\frac{-1}{\gamma_1}}-9\times 2^{-1-\frac{2}{\gamma_1}}+3\times 2^{-1-\frac{3}{\gamma_1}}-\frac{9}{2}\times 7^{\frac{-1}{\gamma_1}}-\frac{1}{2}\times 3^{-1-\frac{2}{\gamma_1}}+2^{-1-\frac{1}{\gamma_1}}\times 3^{\frac{-1}{\gamma_1}}+2^{\frac{-1}{\gamma_1}}\\
&-3^{\frac{-1}{\gamma_1}-1}\Big)\left(3\times 5^{\frac{-1}{\gamma_2}}+7\times 2^{-1-\frac{1}{\gamma_2}}\times 3^{\frac{-1}{\gamma_2}}-9\times 2^{-1-\frac{2}{\gamma_2}}+3\times 2^{-1-\frac{3}{\gamma_2}}-\frac{9}{2}\times 7^{\frac{-1}{\gamma_2}}-\frac{1}{2}\times 3^{-1-\frac{2}{\gamma_2}}\right.\\
&\left.\left.+2^{-1-\frac{1}{\gamma_2}}\times 3^{\frac{-1}{\gamma_2}}+2^{\frac{-1}{\gamma_2}}-3^{\frac{-1}{\gamma_2}-1}\right)\right].
\end{aligned}
$$

We observe that $\rho_{X_1,X_2}=0$, when $\alpha=0$. Moreover, $\alpha=0$ implies that X_1 and X_2 are independent, since SAR(0) corresponds to the independence copula. In Table 1, we have the maximum and minimum values of ρ_{X_1,X_2}, from EP-WD-SAR($\gamma_1,\omega_1,\gamma_2,\omega_2$), as 0.512109 and −0.512006, respectively.

Also, using (2.2) the MGF of X_1 and X_2 is given by

$$
\begin{aligned}
\mathsf{M}_{X_1,X_2}(t_1,t_2)=\;& \sum_{n_1=0}^{\infty}\frac{t_1^{n_1}}{n_1!}\sum_{n_2=0}^{\infty}\frac{t_2^{n_2}}{n_2!}\frac{\omega_1^{n_1}\omega_2^{n_2}}{4}\Gamma\left(1+\frac{n_1}{\gamma_1}\right)\Gamma\left(1+\frac{n_2}{\gamma_2}\right)\left(3\times 2^{\frac{-n_1}{\gamma_1}}-3^{\frac{-n_1}{\gamma_1}}\right)\left(3\times 2^{\frac{-n_2}{\gamma_2}}-3^{\frac{-n_2}{\gamma_2}}\right)\\
&\times\left[1+3\alpha\left(2^{\frac{-n_1}{\gamma_1}}-3^{\frac{-n_1}{\gamma_1}-1}\right)^{-1}\left(2^{\frac{-n_2}{\gamma_2}}-3^{\frac{-n_2}{\gamma_2}-1}\right)^{-1}\left(5^{\frac{-n_1}{\gamma_1}}+2^{\frac{-n_1}{\gamma_1}}-3\times 2^{\frac{-2n_1}{\gamma_1}-1}-3^{\frac{-n_1}{\gamma_1}-1}\right.\right.\\
&-\left.2^{-1-\frac{n_1}{\gamma_1}}\times 3^{\frac{-n_1}{\gamma_1}-1}\right)\left(5^{\frac{-n_2}{\gamma_2}}+2^{\frac{-n_2}{\gamma_2}}-3\times 2^{\frac{-2n_2}{\gamma_2}-1}-3^{\frac{-n_2}{\gamma_2}-1}-2^{-1-\frac{n_2}{\gamma_2}}\times 3^{\frac{-n_2}{\gamma_2}-1}\right)+5\alpha^2
\end{aligned}
$$

$$\times\left(2^{\frac{-n_1}{\gamma_1}}-3^{\frac{-n_1}{\gamma_1}-1}\right)^{-1}\left(2^{\frac{-n_2}{\gamma_2}}-3^{\frac{-n_2}{\gamma_2}-1}\right)^{-1}\left(3\times 5^{\frac{-n_1}{\gamma_1}}+7\times 2^{-1-\frac{n_1}{\gamma_1}}\times 3^{\frac{-n_1}{\gamma_1}}-9\times 2^{-1-\frac{2n_1}{\gamma_1}}\right.$$
$$+\,3\times 2^{-1-\frac{3n_1}{\gamma_1}}-\frac{9}{2}\times 7^{\frac{-n_1}{\gamma_1}}-\frac{1}{2}\times 3^{-1-\frac{2n_1}{\gamma_1}}+2^{-1-\frac{n_1}{\gamma_1}}\times 3^{\frac{-n_1}{\gamma_1}}+2^{\frac{-n_1}{\gamma_1}}-3^{\frac{-n_1}{\gamma_1}-1}\Bigg)\Bigg(3\times 5^{\frac{-n_2}{\gamma_2}}$$
$$+\,7\times 2^{-1-\frac{n_2}{\gamma_2}}\times 3^{\frac{-n_2}{\gamma_2}}-9\times 2^{-1-\frac{2n_2}{\gamma_2}}+3\times 2^{-1-\frac{3n_2}{\gamma_2}}-\frac{9}{2}\times 7^{\frac{-n_2}{\gamma_2}}-\frac{1}{2}\times 3^{-1-\frac{2n_2}{\gamma_2}}+2^{-1-\frac{n_2}{\gamma_2}}$$
$$\left.\left.\times\, 3^{\frac{-n_2}{\gamma_2}}+2^{\frac{-n_2}{\gamma_2}}-3^{\frac{-n_2}{\gamma_2}-1}\right)\right].$$

Table 1. The coefficient of correlation, ρ_{X_1,X_2}, in EP-WD-SAR($\gamma_1,\omega_1,\gamma_2,\omega_2$).

$\alpha=0.2$			$\alpha=0.3$			$\alpha=0.4$			$\alpha=0.529$		
ρ_{X_1,X_2}	γ_1	γ_2	ρ_{X_1,X_2}	γ_1	γ_2	ρ_{X_1,X_2}	γ_1	γ_2	ρ_{X_1,X_2}	γ_1	γ_2
0.172708	1.3	1.1	0.267561	1.4	1.2	0.361303	1.2	1.5	0.484823	1.8	1.2
0.17759	1.3	1.3	0.272987	1.4	1.4	0.363713	1.2	1.7	0.494431	1.8	1.4
0.175401	1.5	1.1	0.270435	1.6	1.2	0.366126	1.4	1.4	0.500034	1.8	1.6
0.180504	1.5	1.3	0.276181	1.6	1.4	0.369965	1.6	1.4	0.503254	1.8	1.8
0.183584	1.5	1.5	0.279623	1.6	1.6	0.374217	1.6	1.6	0.508898	2.2	2.1
0.176991	1.7	1.1	0.272046	1.8	1.2	0.381964	1.9	2	0.509447	2.2	2.2
0.18226	1.7	1.3	0.278041	1.8	1.4	0.383126	1.9	2.9	0.510078	2.4	2.2
0.185464	1.7	1.5	0.281679	1.8	1.6	0.387081	3	3	0.510869	2.4	2.4
0.187441	1.7	1.7	0.283893	1.8	1.8	0.385877	3	4	0.510263	2.6	2.2
0.190714	2.1	1.9	0.272883	2	1.2	0.380975	5	6	0.511191	2.6	2.4
0.191534	2.1	2.1	0.279076	2	1.4	0.3797	5	7	0.511632	2.6	2.6
0.192281	2.5	2.1	0.282872	2	1.6	0.376763	5	10	0.511196	2.8	2.4
0.192818	2.5	2.3	0.285214	2	1.8	0.374596	7	10	0.51174	2.8	2.6
0.193123	2.5	2.5	0.286639	2	2	0.373906	7	11	**0.512109**	**2.886**	**2.886**
$\alpha=-0.2$			$\alpha=-0.3$			$\alpha=-0.4$			$\alpha=-0.529$		
ρ_{X_1,X_2}	γ_1	γ_2	ρ_{X_1,X_2}	γ_1	γ_2	ρ_{X_1,X_2}	γ_1	γ_2	ρ_{X_1,X_2}	γ_1	γ_2
-0.16621	1.3	1.1	-0.25568	1.4	1.2	-0.34233	1.2	1.5	-0.461185	1.8	1.2
-0.17230	1.3	1.3	-0.26334	1.4	1.4	-0.34853	1.2	1.7	-0.475247	1.8	1.4
-0.17015	1.5	1.1	-0.26087	1.6	1.2	-0.34898	1.4	1.4	-0.484605	1.8	1.6
-0.17623	1.5	1.3	-0.26842	1.6	1.4	-0.35617	1.6	1.4	-0.490985	1.8	1.8
-0.18013	1.5	1.5	-0.27338	1.6	1.6	-0.36312	1.6	1.6	-0.503884	2.2	2.1
-0.17279	1.7	1.1	-0.26444	1.8	1.2	-0.3771	1.9	2	-0.505086	2.2	2.2
-0.17884	1.7	1.3	-0.27187	1.8	1.4	-0.38236	1.9	2.9	-0.506885	2.4	2.2
-0.18270	1.7	1.5	-0.27672	1.8	1.6	-0.38705	3	3	-0.508531	2.4	2.4
-0.18523	1.7	1.7	-0.27995	1.8	1.8	-0.38603	3	4	-0.508082	2.6	2.2
-0.18965	2.1	1.9	-0.26694	2	1.2	-0.37830	5	6	-0.509595	2.6	2.4
-0.19071	2.1	2.1	-0.27425	2	1.4	-0.37659	5	7	-0.510541	2.6	2.6
-0.19184	2.5	2.1	-0.27425	2	1.6	-0.37289	5	10	-0.510246	2.8	2.4
-0.1925	2.5	2.3	-0.28213	2	1.8	-0.36880	7	10	-0.511359	3	2.6
-0.19289	2.5	2.5	-0.28423	2	2	-0.36786	7	11	**-0.512006**	**3.014**	**3.014**

3.2. Conditional distribution and concomitants of OSs

After some straightforward algebraic manipulation, the conditional PDF and DF of X_2 given $X_1 = x_1$ are, respectively, expressed as follows:

$$\begin{aligned}
\mathsf{g}_{X_2|X_1}(x_2|x_1) &= \left(\frac{3\gamma_2}{2\omega_2^{\gamma_2}}\right) x_2^{\gamma_2-1} e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}} \left(2 - e^{-\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right)\left[1 + 3\alpha\left(1 + e^{-3\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}} - 3e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right)\right. \\
&\times \left(1 + e^{-3\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}} - 3e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right) + \frac{5\alpha^2}{4}\left(3\left(1 + e^{-3\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}} - 3e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right)^2 - 1\right) \\
&\times \left.\left(3\left(1 + e^{-3\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}} - 3e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right)^2 - 1\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{G}_{X_2|X_1}(x_2|x_1) &= \left(1 + \frac{1}{2}e^{-3\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}} - \frac{3}{2}e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right)\left[1 + 3\alpha\left(1 + e^{-3\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}} - 3e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right)\left(\frac{1}{2}e^{-3\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right.\right. \\
&- \left.\frac{3}{2}e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right) + \frac{5\alpha^2}{4}\left(3\left(1 + e^{-3\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}} - 3e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right)^2 - 1\right)\left(4\left(1 + \frac{1}{2}e^{-3\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right.\right. \\
&- \left.\left.\left.\frac{3}{2}e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right)^2 - 6\left(1 + \frac{1}{2}e^{-3\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}} - \frac{3}{2}e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right) + 2\right)\right].
\end{aligned}$$

Consequently, for the EP-WD-SAR$(\gamma_1, \omega_1, \gamma_2, \omega_2)$ the regression curve for X_2 given $X_1 = x_1$ is

$$\begin{aligned}
\mathsf{E}(X_2|X_1 = x_1) &= \frac{\omega_2}{2}\Gamma\left(1 + \frac{1}{\gamma_2}\right)\left[\left(3 \times 2^{\frac{-1}{\gamma_2}} - 3^{\frac{-1}{\gamma_2}}\right) + 3\alpha\left(1 + e^{-3\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}} - 3e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right)\right. \\
&\times \left(3\left(5^{\frac{-1}{\gamma_2}} - 2^{-1-\frac{1}{\gamma_2}} \times 3^{-1-\frac{1}{\gamma_2}} - 3^{\frac{-1}{\gamma_2}-1} - 3 \times 2^{-1-\frac{2}{\gamma_2}} + 2^{\frac{-1}{\gamma_2}}\right)\right) + \frac{5\alpha^2}{4}\left(3\left(1 + e^{-3\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right.\right. \\
&- \left.\left.3e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right)^2 - 1\right)\left(9\left(2 \times 5^{\frac{-1}{\gamma_2}} + 2^{3-\frac{1}{\gamma_2}} \times 3^{-1-\frac{1}{\gamma_2}} + 2^{\frac{-3}{\gamma_2}} - 3 \times 7^{\frac{-1}{\gamma_2}} - 3 \times 2^{\frac{-2}{\gamma_2}}\right.\right. \\
&+ \left.\left.\left.2^{\frac{-1}{\gamma_2}} - 3^{-2-\frac{2}{\gamma_2}} - 3^{-1-\frac{1}{\gamma_2}}\right)\right)\right].
\end{aligned}$$

The concomitants of OSs arise when one sorts the members of a random sample according to corresponding values of another random sample. More specifically, in collecting any data for an observation, several characteristics are often recorded, and some are considered primary and others can be observed from the primary data automatically. David [16] was among the early authors who popularized the study of this subject. Many studies have been published on the concomitants of the OS model. Researchers such as Abd Elgawad et al. [17], Barakat et al. [18], and Scaria and Nair [19] have studied this issue.

Let $(X_{1,i}, X_{2,i})$, $i = 1, 2, ..., n$, be a random sample from a continuous bivariate DF $\mathsf{G}_{X_1,X_2}(x_1, x_2)$. If we denote $X_{1,s:n}$ as the sth OS of the X_1 sample values, then the X_2 values associated with $X_{1,s:n}$ are called the concomitants of the sth OS and are denoted by $X_{2[s:n]}$, $s = 1, 2, ..., n$. The PDF of the concomitant of the sth OS is given by

$$\mathsf{f}_{X_{2[s:n]}}(x_2) = \int_{-\infty}^{\infty} \mathsf{g}_{X_2|X_1}(x_2|x_1)\mathsf{f}_{1,s:n}(x_1)dx_1,$$

where $\mathsf{f}_{1,s:n}(x_1)$ is the PDF of the sth OS of $X_{1,i}, i = 1, 2, ..., n$. Thus, the PDF of $X_{2[s:n]}$ is given by

$$\begin{aligned}
\mathsf{f}_{X_{2[s:n]}}(x_2) =& \int_{x_1=0}^{\infty} \frac{\mathsf{g}_{X_1,X_2}(x_1,x_2)}{\mathsf{f}_{X_1}(x_1)}\mathsf{f}_{1,s:n}(x_1)dx_1 \\
=& \frac{1}{\beta(s,n-s+1)} \int_{x_1=0}^{\infty} \mathsf{g}_{X_1,X_2}(x_1,x_2)\left[\mathsf{F}_{X_1}(x_1)\right]^{s-1}\left[1-\mathsf{F}_{X_1}(x_1)\right]^{n-s} dx_1 \\
=& \frac{9\gamma_1\gamma_2}{4\omega_1^{\gamma_1}\omega_2^{\gamma_2}\beta(s,n-s+1)} \int_{x_1=0}^{\infty} x_1^{\gamma_1-1} e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}} x_2^{\gamma_2-1} e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}} \left(2-e^{-\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right)\left(2-e^{-\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right) \\
\times& \left[1+3\alpha\left(1+e^{-3\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}-3e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right)\left(1+e^{-3\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}-3e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right)+\frac{5\alpha^2}{4}\right. \\
\times& \left.\left(3\left(1+e^{-3\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}-3e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right)^2-1\right)\left(3\left(1+e^{-3\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}-3e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right)^2-1\right)\right] \\
\times& \left(1+\frac{1}{2}e^{-3\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}-\frac{3}{2}e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right)^{s-1}\left(\frac{3}{2}e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}-\frac{1}{2}e^{-3\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right)^{n-s} dx_1 \\
=& \left(\frac{3\gamma_2}{2\omega_2^{\gamma_2}}\right) x_2^{\gamma_2-1} e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\left(2-e^{-\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right)\left[\left(1-3\Delta_{1,s:n}^{(\alpha)}+\frac{5}{2}\Delta_{2,s:n}^{(\alpha)}\right)+2\left(3\Delta_{1,s:n}^{(\alpha)}-\frac{15}{2}\Delta_{2,s:n}^{(\alpha)}\right)\right. \\
\times& \left.\left(1+\frac{1}{2}e^{-3\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}-\frac{3}{2}e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right)+15\Delta_{2,s:n}^{(\alpha)}\left(1+\frac{1}{2}e^{-3\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}-\frac{3}{2}e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right)^2\right],
\end{aligned}$$

where $\Delta_{1,s:n}^{(\alpha)} = \frac{\alpha(2s-n-1)}{n+1}$ and $\Delta_{2,s:n}^{(\alpha)} = 2\alpha^2\left[1-6\frac{s(n-s+1)}{(n+1)(n+2)}\right]$. The rth moment of $X_{2[s:n]}$ is given by

$$\begin{aligned}
\mu_{X_{2[s:n]}}^{(r)} =& \left(\frac{3\gamma_2}{2\omega_2^{\gamma_2}}\right)\int_{x_2=0}^{\infty} x_2^{\gamma_2+r-1} e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\left(2-e^{-\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right)\left[\left(1-3\Delta_{1,s:n}^{(\alpha)}+\frac{5}{2}\Delta_{2,s:n}^{(\alpha)}\right)+2\left(3\Delta_{1,s:n}^{(\alpha)}-\frac{15}{2}\Delta_{2,s:n}^{(\alpha)}\right)\right. \\
\times& \left.\left(1+\frac{1}{2}e^{-3\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}-\frac{3}{2}e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right)+15\Delta_{2,s:n}^{(\alpha)}\left(1+\frac{1}{2}e^{-3\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}-\frac{3}{2}e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right)^2\right]dx_2 \\
=& \omega_2^r\Gamma\left(1+\frac{r}{\gamma_2}\right)\left[\left(3\times 2^{-1-\frac{r}{\gamma_2}}-\frac{3^{\frac{-r}{\gamma_2}}}{2}\right)\left(1-3\Delta_{1,s:n}^{(\alpha)}+\frac{5}{2}\Delta_{2,s:n}^{(\alpha)}\right)+\left(\frac{3}{2}\times 5^{\frac{-r}{\gamma_2}}-9\times 2^{-2-\frac{2r}{\gamma_2}}+3\right.\right. \\
\times& \left.2^{\frac{-r}{\gamma_2}}-3^{\frac{-r}{\gamma_2}}-2^{-2-\frac{r}{\gamma_2}}\times 3^{\frac{-r}{\gamma_2}}\right)\left(3\Delta_{1,s:n}^{(\alpha)}-\frac{15}{2}\Delta_{2,s:n}^{(\alpha)}\right)+15\Delta_{2,s:n}^{(\alpha)}\left(\frac{9}{2}\times 5^{\frac{-r}{\gamma_2}}+7\times 2^{-3-\frac{r}{\gamma_2}}\times 3^{1-\frac{r}{\gamma_2}}\right. \\
-& \left.\left.27\times 2^{-2-\frac{2r}{\gamma_2}}-\frac{3^{1-\frac{r}{\gamma_2}}}{2}+9\times 2^{-3-\frac{3r}{\gamma_2}}+9\times 2^{-1-\frac{r}{\gamma_2}}-\frac{27}{8}\times 7^{\frac{-r}{\gamma_2}}-\frac{3^{\frac{-2r}{\gamma_2}}}{8}\right)\right].
\end{aligned}$$

3.3. Mean residual life (MRL)

The MRL represents the expected remaining lifetime of a unit, given that it has already survived up to a specific time t. The MRL function is similar to the PDF or the characteristic function for distributions with a finite mean. Notably, the MRL function fully characterizes the distribution and can be inverted to recover it (cf. Guess and Proschan, [20]). The MRL function applies to both parametric and nonparametric models. In biomedical research, it is frequently used in survivorship studies to analyze patient longevity. Additionally, Shanbag and Kotz [21] extended the concept of MRL to RVs with vector values, broadening its applicability across fields. The idea of the MRL for a random vector (X_1, X_2) can be coined as

$$\mathsf{m}(x_1, x_2) = (\mathsf{m}_1(x_1, x_2), \mathsf{m}_2(x_1, x_2)), \tag{3.2}$$

where

$$\mathsf{m}_1(x_1, x_2) = \mathsf{E}(X_1 - x_1 | X_1 \geq x_1, X_2 \geq x_2)$$

and

$$\mathsf{m}_2(x_1, x_2) = \mathsf{E}(X_2 - x_2 | X_1 \geq x_1, X_2 \geq x_2).$$

The expressions for $\mathsf{m}_1(x_1, x_2)$ and $\mathsf{m}_2(x_1, x_2)$ in EP-WD-SAR$(\gamma_1, \omega_1, \gamma_2, \omega_2)$ distribution are obtained as

$$\begin{aligned}\mathsf{m}_1(x_1, x_2) &= \frac{\frac{\omega_1}{2}\Gamma\left(1+\frac{1}{\gamma_1}\right)}{\frac{(D_1+1)}{2}\left[1+\frac{3\alpha D_2 e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}}{2}\left(e^{-\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}-3\right)+\frac{5\alpha^2}{4}\left(3D_2^2-1\right)\left(4\left(\frac{D_1+1}{2}\right)^2-6\left(\frac{D_1+1}{2}\right)+2\right)\right]} \\ &\times \left[\left(3\times 2^{\frac{-1}{\gamma_1}}-3^{\frac{-1}{\gamma_1}}\right)+3\alpha D_2\left(3\left(5^{\frac{-1}{\gamma_1}}-2^{-1-\frac{1}{\gamma_1}}\times 3^{-1-\frac{1}{\gamma_1}}-3^{\frac{-1}{\gamma_1}-1}-3\times 2^{-1-\frac{2}{\gamma_1}}+2^{\frac{-1}{\gamma_1}}\right)\right)+\frac{5\alpha^2}{4}\right. \\ &\left.\times\left(3D_2^2-1\right)\left(9\left(2\times 5^{\frac{-1}{\gamma_1}}+2^{3-\frac{1}{\gamma_1}}\times 3^{-1-\frac{1}{\gamma_1}}+2^{\frac{-3}{\gamma_1}}-3\times 7^{\frac{-1}{\gamma_1}}-3\times 2^{\frac{-2}{\gamma_1}}+2^{\frac{-1}{\gamma_1}}-3^{-2-\frac{2}{\gamma_1}}-3^{-1-\frac{1}{\gamma_1}}\right)\right)\right]\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\mathsf{m}_2(x_1, x_2) &= \frac{\frac{\omega_2}{2}\Gamma\left(1+\frac{1}{\gamma_2}\right)}{\frac{(D_2+1)}{2}\left[1+\frac{3\alpha D_1 e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}}{2}\left(e^{-\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}-3\right)+\frac{5\alpha^2}{4}\left(3D_1^2-1\right)\left(4\left(\frac{D_2+1}{2}\right)^2-6\left(\frac{D_2+1}{2}\right)+2\right)\right]} \\ &\times \left[\left(3\times 2^{\frac{-1}{\gamma_2}}-3^{\frac{-1}{\gamma_2}}\right)+3\alpha D_1\left(3\left(5^{\frac{-1}{\gamma_2}}-2^{-1-\frac{1}{\gamma_2}}\times 3^{-1-\frac{1}{\gamma_2}}-3^{\frac{-1}{\gamma_2}-1}-3\times 2^{-1-\frac{2}{\gamma_2}}+2^{\frac{-1}{\gamma_2}}\right)\right)+\frac{5\alpha^2}{4}\right. \\ &\left.\times\left(3D_1^2-1\right)\left(9\left(2\times 5^{\frac{-1}{\gamma_2}}+2^{3-\frac{1}{\gamma_2}}\times 3^{-1-\frac{1}{\gamma_2}}+2^{\frac{-3}{\gamma_2}}-3\times 7^{\frac{-1}{\gamma_2}}-3\times 2^{\frac{-2}{\gamma_2}}+2^{\frac{-1}{\gamma_2}}-3^{-2-\frac{2}{\gamma_2}}-3^{-1-\frac{1}{\gamma_2}}\right)\right)\right],\end{aligned} \tag{3.4}$$

where $\Gamma(.)$ is the usual gamma function and $D_j = 1 + e^{-3\left(\frac{x_j}{\omega_j}\right)^{\gamma_j}} - 3e^{-2\left(\frac{x_j}{\omega_j}\right)^{\gamma_j}}$, $j = 1, 2$. Substituting (3.3) and (3.4) in (3.2) yields EP-WD-SAR's MRL.

3.4. *Vitality function*

The vitality function is a valuable tool for modeling various aspects of life spans. Kupka and Loo [22] conducted an in-depth study on this function in the context of aging. Kotz and Shanbhag [23] further utilized this concept to derive characterizations of different lifetime distributions. Unlike the hazard rate, which quantifies the instantaneous risk of failure, the vitality function provides a more intuitive measure of the failure pattern by focusing on the expected remaining lifespan. Specifically, for a non-negative RV X, the vitality function is defined as $\mathsf{V}(x) = \mathsf{E}(X|X > x)$. Additionally, the bivariate vitality function for a random vector (X_1, X_2) extends this concept to a multidimensional setting where it is defined on a positive domain as a binomial vector.

$$\mathsf{V}(x_1, x_2) = (\mathsf{V}_1(x_1, x_2), \mathsf{V}_2(x_1, x_2)), \tag{3.5}$$

where

$$\mathsf{V}_1(x_1, x_2) = \mathsf{E}(X_1|X_1 \geq x_1, X_2 \geq x_2)$$

and

$$\mathsf{V}_2(x_1, x_2) = \mathsf{E}(X_2|X_1 \geq x_1, X_2 \geq x_2).$$

For more details, see Sankaran and Nair [24]. Moreover, $\mathsf{V}_i(x_1, x_2)$ is related to $\mathsf{m}_i(x_1, x_2)$ by

$$\mathsf{V}_i(x_1, x_2) = x_i + \mathsf{m}_i(x_1, x_2), \quad i = 1, 2. \tag{3.6}$$

Here, $\mathsf{V}_1(x_1, x_2)$ computes the expected lifetime to the first component as the sum of current age x_1 and the average lifetime remaining to it, assuming the second component has survived past age x_2. $\mathsf{V}_2(x_1, x_2)$ has a similar interpretation. Using (3.3) and (3.4) in (3.6), we obtain $\mathsf{V}_1(x_1, x_2)$ and $\mathsf{V}_2(x_1, x_2)$ of EP-WD-SAR distribution as

$$\begin{aligned}\mathsf{V}_1(x_1, x_2) = x_1 + &\frac{\frac{\omega_1}{2}\Gamma\left(1+\frac{1}{\gamma_1}\right)}{\frac{(D_1+1)}{2}\left[1+\frac{3\alpha D_2 e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}}{2}\left(e^{-\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}-3\right)+\frac{5\alpha^2}{4}\left(3D_2^2-1\right)\left(4\left(\frac{D_1+1}{2}\right)^2-6\left(\frac{D_1+1}{2}\right)+2\right)\right]} \\ &\times\left[\left(3\times 2^{\frac{-1}{\gamma_1}}-3^{\frac{-1}{\gamma_1}}\right)+3\alpha D_2\left(3\left(5^{\frac{-1}{\gamma_1}}-2^{-1-\frac{1}{\gamma_1}}\times 3^{-1-\frac{1}{\gamma_1}}-3^{\frac{-1}{\gamma_1}-1}-3\times 2^{-1-\frac{2}{\gamma_1}}+2^{\frac{-1}{\gamma_1}}\right)\right)+\frac{5\alpha^2}{4}\right. \\ &\left.\times\left(3D_2^2-1\right)\left(9\left(2\times 5^{\frac{-1}{\gamma_1}}+2^{3-\frac{1}{\gamma_1}}\times 3^{-1-\frac{1}{\gamma_1}}+2^{\frac{-3}{\gamma_1}}-3\times 7^{\frac{-1}{\gamma_1}}-3\times 2^{\frac{-2}{\gamma_1}}+2^{\frac{-1}{\gamma_1}}-3^{-2-\frac{2}{\gamma_1}}-3^{-1-\frac{1}{\gamma_1}}\right)\right)\right]\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\mathsf{V}_2(x_1, x_2) = x_2 + &\frac{\frac{\omega_2}{2}\Gamma\left(1+\frac{1}{\gamma_2}\right)}{\frac{(D_2+1)}{2}\left[1+\frac{3\alpha D_1 e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}}{2}\left(e^{-\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}-3\right)+\frac{5\alpha^2}{4}\left(3D_1^2-1\right)\left(4\left(\frac{D_2+1}{2}\right)^2-6\left(\frac{D_2+1}{2}\right)+2\right)\right]} \\ &\times\left[\left(3\times 2^{\frac{-1}{\gamma_2}}-3^{\frac{-1}{\gamma_2}}\right)+3\alpha D_1\left(3\left(5^{\frac{-1}{\gamma_2}}-2^{-1-\frac{1}{\gamma_2}}\times 3^{-1-\frac{1}{\gamma_2}}-3^{\frac{-1}{\gamma_2}-1}-3\times 2^{-1-\frac{2}{\gamma_2}}+2^{\frac{-1}{\gamma_2}}\right)\right)+\frac{5\alpha^2}{4}\right. \\ &\left.\times\left(3D_1^2-1\right)\left(9\left(2\times 5^{\frac{-1}{\gamma_2}}+2^{3-\frac{1}{\gamma_2}}\times 3^{-1-\frac{1}{\gamma_2}}+2^{\frac{-3}{\gamma_2}}-3\times 7^{\frac{-1}{\gamma_2}}-3\times 2^{\frac{-2}{\gamma_2}}+2^{\frac{-1}{\gamma_2}}-3^{-2-\frac{2}{\gamma_2}}-3^{-1-\frac{1}{\gamma_2}}\right)\right)\right].\end{aligned} \tag{3.8}$$

From (3.7) and (3.8), the vitality function defined in (3.5) can be obtained.

4. Reliability and information measures

In this section, we derive reliability measures such as hazard function, RH function, bivariate extropy, bivariate weighted extropy, and bivariate CREX in the context of EP-WD-SAR($\gamma_1, \omega_1, \gamma_2, \omega_2$).

4.1. Reliability function

Sreelakshmi [25] introduced the relationship between copula and bivariate reliability function, which is defined as follows:

$$\mathsf{R}(x_1, x_2) = P(X_1 > x_1, X_2 > x_2) = 1 - \mathsf{F}_{X_1}(x_1) - \mathsf{F}_{X_2}(x_2) + C\left(\mathsf{F}_{X_1}(x_1), \mathsf{F}_{X_2}(x_2)\right).$$

The bivariate reliability function $\mathsf{R}(x_1, x_2)$ for the EP-WD-SAR$(\gamma_1, \omega_1, \gamma_2, \omega_2)$ distribution is

$$\begin{aligned}\mathsf{R}(x_1, x_2) &= \left(\frac{3}{2}e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}} - \frac{1}{2}e^{-3\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right)\left(\frac{3}{2}e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}} - \frac{1}{2}e^{-3\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right)\left[1 + \left(1 + \frac{1}{2}e^{-3\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}} - \frac{3}{2}e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right)\right.\\ &\times \left(1 + \frac{1}{2}e^{-3\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}} - \frac{3}{2}e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right)\left[3\alpha + 5\alpha^2\left(1 + e^{-3\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}} - 3e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right)\right.\\ &\times \left.\left.\left(1 + e^{-3\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}} - 3e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right)\right]\right].\end{aligned}$$

4.2. Hazard function

The bivariate hazard function at a point (x_1, x_2) is defined, according to Basu [26], by $\mathsf{H}(x_1, x_2) = \frac{\mathsf{g}_{X_1,X_2}(x_1,x_2)}{\mathsf{R}(x_1,x_2)}$. Thus, we get

$$\begin{aligned}\mathsf{H}(x_1, x_2) &= \frac{\left(\frac{9\gamma_1\gamma_2}{4\omega_1^{\gamma_1}\omega_2^{\gamma_2}}\right)x_1^{\gamma_1-1}e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}x_2^{\gamma_2-1}e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\left(2 - e^{-\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right)\left(2 - e^{-\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right)}{\left(\frac{3}{2}e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}} - \frac{1}{2}e^{-3\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right)\left(\frac{3}{2}e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}} - \frac{1}{2}e^{-3\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right)\left[1 + \left(\frac{(1+D_1)(1+D_2)}{4}\right)\left[3\alpha + 5\alpha^2 D_1 D_2\right]\right]}\\ &\times \left[1 + 3\alpha D_1 D_2 + \frac{5\alpha^2}{4}\left(3D_1^2 - 1\right)\left(3D_2^2 - 1\right)\right]. \end{aligned} \tag{4.1}$$

One of the key constraints of Basu [26] is that $\mathsf{H}(x_1, x_2)$, as defined by (4.1), is not a vector quantity, as defined by $\mathsf{R}^2 \to \mathsf{R}$. The bivariate hazard function was created in vector form by Johnson and Kotz [27] and Sreelakshmi [25] to get around this restriction.

$$\mathsf{H}(x_1, x_2) = \left(\frac{-\partial \ln \mathsf{R}(x_1, x_2)}{\partial x_1}, \frac{-\partial \ln \mathsf{R}(x_1, x_2)}{\partial x_2}\right), \tag{4.2}$$

where, R denotes the bivariate reliability function for SAR copula. For the SAR copula Vaidyanathan and Sharon [28] studied the elements in the vector (4.2). For the EP-WD-SAR copula, we can, after simple algebra, get the following relations:

$$\frac{-\partial \ln \mathsf{R}(x_1, x_2)}{\partial x_1} = \mathsf{H}(x_1)\left(1 - \frac{3\alpha + 5\alpha^2 D_2\left(2D_1 + 1\right)}{\left(\frac{(1-D_1)(1+D_2)}{4}\right)^{-1} + (3\alpha + 5\alpha^2 D_1 D_2)\left(\left(\frac{1-D_1}{2}\right)^{-1} - 1\right)}\right) \tag{4.3}$$

and

$$\frac{-\partial \ln \mathsf{R}(x_1, x_2)}{\partial x_2} = \mathsf{H}(x_2)\left(1 - \frac{3\alpha + 5\alpha^2 D_1\left(2D_2 + 1\right)}{\left(\frac{(1-D_2)(1+D_1)}{4}\right)^{-1} + (3\alpha + 5\alpha^2 D_1 D_2)\left(\left(\frac{1-D_2}{2}\right)^{-1} - 1\right)}\right), \tag{4.4}$$

where $\mathsf{H}_i(x_i)$ is the hazard function of X_i, $i = 1, 2$, i.e., $\mathsf{H}_i(x_i) = \frac{\mathsf{f}_{X_i}(x_i)}{\mathsf{R}(x_i)}$ and $\mathsf{R}(x_i) = P(X_i > x_i)$. The vector hazard function of the EP-WD-SAR$(\gamma_1, \omega_1, \gamma_2, \omega_2)$ distribution is obtained by substituting (4.3) and (4.4) in (4.2).

4.3. Reversed hazard function

The RH function at a point (x_1, x_2) is defined as $\mathsf{RH}(x_1, x_2) = \frac{\mathsf{g}_{X_1,X_2}(x_1,x_2)}{\mathsf{G}_{X_1,X_2}(x_1,x_2)}$. Thus, we get

$$\mathsf{RH}(x_1, x_2) = \frac{\left(\frac{9\gamma_1\gamma_2}{4\omega_1^{\gamma_1}\omega_2^{\gamma_2}}\right) x_1^{\gamma_1-1} e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}} x_2^{\gamma_2-1} e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}} \left(2 - e^{-\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right)\left(2 - e^{-\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right)}{\left(\frac{(D_1+1)(D_2+1)}{4}\right)\left[1 + \left(\frac{3}{2}e^{-2\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}} - \frac{1}{2}e^{-3\left(\frac{x_1}{\omega_1}\right)^{\gamma_1}}\right)\left(\frac{3}{2}e^{-2\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}} - \frac{1}{2}e^{-3\left(\frac{x_2}{\omega_2}\right)^{\gamma_2}}\right)[3\alpha + 5\alpha^2 D_1 D_2]\right]}$$
$$\times \left[1 + 3\alpha D_1 D_2 + \frac{5\alpha^2}{4}\left(3D_1^2 - 1\right)\left(3D_2^2 - 1\right)\right].$$

All the aforementioned reliability concepts and measures are widely employed in reliability modeling. For the most recent developments in this area, see Xu et al. [29], Xu et al. [30], and Zhuang et al. [31].

4.4. Bivariate extropy, bivariate weighted extropy, and bivariate CREX

Lad et al. [32] developed the idea of extropy as dual to entropy, enabling the comparison of uncertainty between two RVs. Unlike entropy, which focuses on the self-information of outcomes, extropy reflects how spread out the remaining probability mass is, offering a complementary view of randomness and dispersion. In other words, while entropy is high when outcomes are unpredictable, extropy is high when outcomes are predictable. Mathematically, if $P = (p_1, p_2, ..., p_n)$ represents the discrete probability distribution of a discrete RV X, the extropy is defined as $\mathsf{J}(X) = -\sum_{i=1}^{n}(1 - p_i)\ln(1 - p_i)$. This measure captures the dispersion of the complement probabilities $1 - p_i$ offering a dual perspective to entropy. For a non-negative continuous RV X with PDF $\mathsf{f}_X(.)$, the extropy is defined by $\mathsf{J}(X) = -\frac{1}{2}\int_0^\infty \mathsf{f}_X^2(x)dx$.

Studying entropy or extropy for a distribution such as the EP-WD, particularly when modeled using the Sarmanov copula, helps quantify the uncertainty associated with its outcomes. This is especially useful in modeling complex phenomena, such as reliability or survival analysis. The Sarmanov distribution introduces a flexible dependence structure between RVs, and analyzing information measures like entropy or extropy provides insights into how this dependence influences overall uncertainty or predictability. Furthermore, information based criteria such as Akaike information criterion (AIC) and bayesian information criterion (BIC) often involve entropy, especially in maximum entropy frameworks. In contexts that demand greater certainty or tighter predictions, a distribution with lower entropy (or higher extropy) may be preferable. In practical applications involving the EP-WD such as industrial system reliability or survival data extropy can serve as a tool for assessing information gain or minimizing risk, thus supporting decision making in a decision theoretic context.

Due to the difficulty of deriving the entropy for model (1.2), our focus in this subsection is on the extropy. Extropy has several applications, such as scoring the forecasting distributions using the total scoring rule and comparing the uncertainties of two RVs. There are many studies encompassing this; e.g., Husseiny et al. [13].

It is possible to introduce a bivariate version of extropy based on the SAR family in the following theorem.

Theorem 4.1. *Consider two non-negative continuous RVs X_1 and X_2 with the JPDF $g_{X_1,X_2}(x_1,x_2)$, which is defined in* (1.6). *Then, the bivariate version of extropy based on SAR family is given by*

$$\begin{aligned}
J(X_1,X_2) &= \left(1+6\alpha+19\alpha^2+30\alpha^3+25\alpha^4\right)J(X_1)J(X_2)+\alpha\left(1+21\alpha+80\alpha^2+150\alpha^3\right)\psi^{(1)}_{X_1}\psi^{(1)}_{X_2}+18\alpha^2\\
&\times\left(7+135\alpha+800\alpha^2\right)\psi^{(2)}_{X_1}\psi^{(2)}_{X_2}-9\alpha^2\left(14+45\alpha+400\alpha^2\right)\left(\psi^{(2)}_{X_1}\psi^{(1)}_{X_2}+\psi^{(1)}_{X_1}\psi^{(2)}_{X_2}\right)+3\alpha^2\left(8+45\alpha+100\alpha^2\right)\\
&\times\left(\psi^{(2)}_{X_1}J(X_2)+J(X_1)\psi^{(2)}_{X_2}\right)-3\alpha\left(1+8\alpha+20\alpha^2+25\alpha^3\right)\left(\psi^{(1)}_{X_1}J(X_2)+J(X_1)\psi^{(1)}_{X_2}\right)+1080\alpha^3(1+30\alpha)\psi^{(3)}_{X_1}\\
&\times\psi^{(3)}_{X_2}-540\alpha^3(3+40\alpha)\left(\psi^{(3)}_{X_1}\psi^{(2)}_{X_2}+\psi^{(2)}_{X_1}\psi^{(3)}_{X_2}\right)360\alpha^3(2+15\alpha)\left(\psi^{(3)}_{X_1}\psi^{(1)}_{X_2}+\psi^{(1)}_{X_1}\psi^{(3)}_{X_2}\right)\\
&-90\alpha^3(1+5\alpha)\left(\psi^{(3)}_{X_1}J(X_2)+J(X_1)\psi^{(3)}_{X_2}\right)+225\alpha^4\left[\left(\psi^{(4)}_{X_1}J(X_2)+J(X_1)\psi^{(4)}_{X_2}\right)-12\left(\psi^{(1)}_{X_1}\psi^{(4)}_{X_2}+\psi^{(4)}_{X_1}\psi^{(1)}_{X_2}\right)\right.\\
&\left.-72\left(\psi^{(3)}_{X_1}\psi^{(4)}_{X_2}+\psi^{(4)}_{X_1}\psi^{(3)}_{X_2}\right)+36\psi^{(4)}_{X_1}\psi^{(4)}_{X_2}+48\left(\psi^{(4)}_{X_1}\psi^{(2)}_{X_2}+\psi^{(2)}_{X_1}\psi^{(4)}_{X_2}\right)\right],
\end{aligned}$$

where

$$J(X_i)=-\frac{1}{2}E(f_{X_i}(X_i)) \ \ \textit{and} \ \ \psi^{(p)}_{X_i}=E(f_{X_i}(X_i)F^p_{X_i}(X_i)), \ \ i=1,2, \ p=1,2,3,4. \tag{4.5}$$

Proof. Using (1.6), we get

$$\begin{aligned}
J(X_1,X_2) &= \frac{1}{4}E(g_{X_1,X_2}(X_1,X_2))\\
&= \frac{1}{4}\int_0^\infty\int_0^\infty f^2_{X_1}(x_1)f^2_{X_2}(x_2)\Big[1+3\alpha(2F_{X_1}(x_1)-1)(2F_{X_2}(x_2)-1)+\frac{5}{4}\alpha^2(3(2F_{X_1}(x_1)-1)^2-1)\\
&\times\ (3(2F_{X_2}(x_2)-1)^2-1)\Big]^2dx_1dx_2\\
&= \frac{1}{4}\int_0^\infty\int_0^\infty f^2_{X_1}(x_1)f^2_{X_2}(x_2)dx_1dx_2+\frac{9\alpha^2}{4}\int_0^\infty\int_0^\infty f^2_{X_1}(x_1)f^2_{X_2}(x_2)\left(2F_{X_1}(x_1)-1\right)^2\left(2F_{X_2}(x_2)-1\right)^2\\
&\times dx_1dx_2+\frac{25\alpha^4}{64}\int_0^\infty\int_0^\infty f^2_{X_1}(x_1)f^2_{X_2}(x_2)\left(3\left(2F_{X_1}(x_1)-1\right)^2-1\right)^2\left(3\left(2F_{X_2}(x_2)-1\right)^2-1\right)^2dx_1dx_2\\
&+\frac{3\alpha}{2}\int_0^\infty\int_0^\infty f^2_{X_1}(x_1)f^2_{X_2}(x_2)\left(2F_{X_1}(x_1)-1\right)\left(2F_{X_2}(x_2)-1\right)dx_1dx_2+\frac{5\alpha^2}{8}\int_0^\infty\int_0^\infty f^2_{X_1}(x_1)f^2_{X_2}(x_2)\\
&\times\left(3\left(2F_{X_1}(x_1)-1\right)^2-1\right)\left(3\left(2F_{X_2}(x_2)-1\right)^2-1\right)dx_1dx_2+\frac{15\alpha^3}{8}\int_0^\infty\int_0^\infty f^2_{X_1}(x_1)f^2_{X_2}(x_2)\\
&\times\left(2F_{X_1}(x_1)-1\right)\left(2F_{X_2}(x_2)-1\right)\left(3\left(2F_{X_1}(x_1)-1\right)^2-1\right)\left(3\left(2F_{X_2}(x_2)-1\right)^2-1\right)dx_1dx_2.
\end{aligned}$$

Since each bivariate integral in the above formula is separable into the product of its univariate integrals, and by incorporating (4.5) in the above integrations, the required result directly follows. This completes the proof. □

Corolary 4.1. *Let $X_1\sim EP\text{-}WD(\gamma_1,\omega_1)$ and $X_2\sim EP\text{-}WD(\gamma_2,\omega_2)$. Using (4.5) and after simple algebra, we get*

$$J(X_i) = -\frac{\gamma_i}{800\omega_i}\Gamma\left(2-\frac{1}{\gamma_i}\right)\left[225\times 4^{\frac{1}{\gamma_i}} - 144\times 5^{\frac{1}{\gamma_i}} + 25\times 6^{\frac{1}{\gamma_i}}\right],$$

$$\psi^{(1)}_{x_i} = \frac{\gamma_i}{\omega_i}\Gamma\left(2-\frac{1}{\gamma_i}\right)\left[729\times 2^{-6+\frac{2}{\gamma_i}} - 5\times 2^{-6+\frac{1}{\gamma_i}}\times 3^{4+\frac{1}{\gamma_i}} - \frac{729}{4}\times 5^{-2+\frac{1}{\gamma_i}} + \frac{729}{2}\times 7^{-2+\frac{1}{\gamma_i}} - 5103\times 2^{-11+\frac{3}{\gamma_i}} + \frac{3^{2+\frac{2}{\gamma_i}}}{32}\right],$$

$$\psi^{(2)}_{x_i} = \frac{\gamma_i}{\omega_i}\Gamma\left(2-\frac{1}{\gamma_i}\right)\left[9\times 2^{-7+\frac{3}{\gamma_i}} - 11\times 2^{-4+\frac{1}{\gamma_i}}\times 3^{\frac{1}{\gamma_i}} + 9\times 2^{-4+\frac{2}{\gamma_i}} + 3^{\frac{1}{\gamma_i}}\times 2^{-8+\frac{2}{\gamma_i}} - 9\times 5^{-2+\frac{1}{\gamma_i}} + 333\times 5^{-2+\frac{1}{\gamma_i}}\times 2^{-6+\frac{1}{\gamma_i}} + 36\times 7^{-2+\frac{1}{\gamma_i}} - \frac{7}{2}\times 3^{-2+\frac{2}{\gamma_i}} - \frac{45}{8}\times 11^{-2+\frac{1}{\gamma_i}}\right],$$

$$\psi^{(3)}_{x_i} = \frac{\gamma_i}{\omega_i}\Gamma\left(2-\frac{1}{\gamma_i}\right)\left[-55\times 2^{-9+\frac{2}{\gamma_i}}\times 3^{1+\frac{1}{\gamma_i}} - \frac{29}{8}\times 3^{-1+\frac{2}{\gamma_i}} - 17\times 2^{-4+\frac{1}{\gamma_i}}\times 3^{\frac{1}{\gamma_i}} + 9\times 4^{-2+\frac{1}{\gamma_i}} - 9\times 5^{-2+\frac{1}{\gamma_i}} + 513\times 2^{-6+\frac{1}{\gamma_i}}\times 5^{-2+\frac{1}{\gamma_i}} + \frac{1}{32}\times 3^{\frac{1}{\gamma_i}}\times 5^{-2+\frac{1}{\gamma_i}} + 54\times 7^{-2+\frac{1}{\gamma_i}} - 117\times 2^{-7+\frac{1}{\gamma_i}}\times 7^{-2+\frac{1}{\gamma_i}} + 297\times 8^{-3+\frac{1}{\gamma_i}} + \frac{351}{8}\times 11^{-2+\frac{1}{\gamma_i}} + \frac{603}{32}\times 13^{-2+\frac{1}{\gamma_i}}\right],$$

and

$$\psi^{(4)}_{x_i} = \frac{\gamma_i}{\omega_i}\Gamma\left(2-\frac{1}{\gamma_i}\right)\left[45\times 2^{-5+\frac{3}{\gamma_i}} + 477\times 2^{-13+\frac{4}{\gamma_i}} - 23\times 2^{-4+\frac{1}{\gamma_i}}\times 3^{\frac{1}{\gamma_i}} - 85\times 3^{1+\frac{1}{\gamma_i}}\times 4^{-4+\frac{1}{\gamma_i}} + 9\times 4^{-2+\frac{1}{\gamma_i}} - 9\times 5^{-2+\frac{1}{\gamma_i}} + 27\times 2^{-5+\frac{1}{\gamma_i}}\times 5^{-2+\frac{1}{\gamma_i}} - \frac{91}{16}\times 3^{\frac{1}{\gamma_i}}\times 5^{-2+\frac{1}{\gamma_i}} + 72\times 7^{-2+\frac{1}{\gamma_i}} + 5625\times 2^{-8+\frac{1}{\gamma_i}}\times 7^{-2+\frac{1}{\gamma_i}} - 22\times 9^{-1+\frac{1}{\gamma_i}} + 2^{-8+\frac{1}{\gamma_i}}\times 9^{-1+\frac{1}{\gamma_i}} + \frac{837}{4}\times 11^{-2+\frac{1}{\gamma_i}} - \frac{495}{16}\times 13^{-2+\frac{1}{\gamma_i}} - \frac{9}{4}\times 17^{-2+\frac{1}{\gamma_i}}\right],\ i=1,2.$$

In a manner similar to the bivariate extropy, we can introduce bivariate weighted extropy based on the SAR family, as presented in the following theorem.

Theorem 4.2. *Let X_1 and X_2 be non-negative continuous RVs with the JPDF $g_{X_1,X_2}(x_1,x_2)$, which is defined in* (1.6). *Then, the bivariate weighted extropy based on the SAR family is given by*

$$J^w(X_1,X_2) = \left(1+6\alpha+19\alpha^2+30\alpha^3+25\alpha^4\right)J^w(X_1)J^w(X_2)+6\alpha\left(1+21\alpha+80\alpha^2+150\alpha^3\right)\psi^{w(1)}_{X_1}\psi^{w(1)}_{X_2} +18\alpha^2\left(7+135\alpha+800\alpha^2\right)\psi^{w(2)}_{X_1}\psi^{w(2)}_{X_2} - 9\alpha^2\left(14+45\alpha+400\alpha^2\right)\left(\psi^{w(2)}_{X_1}\psi^{w(1)}_{X_2}+\psi^{w(1)}_{X_1}\psi^{w(2)}_{X_2}\right)+3\alpha^2\,(8+45\alpha +100\alpha^2)\left(\psi^{w(2)}_{X_1}J^w(X_2)+J^w(X_1)\psi^{w(2)}_{X_2}\right)-3\alpha\left(1+8\alpha+20\alpha^2+25\alpha^3\right)\left(\psi^{w(1)}_{X_1}J^w(X_2)+J^w(X_1)\psi^{w(1)}_{X_2}\right) +1080\alpha^3(1+30\alpha)\,\psi^{w(3)}_{X_1}\psi^{w(3)}_{X_2} - 540\alpha^3(3+40\alpha)\left(\psi^{w(3)}_{X_1}\psi^{w(2)}_{X_2}+\psi^{w(2)}_{X_1}\psi^{w(3)}_{X_2}\right)+360\alpha^3(2+15\alpha) \times\left(\psi^{w(3)}_{X_1}\psi^{w(1)}_{X_2}+\psi^{w(1)}_{X_1}\psi^{w(3)}_{X_2}\right)-90\alpha^3\,(1+5\alpha)\left(\psi^{w(3)}_{X_1}J^w(X_2)+J^w(X_1)\psi^{w(3)}_{X_2}\right)+225\alpha^4\left[\left(\psi^{w(4)}_{X_1}J^w(X_2) + J^w(X_1)\psi^{w(4)}_{X_2}\right) - 12\left(\psi^{w(1)}_{X_1}\psi^{w(4)}_{X_2}+\psi^{w(4)}_{X_1}\psi^{w(1)}_{X_2}\right) - 72\left(\psi^{w(3)}_{X_1}\psi^{w(4)}_{X_2}+\psi^{w(4)}_{X_1}\psi^{w(3)}_{X_2}\right) + 36\psi^{w(4)}_{X_1}$$

$$\times \psi_{X_2}^{w(4)} + 48\left(\psi_{X_1}^{w(4)}\psi_{X_2}^{w(2)} + \psi_{X_1}^{w(2)}\psi_{X_2}^{w(4)}\right)\Big],$$

where

$$\mathsf{J}^{w}(X_i) = -\frac{1}{2}E(X_i f_{X_i}(X_i)), \text{ and } \psi_{X_i}^{w(p)} = E(X_i f_{X_i}(X_i) F_{X_i}^{p}(X_i)), i = 1, 2, \ p = 1, 2, 3, 4. \tag{4.6}$$

Proof. We obtain the results by applying Theorem 4.1's proving techniques. □

Corolary 4.2. *Let $X_1 \sim$ EP-WD(γ_1, ω_1) and $X_2 \sim$ EP-WD(γ_2, ω_2). Using (4.6) and after simple algebra, we get* $\mathsf{J}^{w}(X_i) = -\frac{53\gamma_i}{400}$, $\psi_{X_i}^{w(1)} = \frac{836497\gamma_i}{5644800}$, $\psi_{X_i}^{w(2)} = \frac{33010783\gamma_i}{341510400}$, $\psi_{X_i}^{w(3)} = \frac{331119317\gamma_i}{4809604800}$, $\psi_{X_i}^{w(4)} = \frac{9238144574639\gamma_i}{177916900761600}$, $i = 1, 2$.

Proposition 4.1. *Let X_1 and X_2 follow the SAR family.*

- *If $X_2 = aX_1 + b$, then $\mathsf{J}(X_2) = \frac{1}{a}\mathsf{J}(X_1)$.*
- *If $\alpha = 0$, then $\mathsf{J}(X_1, X_2) = \mathsf{J}(X_1)\mathsf{J}(X_2)$.*

It is worth noting that, as was shown by Balakrishnan et al. [33], there are distributions with the same extropy, but different weighted extropy. Moreover, distributions exist with the same weighted extropy but different extropy.

Replacing PDF in the extropy function with the survival function, Jahanshahi et al. [34] proposed a new measure of uncertainty of a non-negative continuous RV called CREX. Jahanshahi et al. [34] showed that if two RVs X_1 and X_2 are lifetimes of two systems A and B, and if the CREX of X_1 is less than the CREX of X_2, then system A has less uncertainty than system B.

We can define the bivariate CREX based on the SAR family in the following theorem.

Theorem 4.3. *Let X_1 and X_2 be non-negative continuous RVs with the JDF $\mathsf{G}_{X_1,X_2}(x_1, x_2)$, which is defined in* (1.5)*. Then, the bivariate CREX based on the SAR family is given by*

$$\mathsf{J}^{c}(X_1, X_2) = \mathsf{J}^{c}(X_1)\,\mathsf{J}^{c}(X_2) + \frac{\alpha^2}{4}\left(49 + 30\alpha + 25\alpha^2\right)\phi_{X_1}^{(2)}\phi_{X_2}^{(2)} + \frac{\alpha}{2}(3 + 5\alpha)\,\phi_{X_1}^{(1)}\phi_{X_2}^{(1)}$$
$$-5\alpha^2\left(\phi_{X_1}^{(2)}\phi_{X_2}^{(1)} + \phi_{X_1}^{(1)}\phi_{X_2}^{(2)}\right) + 10\alpha^3(3 + 10\alpha)\,\phi_{X_1}^{(3)}\phi_{X_2}^{(3)} - 5\alpha^3(3 + 5\alpha)\left(\phi_{X_1}^{(3)}\phi_{X_2}^{(2)} + \phi_{X_1}^{(2)}\phi_{X_2}^{(3)}\right)$$
$$+25\alpha^4\left(4\phi_{X_1}^{(4)}\phi_{X_2}^{(4)} - 4\left(\phi_{X_1}^{(4)}\phi_{X_2}^{(3)} + \phi_{X_1}^{(3)}\phi_{X_2}^{(4)}\right) + \left(\phi_{X_1}^{(4)}\phi_{X_2}^{(2)} + \phi_{X_1}^{(2)}\phi_{X_2}^{(4)}\right)\right),$$

where

$$\mathsf{J}^{c}(X_i) = -\frac{1}{2}\int_0^\infty \left(\overline{\mathsf{F}}_{X_i}(x_i)\right)^2 dx_i \text{ and } \phi_{X_i}^{(p)} = \int_0^\infty \left(\overline{\mathsf{F}}_{X_i}(x_i)\right)^2 \mathsf{F}_{X_i}^{p}(x_i) dx_i, \ i = 1, 2, \ p = 1, 2, 3, 4. \tag{4.7}$$

Proof. Using (1.5), we get

$$\begin{aligned}
\mathsf{J}^{c}(X_1, X_2) &= \frac{1}{4}\int_0^\infty \int_0^\infty \left(\overline{\mathsf{G}}_{X_1,X_2}(x_1, x_2)\right)^2 dx_1 dx_2 \\
&= \frac{1}{4}\int_0^\infty \int_0^\infty (\overline{\mathsf{F}}_{X_1}(x_1))^2 (\overline{\mathsf{F}}_{X_2}(x_2))^2 \Big[1 + 3\alpha \mathsf{F}_{X_1}(x_1)\mathsf{F}_{X_2}(x_2) + 5\alpha^2(2\mathsf{F}_{X_1}(x_1) - 1)(2\mathsf{F}_{X_2}(x_2) - 1) \\
&\times \ \mathsf{F}_{X_1}(x_1)\mathsf{F}_{X_2}(x_2)\Big]^2 dx_1 dx_2
\end{aligned}$$

$$= \frac{1}{4}\int_0^\infty\int_0^\infty (\overline{\mathsf{F}}_{X_1}(x_1))^2(\overline{\mathsf{F}}_{X_2}(x_2))^2 dx_1 dx_2 + \frac{9\alpha^2}{4}\int_0^\infty\int_0^\infty \mathsf{F}^2_{X_1}(x_1)\mathsf{F}^2_{X_2}(x_2)(\overline{\mathsf{F}}_{X_1}(x_1))^2(\overline{\mathsf{F}}_{X_2}(x_2))^2$$
$$\times\ dx_1 dx_2 + \frac{25\alpha^4}{4}\int_0^\infty\int_0^\infty \mathsf{F}^2_{X_1}(x_1)\mathsf{F}^2_{X_2}(x_2)(\overline{\mathsf{F}}_{X_1}(x_1))^2(\overline{\mathsf{F}}_{X_2}(x_2))^2(2\mathsf{F}_{X_1}(x_1)-1)^2(2\mathsf{F}_{X_2}(x_2)-1)^2$$
$$\times\ dx_1 dx_2 + \frac{3\alpha}{2}\int_0^\infty\int_0^\infty \mathsf{F}_{X_1}(x_1)\mathsf{F}_{X_2}(x_2)(\overline{\mathsf{F}}_{X_1}(x_1))^2(\overline{\mathsf{F}}_{X_2}(x_2))^2 dx_1 dx_2 + \frac{10\alpha^2}{4}\int_0^\infty\int_0^\infty \mathsf{F}_{X_1}(x_1)$$
$$\times\ \mathsf{F}_{X_2}(x_2)(\overline{\mathsf{F}}_{X_1}(x_1))^2(\overline{\mathsf{F}}_{X_2}(x_2))^2(2\mathsf{F}_{X_1}(x_1)-1)(2\mathsf{F}_{X_2}(x_2)-1)dx_1 dx_2 + \frac{30\alpha^3}{4}\int_0^\infty\int_0^\infty \mathsf{F}^2_{X_1}(x_1)$$
$$\times\ \mathsf{F}^2_{X_2}(x_2)(\overline{\mathsf{F}}_{X_1}(x_1))^2(\overline{\mathsf{F}}_{X_2}(x_2))^2(2\mathsf{F}_{X_1}(x_1)-1)(2\mathsf{F}_{X_2}(x_2)-1)dx_1 dx_2.$$

Since each bivariate integral in the formula above can be divided into the product of its univariate integrals, the required result can be obtained immediately by incorporating (4.7) in the integrations above. The proof is now complete. □

Corolary 4.3. *Let* $X_1 \sim EP\text{-}WD(\gamma_1, \omega_1)$ *and* $X_2 \sim EP\text{-}WD(\gamma_2, \omega_2)$. *Using (4.7) and after simple algebra, we get* $\mathsf{J}^c(X_i) = -\frac{\Gamma\left(1+\frac{1}{\gamma_i}\right)}{2}\left[15^{\frac{-1}{\gamma_i}} \times 4^{-\frac{1+\gamma_i}{\gamma_i}}\left(-2^{1+\frac{2}{\gamma_i}} \times 3^{1+\frac{1}{\gamma_i}} + 3^{2+\frac{1}{\gamma_i}} \times 5^{\frac{1}{\gamma_i}} + 10^{\frac{1}{\gamma_i}}\right)\right]$,

$$\phi^{(1)}_{X_i} = \Gamma\left(1+\frac{1}{\gamma_i}\right)\left[8^{-\frac{1+\gamma_i}{\gamma_i}} \times 315^{\frac{-1}{\gamma_i}}\left(-2^{2+\frac{3}{\gamma_i}} \times 3^{1+\frac{2}{\gamma_i}} \times 7^{\frac{1}{\gamma_i}} - 9^{1+\frac{1}{\gamma_i}} \times 35^{\frac{1}{\gamma_i}} + 18^{1+\frac{1}{\gamma_i}} \times 35^{\frac{1}{\gamma_i}} + 3^{3+\frac{2}{\gamma_i}}\right.\right.$$
$$\left.\left. \times 40^{\frac{1}{\gamma_i}} - 5^{2+\frac{1}{\gamma_i}} \times 84^{\frac{1}{\gamma_i}} + 280^{\frac{1}{\gamma_i}}\right)\right],$$

$$\phi^{(2)}_{X_i} = \Gamma\left(1+\frac{1}{\gamma_i}\right)\left[2^{-4-\frac{3}{\gamma_i}} \times 3465^{\frac{-1}{\gamma_i}}\left(-2^{2+\frac{3}{\gamma_i}} \times 3^{1+\frac{2}{\gamma_i}} \times 35^{\frac{1}{\gamma_i}} + 2^{2+\frac{3}{\gamma_i}} \times 3^{3+\frac{2}{\gamma_i}} \times 55^{\frac{1}{\gamma_i}} + 2^{1+\frac{2}{\gamma_i}} \times 3^{3+\frac{2}{\gamma_i}}\right.\right.$$
$$\times 77^{\frac{1}{\gamma_i}} - 3^{1+\frac{2}{\gamma_i}} \times 8^{1+\frac{1}{\gamma_i}} \times 77^{\frac{1}{\gamma_i}} + 45^{1+\frac{1}{\gamma_i}} \times 77^{\frac{1}{\gamma_i}} - 13 \times 8^{1+\frac{1}{\gamma_i}} \times 385^{\frac{1}{\gamma_i}} + 2^{2+\frac{1}{\gamma_i}} \times 9^{1+\frac{1}{\gamma_i}} \times 385^{\frac{1}{\gamma_i}} - 13$$
$$\left.\left. \times 2^{3+\frac{2}{\gamma_i}} \times 1155^{\frac{1}{\gamma_i}} + 2310^{\frac{1}{\gamma_i}}\right)\right],$$

$$\phi^{(3)}_{X_i} = \Gamma\left(1+\frac{1}{\gamma_i}\right)\left[2^{-5-\frac{3}{\gamma_i}} \times 45045^{\frac{-1}{\gamma_i}}\left(2^{1+\frac{3}{\gamma_i}} \times 45^{1+\frac{1}{\gamma_i}} \times 77^{\frac{1}{\gamma_i}} - 2^{3+\frac{1}{\gamma_i}} \times 33^{1+\frac{1}{\gamma_i}} \times 455^{\frac{1}{\gamma_i}} - 3^{1+\frac{2}{\gamma_i}}\right.\right.$$
$$\times 5^{1+\frac{1}{\gamma_i}} \times 572^{\frac{1}{\gamma_i}} + 2 \times 3^{3+\frac{2}{\gamma_i}} \times 7^{1+\frac{1}{\gamma_i}} \times 715^{\frac{1}{\gamma_i}} + 2^{2+\frac{3}{\gamma_i}} \times 9^{2+\frac{1}{\gamma_i}} \times 715^{\frac{1}{\gamma_i}} - 2^{4+\frac{3}{\gamma_i}} \times 3^{1+\frac{2}{\gamma_i}} \times 1001^{\frac{1}{\gamma_i}}$$
$$+37 \times 9^{1+\frac{1}{\gamma_i}} \times 3640^{\frac{1}{\gamma_i}} + 9^{2+\frac{1}{\gamma_i}} \times 4004^{\frac{1}{\gamma_i}} - 159 \times 2^{2+\frac{3}{\gamma_i}} \times 5005^{\frac{1}{\gamma_i}} + 2^{3+\frac{1}{\gamma_i}} \times 9^{1+\frac{1}{\gamma_i}} \times 5005^{\frac{1}{\gamma_i}} - 79$$
$$\left.\left. \times 4^{1+\frac{1}{\gamma_i}} \times 15015^{\frac{1}{\gamma_i}} + 24024^{\frac{1}{\gamma_i}}\right)\right],$$

and

$$\phi^{(4)}_{X_i} = \Gamma\left(1+\frac{1}{\gamma_i}\right)\left[4^{-3-\frac{2}{\gamma_i}} \times 765765^{\frac{-1}{\gamma_i}}\left(-2^{1+\frac{4}{\gamma_i}} \times 9^{1+\frac{1}{\gamma_i}} \times 5005^{\frac{1}{\gamma_i}} - 41 \times 2^{1+\frac{4}{\gamma_i}} \times 9^{1+\frac{1}{\gamma_i}} \times 6545^{\frac{1}{\gamma_i}}\right.\right.$$
$$-19 \times 4^{1+\frac{2}{\gamma_i}} \times 7^{1+\frac{1}{\gamma_i}} \times 7293^{\frac{1}{\gamma_i}} + 41 \times 2^{3+\frac{4}{\gamma_i}} \times 9^{1+\frac{1}{\gamma_i}} \times 7735^{\frac{1}{\gamma_i}} + 2^{5+\frac{4}{\gamma_i}} \times 3^{3+\frac{2}{\gamma_i}} \times 12155^{\frac{1}{\gamma_i}} - 2^{5+\frac{4}{\gamma_i}}$$
$$\times 3^{1+\frac{2}{\gamma_i}} \times 17017^{\frac{1}{\gamma_i}} - 2^{9+\frac{4}{\gamma_i}} \times 5^{1+\frac{1}{\gamma_i}} \times 17017^{\frac{1}{\gamma_i}} + 3^{3+\frac{2}{\gamma_i}} \times 5^{1+\frac{1}{\gamma_i}} \times 17017^{\frac{1}{\gamma_i}} - 8^{1+\frac{1}{\gamma_i}} \times 9^{2+\frac{1}{\gamma_i}} \times 17017^{\frac{1}{\gamma_i}}$$
$$+ 73 \times 3^{1+\frac{2}{\gamma_i}} \times 5^{1+\frac{1}{\gamma_i}} \times 19448^{\frac{1}{\gamma_i}} - 67 \times 21^{1+\frac{1}{\gamma_i}} \times 48620^{\frac{1}{\gamma_i}} + 23 \times 2^{3+\frac{1}{\gamma_i}} \times 9^{1+\frac{1}{\gamma_i}} \times 85085^{\frac{1}{\gamma_i}} + 4^{2+\frac{1}{\gamma_i}}$$
$$\left.\left. \times 9^{1+\frac{1}{\gamma_i}} \times 85085^{\frac{1}{\gamma_i}} - 53 \times 2^{4+\frac{3}{\gamma_i}} \times 255255^{\frac{1}{\gamma_i}} + 680680^{\frac{1}{\gamma_i}}\right)\right],\ i = 1, 2.$$

Corolary 4.4. *It is worth noting that if* $(X_1, X_2) \sim$*EP-WD-SAR*$(\gamma_1, \omega_1, \gamma_2, \omega_2)$, *Corollaries 4.2 and 4.3 reveal that both* $\mathsf{J}^w(X_1, X_2)$ *(as well as* $\mathsf{J}^w(X_i),\ i = 1,2$*) and* $\mathsf{J}^c(X_1, X_2)$ *(as well as* $\mathsf{J}^c(X_i),\ i = 1,2$*) are independent on the shape parameters* γ_1 *and* γ_2.

5. Methods of estimation

In this section, we discuss two estimation methods for estimating the unknown parameters of the EP-WD-SAR distribution: The ML and Bayesian estimations. Moreover, we construct asymptotic CIs using the Fisher information matrix (FIM) for the model's parameters.

5.1. The ML estimation

The ML method is a widely used and important statistical technique. It provides parameter estimates with desirable statistical properties, including consistency, asymptotic unbiasedness, efficiency, and asymptotic normality. To obtain these estimates, the ML method maximizes the likelihood of the observed sample data. The log-likelihood function $\ln L$ is derived using the PDF given in (2.2).

$$\begin{aligned}\ln L = {}& n\ln\left(\frac{3\gamma_1}{2\omega_1^{\gamma_1}}\right) + n\ln\left(\frac{3\gamma_2}{2\omega_2^{\gamma_2}}\right) + \sum_{i=1}^{n}\ln\left(x_{1,i}^{\gamma_1-1}\right) + \sum_{i=1}^{n}\ln\left(x_{2,i}^{\gamma_2-1}\right) - 2\sum_{i=1}^{n}\left(\frac{x_{1,i}}{\omega_1}\right)^{\gamma_1} - 2\sum_{i=1}^{n}\left(\frac{x_{2,i}}{\omega_2}\right)^{\gamma_2}\\ &+ \sum_{i=1}^{n}\ln\left(2 - e^{-\left(\frac{x_{1,i}}{\omega_1}\right)^{\gamma_1}}\right) + \sum_{i=1}^{n}\ln\left(2 - e^{-\left(\frac{x_{2,i}}{\omega_2}\right)^{\gamma_2}}\right) + \sum_{i=1}^{n}\ln\left[1 + 3\alpha\left(1 + e^{-3\left(\frac{x_{1,i}}{\omega_1}\right)^{\gamma_1}} - 3e^{-2\left(\frac{x_{1,i}}{\omega_1}\right)^{\gamma_1}}\right)\right.\\ &\times\left(1 + e^{-3\left(\frac{x_{2,i}}{\omega_2}\right)^{\gamma_2}} - 3e^{-2\left(\frac{x_{2,i}}{\omega_2}\right)^{\gamma_2}}\right) + \frac{5\alpha^2}{4}\left(3\left(1 + e^{-3\left(\frac{x_{1,i}}{\omega_1}\right)^{\gamma_1}} - 3e^{-2\left(\frac{x_{1,i}}{\omega_1}\right)^{\gamma_1}}\right)^2 - 1\right)\left(3\left(1 + e^{-3\left(\frac{x_{2,i}}{\omega_2}\right)^{\gamma_2}}\right.\right.\\ &\left.\left.\left.- 3e^{-2\left(\frac{x_{2,i}}{\omega_2}\right)^{\gamma_2}}\right)^2 - 1\right)\right].\end{aligned}$$

We obtain the following normal equations by partially differentiating $\ln L$ with respect to the vector of the parameters $\Omega = (\gamma_1, \gamma_2, \omega_1, \omega_2, \alpha)$ and equating them to zero. The following are those derivatives:

$$\begin{aligned}\frac{\partial \ln L(\Omega)}{\partial\gamma_l} = {}& n\left(\frac{1}{\gamma_l} - \ln(\omega_l)\right) + \sum_{i=1}^{n}\ln\left(x_{l,i}\right) - 2\sum_{i=1}^{n}\left(\frac{x_{l,i}}{\omega_l}\right)^{\gamma_l}\ln\left(\frac{x_{l,i}}{\omega_l}\right) - \sum_{i=1}^{n}\frac{e^{-\left(\frac{x_{l,i}}{\omega_l}\right)^{\gamma_l}}\left(\frac{x_{l,i}}{\omega_l}\right)^{\gamma_l}\ln\left(\frac{x_{l,i}}{\omega_l}\right)}{2 - e^{-\left(\frac{x_{l,i}}{\omega_l}\right)^{\gamma_l}}}\\ &+ \sum_{i=1}^{n}\frac{9\alpha\left(\frac{x_{l,i}}{\omega_l}\right)^{\gamma_l}e^{-2\left(\frac{x_{l,i}}{\omega_l}\right)^{\gamma_l}}\left(e^{-\left(\frac{x_{l,i}}{\omega_l}\right)^{\gamma_l}} - 2\right)\left[D_{j,i} + \frac{5\alpha}{2}D_{l,i}\left(3D_{j,i} - 1\right)\right]\ln\left(\frac{x_{l,i}}{\omega_l}\right)}{1 + 3\alpha D_{l,i}D_{j,i} + \frac{5\alpha^2}{4}\left(3D_{l,i}^2 - 1\right)\left(3D_{j,i}^2 - 1\right)},\end{aligned}$$

$$\begin{aligned}\frac{\partial \ln L(\Omega)}{\partial\omega_l} = {}& \frac{-n\gamma_l}{\omega_l} + \frac{2\gamma_l}{\omega_l^2}\sum_{i=1}^{n}x_{l,i}\left(\frac{x_{l,i}}{\omega_l}\right)^{\gamma_l-1} - \sum_{i=1}^{n}\frac{\gamma_l x_{l,i}e^{-\left(\frac{x_{l,i}}{\omega_l}\right)^{\gamma_l}}\left(\frac{x_{l,i}}{\omega_l}\right)^{\gamma_l-1}}{\omega_l^2\left(2 - e^{-\left(\frac{x_{l,i}}{\omega_l}\right)^{\gamma_l}}\right)}\\ &+ \sum_{i=1}^{n}\frac{9\alpha\gamma_l x_{l,i}e^{-2\left(\frac{x_{l,i}}{\omega_l}\right)^{\gamma_l}}\left(\frac{x_{l,i}}{\omega_l}\right)^{\gamma_l-1}\omega_l^{-2}\left(e^{-\left(\frac{x_{l,i}}{\omega_l}\right)^{\gamma_l}} - 2\right)\left[D_{j,i} + \frac{5\alpha}{2}D_{l,i}\left(3D_{j,i}^2 - 1\right)\right]}{1 + 3\alpha D_{l,i}D_{j,i} + \frac{5\alpha^2}{4}\left(3D_{l,i}^2 - 1\right)\left(3D_{j,i}^2 - 1\right)},\end{aligned}$$

and

$$\frac{\partial \ln L(\Omega)}{\partial \alpha} = \sum_{i=1}^{n} \frac{3D_{l,i}D_{j,i} + \frac{5\alpha}{2}\left(3D_{l,i}^2 - 1\right)\left(3D_{j,i}^2 - 1\right)}{1 + 3\alpha D_{l,i}D_{j,i} + \frac{5\alpha^2}{4}\left(3D_{l,i}^2 - 1\right)\left(3D_{j,i}^2 - 1\right)},$$

where $D_{i,k} = 1 + e^{-3\left(\frac{x_{i,k}}{\omega_k}\right)^{\gamma_k}} - 3e^{-2\left(\frac{x_{i,k}}{\omega_k}\right)^{\gamma_k}}, \;\; k = l, j; \;\; l, j = 1, 2,$ and $l \neq j$.

Because the likelihood equations for γ_1, γ_2, ω_1, ω_2, and α are nonlinear and challenging to solve analytically, numerical methods like the Newton-Raphson method can be employed to find the ML estimates (MLE) of the distribution parameters γ_1, γ_2, ω_1, ω_2, and α (see Tables 2–5 in Section 6).

Existence and uniqueness of the MLE

The existence and uniqueness of the MLE of parameter vector $\Omega = (\gamma_1, \omega_1, \gamma_2, \omega_2, \alpha) \in \Theta$ is a fundamental issue in the statistical analysis of the model. If the parameter space Θ is restricted to a closed and bounded subset (i.e., a compact set), then the log-likelihood function being continuous as a result of the density being strictly positive and continuous in the parameters satisfies the conditions of the Weierstrass extreme value theorem. This theorem guarantees that any continuous function defined on a compact set attains its maximum and minimum. Therefore, the MLE exists under these conditions.

The uniqueness of the MLE is typically local and relies on standard regularity conditions and concavity properties of the log-likelihood function. In particular:

(1) Identifiability: The model is identifiable because the EP-WD marginals are strictly increasing and distinguishable for different values of γ_i and ω_i, $i = 1, 2$. Additionally, the copula parameter α influences only the dependence structure and not the marginals, further supporting identifiability.
(2) Local concavity: The log-likelihood function is twice differentiable concerning Ω, and the observed information matrix (i.e., the Hessian) is negative definite in a neighborhood of the true parameter value under regularity conditions. This implies local concavity and hence local uniqueness of the MLE.

Initialization and stability in numerical optimization

To address the sensitivity of the Newton-Raphson method to initial values, we propose some or all the following stabilization strategies:
(1) Use the method of moments to obtain rough initial estimates for the marginal parameters γ_i and $\omega_i, i = 1, 2$.
(2) Fit each EP-WD marginal independently using univariate MLE to get more refined starting values.
(3) Use a pseudo-likelihood approach to initialize the dependence parameter α.
(4) Fix some parameters temporarily and estimate the rest, simplifying the initial optimization landscape.

For a comprehensive treatment of the theoretical issues discussed above, including MLE theory for copula models, parameter identifiability, and computational aspects such as initialization and optimization strategies, see Joe [35]. This work is widely regarded as one of the most authoritative modern references on copula modeling.

5.2. Bayesian estimation

The Bayesian estimation technique is a robust method for inferring unknown parameters from observable data. By leveraging Bayes' theorem, a fundamental concept in probability theory, the method continuously updates the probability of a hypothesis as new information becomes available. Unlike traditional MLE, Bayesian estimation incorporates prior knowledge, offering distinct advantages such as improved parameter inference and uncertainty quantification. A critical aspect of this approach is selecting appropriate prior PDF and hyperparameter values that align with our prior beliefs about the data. For parameters ω_l, $l = 1, 2$, we adopt gamma-independent priors, specifically:

$$\Pi_l(\omega_l) \propto \frac{b_l^{a_l}}{\Gamma(a_l)} \omega_l^{a_l-1} e^{-b_l\omega_l}, \quad \omega_l > 0, a_l, b_l > 0, \quad l = 1, 2.$$

Here, the copula parameter α has uniform prior distribution. The prior JPDF is given by

$$\Pi(\omega_l, \omega_2) \propto \frac{b_1^{a_1} b_2^{a_2}}{\Gamma(a_1)\Gamma(a_2)} \omega_1^{a_1-1} e^{-b_1\omega_1} \omega_2^{a_2-1} e^{-b_2\omega_2}.$$

The estimate obtained from the likelihood method, along with its variance-covariance matrix, can be used to guide the elicitation of hyperparameters for the independent joint prior. Specifically, the mean and variance of the gamma prior serve as representations of the derived hyperparameters. For further details, refer to Gupta and Kundu [36], Dey et al. [37], and Hamdy and Almetwally [38]. Parameter γ_l, where $l = 1, 2$, in the EP-WD-SAR model should be well-defined and strictly positive. The likelihood function is given by

$$\begin{aligned}
L(\Omega) = &\left(\frac{3\gamma_1}{2\omega_1^{\gamma_1}}\right)^n \prod_{i=1}^{n} x_{i,1}^{\gamma_1-1} e^{-2\left(\frac{x_{i,1}}{\omega_1}\right)^{\gamma_1}} \left(2 - e^{-\left(\frac{x_{i,1}}{\omega_1}\right)^{\gamma_1}}\right) \left(\frac{3\gamma_2}{2\omega_2^{\gamma_2}}\right)^n \prod_{i=1}^{n} x_{i,2}^{\gamma_2-1} e^{-2\left(\frac{x_{i,2}}{\omega_2}\right)^{\gamma_2}} \left(2 - e^{-\left(\frac{x_{i,2}}{\omega_2}\right)^{\gamma_2}}\right) \\
&\times \prod_{i=1}^{n} \left[1 + 3\alpha \left(1 + e^{-3\left(\frac{x_{i,1}}{\omega_1}\right)^{\gamma_1}} - 3e^{-2\left(\frac{x_{i,1}}{\omega_1}\right)^{\gamma_1}}\right) \left(1 + e^{-3\left(\frac{x_{i,2}}{\omega_2}\right)^{\gamma_2}} - 3e^{-2\left(\frac{x_{i,2}}{\omega_2}\right)^{\gamma_2}}\right) + \frac{5\alpha^2}{4}\right. \\
&\left.\times \left(3\left(1 + e^{-3\left(\frac{x_{i,1}}{\omega_1}\right)^{\gamma_1}} - 3e^{-2\left(\frac{x_{i,1}}{\omega_1}\right)^{\gamma_1}}\right)^2 - 1\right) \left(3\left(1 + e^{-3\left(\frac{x_{i,2}}{\omega_2}\right)^{\gamma_2}} - 3e^{-2\left(\frac{x_{i,2}}{\omega_2}\right)^{\gamma_2}}\right)^2 - 1\right)\right].
\end{aligned}$$

The corresponding posterior density is given as follows:

$$\begin{aligned}
\Pi(\Omega|x_1, x_2) \propto &\frac{b_1^{a_1} b_2^{a_2}}{\Gamma(a_1)\Gamma(a_2)} \omega_1^{a_1-1} e^{-b_1\omega_1} \omega_2^{a_2-1} e^{-b_2\omega_2} \left(\frac{3\gamma_1}{2\omega_1^{\gamma_1}}\right)^n \prod_{i=1}^{n} x_{i,1}^{\gamma_1-1} e^{-2\left(\frac{x_{i,1}}{\omega_1}\right)^{\gamma_1}} \left(2 - e^{-\left(\frac{x_{i,1}}{\omega_1}\right)^{\gamma_1}}\right) \left(\frac{3\gamma_2}{2\omega_2^{\gamma_2}}\right)^n \\
&\times \prod_{i=1}^{n} x_{i,2}^{\gamma_2-1} e^{-2\left(\frac{x_{i,2}}{\omega_2}\right)^{\gamma_2}} \left(2 - e^{-\left(\frac{x_{i,2}}{\omega_2}\right)^{\gamma_2}}\right) \prod_{i=1}^{n} \left[1 + 3\alpha \left(1 + e^{-3\left(\frac{x_{i,1}}{\omega_1}\right)^{\gamma_1}} - 3e^{-2\left(\frac{x_{i,1}}{\omega_1}\right)^{\gamma_1}}\right) \left(1 + e^{-3\left(\frac{x_{i,2}}{\omega_2}\right)^{\gamma_2}}\right.\right. \\
&\left.\left. -3e^{-2\left(\frac{x_{i,2}}{\omega_2}\right)^{\gamma_2}}\right) + \frac{5\alpha^2}{4} \left(3\left(1 + e^{-3\left(\frac{x_{i,1}}{\omega_1}\right)^{\gamma_1}} - 3e^{-2\left(\frac{x_{i,1}}{\omega_1}\right)^{\gamma_1}}\right)^2 - 1\right) \left(3\left(1 + e^{-3\left(\frac{x_{i,2}}{\omega_2}\right)^{\gamma_2}} - 3e^{-2\left(\frac{x_{i,2}}{\omega_2}\right)^{\gamma_2}}\right)^2 - 1\right)\right].
\end{aligned}$$

The marginal posterior distributions $\Pi(\omega_l|\gamma_l, \alpha, x_1, x_2)$ and $\Pi(\alpha|\gamma_l, \omega_l, x_1, x_2)$ of the parameters ω_l, $l = 1, 2$, and α may be found by integrating out the nuisance parameters from the posterior distribution $\Pi(\Omega|x_1, x_2)$ as follows:

$$\Pi(\omega_l|x_1,x_2) \propto \omega_l^{a_l-1}e^{-b_l\omega_l}\left(\omega_l^{\gamma_l}\right)^{-n}\prod_{i=1}^{n}x_{i,l}^{\gamma_l-1}e^{-2\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\left(2-e^{-\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\right)\prod_{i=1}^{n}\left[1+3\alpha\left(1+e^{-3\left(\frac{x_{i,1}}{\omega_1}\right)^{\gamma_1}}\right.\right.$$
$$\left.-3e^{-2\left(\frac{x_{i,1}}{\omega_1}\right)^{\gamma_1}}\right)\left(1+e^{-3\left(\frac{x_{i,2}}{\omega_2}\right)^{\gamma_2}}-3e^{-2\left(\frac{x_{i,2}}{\omega_2}\right)^{\gamma_2}}\right)+\frac{5\alpha^2}{4}\left(3\left(1+e^{-3\left(\frac{x_{i,1}}{\omega_1}\right)^{\gamma_1}}-3e^{-2\left(\frac{x_{i,1}}{\omega_1}\right)^{\gamma_1}}\right)^2-1\right)$$
$$\left.\times\left(3\left(1+e^{-3\left(\frac{x_{i,2}}{\omega_2}\right)^{\gamma_2}}-3e^{-2\left(\frac{x_{i,2}}{\omega_2}\right)^{\gamma_2}}\right)^2-1\right)\right]$$

and

$$\Pi(\alpha|x_1,x_2) \propto \prod_{i=1}^{n}\left[1+3\alpha\left(1+e^{-3\left(\frac{x_{i,1}}{\omega_1}\right)^{\gamma_1}}-3e^{-2\left(\frac{x_{i,1}}{\omega_1}\right)^{\gamma_1}}\right)\left(1+e^{-3\left(\frac{x_{i,2}}{\omega_2}\right)^{\gamma_2}}-3e^{-2\left(\frac{x_{i,2}}{\omega_2}\right)^{\gamma_2}}\right)+\frac{5\alpha^2}{4}\right.$$
$$\left.\times\left(3\left(1+e^{-3\left(\frac{x_{i,1}}{\omega_1}\right)^{\gamma_1}}-3e^{-2\left(\frac{x_{i,1}}{\omega_1}\right)^{\gamma_1}}\right)^2-1\right)\left(3\left(1+e^{-3\left(\frac{x_{i,2}}{\omega_2}\right)^{\gamma_2}}-3e^{-2\left(\frac{x_{i,2}}{\omega_2}\right)^{\gamma_2}}\right)^2-1\right)\right],$$

where $l = 1, 2$. We use the well-known squared error loss function that yields the posterior means as the Bayes estimates of Ω, say $(\hat{\omega}_l, \hat{\alpha})$, given by $\hat{\omega}_l = \int_0^{\infty}\omega_l\Pi(\omega_l|x_1,x_2)d\omega_l$ and $\hat{\alpha} = \int_{-0.529}^{0.529}\alpha\Pi(\alpha|x_1,x_2)d\alpha$. The preceding integrals cannot be obtained explicitly. Because of that, we use the Monte Carlo method to find an approximate value of these integrals; see Tables 2–5.

5.3. Asymptotic confidence intervals

Based on the asymptotic normality of the MLE, the FIM is frequently used to create asymptotic CIs for the unknown parameters in Ω. Under specific assumptions of regularity, the MLE denoted as $\widehat{\Omega}$ is distributed according to the normal distribution. As the sample size n tends to infinity, the distribution of the estimator $\widehat{\Omega} = (\widehat{\gamma_1}, \widehat{\gamma_2}, \widehat{\omega_1}, \widehat{\omega_2}, \widehat{\alpha})$ approaches a normal distribution with mean Ω and covariance matrix equal to the inverse of the FIM, denoted as $\boldsymbol{I}^{-1}(\Omega)$. The FIM consists of the negative expected values of the second-order derivatives of $\ln L$, as expressed by

$$\boldsymbol{I}(\Omega) = -E\begin{bmatrix} I_{\hat{\gamma}_1\hat{\gamma}_1} & & & & \\ I_{\hat{\omega}_1\hat{\gamma}_1} & I_{\hat{\omega}_1\hat{\omega}_1} & & & \\ I_{\hat{\gamma}_2\hat{\omega}_1} & I_{\hat{\gamma}_2\hat{\gamma}_1} & I_{\hat{\gamma}_2\hat{\gamma}_2} & & \\ I_{\hat{\omega}_2\hat{\gamma}_1} & I_{\hat{\omega}_2\hat{\omega}_1} & I_{\hat{\omega}_2\hat{\gamma}_2} & I_{\hat{\omega}_2\hat{\omega}_2} & \\ I_{\hat{\alpha}\hat{\gamma}_1} & I_{\hat{\alpha}\hat{\gamma}_2} & I_{\hat{\alpha}\hat{\omega}_1} & I_{\hat{\alpha}\hat{\omega}_2} & I_{\hat{\alpha}\hat{\alpha}} \end{bmatrix}. \tag{5.1}$$

The elements of the matrix in (5.1) are given by

$$I_{\gamma_l\gamma_l} = \frac{\partial^2 L(\Omega)}{\partial\gamma_l^2} = \frac{-n}{\gamma_l^2} - 2\sum_{i=1}^{n}\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}\ln\left(\frac{x_{i,l}}{\omega_l}\right)^2 + \sum_{i=1}^{n}\frac{\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}\left(1+2e^{\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\left(\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}-1\right)\right)\ln\left(\frac{x_{i,l}}{\omega_l}\right)^2}{\left(1-2e^{\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\right)^2}$$
$$-\sum_{i=1}^{n}\frac{\left(9\alpha e^{-2\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}\left(2-e^{-\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\right)\left[D_{i,j}+\frac{5\alpha}{2}D_{i,l}\left(3D_{i,j}^2-1\right)\right]\right)^2\ln\left(\frac{x_{i,l}}{\omega_l}\right)^2}{\left[1+3\alpha D_{i,l}D_{i,j}+\frac{5\alpha^2}{4}(3D_{i,l}^2-1)(3D_{i,j}^2-1)\right]^2}$$

$$+\sum_{i=1}^{n}\frac{9\alpha e^{-2\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}\left(2-e^{-\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}+3e^{-\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}-4\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}\right)\ln\left(\frac{x_{i,l}}{\omega_l}\right)^2}{1+3\alpha D_{i,l}D_{i,j}+\frac{5\alpha^2}{4}(3D_{i,l}^2-1)(3D_{i,j}^2-1)}$$
$$\times\left[D_{i,j}+\frac{5\alpha}{2}D_{i,l}\left(3D_{i,j}^2-1\right)\right]+\frac{15\alpha^2}{2}\left(3D_{i,j}^2-1\right)\left(3e^{-2\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}\left(2-e^{-\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\right)\ln\left(\frac{x_{i,l}}{\omega_l}\right)\right)^2,$$

$$I_{\omega_l\omega_l}=\frac{n\gamma_l}{\omega_l^2}-\frac{2\gamma_l(1+\gamma_l)}{\omega_l^2}\sum_{i=1}^{n}\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}-\sum_{i=1}^{n}\frac{\gamma_l\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}\left(1+\gamma_l+2e^{\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\left(-1+\gamma_l\left(\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}-1\right)\right)\right)}{\omega_l^2\left(1-2e^{\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\right)^2}$$
$$-\sum_{i=1}^{n}\frac{\left(\frac{9\alpha}{\omega_l^2}\,x_{i,l}\,e^{-2\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\gamma_l\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l-1}\left(e^{-\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}-2\right)\left[D_{i,j}+\frac{5\alpha}{2}D_{i,l}\left(3D_{i,j}^2-1\right)\right]\right)^2}{\left[1+3\alpha D_{i,l}D_{i,j}+\frac{5\alpha^2}{4}(3D_{i,l}^2-1)(3D_{i,j}^2-1)\right]^2}$$
$$+\sum_{i=1}^{n}\frac{\frac{9\alpha}{\omega_l^3}\,x_{i,l}\,e^{-2\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\gamma_l\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l-1}\left(\gamma_l\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}\left(3e^{-\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}-4\right)+(\gamma_l+1)\left(2-e^{-\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\right)\right)}{1+3\alpha D_{i,l}D_{i,j}+\frac{5\alpha^2}{4}(3D_{i,l}^2-1)(3D_{i,j}^2-1)}$$
$$\times\left[D_{i,j}+\frac{5\alpha}{2}D_{i,l}\left(3D_{i,j}^2-1\right)\right]+\frac{15\alpha^2}{2}\left(3D_{i,j}^2-1\right)\left(\frac{3}{\omega_l^2}e^{-2\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\gamma_l x_{i,l}\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l-1}\left(e^{-2\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}-2\right)\right)^2,$$

$$I_{\omega_l\gamma_l}=\frac{-n}{\omega_l}-\sum_{i=1}^{n}\frac{\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}\left(1-2e^{\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}+\gamma_l\left(1+2e^{\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\left(-1+\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}\right)\right)\ln\left(\frac{x_{i,l}}{\omega_l}\right)\right)}{\omega_l\left(1-2e^{\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\right)^2}+\frac{2}{\omega_l}\sum_{i=1}^{n}\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}$$
$$\times\left(1+\gamma_l\ln\left(\frac{x_{i,l}}{\omega_l}\right)\right)-\sum_{i=1}^{n}\frac{\frac{9\alpha}{\omega_l^2}\,x_{i,l}\,e^{-2\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\gamma_l\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l-1}\left(e^{-\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}-2\right)\left[D_{i,j}+\frac{5\alpha}{2}D_{i,l}\left(3D_{i,j}^2-1\right)\right]}{\left[1+3\alpha D_{i,l}D_{i,j}+\frac{5\alpha^2}{4}(3D_{i,l}^2-1)(3D_{i,j}^2-1)\right]^2}$$
$$\times\;9\alpha e^{-2\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}\left(2-e^{-2\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\right)\left[D_{i,j}+\frac{5\alpha}{2}D_{i,l}\left(3D_{i,j}^2-1\right)\right]\ln\left(\frac{x_{i,l}}{\omega_l}\right)$$

$$+\sum_{i=1}^{n}\frac{\frac{9\alpha}{\omega_l}\,e^{-2\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}\left(\left(e^{-\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}-2\right)\left(1+\gamma_l\ln\left(\frac{x_{i,l}}{\omega_l}\right)\right)-\gamma_l\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}\left(3e^{-\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}-4\right)\ln\left(\frac{x_{i,l}}{\omega_l}\right)\right)}{1+3\alpha D_{i,l}D_{i,j}+\frac{5\alpha^2}{4}(3D_{i,l}^2-1)(3D_{i,j}^2-1)}$$
$$\times\left[D_{i,j}+\frac{5\alpha}{2}D_{i,l}\left(3D_{i,j}^2-1\right)\right]-\frac{15\alpha^2}{2}\left(3D_{i,j}^2-1\right)\frac{9\gamma_l\left(e^{-2\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\right)^2\left(\frac{x_{i,l}}{\omega_l}\right)^{2\gamma_l}\left(e^{-\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}-2\right)^2\ln\left(\frac{x_{i,l}}{\omega_l}\right)}{\omega_l},$$

$$I_{\gamma_j\omega_l}=-\sum_{i=1}^{n}\frac{\frac{9\alpha\gamma_l}{\omega_l}\ e^{-2\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}\left(e^{-\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}-2\right)\left[D_{i,j}+\frac{5\alpha}{2}D_{i,l}\left(3D_{i,j}^2-1\right)\right]}{\left[1+3\alpha D_{i,l}D_{i,j}+\frac{5\alpha^2}{4}(3D_{i,l}^2-1)(3D_{i,j}^2-1)\right]^2}$$
$$\times\ 9\alpha e^{-2\left(\frac{x_{i,j}}{\omega_j}\right)^{\gamma_j}}\left(\frac{x_{i,j}}{\omega_j}\right)^{\gamma_j}\left(2-e^{-\left(\frac{x_{i,j}}{\omega_j}\right)^{\gamma_j}}\right)\left[D_{i,l}+\frac{5\alpha}{2}D_{i,j}\left(3D_{i,l}^2-1\right)\right]\ln\left(\frac{x_{i,j}}{\omega_j}\right)$$
$$-\ \sum_{i=1}^{n}\frac{\frac{27\alpha\gamma_l}{\omega_l}e^{-2\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}\left(e^{-\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}-2\right)\left(e^{-\left(\frac{x_{i,j}}{\omega_j}\right)^{\gamma_j}}-2\right)\left(\frac{x_{i,j}}{\omega_j}\right)^{\gamma_j}e^{-2\left(\frac{x_{i,j}}{\omega_j}\right)^{\gamma_j}}\left[1+15\alpha D_{i,l}D_{i,j}\right]\ln\left(\frac{x_{i,j}}{\omega_j}\right)}{1+3\alpha D_{i,l}D_{i,j}+\frac{5\alpha^2}{4}(3D_{i,l}^2-1)(3D_{i,j}^2-1)},$$

$$I_{\gamma_j\gamma_l}=-\sum_{i=1}^{n}\frac{9\alpha\ e^{-2\left(\frac{x_{i,j}}{\omega_j}\right)^{\gamma_j}}\left(\frac{x_{i,j}}{\omega_j}\right)^{\gamma_j}\left(2-e^{-\left(\frac{x_{i,j}}{\omega_j}\right)^{\gamma_j}}\right)\left[D_{i,l}+\frac{5\alpha}{2}D_{i,j}\left(3D_{i,l}^2-1\right)\right]\ln\left(\frac{x_{i,j}}{\omega_j}\right)}{\left[1+3\alpha D_{i,l}D_{i,j}+\frac{5\alpha^2}{4}(3D_{i,l}^2-1)(3D_{i,j}^2-1)\right]^2}$$
$$\times\ 9\alpha e^{-2\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}\left(2-e^{-\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\right)\left[D_{i,j}+\frac{5\alpha}{2}D_{i,l}\left(3D_{i,j}^2-1\right)\right]\ln\left(\frac{x_{i,l}}{\omega_l}\right)$$
$$+\ \sum_{i=1}^{n}\frac{27\alpha\ e^{-2\left(\frac{x_{i,j}}{\omega_j}\right)^{\gamma_j}}\left(\frac{x_{i,j}}{\omega_j}\right)^{\gamma_j}\left(2-e^{-\left(\frac{x_{i,j}}{\omega_j}\right)^{\gamma_j}}\right)\left(2-e^{-\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\right)\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}e^{-2\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\left(\ln\left(\frac{x_{i,j}}{\omega_j}\right)\right)\ln\left(\frac{x_{i,l}}{\omega_l}\right)}{1+3\alpha D_{i,l}D_{i,j}+\frac{5\alpha^2}{4}(3D_{i,l}^2-1)(3D_{i,j}^2-1)}$$
$$\times\ \left[1+15\alpha D_{i,l}D_{i,j}\right],$$

$$I_{\omega_j\gamma_l}=\sum_{i=1}^{n}\frac{\frac{9\gamma_j\alpha}{\omega_j}\ e^{-2\left(\frac{x_{i,j}}{\omega_j}\right)^{\gamma_j}}\left(\frac{x_{i,j}}{\omega_j}\right)^{\gamma_j}\left(2-e^{-\left(\frac{x_{i,j}}{\omega_j}\right)^{\gamma_j}}\right)\left[D_{i,l}+\frac{5\alpha}{2}D_{i,j}\left(3D_{i,l}^2-1\right)\right]}{\left[1+3\alpha D_{i,l}D_{i,j}+\frac{5\alpha^2}{4}(3D_{i,l}^2-1)(3D_{i,j}^2-1)\right]^2}$$
$$\times\ 9\alpha e^{-2\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}\left(2-e^{-\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\right)\left[D_{i,j}+\frac{5\alpha}{2}D_{i,l}\left(3D_{i,j}^2-1\right)\right]\ln\left(\frac{x_{i,l}}{\omega_l}\right)$$
$$-\ \sum_{i=1}^{n}\frac{\frac{27\gamma_j\alpha}{\omega_j}\ e^{-2\left(\frac{x_{i,j}}{\omega_j}\right)^{\gamma_j}}\left(\frac{x_{i,j}}{\omega_j}\right)^{\gamma_j}\left(2-e^{-\left(\frac{x_{i,j}}{\omega_j}\right)^{\gamma_j}}\right)\left(2-e^{-\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\right)\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}e^{-2\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\ln\left(\frac{x_{i,l}}{\omega_l}\right)}{1+3\alpha D_{i,l}D_{i,j}+\frac{5\alpha^2}{4}(3D_{i,l}^2-1)(3D_{i,j}^2-1)}$$
$$\times\ \left[1+15\alpha D_{i,l}D_{i,j}\right],$$

$$I_{\omega_j\omega_l}=\sum_{i=1}^{n}\frac{\frac{9\gamma_j\alpha}{\omega_j}\ e^{-2\left(\frac{x_{i,j}}{\omega_j}\right)^{\gamma_j}}\left(\frac{x_{i,j}}{\omega_j}\right)^{\gamma_j}\left(2-e^{-\left(\frac{x_{i,j}}{\omega_j}\right)^{\gamma_j}}\right)\left[D_{i,l}+\frac{5\alpha}{2}D_{i,j}\left(3D_{i,l}^2-1\right)\right]}{\left[1+3\alpha D_{i,l}D_{i,j}+\frac{5\alpha^2}{4}(3D_{i,l}^2-1)(3D_{i,j}^2-1)\right]^2}$$
$$\times\ \frac{9\alpha\gamma_l}{\omega_l}e^{-2\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}\left(e^{-\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}-2\right)\left[D_{i,j}+\frac{5\alpha}{2}D_{i,l}\left(3D_{i,j}^2-1\right)\right]$$
$$+\ \sum_{i=1}^{n}\frac{\frac{27\gamma_j\gamma_l\alpha}{\omega_l\omega_j}\ e^{-2\left(\frac{x_{i,j}}{\omega_j}\right)^{\gamma_j}}\left(\frac{x_{i,j}}{\omega_j}\right)^{\gamma_j}\left(2-e^{-\left(\frac{x_{i,j}}{\omega_j}\right)^{\gamma_j}}\right)\left(2-e^{-\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\right)\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}e^{-2\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}}{1+3\alpha D_{i,l}D_{i,j}+\frac{5\alpha^2}{4}(3D_{i,l}^2-1)(3D_{i,j}^2-1)}$$

$$\times \left[1 + 15\alpha D_{i,l} D_{i,j}\right],$$

$$I_{\alpha\gamma_l} = \sum_{i=1}^{n} \frac{9\, e^{-2\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}} \left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l} \left(2 - e^{-\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\right) \left[D_{i,j} + 5\alpha D_{i,l}\left(3D_{i,j}^2 - 1\right)\right] \ln\left(\frac{x_{i,l}}{\omega_l}\right)}{1 + 3\alpha D_{i,l} D_{i,j} + \frac{5\alpha^2}{4}(3D_{i,l}^2 - 1)(3D_{i,j}^2 - 1)}$$
$$- \sum_{i=1}^{n} \frac{9\alpha \left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l} e^{-2\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}} \left(3D_{i,l} D_{i,j} + \frac{5\alpha}{2}(3D_{i,l}^2 - 1)(3D_{i,j}^2 - 1)\right) \left(2 - e^{-\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\right) \ln\left(\frac{x_{i,l}}{\omega_l}\right)}{\left[1 + 3\alpha D_{i,l} D_{i,j} + \frac{5\alpha^2}{4}(3D_{i,l}^2 - 1)(3D_{i,j}^2 - 1)\right]^2}$$
$$\times \left[D_{i,j} + \frac{5\alpha}{2} D_{i,l}\left(3D_{i,j}^2 - 1\right)\right],$$

$$I_{\alpha\omega_l} = \sum_{i=1}^{n} \frac{\frac{9\gamma_l}{\omega_l}\, e^{-2\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}} \left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l} \left(e^{-\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}} - 2\right) \left[D_{i,j} + 5\alpha D_{i,l}\left(3D_{i,j}^2 - 1\right)\right]}{1 + 3\alpha D_{i,l} D_{i,j} + \frac{5\alpha^2}{4}(3D_{i,l}^2 - 1)(3D_{i,j}^2 - 1)}$$
$$+ \sum_{i=1}^{n} \frac{\frac{9\gamma_l\alpha}{\omega_l} \left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l} e^{-2\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}} \left(3D_{i,l} D_{i,j} + \frac{5\alpha}{2}(3D_{i,l}^2 - 1)(3D_{i,j}^2 - 1)\right) \left(2 - e^{-\left(\frac{x_{i,l}}{\omega_l}\right)^{\gamma_l}}\right)}{\left[1 + 3\alpha D_{i,l} D_{i,j} + \frac{5\alpha^2}{4}(3D_{i,l}^2 - 1)(3D_{i,j}^2 - 1)\right]^2}$$
$$\times \left[D_{i,j} + \frac{5\alpha}{2} D_{i,l}\left(3D_{i,j}^2 - 1\right)\right],$$

$$I_{\alpha\alpha} = -\sum_{i=1}^{n} \frac{\left(3D_{i,l} D_{i,j} + \frac{5\alpha}{2}(3D_{i,l}^2 - 1)(3D_{i,j}^2 - 1)\right)^2}{\left[1 + 3\alpha D_{i,l} D_{i,j} + \frac{5\alpha^2}{4}(3D_{i,l}^2 - 1)(3D_{i,j}^2 - 1)\right]^2} + \sum_{i=1}^{n} \frac{\frac{5}{2}\left(3D_{i,l}^2 - 1\right)\left(3D_{i,j}^2 - 1\right)}{1 + 3\alpha D_{i,l} D_{i,j} + \frac{5\alpha^2}{4}(3D_{i,l}^2 - 1)(3D_{i,j}^2 - 1)},$$

where $l, j = 1, 2,\ l \neq j$. According to Jia et al. [39], in cases where it is challenging to obtain the anticipated values directly, an alternative approach is to estimate them by calculating the negative second-order derivatives of the natural logarithm of the likelihood function, evaluated at the MLE. Hence, we approximate the expected values by the negatives of the second-order derivatives evaluated at the MLE $\widehat{\Omega}$. Thus, the estimated value of the FIM is $\boldsymbol{I}(\widehat{\Omega})$. Moreover, a $100(1-\beta)$ asymptotic CIs for parameter Ω can be constructed as:

$$\widehat{\gamma_l} \pm Z_{\frac{\beta}{2}} \sqrt{J_{\widehat{\gamma_l}\widehat{\gamma_l}}}\,; \qquad \widehat{\omega_l} \pm Z_{\frac{\beta}{2}} \sqrt{J_{\widehat{\omega_l}\widehat{\omega_l}}}\,; \qquad \widehat{\alpha} \pm Z_{\frac{\beta}{2}} \sqrt{J_{\widehat{\alpha}\widehat{\alpha}}}\,, \quad l = 1, 2,$$

where $J_{y_l y_l}$ is the element in $\boldsymbol{I}^{-1}(\widehat{\Omega})$, that corresponding to the element $I y_l y_l$ in the estimated matrix $\boldsymbol{I}(\widehat{\Omega})$, $y_l = \widehat{\gamma_l}, \widehat{\omega_l}, \widehat{\alpha}$, and $Z_{\frac{\beta}{2}}$ is the percentile of the standard normal distribution with right tail probability $\frac{\beta}{2}$.

5.4. Bootstrap confidence interval

Bootstrapping is a general technique for drawing statistical inferences that entails resampling the data to produce a sampling distribution for a statistic. The idea of bootstrapping was first introduced by Efron [40], and it describes the technique of using sample data as a population from which repeat samples are taken. The bootstrap method is recognized as an alternative to the asymptotic approaches since it has been shown to work well in various situations. The bootstrap method is proposed by Kotz et al. [41] as an alternative method for constructing a CI. The algorithm of the CI for the unknown parameters using the bootstrap method is illustrated below:

- Determine ML estimators for the parameters in the EP-WD-SAR distribution, using the results of Subsection 5.1, $\widehat{\omega}_1 = s_1(\underline{t}_1, \underline{t}_2)$, $\widehat{\omega}_2 = s_2(\underline{t}_1, \underline{t}_2)$ and $\widehat{\alpha} = s_3(\underline{t}_1, \underline{t}_2)$, where $\{(\underline{t}_1, \underline{t}_2)\} = \{(t_{11}, t_{12}), (t_{21}, t_{22}), ..., (t_{n1}, t_{n2})\}$.
- Select B independent bootstrap bivariate samples $\{(\underline{t}_1^b, \underline{t}_2^b)\} = \{(t_{11}^b, t_{12}^b), (t_{21}^b, t_{22}^b), ..., (t_{n1}^b, t_{n2}^b)\}$, $b = 1, 2, ..., B$, from the original bivariate sample $\{(\underline{t}_1, \underline{t}_2)\} = \{(t_{11}, t_{12}), (t_{21}, t_{22}), ..., (t_{n1}, t_{n2})\}$. Mean squared error (MSE) is estimated using $B = 2000$.
- Evaluate the bootstrap estimate for each bootstrap sample: $\omega_1^*(b) = s_1(\underline{t}_1^b, \underline{t}_2^b)$, $\omega_2^*(b) = s_2(\underline{t}_1^b, \underline{t}_2^b)$, and $\alpha^*(b) = s_3(\underline{t}_1^b, \underline{t}_2^b)$, $b = 1, 2, ..., B$.
- Estimate MSE$(\widehat{\omega_l})$, MSE$(\widehat{\omega_2})$, and MSE$(\widehat{\alpha})$:

$$\widehat{MSE}_{B\omega_1} = \frac{1}{B}\sum_{b=1}^{B}(\omega_1^*(b) - \omega_1^*(.))^2, \quad \omega_1^*(.) = \frac{\sum_{b=1}^{B}\omega_1^*(b)}{B},$$

$$\widehat{MSE}_{B\omega_2} = \frac{1}{B}\sum_{b=1}^{B}(\omega_2^*(b) - \omega_2^*(.))^2, \quad \omega_2^*(.) = \frac{\sum_{b=1}^{B}\omega_2^*(b)}{B},$$

and

$$\widehat{MSE}_{B\alpha} = \frac{1}{B}\sum_{b=1}^{B}(\alpha^*(b) - \alpha^*(.))^2, \quad \alpha^*(.) = \frac{\sum_{b=1}^{B}\alpha^*(b)}{B}.$$

CIs are given by:

$$\left(\omega_1^*(.) \pm Z_{\frac{\beta}{2}}\sqrt{\widehat{MSE}_{B\omega_1}}\right), \quad \left(\omega_2^*(.) \pm Z_{\frac{\beta}{2}}\sqrt{\widehat{MSE}_{B\omega_2}}\right),$$

and

$$\left(\alpha^*(.) \pm Z_{\frac{\beta}{2}}\sqrt{\widehat{MSE}_{B\alpha}}\right).$$

6. Simulation

A numerical comparison of the ML, Bayesian, and bootstrap estimates is presented in this section. Analytically derived results and estimation techniques' performance are evaluated. After 1000 samples were collected from EP-WD-SAR distribution, the Mathcad package was used. Parameter values are defined as follows:

- In Table 2: ω_1=0.2, ω_2=2, $\alpha = 0.4, -0.3$, and in Table 3: ω_1=4, ω_2=0.2, $\alpha = 0.4, -0.3$.
- In Table 4: ω_1=0.5, ω_2=3, $\alpha = 0.2, -0.4$, and in Table 5: ω_1=1, ω_2=0.2, $\alpha = 0.2, -0.4$.

The sample sizes are as follows $n = 20, 50, 100,$ and 150. The simulation results of bias, MSE on 5000 iterations of Monte Carlo simulation, are shown in Tables 2–5. The results of the asymptotic CI and bootstrap CI are tabulated in Table 6. The following conclusions can be drawn from Tables 2–6:

- It can be shown that the ML and Bayesian estimates of unknown parameters are fairly good in terms of Bias and MSE.
- With an increase in sample size, the MSEs decrease, and the estimated values of the parameters approach the actual values.
- For $\alpha = -0.3$, both the ML and Bayesian estimate values are smaller than in the case of $\alpha = 0.4$.
- For $\alpha = -0.4$, both the ML and Bayesian estimate values are smaller than in the case of $\alpha = 0.2$.
- The length of the bootstrap CI is smaller than the asymptotic CI.

Figure 2 displays some selected scatterplots of the simulated data from Tables 2–5.

Table 2. ML and Bayesian estimation methods for the parameters of the EP-WD-SAR model.

With known parameters $\gamma_1 = 1$, $\gamma_2 = 2$									
$\omega_1 = 0.2, \omega_2 = 2$									
α		0.4				−0.3			
		ML estimate		Bayesian estimate		ML estimate		Bayesian estimate	
n		*Bias*	*MSE*	*Bias*	*MSE*	*Bias*	*MSE*	*Bias*	*MSE*
20	ω_1	0.068	0.298	0.066	0.295	0.07	0.315	0.06	0.311
	ω_2	0.055	0.334	0.053	0.332	0.076	0.321	0.073	0.315
	α	0.064	0.316	0.062	0.313	0.074	0.457	0.071	0.452
50	ω_1	0.059	0.289	0.057	0.282	0.071	0.299	0.068	0.296
	ω_2	0.047	0.321	0.045	0.319	0.067	0.311	0.065	0.305
	α	0.051	0.287	0.048	0.284	0.069	0.421	0.063	0.417
100	ω_1	0.046	0.256	0.044	0.255	0.067	0.289	0.062	0.281
	ω_2	0.036	0.312	0.034	0.310	0.057	0.289	0.054	0.283
	α	0.047	0.245	0.045	0.244	0.059	0.401	0.053	0.398
150	ω_1	0.038	0.249	0.036	0.247	0.058	0.277	0.057	0.276
	ω_2	0.031	0.308	0.028	0.302	0.049	0.278	0.044	0.275
	α	0.038	0.234	0.036	0.231	0.053	0.389	0.051	0.386
200	ω_1	0.028	0.234	0.025	0.232	0.045	0.267	0.044	0.264
	ω_2	0.021	0.298	0.018	0.296	0.035	0.265	0.033	0.263
	α	0.011	0.221	0.008	0.218	0.049	0.378	0.047	0.376

Table 3. ML and Bayesian estimation methods for the parameters of the EP-WD-SAR model.

With known parameters $\gamma_1 = 1$, $\gamma_2 = 2$									
$\omega_1 = 4, \omega_2 = 0.2$									
α		0.4				−0.3			
		ML estimate		Bayesian estimate		ML estimate		Bayesian estimate	
n		*Bias*	*MSE*	*Bias*	*MSE*	*Bias*	*MSE*	*Bias*	*MSE*
20	ω_1	0.085	0.355	0.081	0.354	0.088	0.257	0.084	0.255
	ω_2	0.088	0.399	0.083	0.394	0.112	0.384	0.108	0.383
	α	0.078	0.303	0.075	0.299	0.085	0.345	0.083	0.342
50	ω_1	0.075	0.323	0.071	0.321	0.076	0.297	0.074	0.296
	ω_2	0.074	0.316	0.072	0.314	0.084	0.357	0.081	0.352
	α	0.068	0.283	0.064	0.281	0.079	0.361	0.075	0.357
100	ω_1	0.072	0.287	0.070	0.284	0.066	0.268	0.062	0.265
	ω_2	0.069	0.289	0.064	0.285	0.071	0.299	0.068	0.296
	α	0.057	0.251	0.053	0.249	0.067	0.343	0.063	0.341
150	ω_1	0.061	0.277	0.057	0.275	0.061	0.241	0.056	0.240
	ω_2	0.059	0.278	0.053	0.274	0.063	0.279	0.061	0.274
	α	0.055	0.244	0.052	0.241	0.053	0.331	0.050	0.327
200	ω_1	0.056	0.256	0.053	0.252	0.054	0.239	0.052	0.233
	ω_2	0.046	0.252	0.043	0.248	0.057	0.221	0.053	0.219
	α	0.047	0.235	0.042	0.233	0.047	0.301	0.045	0.301

Table 4. ML and Bayesian estimation methods for the parameters of the EP-WD-SAR model.

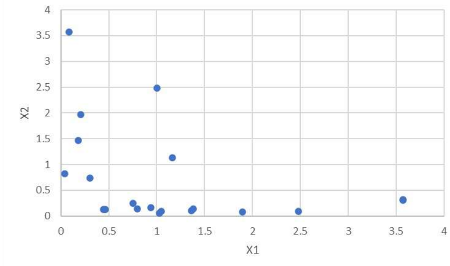
With known parameters $\gamma_1 = 2$, $\gamma_2 = 1$									
$\omega_1 = 0.5$, $\omega_2 = 3$									
α		0.2				−0.4			
		ML estimate		Bayesian estimate		ML estimate		Bayesian estimate	
n		*Bias*	*MSE*	*Bias*	*MSE*	*Bias*	*MSE*	*Bias*	*MSE*
20	ω_1	0.087	0.448	0.084	0.443	0.068	0.379	0.062	0.371
	ω_2	0.059	0.288	0.056	0.283	0.079	0.304	0.073	0.299
	α	0.065	0.342	0.061	0.340	0.069	0.277	0.064	0.275
50	ω_1	0.078	0.355	0.074	0.351	0.057	0.367	0.051	0.362
	ω_2	0.043	0.267	0.041	0.262	0.065	0.272	0.062	0.268
	α	0.058	0.336	0.054	0.333	0.059	0.248	0.054	0.242
100	ω_1	0.065	0.328	0.061	0.326	0.045	0.355	0.041	0.351
	ω_2	0.038	0.221	0.034	0.216	0.054	0.264	0.052	0.260
	α	0.049	0.324	0.046	0.321	0.066	0.256	0.060	0.252
150	ω_1	0.056	0.311	0.052	0.308	0.043	0.347	0.038	0.344
	ω_2	0.025	0.211	0.023	0.205	0.042	0.231	0.038	0.228
	α	0.032	0.315	0.030	0.310	0.053	0.244	0.050	0.240
200	ω_1	0.045	0.291	0.041	0.287	0.039	0.334	0.037	0.330
	ω_2	0.015	0.201	0.011	0.197	0.035	0.245	0.030	0.241
	α	0.025	0.311	0.022	0.305	0.069	0.238	0.061	0.232

Table 5. ML and Bayesian estimation methods for the parameters of the EP-WD-SAR model.

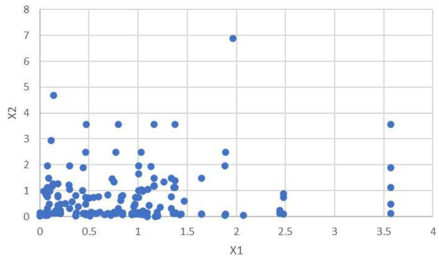
With known parameters $\gamma_1 = 2$, $\gamma_2 = 1$									
$\omega_1 = 1$, $\omega_2 = 0.2$									
α		0.2				−0.4			
		ML estimate		Bayesian estimate		ML estimate		Bayesian estimate	
n		*Bias*	*MSE*	*Bias*	*MSE*	*Bias*	*MSE*	*Bias*	*MSE*
20	ω_1	0.081	0.347	0.079	0.342	0.082	0.365	0.078	0.364
	ω_2	0.096	0.278	0.092	0.272	0.089	0.444	0.084	0.441
	α	0.065	0.287	0.061	0.284	0.079	0.277	0.073	0.276
50	ω_1	0.076	0.321	0.070	0.318	0.077	0.354	0.075	0.351
	ω_2	0.088	0.264	0.084	0.260	0.079	0.337	0.074	0.333
	α	0.054	0.267	0.051	0.262	0.075	0.272	0.072	0.270
100	ω_1	0.071	0.312	0.068	0.303	0.067	0.279	0.0617	0.274
	ω_2	0.075	0.255	0.071	0.252	0.072	0.265	0.070	0.261
	α	0.075	0.252	0.070	0.246	0.062	0.244	0.058	0.241
150	ω_1	0.051	0.282	0.047	0.278	0.052	0.321	0.049	0.319
	ω_2	0.055	0.243	0.052	0.234	0.065	0.211	0.060	0.203
	α	0.064	0.213	0.061	0.209	0.052	0.242	0.045	0.238
200	ω_1	0.065	0.265	0.062	0.260	0.043	0.224	0.039	0.220
	ω_2	0.071	0.234	0.069	0.231	0.054	0.211	0.050	0.208
	α	0.042	0.204	0.037	0.201	0.048	0.257	0.043	0.253

Table 6. Lower and upper bootstrap CI limits for the EP-WD-SAR model.

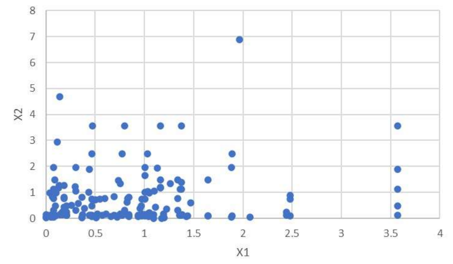
With known parameters $\gamma_1 = 1$, $\gamma_2 = 2$										
		$\omega_1 = 0.2$ $\omega_2 = 2$					$\omega_1 = 4$ $\omega_2 = 0.2$			
	α	0.4		-0.3			0.4		-0.3	
n		*CI*	*Bootstrap CI*	*CI*	*Bootstrap CI*		*CI*	*Bootstrap CI*	*CI*	*Bootstrap CI*
20	$\omega_1 = 0.2$	(0.134,0.564)	(0.139,0.532)	(0.154,0.421)	(0.159,0.411)	$\omega_1 = 4$	(2.327,5.121)	(2.768,4.987)	(2.453,5.011)	(2.873,4.977)
	$\omega_2 = 2$	(1.345,2.543)	(1.443,2.576)	(1.543,2.654)	(1.623,2.678)	$\omega_2 = 0.2$	(0.101,0.389)	(0.111,0.395)	(0.105,0.399)	(0.115,0.401)
	$\alpha = 0.4$	(0.356,0.632)	(0.365,0.622)	(0.376,0.639)	(0.379,0.611)	$\alpha = 0.4$	(0.311,0.499)	(0.314,0.489)	(0.315,0.497)	(0.319,0.485)
	$\alpha = -0.3$	(-0.432,-0.134)	(-0.422,-0.165)	(-0.429,-0.167)	(-0.421,-0.155)	$\alpha = -0.3$	(-0.532,-0.234)	(-0.522,-0.265)	(-0.529,-0.267)	(-0.521,-0.255)
50	$\omega_1 = 0.2$	(0.144,0.544)	(0.146,0.523)	(0.156,0.412)	(0.157,0.401)	$\omega_1 = 4$	(2.451,5.011)	(2.888, 4.784)	(2.555,4.933)	(2.953,4.864)
	$\omega_2 = 2$	(1.415,2.541)	(1.454,2.567)	(1.552,2.645)	(1.645,2.665)	$\omega_2 = 0.2$	(0.104,0.375)	(0.119,0.383)	(0.109,0.374)	(0.122,0.400)
	$\alpha = 0.4$	(0.378,0.629)	(0.367,0.620)	(0.379,0.631)	(0.381,0.607)	$\alpha = 0.4$	(0.315,0.487)	(0.322,0.481)	(0.319,0.491)	(0.325,0.481)
	$\alpha = -0.3$	(-0.454,-0.144)	(-0.444,-0.176)	(-0.437,-0.181)	(-0.431,-0.165)	$\alpha = -0.3$	(-0.554,-0.224)	(-0.544,-0.276)	(-0.537,-0.281)	(-0.531,-0.265)
100	$\omega_1 = 0.2$	(0.146,0.521)	(0.149,0.501)	(0.159,0.402)	(0.160,0.399)	$\omega_1 = 4$	(2.642,4.912)	(2.932, 4.654)	(2.698,4.845)	(3.211,4.599)
	$\omega_2 = 2$	(1.433,2.532)	(1.467,2.543)	(1.587,2.632)	(1.654,2.643)	$\omega_2 = 0.2$	(0.107,0.371)	(0.122,0.379)	(0.113,0.369)	(0.125,0.399)
	$\alpha = 0.4$	(0.382,0.625)	(0.370,0.618)	(0.385,0.626)	(0.384,0.601)	$\alpha = 0.4$	(0.321,0.477)	(0.327,0.479)	(0.325,0.486)	(0.328,0.479)
	$\alpha = -0.3$	(-0.458,-0.149)	(-0.456,-0.178)	(-0.443,-0.189)	(-0.439,-0.171)	$\alpha = -0.3$	(-0.558,-0.229)	(-0.556,-0.278)	(-0.543,-0.289)	(-0.539,-0.271)
150	$\omega_1 = 0.2$	(0.149,0.511)	(0.154,0.499)	(0.164,0.398)	(0.165,0.388)	$\omega_1 = 4$	(2.732,4.823)	(3.111, 4.594)	(2.745,4.745)	(3.342,4.589)
	$\omega_2 = 2$	(1.438,2.567)	(1.476,2.538)	(1.617,2.643)	(1.665,2.621)	$\omega_2 = 0.2$	(0.114,0.369)	(0.125,0.375)	(0.119,0.361)	(0.129,0.381)
	$\alpha = 0.4$	(0.388,0.614)	(0.373,0.612)	(0.390,0.612)	(0.389,0.589)	$\alpha = 0.4$	(0.331,0.471)	(0.333,0.468)	(0.336,0.468)	(0.335,0.461)
	$\alpha = -0.3$	(-0.461,-0.155)	(-0.460,-0.185)	(-0.450,-0.195)	(-0.445,-0.180)	$\alpha = -0.3$	(-0.561,-0.231)	(-0.560,-0.283)	(-0.550,-0.295)	(-0.545,-0.280)
With known parameters $\gamma_1 = 2$, $\gamma_2 = 1$										
		$\omega_1 = 0.5$ $\omega_2 = 3$					$\omega_1 = 1$ $\omega_2 = 0.2$			
	α	0.2		-0.4			0.2		-0.4	
n		*CI*	*Bootstrap CI*	*CI*	*Bootstrap CI*		*CI*	*Bootstrap CI*	*CI*	*Bootstrap CI*
20	$\omega_1 = 0.5$	(0.366,0.632)	(0.366,0.622)	(0.378,0.639)	(0.381,0.611)	$\omega_1 = 1$	(0.346,1.544)	(0.444,1.577)	(0.544,1.655)	(0.624,1.679)
	$\omega_2 = 3$	(2.345,3.543)	(2.443,3.576)	(2.543,3.654)	(2.623,3.678)	$\omega_1 = 2$	(1.346,2.544)	(1.444,2.577)	(1.544,2.655)	(1.624,2.679)
	$\alpha = 0.2$	(0.101,0.389)	(0.111,0.395)	(0.105,0.399)	(0.115,0.401)	$\omega_1 = 0.2$	(0.111,0.299)	(0.114,0.289)	(0.115,0.297)	(0.119,0.285)
	$\alpha = -0.4$	(-0.522,-0.324)	(-0.532,-0.365)	(-0.528,-0.367)	(-0.529-0.355)	$\omega_1 = -0.4$	(-0.422,-0.314)	(-0.432,-0.355)	(-0.428,-0.357)	(-0.429-0.345)
50	$\omega_1 = 0.5$	(0.379,0.629)	(0.369,0.620)	(0.381,0.631)	(0.383,0.607)	$\omega_1 = 1$	(0.416,1.542)	(0.455,1.568)	(0.553,1.646)	(0.646,1.666)
	$\omega_2 = 3$	(2.415,3.541)	(2.454,3.567)	(2.552,3.645)	(2.645,3.665)	$\omega_1 = 2$	(1.416,2.542)	(1.455,2.568)	(1.553,2.646)	(1.646,2.666)
	$\alpha = 0.2$	(0.104,0.375)	(0.119,0.383)	(0.109,0.374)	(0.122,0.400)	$\omega_1 = 0.2$	(0.115,0.287)	(0.122,0.281)	(0.119,0.291)	(0.125,0.281)
	$\alpha = -0.4$	(-0.534,-0.327)	(-0.541,-0.376)	(-0.536,-0.381)	(-0.539,-0.365)	$\omega_1 = -0.4$	(-0.434,-0.317)	(-0.441,-0.356)	(-0.436,-0.361)	(-0.439,-0.355)
100	$\omega_1 = 0.5$	(0.386,0.625)	(0.372,0.618)	(0.386,0.626)	(0.387,0.601)	$\omega_1 = 1$	(0.434,1.533)	(0468,1.544)	(0.588,1.633)	(0.655,1.644)
	$\omega_2 = 3$	(2.433,3.532)	(2.467,3.543)	(2.587,3.632)	(2.654,3.643)	$\omega_1 = 2$	(1.434,2.533)	(1.468,1.544)	(1.588,2.633)	(1.655,2.644)
	$\alpha = 0.2$	(0.107,0.371)	(0.122,0.379)	(0.113,0.369)	(0.125,0.399)	$\omega_1 = 0.2$	(0.121,0.277)	(0.127,0.279)	(0.125,0.286)	(0.128,0.279)
	$\alpha = -0.4$	(-0.543,-0.329)	(-0.546,-0.378)	(-0.548-0.389)	(-0.541,-0.371)	$\omega_1 = -0.4$	(-0.443,-0.319)	(-0.446,-0.358)	(-0.448-0.369)	(-0.441,-0.361)
150	$\omega_1 = 0.5$	(0.389,0.614)	(0.374,0.612)	(0.393,0.612)	(0.389,0.589)	$\omega_1 = 1$	(0.439,1.568)	(0.477,1.539)	(0.618,1.644)	(0.666,1.622)
	$\omega_2 = 3$	(2.438,3.567)	(2.476,3.538)	(2.617,3.643)	(2.665,3.621)	$\omega_1 = 2$	(1.439,2.568)	(1.477,2.539)	(1.618,2.644)	(1.666,2.622)
	$\alpha = 0.2$	(0.114,0.369)	(0.125,0.375)	(0.119,0.361)	(0.129,0.381)	$\omega_1 = 0.2$	(0.131,0.271)	(0.133,0.268)	(0.136,0.268)	(0.135,0.261)
	$\alpha = -0.4$	(-0.551,-0.331)	(-0.555,-0.383)	(-0.554,-0.395)	(-0.548,-0.380)	$\omega_1 = -0.4$	(-0.451,-0.321)	(-0.455,-0.363)	(-0.454,-0.375)	(-0.448,-0.370)



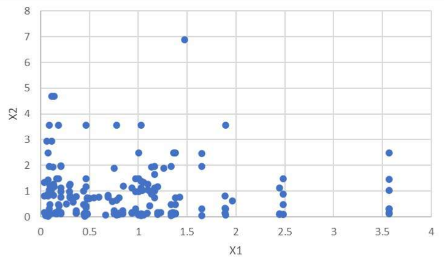
The simulated data from Table 2 at $n = 20$



The simulated data from Table 3 at $n = 50$



The simulated data from Table 4 at $n = 150$



The simulated data from Table 5 at $n = 200$

Figure 2. Some selected scatterplots of the simulated data.

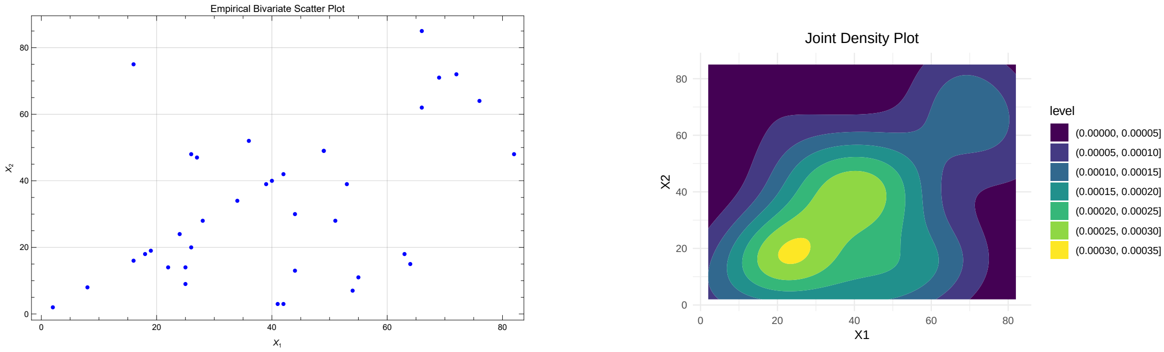
7. Real data applications

Example 7.1 (UEFA Champion's League data)**.** *This dataset, sourced from Meintanis [42], pertains to 37 soccer matches (i.e., $n = 37$). It includes instances where the home team scores at least one goal and instances where a goal is scored directly from a penalty kick, foul kick, or any other direct kick collectively referred to as kick goals by any team under consideration. Let X_1 denote the time in minutes when the first kick goal is scored by any team, and let X_2 denote the time in minutes when the first goal of any type is scored by the home team. The data (X_1, X_2), $i = 1, 2, \ldots, 37$, were fitted using the EP-WD-SAR model with parameters $(\gamma_1, \omega_1, \gamma_2, \omega_2)$. The MLE of the parameters are: $\widehat{\gamma_1} = 2.053$, $\widehat{\omega_1} = 60.754$, $\widehat{\gamma_2} = 1.373$, $\widehat{\omega_2} = 53.895$, and $\widehat{\alpha} = 0.429$.*

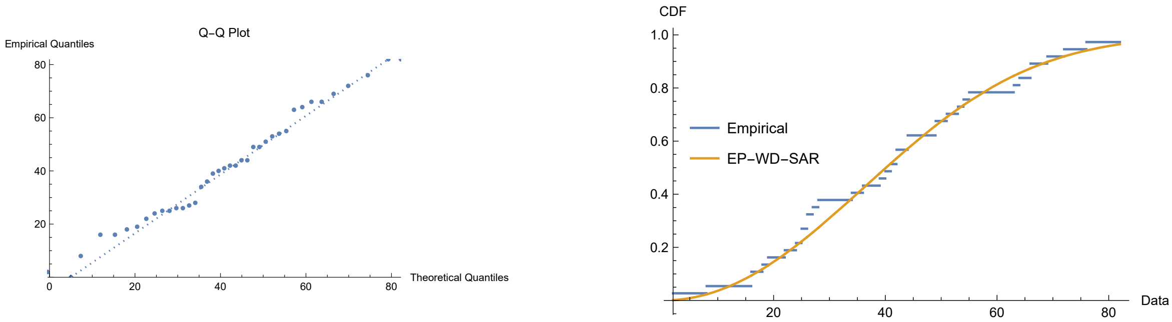
The EP-WD-SAR model was compared with other models, namely the Weibull distribution-SAR (WD-SAR), Epanechnikov-exponential distribution-SAR (EED-SAR), and exponential distribution-SAR (ED-SAR). Among all considered models, the EP-WD-SAR yielded the lowest values for the AIC, corrected AIC (AICc), BIC, Hannan–Quinn Information Criterion (HQIC), and consistent AIC (CAIC). The results are presented in Table 7. Figure 3 presents several statistical visualizations, including a scatterplot of the empirical data, a joint density plot, an empirical DF plot, and a Q–Q plot. Collectively, these plots support the conclusion that the EP-WD-SAR model provides a good fit to the data.

Table 7. $-\ln L$, AIC, AICc, BIC, HQIC, and CAIC.

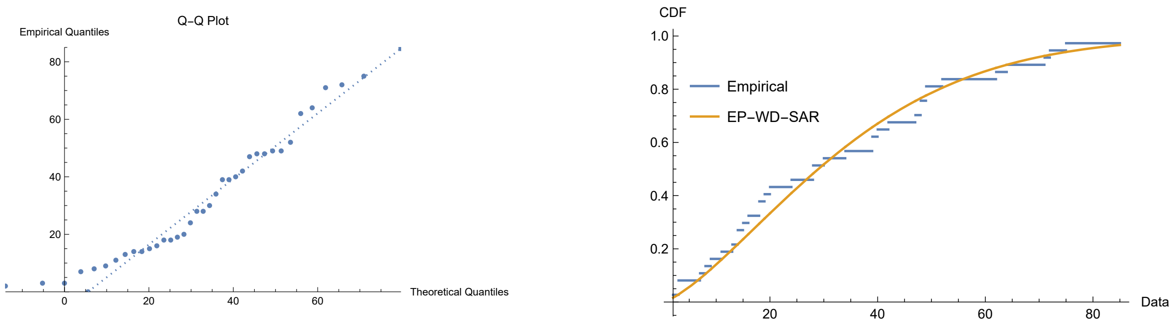
	$-\ln L$	AIC	AICc	BIC	HQIC	CAIC
EP-WD-SAR	-321.833	653.667	655.602	661.722	656.507	655.602
WD-SAR	-325.510	661.021	662.956	669.075	663.86	662.956
EED-SAR	-332.925	671.851	672.578	676.683	673.554	672.578
ED-SAR	-334.955	675.909	676.637	680.742	677.613	676.637



(a) The scatterplot of the empirical data and the joint density plot.



(b) UEFA Champion's League data (2005-2006).



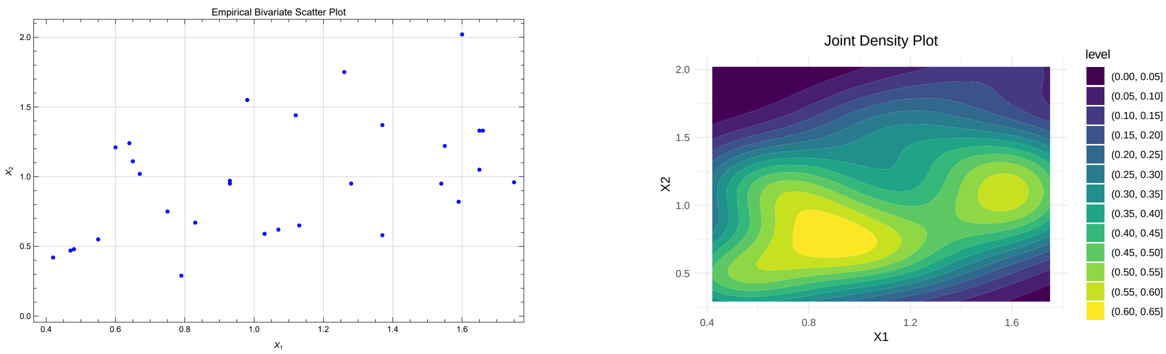
(c) UEFA Champion's League data (2004-2005)

Figure 3. Some summary plots of UEFA Champion's League data.

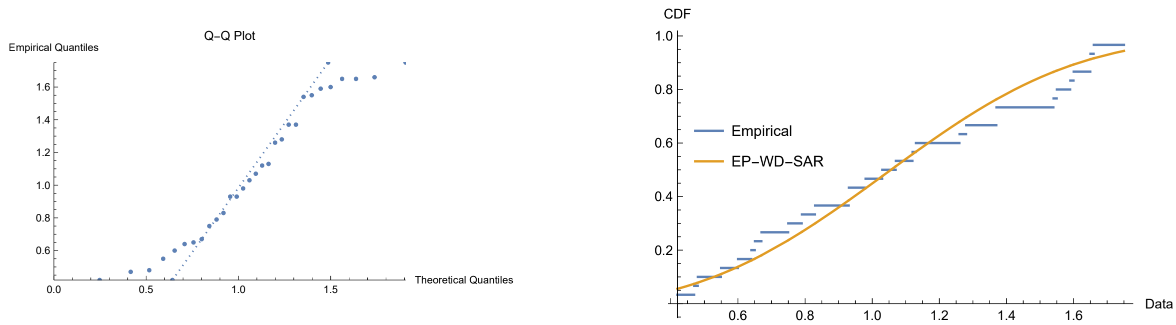
Example 7.2 (Cholesterol data set)**.** *This dataset includes cholesterol levels measured at 5 and 25 weeks after treatment in 30 patients (see, Shoaee [43]). Let* X_1 *denote the transformed cholesterol levels after 5 weeks of treatment, and let* X_2 *denote the transformed cholesterol levels after 25 weeks of treatment. The data* $(X_1, X_2),\ i = 1, 2, \ldots, 30,$ *were fitted using the EP-WD-SAR*$(\gamma_1, \omega_1, \gamma_2, \omega_2)$*. The MLE of parameters are* $\widehat{\gamma_1} = 2.629, \widehat{\omega_1} = 1.460, \widehat{\gamma_2} = 2.353,\ \widehat{\omega_2} = 1.340,$ *and* $\widehat{\alpha} = 0.398$*. Based on AIC and BIC, EP-WD-SAR has been compared to the Chen distribution-SAR (CHD-SAR), EED-SAR, and ED-SAR. The results are displayed in Table 8 and indicate that the EP-WD-SAR has the lowest AIC, AICc, BIC, HQIC, and CAIC when compared to other SAR distributions. Figure 4 presents several statistical visualizations, including a scatterplot of the empirical data, a joint density plot, an empirical DF plot, and a Q–Q plot. Collectively, these plots support the conclusion that the EP-WD-SAR model provides a good fit to the data.*

Table 8. $-\ln L$, AIC, AICc, BIC, HQIC, and CAIC.

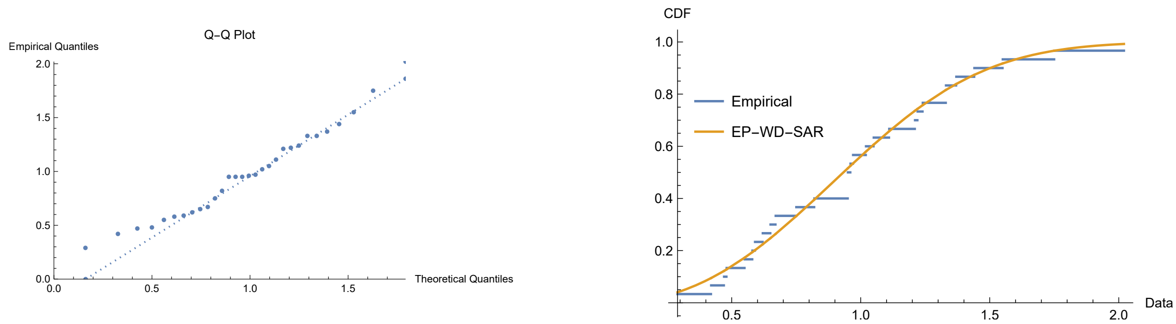
	$-\ln L$	AIC	AICc	BIC	HQIC	CAIC
EP-WD-SAR	-27.418	64.8365	67.3365	71.8424	67.0777	67.3365
CHD-SAR	-29.516	69.0326	71.5326	76.0385	71.2738	71.5326
EED-SAR	-51.071	108.141	109.064	112.345	109.486	109.064
ED-SAR	-54.121	114.242	115.166	118.446	115.587	115.166



(a) The scatterplot of the empirical data and the joint density plot.



(b) The transformed cholesterol levels of 30 patients after 5 weeks of treatment.



(c) The transformed cholesterol levels of 30 patients after 25 weeks of treatment.

Figure 4. Some summary plots of Cholesterol data set.

8. Conclusions

We introduce a novel bivariate distribution model, the EP-WD-SAR model, which is based on the SAR copula. By combining the Epanechnikov (EP) and Weibull distributions (WD) with the SAR copula, this model offers a significant contribution to bivariate statistical modeling. The EP-WD-SAR model stands out as an innovative framework for analyzing bivariate data, demonstrating originality and importance in this field. The model's correlation coefficient between X_1 and X_2 ranges from -0.512006 to 0.512109, enabling it to capture positive and negative dependencies effectively.

Additionally, we investigated key reliability measures such as the vitality function, MRL function, and hazard function, providing valuable insights into the model's behavior. To estimate the model parameters, both Bayesian and ML methods were employed, with Bayesian estimation demonstrating superior performance. A Monte Carlo simulation was used to evaluate the estimators, and asymptotic confidence intervals were derived for the likelihood estimates. Furthermore, bootstrap confidence intervals were computed. Finally, the versatility and practical applicability of the EP-WD-SAR model were validated through its analysis of real-world data sets.

Author contributions

G. M. Mansour, M. A. Abd Elgawad, A. S. Al-Moisheer, H. M. Barakat, M. A. Alawady, I. A. Husseiny, M. O. Mohamed: Methodology, conceptualization, investigation, software, resources, writing-original draft, writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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