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Research article

Semi-neat rings: A generalization of neat ring structures

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Abstract: This article introduces the concept of semi-neat rings, a generalization of neat rings, defined as rings whose nontrivial homomorphic images are semiclean. The study establishes fundamental properties of these rings, exploring their relationships with clean, neat, and semiclean rings. A classification of semi-neat FGC (Finitely Generated Cyclic) rings is provided, and the paper demonstrates that torch rings are semi-neat but not neat, illustrating the broader scope of the semi-neat concept. The article also investigates conditions under which the corner rings of semi-neat rings retained semi-neatness and explored extensions of the concept, such as weakly semi-neat rings. Constructed examples showed the existence of semi-neat rings that were not weakly neat, highlighting distinctions among related ring classes. The findings expanded the understanding of ring structures by demonstrating how semi-neat and weakly semi-neat rings were closed under homomorphic images and direct products. Applications to polynomial rings, triangular matrix rings, and torch rings revealed the utility of these generalized classes in algebra. The paper not only broadens the theoretical framework for neat and clean rings but also provides a foundation for further research on generalized ring properties and their implications in algebraic studies.

Keywords: clean ring; neat ring; weakly clean ring; semi-clean ring; semi-neat ring; weakly semi-neat ring
Mathematics Subject Classification: 13A99, 13F99

1. Introduction

Throughout this paper, assume R is a commutative ring with unity $1_R \neq 0$, unless otherwise specified. For a ring R, we indicate by $Id(R)$, $U(R)$ and $Per(R)$, the set of all idempotent, units and periodic of R, respectively. An element $e \in R$ is idempotent if $e^2 = e$, and an element $p \in R$ is periodic if there are $l < m \in \mathbb{N}$ such that $p^l = p^m$. Moreover, $N(R)$ and $J(R)$ represent the nil-radical and the Jacobson radical of R respectively. In other words, $N(R)$ is the ideal of all nilpotent elements of R, since an element $a \in R$ is nilpotent if $a^n = 0$ for some $n \in \mathbb{Z}^+$. Recall that every idempotent or nilpotent element is periodic. In addition, a group ring denoted by $R[G]$ or RG is a set of formal sums on the form $\{\sum_{i=1}^{n} r_i g_i : r_i \in R, g_i \in G\}$.

A ring R is said to be neat if every proper homomorphic image is clean. More generally, a ring is called clean if each element can be written as the sum of a unit and an idempotent. McGovern introduced the concept of neat rings [1]. Also, in the same reference, see the history of commutative clean rings. Many researchers have studied the related classes of clean rings, such as uniquely clean rings and weakly clean rings [2,3]. Recall that a ring R is called to be weakly clean if every element of R can be written as either the sum or difference of a unit and an idempotent. In addition a ring R is said to be nil-clean if each of its elements can be expressed as the sum of a nilpotent element and an idempotent element [4]. Therefore, it is reasonable to study rings in which every element can be expressed either as the sum or the difference of a nilpotent and an idempotent element. Such rings are referred to as weakly nil-clean rings in [5]. In 2003, Ye extended the concept of clean rings to achieve a semiclean ring if each element of R can be written as $r = p + u$, where $u \in U(R)$ and $p \in Per(R)$ [6]. Moreover, there have been several studies on semiclean rings where researchers explored this concept widely from different perspectives [7,8]. Some of them presented general characteristics of semiclean and strongly semiclean rings [9]. Other researchers investigated the relationship between semiclean ring and weakly clean ring [10]. Recently, Klinger et al. generalized Ye's theorems on semiclean group rings mentioned in example 2.11. They evidenced that the group ring RG is semi-clean where R is local if and only if G is a torsion abelian group [11]. According to Zhou, characterized semiclean commutative group rings [12]. An application of neat rings over group rings is a research conducted by Udar et al. [13]. They showed that when RG is neat but not clean, over R is a field. Also, if R is not a field, then necessary conditions that are not sufficient are found for a commutative RG to be neat but not clean.

Let R be a ring, R is FGC ring if every finitely generated module is isomorphic to direct sum of cyclic. This category of rings was first studied by Kaplansky [14]. For more details on FGC rings, see [15]. Further, a ring is Bézout, where every finitely generated ideal is principal. An essential property of a clean ring is closed under the homomorphic image. This property has led McGovern to define neat rings. There are many classes of clean rings, such as nil-clean, weakly clean, weakly nil-clean and semi-clean. Furthermore, these classes of clean rings are closed under a homomorphic image. Recently, Samiei and Danchev defined and investigated a commutative nil-neat and weakly nil-neat rings [16,17]. Motivated by all the above studies, this paper introduces two new classes of rings: Semi-neat rings and weakly semi-neat rings, which serve as natural generalizations of neat rings. By extending the definition of neat rings, we aim to establish a broader framework that captures a wider range of element decompositions within ring theory. These newly proposed structures not only enrich the existing hierarchy of clean-type rings but also open avenues for further exploration of their connections to other algebraic systems, such as Bézout rings, torch rings and weakly clean rings. The development of semi-neat rings is expected to contribute meaningfully to the structural understanding of rings and to the broader advancement of abstract algebra and ring theory.

2. Definition and background

Building upon this foundation and the extensive study of clean-type rings, we propose a new generalization: The semi-neat ring. This framework sets the stage for a deeper exploration of

semi-neat rings and their position within the established hierarchy of clean-related structures.
Definition 2.1. A ring R is called a semi-neat ring if every proper homomorphic image is semiclean.

The relationships between the main concepts in this study are shown in Figure 1.

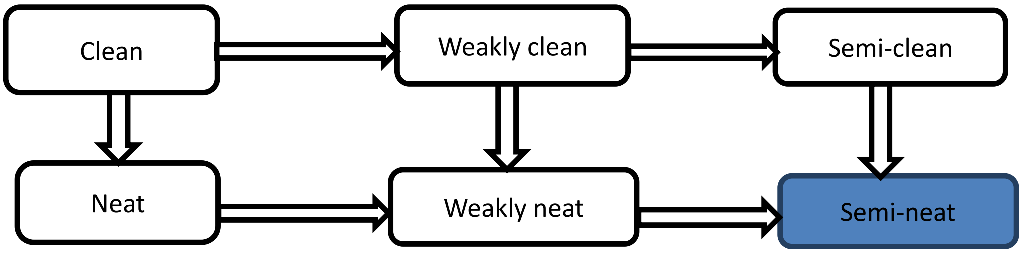

Figure 1. The main concept of the proposed study.

Figure 1 illustrates the relationships among the key ring structures explored in this study, including clean rings, weakly clean rings, semi-clean rings, neat rings, weakly neat rings, and the newly introduced class of semi-neat rings. These concepts are organized to highlight their structural connections. This diagram serves to contextualize semi-neat rings within the broader framework of clean-type and neat-type ring structures, emphasizing their role as a natural generalization in the ongoing development of ring theory.

3. Main properties

Proposition 3.1. Let R be a ring. The following statements are correct:

i) Every clean ring, semiclean ring, and neat ring are semi-neat rings.

ii) A semi-neat ring is closed under homomorphic image.

Proof. i) Since every idempotent is periodic, hence a clean ring R is a semiclean ring. Then, by Proposition 2.1 in [6], it is semiclean closed under a homomorphic image. Thus, R is a semi-neat ring. Moreover, if R is a neat ring, that means every proper homomorphic image is clean, such that clean rings are semiclean rings. Consequently, R is a semi-neat ring.

ii) If R is a semi-neat ring. Hence, every nontrivial homomorphic image is a semiclean ring. By using part i), the result holds.

There are many examples of semi-neat rings, such as filed, factor rings over maximal ideal, valuation ring, discrete valuation ring, local ring, pointed, von Neumann regular ring, Boolean ring, and zero-dimensional ring, which are originally neat (clean) rings. We will later in this paper provide an example to show that some semi-neat rings are not neat rings.

Proposition 3.2. For a ring R, the following are equivalent:

i) R is a semi-neat ring.

ii) R/rR is semi-clean for every nonzero $r \in R$.

iii) R/rR semi-neat for every $r \in R$.

iv) $R/N(R)$ is a semi-clean ring such that $N(R) \neq 0$. In particular, R/I is a semi-clean for every nonzero semiprime ideal.

Proof. i) $\Longrightarrow$ ii) Let R be a semi-neat ring, for a nonzero $r \in R$,then rR is a principal ideal of R. R/rR is a proper homomorphic image of the semi-neat ring. Thus, R/rR is a semi-clean ring.

ii) $\Longrightarrow$ iii) is clear by using the previous proposition.

iii) $\Longrightarrow$ i) Choose $r = 0$.

i) $\Leftrightarrow$ iv) If R is a semi-neat ring, then every proper homomorphic image is a semi-clean ring. Therefore, one of them $R/N(R)$, where $N(R) \neq 0$. Conversely, if $R/N(R)$ is a semi-clean ring, $\forall$ $r \in R$, $r + N(R) \in R/N(R)$ such that where $r + N(R) = \big(u + N(R)\big) + (x + N(R))$ such that $u + N(R) \in U(R/N(R))$ and $x + N(R) \in Per(R/N(R))$. Then, $u \in U(R)$ and by Proposition 2.1 in [7], the periodic in R can be lifted module $N(R)$. Thus, R is a semi-clean ring, using Proposition 2.2 in [6]. In particular, every prime ideal is a semiprime ideal. Hence, $I \subseteq N(R)$.

Proposition 3.3. If R is a semi-neat ring that is not semi-clean, then R is a reduced ring.

Proof. Let R be a semi-neat ring such that R is not semi-clean. Assume R is not semiprime; that means the zero ideal is not a semiprime ideal; $N(R) \neq 0$.Thus, $R/N(R)$ is semi-clean. Therefore, R is semi-clean by the previous proposition, which is a contradiction. So, R is reduced.

Theorem 3.4. For decomposable ring R, then R is a semi-neat ring if and only if R is semi-clean.

Proof. Suppose that R is decomposable; that implies $R = I \bigoplus J$ such that I and J are ideals in R. Now, if R is semi-neat, then $I \cong R/J$ is semi-clean. Similarly, J is semi-clean. Thus, R is a direct product of semi-clean rings. Consequently, by Theorem 2.5 in [10], R is semi-clean. Conversely, this is shown by Proposition 3.1.

Theorem 3.5. If D is a domain of dimension equal to 1, then D is semi-neat. In particular, the principal ideal domain PID is semi-neat.

Proof. By using Proposition 2.4 in [1] and Proposition 3.1, the result holds. In addition, every proper homomorphic image of D has dimension equal to zero. Also, a zero-dimensional ring is semi-clean (clean).

Corollary 3.6. Any Euclidean domain is a semi-neat ring. In particular, the set of all Gaussian integers $\mathbb{Z}[i]$ is a semi-neat ring.

Proof. Clear since a Euclidean domain is a PID. Now, using the previous theorem.

If the finite direct product of rings is a semi-neat ring, then each factor is a semi-neat ring. However, unless each factor ring is a semi-clean ring, the converse is not true. The next theorem will make this proposition very clear.

Theorem 3.7. A finite direct product of rings is a semi-neat ring if and only if each factor is a semi-clean ring.

Proof. Assume that $R = \prod_{i\in\mathbb{N}} R_i = R_1 \times R_2 \times \ldots \times R_N$ is a semi-neat ring. Define the ring epimorphism $\Phi_i: \prod_{i\in\mathbb{N}} R_i \rightarrow R_i$, where $\Phi_i((r_i))_{i\in\mathbb{N}}$. Clearly, each R_i is a homomorphic image of a semi-neat ring. Thus, each R_i is a semi-neat (semi-clean) ring, Proposition 3.1. Conversely, if R_i is a semi-clean ring for each $i \in \mathbb{N}$. By using Theorem 2.5 in [10], hence R is a semi-clean (semi-neat) ring.

Proposition 3.8. The ring $R[[x]]$ is semi-neat if and only if R is semi-clean.

Proof. Suppose $R[[x]]$ be a semi-neat ring. Hence, every proper homomorphic image of $R[[x\,]]$ is a semi-clean ring. Thus, $R \cong R[[x]]/\langle x\rangle$ is a semi-clean. Also, R is a semi-neat. Conversely, if R is a semi-clean ring. By using Proposition 3.3 in [6], $R[[x]]$ is a semi-clean ring. So, $R[[x]]$is a semi-neat ring.

Example 3.9.

1) Let $R = (\mathbb{Z}, +,\cdot)$ is a semi-neat (neat), since $\mathbb{Z}$ is a nonlocal *PID*. Also, R is not semiclean since 3 or 4 cannot be written as a sum of a periodic and a unit where $Per(\mathbb{Z}) = \{-1,0,1\}$ and $U(\mathbb{Z}) = \{1, -1\}$. In addition, $N(\mathbb{Z}) = \{0\}$ application Proposition 3.3.

2) The polynomial ring $\mathbb{Q}[x]$ is a semi-neat ring, which is not a semiclean, then $\mathbb{Q}[x]$ is a reduced ring by Proposition 3.3, so that $N(\mathbb{Q}[x]) = \big\{f(x) \in \mathbb{Q}[x]: \big(f(x)\big)^n = 0, for\ some\ n > 0\big\} = \{0\}$. In general, the polynomial ring over field $F[x]$ is semi-neat (neat) but is never

semiclean (clean).

3) The polynomial ring $\mathbb{Z}[x]$ is not a semi-neat ring because $\mathbb{Z}[x]/\langle x\rangle \cong \mathbb{Z}$ is a proper homomorphic image of $\mathbb{Z}[x]$ and $\mathbb{Z}$ is not a semiclean ring. Two points should be mentioned here: The first one, $\mathbb{Z}[x]$ is a unique factorization domain *UFD*. The second one, $\mathbb{Z}[x]$ is a greatest common divisor *GCD*-domain, which is not a *Bézout* domain. This example explains that not all *UFD* and *GCD* are semi-neat rings.

4) $R = F[x, y]$ is a *UFD*, where F is a field and x and y be indeterminates. Then R is never semi-neat since $R/xR \cong F[y]$ is never semiclean.

The polynomial ring $R[x]$ is never semiclean for any commutative ring R; this result is clear from Example 3.2 in [6], since $U(R[x]) = \{r_0 + r_1x + \cdots + r_nx^n \mid r_0 \in U(R), r_i \in N(R)\ for\ i = 1, \dots, n\}$, $Per(R[x]) = Per(R)$. Thus, R is a reduced ring. Based on Proposition 3.3 and previous example (2) leads us to this remark.

Remark 3.10. The polynomials ring $R[x]$ over a semi-clean ring R are semi-neat which $N(R[x]) = 0$; $R[x]$ is a reduced ring.

Example 3.11. This example shows the existence of semi-neat rings that are not neat rings. The group ring $\mathbb{Z}_p[G]$ is semi-neat where p is any prime integer and G is a cyclic group of order 3. Further, by utilizing Corollary 2.8 in [18] for $p \equiv 1\ modulo\ 3$, $\mathbb{Z}_p[G]$ is not clean. By Theorem 3.1 in [6], $\mathbb{Z}_p[G]$ is semiclean; hence, it is semi-neat. Although $\mathbb{Z}_p[G]$ is not neat with $p \neq 3$; see Example 3.5 in [13]. For instance, in $\mathbb{Z}_{(7)}C_3$, we obtain the next Peirce decomposition: $\mathbb{Z}_{(7)}C_3 \cong \mathbb{Z}_{(7)}C_3e \oplus \mathbb{Z}_{(7)}C_3f$, where $e = \frac{1}{3}(1 + a + a^2)$ and $f = \frac{1}{3}(2 - a - a^2)$ are the only nontrivial idempotents of $\mathbb{Z}_{(7)}C_3$ by Proposition 3.1 in [6] such that a generates C_3. Now, let $\alpha = \frac{1}{3}(1 + 2a - 3a^2)$; hence, $\alpha \in \mathbb{Z}_{(7)}C_3f$ can't be written as the sum of a unit and an idempotent. Therefore, α is not clean in $\mathbb{Z}_{(7)}C_3f$. Consequently, $\mathbb{Z}_{(7)}C_3f \cong \mathbb{Z}_{(7)}C_3 / \mathbb{Z}_{(7)}C_3e$ is not clean. Thus, $\mathbb{Z}_{(7)}C_3$ is never neat, but it is semi-neat.

4. Classify semi-neat FGC rings

In order to explain how to classify semi-neat FGC rings, we mention some lemmas and theorems. We say that R is a *PM* ring if every prime ideal is contained in a unique maximal ideal. However, every non-zero prime ideal is contained in a unique maximal ideal called PM^*. Let $V(I)$ represent the set of maximum ideals of R that contain I, where I is an ideal of R. Let D be an integral domain; it is said to be $h-local$ if it is a PM^* and for any non-zero ideal I, $V(I)$ is finite.

A ring R satisfies the following properties: (i) R has at least two maximal ideals; (ii) R has a unique minimal prime ideal P such that $P \neq 0$ and also P is uniserial as an R-module; (iii) R/P is an $h-local$ domain; (iv) R is a locally almost maximal *Bézout* ring. It is referred to as a *torch* ring [15]. Moreover, McGovern, in [1], proved a *torch* ring is never a neat ring, but in this paper we prove a *torch* ring is a semi-neat.

Theorem 4.1. (Brandal [15, Theorem 9.1; page 64]) A ring R is an FGC if and only if R is a finite direct product of the following classes of rings: Maximal valuation rings, almost-maximal *Bézout* domains, and *torch* rings.

Corollary 4.2. A maximal ring is semi-clean. Furthermore, an almost maximal ring is semi-neat.

Proof. Clear by Theorem Zelinsky: If the ring R is maximal, then R is a finite direct product of local rings. Thus, R is a clean ring. Consequently, R is a semi-clean ring and also it is semi-neat. Since R is maximal, that means R is a linearly compact R-module. In addition, R/I is a linearly compact R-module because a linearly compact R-module is closed under a homomorphic image.

Therefore, if R is an almost maximal ring, R is neat; it is also semi-neat.

Lemma 4.3. (Brandal [15, Lemma 2.4]) For I is an ideal of R. If $V(I)$ is finite, then the factor R/I is a direct sum of indecomposable modules as the form R/J such that $I \leq J$.

Lemma 4.4. (Brandal [15, Proposition 2.5]) Assume that $V(I)$ is finite for any ideal I in R. Then R/I is indecomposable if and only if for all nontrivial partitions $\mathcal{V}_1, \mathcal{V}_2$ of $V(I)$ there are $M_1 \in \mathcal{V}_1$, $M_2 \in \mathcal{V}_2$ and a prime ideal P of R such that $I \subseteq P \subseteq M_1 \cap M_2$.

Lemma 4.5. (McGovern [1, Proposition 3.13]) Assume that R/I is a PM-ring and that $V(I)$ is finite. In such a case, R/I is a finite direct product of local rings

Theorem 4.6. An $h-local$ domain is a semi-neat. Furthermore, FGC-domain is semi-neat.

Proof. Let D be an $h-local$ domain or FGC-domain, then D is neat [1]; Theorem 3.15 and Corollary 3.9. Consequently, D is semi-neat by Proposition 3.1.

We know that an integral domain is clean if and only if it is local. But if it is semi-clean and not clean, this does not necessary hold since every element in a semi-clean domain is either a unit or sum of two units [7]. Thus, it has at least two maximal ideal. This is demonstrated in the following example.

Example 4.7. For $R = \mathbb{Z}_3 \cap \mathbb{Z}_5 = \left\{ \frac{\mathrm{m}}{n} \in \mathrm{Q} : 3 \nmid \mathrm{n}, 5 \nmid \mathrm{n} \right\}$. To clarify further $2 \in U(R)$ also, the only maximal ideals of R are M_1 and M_2, such that $M_1 = \left\{ \frac{\mathrm{m}}{n} \in \mathrm{Q} : \mathrm{m} \in \langle 3 \rangle, 3 \nmid \mathrm{n} \text{ and } 5 \nmid \mathrm{n} \right\}$ and $M_2 = \left\{ \frac{\mathrm{m}}{n} \in \mathrm{Q} : \mathrm{m} \in \langle 5 \rangle, 3 \nmid \mathrm{n} \text{ and } 5 \nmid \mathrm{n} \right\}$. By using Theorem 16 in [2], R is a weakly clean (semi-clean) ring, but R is not a clean ring. Since $R \subset \mathrm{Q}$, hence R is indecomposable but not local.

Theorem 4.8. A Torch ring is semi-neat but not neat.

Proof. Let R be a torch ring, so R is a nonlocal ring such that a unique minimal prime ideal P is a non-zero ideal and uniserial R-module. If R is a neat ring, hence R/P is a domain and a homomorphic image of R, so it is an indecomposable clean ring. Thus, by Theorem 3 in [2], R is a local ring; this is a contradiction by definition of a torch ring. Consequently, R is a never-neat ring. Now, $N(R) \neq 0$ is a unique minimal prime ideal such that the intersection of distinct minimal prime ideals can never be a prime ideal because if $p_1 \cap \ldots \cap p_n = p$ then $p_1 \ldots p_n \subset p$ so $p_i \subset p$ for some i. This means $p = p_i$. Since $R/N(R)$ is a $h-$local domain, defined $S := R/N(R), J(S) \neq 0$, therefore $S/J(S)$ is a proper homomorphic image of $h-$local domain. Consequently, $S/J(S)$ is PM^* and $V(J(S))$ is finite. By Lemma 4.3, $S/J(S)$ is a direct sum of indecomposable modules of the form S/I such that $J(S) \leq I$. By Lemma 4.4, for all nontrivial partitions $\mathcal{V}_1, \mathcal{V}_2$ of $V(I)$, there are two maximal ideals $M_1 \in \mathcal{V}_1$, $M_2 \in \mathcal{V}_2$ and a prime ideal P of R such that $I \subseteq P \subseteq M_1 \cap M_2$. This means P belongs to both of them. This is a contradiction, since $S/J(S)$ is a PM^* ring. Then, by Lemma 4.5, $S/J(S)$ is a finite direct product of local rings, so $S/J(S)$ is a clean ring. Thus, S is semi-clean using Theorem 3.13 in [19]. Therefore, $R/N(R)$ is semi-clean. Consequently, R is a semi-neat. To clarify more, if R is a torch ring, then R is a locally almost maximal Bézout ring. Thus, R_M is neat for all maximal ideals M of R.

Assume R is a FGC ring. R is clean if and only if R is a finite direct product of local rings, that means R is a finite direct product of almost maximal valuation rings. Moreover, R is neat but not clean if and only if R is an almost maximal *Bézout* domain (not local). Thus, a FGC domain is neat [1]. At this point, a natural question is whether FGC rings are semi-neat. We know FGC-domain is semi-neat by using Theorem 4.6. According to Theorem 4.1, that is obvious any FGC ring R is a finite direct product of maximal valuation rings, almost maximal Bézout domains, and torch rings. The first class is clean, the second class is neat, but the third class is semi-neat never neat. We know that consider R be a reduced ring. Then R is an FGC ring if and only if R is a direct product of finitely many almost-maximal *Bézout* domains by Theorem 8 in [20]. According the information above we demonstrate that, if R is a torch ring (semi-neat) under certain conditions

$N(R) \neq 0$, that means R is a not reduced ring, hence R is semi-clean.
Theorem 4.9. Suppose R is an FGC-ring. R is a semi-neat ring purely (not neat) if and only if R is a torch ring.
Proof. It suffices to show that if R is semi-neat but not neat then it is a torch ring. Assume R is a semi-neat FGC ring but not neat. Write $R = R_1 \times \cdots \times R_n$, where each R_i is one of the adequate classes of rings from Theorem 4.1. Since R is semi-neat each R_i is semi-clean but not clean. By Theorem 4.1, it follows that none of the R_i is neither a maximal valuation ring nor an almost maximal Bézout domain and hence $each\, R_i$ is torch rings. Thus, R is a finite direct product of semi-clean (not clean), we obtain that R is a torch ring.
Corollary 4.10. An FGC-ring is semi-neat.
Proof. Any FGC-ring R is obviously a finite direct product of maximal valuation rings, almost maximal Bézout domains, and torch rings, as stated in Theorem 4.1. First class is clean; second class is neat; third class is never neat but is semi-neat. By Proposition 3.1, Theorems 4.6 and 4.9 are achieved. It must be noted here that when FGC is of the third class, each factor is semi-neat (Theorem 4.8). Also, they are semi-clean due to these factors which are torch rings $N(R_i) \neq 0$. Thus, R is semi-neat by Theorem 3.7.

5. Weakly semi-neat rings

Definition 5.1. A ring R is called a weakly neat if every proper homomorphic image is weakly clean.

We recognize that every weakly clean is a semi-clean [10]. Thus, every weakly clean ring, semi-clean ring, or weakly neat ring is a semi-neat ring. Now we give some examples to show the existence of semi-neat rings, not weakly neat rings.
Example 5.2. For $R = \mathbb{Z}_3 \cap \mathbb{Z}_5 = \left\{ \frac{\mathrm{m}}{n} \in \mathrm{Q} : 3 \nmid \mathrm{n}, 5 \nmid \mathrm{n} \right\}$. Since $\left(\frac{3}{2}, \frac{5}{2}\right) \in R \times R$ is not weakly clean. But, subtracting $(1, -1)$ from this gives a unit. By using Theorem 3.7, $R \times R$ is a semi-neat ring. To clarify further $2 \in U(R)$ also, the only maximal ideals of R are M_1 and M_2 such that $M_1 = \left\{ \frac{\mathrm{m}}{n} \in \mathrm{Q} : \mathrm{m} \in \langle 3 \rangle, 3 \nmid \mathrm{n} \text{ and } 5 \nmid \mathrm{n} \right\}$ and $M_2 = \left\{ \frac{\mathrm{m}}{n} \in \mathrm{Q} : \mathrm{m} \in \langle 5 \rangle, 3 \nmid \mathrm{n} \text{ and } 5 \nmid \mathrm{n} \right\}$. Using Theorem 16 in [2], R is a weakly clean ring but R is not a clean ring since $R \subset \mathrm{Q}$, then R is indecomposable but not local. Now, $R \times R$ is a direct product of two weakly clean rings, hence $R \times R$ is semi-neat (semi-clean). However, $R \times R$ is not weakly clean since R is not clean by Theorem 1.7 in [3] at most one R_i is not clean not both of them. Similarly, for $S = \mathbb{Z}_5 \cap \mathbb{Z}_7 = \left\{ \frac{\mathrm{m}}{n} \in \mathrm{Q} : 5 \nmid \mathrm{n}, 7 \nmid \mathrm{n} \right\}$ since $\left(\frac{5}{2}, \frac{7}{2}\right) \in S \times S$ is a semi-neat ring but it's not weakly clean. Also, subtracting $(1, -1)$ from this gives a unit. Moreover, let $T = R \times S$ is a semi-neat ring but it's not weakly clean. In general, the ring $\mathbb{Z}_p \cap \mathbb{Z}_q = \left\{ \frac{\mathrm{m}}{n} \in \mathrm{Q} : \mathrm{p} \nmid \mathrm{n}, \mathrm{q} \nmid \mathrm{n} \right\}$ is not clean for p, q distinct primes. As an illustration, the element $p/(p - q)$ is not clean. However, these rings are always weakly clean. Let's say that m/n is not weakly clean. In that case, every m/n and $(m/n) \pm 1$ is a non-unit. In other words, m, $m - n$ and $m + n$ are all multiples of p or q. If p divides two of them, then there is a contradiction: p divides n.
Example 5.3. Define $R = R_1 \times R_2$ is a direct product of two weakly clean rings, so R_1 and R_2 are semi-clean, hence R is semi-clean (semi-neat) since the class of semi-clean rings is closed under products. Furthermore, only if at most one of R_1 or R_2 is not clean will R be considered weakly clean. In other words, only if at least one of R_1 or R_2 is clean will R be considered weakly clean.
This is general example of a semi-neat ring not weakly neat ring. We provide the following

proposition in order to clarify the idea of why it could not be weakly neat.

Proposition 5.4. For R decomposable as $R = I \oplus J$ such that I or J is clean. Then R is weakly neat if and only if R is weakly clean.

Proof. Suppose that R is decomposable, that implies $R = I \oplus J$ such that I and J are ideals in R. $(\Rightarrow)$. If R is weakly neat then $I \cong R/J$ is weakly clean. Similarly, J is weakly clean. Hence, R is a direct sum of weakly clean rings also I or J is clean. Consequently, by using Theorem 1.7 in [3], R is weakly clean. $(\Leftarrow)$. Obvious because the weakly clean rings are closed homomorphic image. Thus, each weakly clean is weakly neat and hence semi-neat.

From the preceding, we can define a weakly semi-clean ring R if each $x \in R$ can be written as $x = u + p$ or $x = u - p$ where $u \in U(R)$ and $p \in Per(R)$ specially l is even and m is odd and vice visa. Since if l and m are both even or odd then $-p = p$; $(-p)^l = (-1)^l p^l = p^m = (-p)^m$. Based on the investigation of the closure under homomorphic image for these classes, that leads us to a definition of a weakly semi-neat ring, which is similar to definition of weakly neat, that is: Every nontrivial homomorphic image is weakly semi-clean. Now, we discuss the basic properties of weakly semi-neat rings, such that the proofs of these properties are also the proofs of the properties of semi-neat rings.

Theorem 5.5. Let R be a ring. Then, the following statements are valid:

i) Every clean ring, weakly clean ring, semi-clean ring, weakly semi-clean ring, neat ring, semi-neat ring and weakly neat ring is weakly semi-neat.

ii) A weakly semi-neat ring is closed under homomorphic image.

The following chain explains the relations between previous rings:

$$\textbf{weakly neat} \subseteq \textbf{semineat} \subseteq \textbf{weakly semineat}.$$

Example 5.6. $\mathbb{Z}_{(p)}[\sqrt{d}\,] = \{a_0 + a_1\sqrt{d} \mid a_0, a_1 \in \mathbb{Z}_{(p)}\}$ for prime number p and a square-free integer d. These rings can be represented as $\frac{k_1 + k_2\sqrt{d}}{m}$, k_1, k_2, $m \in \mathbb{Z}$, $p \nmid m$. The $\frac{k_1 + k_2\sqrt{d}}{m}$ in $\mathbb{Z}_{(p)}[\sqrt{d}\,]$ is a unit if and only if $p \nmid {k_1}^2 - d{k_2}^2$. Let $R = \mathbb{Z}_{(p)}[\sqrt{d}\,]$ where p prime is weakly neat and semi-neat (need not be clean), as stated in Theorem 3.5 in [10]. These are subrings of $\mathbb{R}$. The ring $(\mathbb{Z}_3 \cap \mathbb{Z}_5)[i]$ is a semi-clean (semi-neat ring). But it is not a weakly clean ring. The ring $(\mathbb{Z}_2 \cap \mathbb{Z}_p)[i]$ for an odd prime p, is not semi-clean but is semi-neat.

Proposition 5.7. A finite direct product of rings is a weakly semi-neat ring if and only if each factor is a weakly semi-clean ring.

Proof. Based on this property, the class of weakly semi-clean rings is closed under finite products.

Remark 5.8. One limitation of this study is that we have been unable to construct an example to show the existence of weakly semi-neat rings, not weakly semi-clean rings.

6. Some extensions of semi-neat and weakly semi-neat rings

Ye's Theorem 4.1 in [6] established the following: If e is an idempotent element in a ring R such that $(1-e)R(1-e)$ and eRe are both semiclean rings, then R is also semiclean. This result can be extended to semi-neat rings.

Theorem 6.1. Let R be a ring and $e \in Id(R)$. If the corner rings eRe and $(1-e)R(1-e)$ are semi-clean, then R is a semi-neat ring.

Proof. Directly use Theorem 4.1 in [6] and Proposition 3.1.

Theorem 6.2. If $1 = e_1 + e_2 + \cdots + e_n$ in a ring R where the e_i are orthogonal idempotent and each e_iRe_i is semi-clean, then R is semi-neat.

Proof. Applying the Theorem 6.1 with an inductive logic, we get this result. If $n = 2$ and $1 = e_1 + e_2$, hence $e_2 = 1 - e_1$. Since e_1Re_1 and e_2Re_2 are semi-clean. By previous theorem R is a semi-neat ring. Now, if $n = 3$ and $1 = e_1 + e_2 + e_3$, then $e_3 = 1 - (e_1 + e_2)$. Clearly, $f = (e_1 + e_2) \in Id(R)$, hence fRf is semi-clean by given. Thus, R is semi-neat. Follow the same process to get the result.

This theorem directly leads to the following three results.

Corollary 6.3. The matrix ring $M_n(R)$ is semi-neat over semiclean ring R.

Proof. Let R be a semiclean ring and I_i be a matrix with rank $n \times n$ for each $i = 1,2, \dots, n$ such that $x_{ii} = 1$, otherwise $x_{ij} = 0$. It is clear that $I = I_1 + I_2 + \cdots + I_n$ and also for each i, I_i are orthogonal idempotents of $M_n(R)$. Moreover, the ring $I_i(M_n(R))I_i \cong R, \forall i$. Given that R is semiclean, then $I_i(M_n(R))I_i, \forall i$. By previous theorem, the result holds.

Corollary 6.4. If $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ are modules and $End\ (M_i)$ is semiclean for each i, then $End\ (M)$ is semi-neat.

Theorem 6.2 and property closeness under homomorphic image for semi-neat (semi-clean) rings.

Corollary 6.5. If A and B are rings and $M = {}_BM_A$ is a bimodule. The formal triangular matrix ring $T = \begin{bmatrix} A & 0 \\ M & B \end{bmatrix}$ is semi-neat if and only if both A and B are semiclean.

In particular, for each $n \geq 2$, a ring R is considered semiclean if and only if the ring T of $n \times n$ upper (lower) triangular matrices over R is semi-neat. By using Proposition 4.1 in [6].

It is observed in that the the corner rings for a ring R are eRe and $(1 - e)R(1 - e)$ are clean (semi-clean), so R is clean (semi-clean). However, the converse is not true in general, see ([21, Example 3.1]). Furthermore, the converse is true for the class of weakly clean rings. Weakly clean rings are known to include clean rings as a proper subclass and to be a subclass of semi-clean rings, see ([22, proposition 2.5]). It is essential to keep in mind that R in this theorem is not necessarily commutative. Now, we are going to reflect this result on semi-neat rings and weakly semi-neat rings.

Theorem 6.6. Let R be a ring and $e \in Id(R)$. If R is weakly clean, then the corner rings eRe and $(1 - e)R(1 - e)$ are semi-neat (weakly semi-neat).

Proof. Let R be a weakly clean ring and $e \in Id(R)$. Then eRe and $(1 - e)R(1 - e)$ are weakly clean by Proposition 2.5 in [22]. Consequently, eRe and $(1 - e)R(1 - e)$ are weakly neat, semi-neat and weakly semi-neat by using Theorem 5.5.

The decomposition of R-modules is closely related to the idempotent of R. As right modules, $R = I \oplus J$ if and only if there is a unique idempotent e where $(1 - e)R = J$ and $eR = I$. Consequently, every direct summand of R is generated by an idempotent. An idempotent e in R is called a central idempotent if $er = re$ for all r in R. Also, an idempotent e in a ring R with unity is central if and if $eR(1 - e) = (1 - e)Re = 0$. In addition, a ring whose all idempotents are central is called an abelian ring. Those rings need not be commutative. If e is a central idempotent then the corner ring $eRe = Re = eR$ is a ring with multiplicative identity e. The direct decompositions of R as a module are determined by idempotents, similarly the central idempotent of R determine the decompositions of R as a direct sum of rings. Specifically, for each central idempotent e in R leads to a decomposition of R as a direct sum of the corner rings eRe and $(1 - e)R(1 - e)$. Now, we introduce some special cases in which the convers of the Theorem 6.1 is valid.

Corollary 6.7. For R a commutative decomposable ring with unity 1 and $e \in Id(R)$. If R is weakly neat, then the corner rings eRe and $(1 - e)R(1 - e)$ are semi-neat (weakly semi-neat).

Proof. Let R be a commutative decomposable weakly neat ring and $e \in Id(R)$. Then R is a semi-neat (weakly semi-neat) by Theorem 5.5. Every idempotent in a commutative ring R is central. Thus, the direct decompositions of R as a module are determined by idempotents, eRe and $(1-e)R(1-e)$ are semi-clean and semi-neat (weakly semi-neat).

Corollary 6.8. A ring R in which all idempotent are central and $e \in Id(R)$. If R is semi-neat, then the corner rings eRe and $(1-e)R(1-e)$ are semi-clean.

Proof. For each central idempotent e in R leads to a decomposition of R as a direct sum of the corner rings eRe and $(1-e)R(1-e)$.

7. Discussion

The introduction of semi-neat rings in this study represents a significant step forward in the generalization of ring theory. Building on the foundational work on neat and semiclean rings, the concept of semi-neat rings addresses the limitations of neat rings by including rings whose nontrivial homomorphic images exhibit semiclean properties. This extension not only broadens the applicability of ring theory but also facilitates a more comprehensive understanding of the relationships between existing ring classes. The study demonstrates that semi-neat rings retain essential closure properties under homomorphic images and direct products, aligning them with the well-established behaviors of clean and semiclean rings. Furthermore, the extension to weakly semi-neat rings introduces a nuanced generalization that links weakly neat, semi-clean, and semi-neat rings, enriching the landscape of generalized ring theory.

The classification of semi-neat FGC rings and corner rings further highlight the versatility of semi-neat rings. The study's detailed analysis of examples, such as torch rings, which are semi-neat but not neat, reveals the broader scope of this new class compared to traditional neat rings. Moreover, the study addresses the distinctions between semi-neat and weakly semi-neat rings, presenting examples that illustrate the divergence between these classes. The potential for further research into weakly semi-neat rings, particularly in identifying unique cases where weakly neat properties differ from semiclean behaviors, opens new avenues for exploration. By establishing these foundational principles, the study paves the way for future advancements in the theory and applications of generalized ring classes, with implications for both pure and applied algebra.

8. Limitations and future research directions

This study introduces and examines the concepts of semi-neat and weakly semi-neat rings, primarily within the framework of commutative algebra. A key limitation lies in this focus, which restricts the generalizability of results to non-commutative settings; although some results extend to structures like triangular matrix rings, a full non-commutative treatment remains undeveloped. Another limitation is the lack of explicit counterexamples distinguishing weakly semi-neat rings from weakly semi-clean ones, leaving some theoretical distinctions unverified.

Future research should extend these investigations to non-commutative and non-unital rings, as well as to ring extensions such as power series and skew polynomial rings, to assess structural robustness and closure properties. Additionally, potential applications in coding theory, module decomposition, and fuzzy theory. The generalizations introduced in this article, particularly semi-neat and weakly semi-neat rings, align with ongoing trends in algebraic research that explore structural flexibility and generalized operations as core themes that are also present in fuzzy algebraic systems and uncertainty modeling. For example, studies such as those on k-folded N-structures in semigroups and crossing cubic Lie algebras delve into algebraic behavior under relaxed, multi-valued, or non-deterministic operations, often to model complex systems or imprecise information. Thus, while this study is grounded in classical ring theory, the structural generalizations it proposes may serve as a bridge to fuzzy or uncertain algebraic frameworks, warranting further exploration [23,24].

9. Conclusions

This study introduces and explores the concepts of semi-neat and weakly semi-neat rings, extending the theoretical framework of neat and semiclean rings. By defining semi-neat rings as those whose nontrivial homomorphic images are semiclean, we bridge the gap between neat rings and broader ring structures, capturing a wider range of algebraic properties. The findings demonstrate that semi-neat rings inherit essential closure properties under homomorphic images and direct products, providing a solid foundation for their integration into existing ring theory. The classification of semi-neat FGC rings, the analysis of corner rings, and the exploration of their relationship with other ring classes underscore the versatility and significance of these new concepts. Notable examples, such as torch rings, highlight the distinct characteristics of semi-neat rings compared to neat rings, while weakly semi-neat rings provide a nuanced extension that warrants further study. This research not only enriches the understanding of generalized ring properties but also opens new avenues for theoretical advancements and practical applications in algebra in related fields like module theory and coding theory.

Author contributions

Abdallah A. Abukeshek: Conceptualization, formal analysis, investigation, data curation, visualization, writing original draft and editing; Andrew Rajah: Proposal supervision, conceptualization, investigation, validation, review. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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