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Theory article

Nonexistence results of nonnegative solutions of elliptic equations and systems on the Heisenberg group

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Abstract: This paper establishes sharp nonexistence criteria for nonnegative solutions to a class of quasilinear elliptic inequalities and divergence-type systems in the subelliptic framework of the Heisenberg group H^n. By developing an optimized test function methodology adapted to the stratified Lie group structure, nonexistence is established through a contradiction argument based on maximum principle-type inequalities. The analysis contributes new insights into the role of sub-Riemannian geometry in constraining the solution behavior for degenerate elliptic operators.

Keywords: quasilinear elliptic inequalities; elliptic systems; Heisenberg group
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1. Introduction

The purpose of this article is to establish the nonexistence results of nontrivial solutions of elliptic inequalities and systems in the Heisenberg group.

In this paper, we study the nonexistence of entire nontrivial weak solutions of quasilinear elliptic inequalities of the following type:

$$L_{\mathcal{A}}u = -\mathrm{div}_H(\mathcal{A}(\xi, u, \nabla_H u)) \geq (K(\xi) * u^p)u^q,\ \xi \in H^n, \tag{1.1}$$

where $p, q > 0$, and the map $\mathcal{A} : H^n \times \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a Carathéodory function with $\mathcal{A}(\xi, z, 0) = 0, \mathcal{A}(\xi, z, \eta) \cdot \eta \geq 0$, for every $(\xi, z, \eta) \in H^n \times [0, \infty) \times \mathbb{R}^{2n}$. In addition, $L_{\mathcal{A}}$ is assumed to be a weakly $m-$coercive operator, that is, there exists a constant $c_0 > 0$ and an exponent $m > 1$, such that the inequality

$$\mathcal{A}(\xi, z, \eta) \cdot \eta \geq c_0|\mathcal{A}(\xi, z, \eta)|^{m'}, \tag{1.2}$$

holds, where $m' = \frac{m}{m-1}$. The function $K \in C(H^n \setminus \{0\})$, $K > 0$ satisfies $\lim_{\xi\to 0} K(\xi) > 0$, and there exists

$\rho > 0$ and $0 < \beta < \frac{m}{2}$ such that

$$K(\xi) \geq c|\xi|^{-\beta}, \tag{1.3}$$

for all $\xi \in H^n, |\xi| > \rho$, where $c > 0$ is a positive constant. The quantity $K * u^p$ is defined as a standard convolution operator on the Heisenberg group by the following:

$$K * u^p(\xi) = \int_{H^n} K(\eta^{-1} \circ \xi)u^p(\eta)d\eta, \text{ for all } \xi \in H^n.$$

In recent years, increasing attention has been given to the analysis and partial differential equations (PDEs) on the Heisenberg group. In particular, the classification of solutions to elliptic equations and systems has been extensively studied by mathematicians. For instance, B. Ahmad, A. Alsaedi and M. kirane [1] demonstrated the nonexistence of global solutions for a class of nonlocal spatial evolution equations on the Heisenberg group by constructing suitable test functions. M. Jleli, M. Kirane and B. Samet [2] extended such nonexistence results to a broader framework of evolution equations, revealing profound connections between nonlinear terms and the group structure. In the context of elliptic equations, I. Birindelli [3] established Liouville-type theorems for the Laplace operator on the Heisenberg group by deriving decay estimates for superharmonic functions. A. Kassymov and D. Suragan [4] analyzed nonlinear equations with Hardy potentials and investigated the multiplicity of positive solutions in relation to potential parameters using variational methods.

In an influential paper [5], Gidas and Spruck discovered that the semilinear equation

$$-\Delta u = u^p \text{ in } \mathbb{R}^n, n \geq 3,$$

has no C^2 positive solutions for $1 \leq p < \frac{n+2}{n-2}$ and the upper exponent $\frac{n+2}{n-2}$ is sharp.

Recently, T. Godoy [6] considered the positive solutions of nonpositone sublinear elliptic problems. The nonexistence results for solutions of inequalities of the type

$$\Delta_H(au) + |u|^p \leq 0,$$

in H^n, with $a \in L^\infty$ was studied in [7]. The quasilinear elliptic inequality

$$\text{div}(A(|\nabla u|)\nabla u) \geq f(u),$$

was discussed in [8, 9] in connection with the strong maximum principle and the compact support principle.

The work [10] dealt with noncoercive elliptic systems of quasilinear elliptic inequalities of the following type:

$$\begin{cases} -\text{div}(h_1(x)A(|\nabla u|\nabla u)) \geq f(x, u, v, \nabla u, \nabla v), & \text{in } \mathbb{R}^n, \\ -\text{div}(h_2(x)B(|\nabla v|\nabla v)) \geq g(x, u, v, \nabla u, \nabla v), & \text{in } \mathbb{R}^n. \end{cases} \tag{1.4}$$

To the best of our knowledge, the first results that dealt with quasilinear elliptic inequalities that featured nonlocal terms appeared in [11]. The authors in [11] obtained local estimates and Liouville type results for the following:

$$-\text{div}[\mathcal{A}(x, u, \nabla u)] \geq K * u^q \ \text{ in } \mathbb{R}^n,$$

where $K \geq 0$ and $q > 0$.

The study of functional inequalities and variational problems on the Heisenberg group has been a central topic in geometric analysis and nonlinear partial differential equations. A foundational contribution in this direction was made by Jerison and Lee [12], who established sharp extremals for the Sobolev inequality on the Heisenberg group and laid crucial groundwork for the CR Yamabe problem. Their work has inspired extensive research into related inequalities and variational frameworks in sub-Riemannian settings. For instance, J. Dou, P. Niu, and Z. Yuan [13] investigated a Hardy-type inequality with remainder terms and explored its applications to weighted eigenvalue problems on stratified Lie groups. Recent advances have focused on nonlinear elliptic problems with critical growth. In particular, X. Sun, Y. Song, and S. Liang [14] studied the critical Choquard-Kirchhoff problem on the Heisenberg group, addressing existence and multiplicity results under nonlocal-to-local transitions. Y. Hu [15] studied the Hardy-Littlewood-Sobolev inequalities with weights, which extended classical duality principles to the subelliptic context and uncovered new phenomena tied to the stratified structure.

There are many interesting results about sub-Laplacian and $p-$ sub-Laplacian equations on the Heisenberg group (see [16–18]). In particular, in [19], the authors studied the Kirchhoff elliptic, parabolic, and hyperbolic-type equations on the Heisenberg group. In addition, the analogous results were transferred to the cases of systems.

Later, Y. Zheng [20] established Liouville theorems for the following system of differential inequalities:

$$\begin{cases} \Delta_H u^{m_1} + |\eta|_H^{\gamma_1} |v|^p \leq 0, \\ \Delta_H v^{m_2} + |\eta|_H^{\gamma_2} |u|^q \leq 0, \end{cases} \tag{1.5}$$

on different unbounded open domains of the Heisenberg group H^n, including the whole space, and the half space of H^n.

Our work is motivated by a recent paper [21], where the authors proved the existence and nonexistence of positive solutions for the following quasilinear elliptic inequality:

$$-\text{div}[\mathcal{A}(x, u, \nabla u)] \geq (I_\alpha * u^p) u^q, \text{ in } \Omega,$$

where $\Omega \subset \mathbb{R}^n, n \geq 1$, is an open set. I_α stands for the Riesz potential of order $\alpha \in (0, n), p > 0$, and $q \in \mathbb{R}$.

The paper is organized as follows: In Section 2, we review some basic notations on the Heisenberg group; and in Section 3, we give the proof of our main results.

Throughout the paper, we denote some positive constants by $C, C_1, C_2 \cdots$, which may vary from line to line.

2. Preliminaries

Before describing our results, we need to recall some notions on the Heisenberg group, see also [22–24].

The $2n+1$-dimensional Heisenberg group H^n is $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ endowed with group law "$\circ$"

$$\hat{\xi} \circ \xi := \left(x + \hat{x}, y + \hat{y}, t + \hat{t} + 2 \sum_{i=1}^{n} x_i \hat{y}_i - y_i \hat{x}_i \right),$$

for any $\xi = (x, y, t), \hat{\xi} = (\hat{x}, \hat{y}, \hat{t})$ in H^n, with $x = (x_1, \cdots, x_n)$, $\hat{x} = (\hat{x}_1, \cdots, \hat{x}_n)$, $y = (y_1, \cdots, y_n)$, and $\hat{y} = (\hat{y}_1, \cdots, \hat{y}_n)$ denoting the elements of $\mathbb{R}^n$.

The $2n+1-$dimension Heisenberg algebra is the Lie algebra spanned by the following left-invariant vector fields:

$$\begin{aligned} X_i &:= \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad i = 1, \cdots, n, \\ Y_i &:= \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \quad i = 1, \cdots, n, \\ T &:= \frac{\partial}{\partial t}. \end{aligned} \tag{2.1}$$

Then, the Heisenberg gradient, or the horizontal gradient of a regular function u, is defined by the following:

$$\nabla_H u := (X_1 u, \cdots, X_n u, Y_1 u, \cdots, Y_n u).$$

Given the real number $\lambda > 0$, a natural dilation on the Heisenberg group is given by the following:

$$\delta_\lambda(\xi) = (\lambda x, \lambda y, \lambda^2 t), \tag{2.2}$$

whose Jacobian determinant is λ^Q, where $Q = 2n + 2$ stands for the homogeneous dimension of H^n. The norm on H^n is defined by

$$|\xi|_H := \rho(\xi) = \left[\left(\sum_{i=1}^{n} x_i^2 + y_i^2 \right)^2 + t^2 \right]^{\frac{1}{4}}, \tag{2.3}$$

and the associated Heisenberg distance

$$d_H(\xi, \hat{\xi}) = \rho(\hat{\xi}^{-1} \circ \xi),$$

where $\hat{\xi}^{-1}$ is the inverse of $\hat{\xi}$ with respect to "$\circ$" (i.e., $\hat{\xi}^{-1} = -\hat{\xi}$).

The open ball of radius R centered at ξ is the following set:

$$D_R(\xi) := \{\eta \in H^n | d_H(\xi, \eta) < R\}.$$

It important to note that

$$|D_R(\xi)| = |D_R(0)| = |D_1(0)| R^Q,$$

where $|D_1(0)|$ is the volume of the unit Heisenberg ball under a Haar measure, which is equivalent to the $2n + 1$ dimensional Lebesgue measure of $\mathbb{R}^{2n+1}$.

Define

$$W^{1,1}_{loc}(\Omega) = \{u : \Omega \to \mathbb{R} | u, \nabla_H u \in L^{1,1}_{loc}(\Omega)\},$$

and let $W^{1,1}_c(\Omega)$ be the subspace of $W^{1,1}_{loc}(\Omega)$ of functions with a compact support.

3. Main results

Let us formulate a definition of the weak solution of Eq (1.1).

Definition 3.1. *A locally integrable nonnegative function $u \in C(\Omega) \cap W^{1,m}_{loc}(\Omega)$ is called a entire weak solution of (1.1) in $\Omega = H^n$, if*
(i) $u > 0$, $\mathcal{A}(\xi, u, \nabla_H u) \in L^{m'}_{loc}(\Omega)^{2n}$;
(ii) $(K * u^p)u^q \in L^1_{loc}(\Omega)$;
(iii) For any $\psi \in C^\infty_c(\Omega), \psi \geq 0$, we have

$$\int_\Omega \mathcal{A}(\xi, u, \nabla_H u) \cdot \nabla_H \psi d\xi \geq \int_\Omega (K * u^p)u^q \psi d\xi. \tag{3.1}$$

Our main result that concerns (1.1) is as follows.

Theorem 3.2. *Assume (1.2) and (1.3), then, every entire weak solution of (1.1) is necessarily constant when*

$$2\max\{1, m-1\} < p + q < \frac{2(m-1)\beta}{Q - m + \beta}. \tag{3.2}$$

We start with the following result, which provides an extra local integrability of u.

Lemma 3.3. *Let u be a nonnegative solution of (1.1), then $u^{\frac{p+q}{2}} \in L^1_{loc}(H^n)$.*

Proof. Let $R > \rho$ be large enough for $\xi \in D_R(0)$, additionally, using (1.3), we obtain the following:

$$\begin{aligned}
(K * u^p)(\xi) &\geq \int_{D_R(0)} K(\eta^{-1} \circ \xi)u^p(\eta)d\eta \\
&= \int_{|\eta^{-1}\circ\xi| \leq \rho} K(\eta^{-1} \circ \xi)u^p(\eta)d\eta \\
&\quad + \int_{|\eta^{-1}\circ\xi| > \rho, |\eta| < R, |\xi| < R} K(\eta^{-1} \circ \xi)u^p(\eta)d\eta \\
&\geq \inf_{z \in D_1(0)} K(z) \int_{|\eta^{-1}\circ\xi| \leq \rho} u^p(\eta)d\eta \\
&\quad + c\int_{|\eta^{-1}\circ\xi| > \rho, |\eta| < R, |\xi| < R} |\eta^{-1} \circ \xi|^{-\beta} u^p(\eta)d\eta \\
&\geq \inf_{z \in D_1(0)} K(z) \int_{|\eta^{-1}\circ\xi| \leq \rho} u^p(\eta)d\eta \\
&\quad + c(2R)^{-\beta} \int_{|\eta^{-1}\circ\xi| > \rho, |\eta| < R} u^p(\eta)d\eta \\
&\geq CR^{-\beta}\left[\int_{|\eta^{-1}\circ\xi| \leq \rho, |\eta| < R} u^p(\eta)d\eta + \int_{|\eta^{-1}\circ\xi| > \rho, |\eta| < R} u^p(\eta)d\eta\right] \\
&\geq CR^{-\beta} \int_{D_R(0)} u^p(\eta)d\eta.
\end{aligned} \tag{3.3}$$

Consequently, using (3.3) and the Hölder inequality, we deduce the following:

$$\infty > \int_{D_R(0)} (K * u^p(\xi))u^q(\xi)d\xi \geq CR^{-\beta}\left(\int_{D_R(0)} u^p(\xi)d\xi\right)\left(\int_{D_R(0)} u^q(\xi)d\xi\right)$$
$$\geq CR^{-\beta}\left(\int_{D_R(0)} u^{\frac{p+q}{2}}(\xi)d\xi\right)^2,$$

which shows $u^{\frac{p+q}{2}} \in L^1(D_R(0))$. This completes the proof of the lemma.

Lemma 3.4. *Let u be a nonnegative nontrivial weak solution to (1.1), for every $\tau > 0$, define*

$$u_\tau := u + \tau.$$

Then, for any $d \in (0, 1)$, $k > m$, and $\phi \in C_c^1(H^n; \mathbb{R}^+)$, the following holds

$$\int_{H^n} (K * u^p)u^{q-d}\phi^k d\xi + c_\epsilon \int_{H^n} u^{-d-1}\phi^k |\mathcal{A}(\xi, u, \nabla_H u)|^{m'} d\xi$$
$$\leq d_\epsilon \int_{H^n} u^{-d+m-1}\phi^{k-m}|\nabla_H \phi|^m d\xi, \tag{3.4}$$

for some positive constants c_ϵ and d_ϵ, which is given by

$$c_\epsilon = dc_0 - \frac{k\epsilon^{m'}}{m'},\ d_\epsilon = \frac{k}{m\epsilon^m},$$

for $1 < \epsilon < (dc_0m')^{\frac{1}{m'}}$, where c_0 and m are given in (1.2).

Proof. Let $\varepsilon > 0$ be sufficiently small, and let $(\zeta_\varepsilon)_{\varepsilon>0}$ be a standard family of mollifiers, that is,

$$\zeta_\varepsilon \in C_0^\infty(H^n; \mathbb{R}_0^+),\ \text{supp}(\zeta_\varepsilon) \subset D_\varepsilon(0),\ \int \zeta_\varepsilon = 1.$$

We define

$$\tilde{u} = \tau + \int_{H^n} \zeta_\varepsilon(\eta^{-1} \circ \xi)u(\eta)d\eta,$$

for $\xi \in H^n$. Clearly, $\tilde{u}, u_\tau > 0$. In particular, since $u \in L^1_{loc}(H^n)$, we have $\tilde{u} \in C^1(H^n)$, which shows that the function $\phi = \tilde{u}^{-d}\phi^k \geq 0$ can be used as a test function in the weak formulation of (1.1), so that

$$\int_{H^n} (K * u^p)u^q\tilde{u}^{-d}\phi^k d\xi + d\int_{H^n} \tilde{u}^{-d-1}\phi^k\mathcal{A}(\xi, u, \nabla_H u) \cdot \nabla_H \tilde{u} d\xi$$
$$\leq k\int_{H^n} \tilde{u}^{-d}\phi^{k-1}\mathcal{A}(\xi, u, \nabla_H u) \cdot \nabla_H \phi d\xi,$$

where k positive will be chosen later. Since $\tilde{u} \to u_\tau$ in $L^1_{loc}(H^n)$ as $\varepsilon \to 0$, then, by the Lebesgue dominated convergence theorem and being $\nabla_H u_\tau = \nabla_H u$, we arrive at the following:

$$\int_{H^n} (K * u^p)u^q u_\tau^{-d}\phi^k d\xi + d\int_{H^n} u_\tau^{-d-1}\phi^k\mathcal{A}(\xi, u, \nabla_H u) \cdot \nabla_H u d\xi$$

$$\leq k\int_{H^n} u_\tau^{-d}\phi^{k-1}\mathcal{A}(\xi,u,\nabla_H u)\cdot\nabla_H\phi d\xi.$$

Applying condition (1.2), we deduce the following:

$$\begin{aligned}&\int_{H^n}(K*u^p)u^q u_\tau^{-d}\phi^k d\xi + dc_0\int_{H^n} u_\tau^{-d-1}\phi^k|\mathcal{A}(\xi,u,\nabla_H u)|^{m'}d\xi\\ &\leq k\int_{H^n} u_\tau^{-d}\phi^{k-1}|\mathcal{A}(\xi,u,\nabla_H u)||\nabla_H\phi|d\xi.\end{aligned}\tag{3.5}$$

Now, by using the Young's inequality with exponents $\epsilon,\epsilon'>1$, we obtain the following:

$$\begin{aligned}&\int_{H^n} u_\tau^{-d}\phi^{k-1}|\mathcal{A}(\xi,u,\nabla_H u)||\nabla_H\phi|d\xi\\ &=\int_{H^n}\epsilon u_\tau^{-\frac{1+d}{m'}}|\mathcal{A}(\xi,u,\nabla_H u)|\phi^{\frac{k}{m'}}\epsilon^{-1}u_\tau^{-d+\frac{d+1}{m'}}\phi^{\frac{(m'-1)k}{m'}-1}|\nabla_H\phi|d\xi\\ &\leq\frac{\epsilon^{m'}}{m'}\int_{H^n}u_\tau^{-d-1}|\mathcal{A}(\xi,u,\nabla_H u)|^{m'}\phi^k d\xi\\ &\quad+\frac{1}{m\epsilon^m}\int_{H^n}u_\tau^{-d+\frac{1}{m'-1}}\phi^{k-m}|\nabla_H\phi|^m d\xi.\end{aligned}$$

Consequently, by inserting this estimation in the right hand side of (3.5), we obtain the following:

$$\begin{aligned}&\int_{H^n}(K*u^p)u^q u_\tau^{-d}\phi^k d\xi + (dc_0-\frac{k\epsilon^{m'}}{m'})\int_{H^n}u_\tau^{-d-1}\phi^k|\mathcal{A}(\xi,u,\nabla_H u)|^{m'}d\xi\\ &\leq\frac{1}{m\epsilon^m}\int_{H^n}u_\tau^{-d+\frac{1}{m'-1}}\phi^{k-m}|\nabla_H\phi|^m d\xi.\end{aligned}\tag{3.6}$$

An application of Fatou's Lemma when $\tau\to 0$ in (3.6) immediately gives (3.4).

Lemma 3.5. *Let $l>1$ and $u\geq 0$ be a solution of (1.1) such that $u^l\in L^1_{loc}(H^n)$, if we define*

$$J=\int_{H^n}u^l(\xi)\phi^k(\xi)d\xi,\tag{3.7}$$

then

$$\begin{aligned}&\int_{H^n}(K*u^p)u^q\phi^k d\xi\\ &\leq C_\epsilon J^{\frac{m-1}{l}}\left(\int_{H^n}\phi^{k-\frac{lm}{l-m+d+1}}|\nabla_H\phi|^{\frac{lm}{l-m+d+1}}d\xi\right)^{\frac{(l-m+d+1)(m-1)}{lm}}\\ &\quad\times\left(\int_{H^n}\phi^{k-\frac{lm}{l-m+1-d(m-1)}}|\nabla_H\phi|^{\frac{lm}{l-m+1-d(m-1)}}d\xi\right)^{\frac{l-m+1-d(m-1)}{lm}}.\end{aligned}\tag{3.8}$$

Proof. To claim the inequality (3.8), we multiply the inequality (1.1) by ϕ^k, and find the following:

$$\int_{H^n}(K*u^p)u^q\phi^k d\xi\leq\int_{\mathrm{supp}(\nabla_H\phi)}|\mathcal{A}(\xi,u,\nabla_H u)||\nabla_H\phi^k|d\xi.\tag{3.9}$$

Using the Hölder inequality with exponents $m, m' > 1$, we obtain the following:

$$\begin{aligned}&\int_{H^n} |\mathcal{A}(\xi, u, \nabla_H u)||\nabla_H \phi^k| d\xi \\ &\leq k \left(\int_{H^n} u^{-d-1} \phi^k |\mathcal{A}(\xi, u, \nabla_H u)|^{m'} d\xi \right)^{\frac{1}{m'}} \\ &\quad \times \left(\int_{H^n} u^{(d+1)(m-1)} \phi^{k-m} |\nabla_H \phi|^m d\xi \right)^{\frac{1}{m}}.\end{aligned}$$

Moreover, by using Lemma 3.4, we have

$$\begin{aligned}&\int_{H^n} |\mathcal{A}(\xi, u, \nabla_H u)||\nabla_H \phi^k| d\xi \\ &\leq k (\frac{d_\epsilon}{c_\epsilon})^{\frac{1}{m'}} \left(\int_{H^n} u^{-d+m-1} \phi^{k-m} |\nabla_H \phi|^m d\xi \right)^{\frac{1}{m'}} \\ &\quad \times \left(\int_{H^n} u^{(d+1)(m-1)} \phi^{k-m} |\nabla_H \phi|^m d\xi \right)^{\frac{1}{m}}. \end{aligned} \tag{3.10}$$

Now, we use the Hölder inequality in all the factors of the right hand side of (3.10), so that

$$\begin{aligned}&\int_{H^n} u^{-d+m-1} \phi^{k-m} |\nabla_H \phi|^m d\xi \\ &\leq \left(\int_{\mathrm{supp}(\nabla_H \phi)} u^l \phi^k d\xi \right)^{\frac{1}{\theta}} \left(\int_{H^n} \phi^{k-m\theta'} |\nabla_H \phi|^{m\theta'} d\xi \right)^{\frac{1}{\theta'}}, \end{aligned} \tag{3.11}$$

where $\theta = \frac{l}{m-d-1}$, $\theta' = \frac{l}{l-m+d+1}$.

$$\begin{aligned}&\int_{H^n} u^{(d+1)(m-1)} \phi^{k-m} |\nabla_H \phi|^m d\xi \\ &\leq \left(\int_{\mathrm{supp}(\nabla_H \phi)} u^l \phi^k d\xi \right)^{\frac{1}{\kappa}} \left(\int_{H^n} \phi^{k-m\kappa'} |\nabla_H \phi|^{m\kappa'} d\xi \right)^{\frac{1}{\kappa'}}, \end{aligned} \tag{3.12}$$

where $\kappa = \frac{l}{(d+1)(m-1)}$, $\kappa' = \frac{l}{l-m+1-d(m-1)}$.
Consequently, by inserting (3.11) and (3.12) into (3.10), we compute the following:

$$\begin{aligned}&\int_{H^n} |\mathcal{A}(\xi, u, \nabla_H u)||\nabla_H \phi^k| d\xi \\ &\leq C_\epsilon \left(\int_{H^n} u^l \phi^k d\xi \right)^{\frac{1}{\theta m'}} \left(\int_{H^n} \phi^{k-m\theta'} |\nabla_H \phi|^{m\theta'} d\xi \right)^{\frac{1}{m'\theta'}} \\ &\quad \times \left(\int_{H^n} u^l \phi^k d\xi \right)^{\frac{1}{\kappa m}} \left(\int_{H^n} \phi^{k-m\kappa'} |\nabla_H \phi|^{m\kappa'} d\xi \right)^{\frac{1}{m\kappa'}} \\ &= C_\epsilon \left(\int_{H^n} u^l \phi^k d\xi \right)^{\frac{1}{\theta m'} + \frac{1}{m\kappa}} \left(\int_{H^n} \phi^{k-m\theta'} |\nabla_H \phi|^{m\theta'} d\xi \right)^{\frac{1}{m'\theta'}}\end{aligned}$$

$$\times\left(\int_{H^n}\phi^{k-m\kappa'}|\nabla_H\phi|^{m\kappa'}d\xi\right)^{\frac{1}{m\kappa'}},$$

where $C_\epsilon = k(\frac{d_\epsilon}{c_\epsilon})^{\frac{1}{m'}}$. Finally, from (3.9), we have

$$\begin{aligned}
&\int_{H^n}(K * u^p)u^q\phi^k d\xi\\
&\leq C_\epsilon\left(\int_{H^n}u^l\phi^k d\xi\right)^{\frac{1}{\theta m'}+\frac{1}{m\kappa}}\left(\int_{H^n}\phi^{k-m\theta'}|\nabla_H\phi|^{m\theta'}d\xi\right)^{\frac{1}{m'\theta'}}\\
&\quad\times\left(\int_{H^n}\phi^{k-m\kappa'}|\nabla_H\phi|^{m\kappa'}d\xi\right)^{\frac{1}{m\kappa'}}.
\end{aligned} \tag{3.13}$$

We obtain

$$\frac{1}{\theta m'}+\frac{1}{m\kappa}=\frac{m-1}{l},\ m\theta'=\frac{lm}{l-m+d+1},\ m\kappa'=\frac{lm}{l-m+1-d(m-1)},$$
$$\frac{1}{m'\theta'}=\frac{(l-m+d+1)(m-1)}{lm},\ \frac{1}{m\kappa'}=\frac{l-m+1-d(m-1)}{lm}.$$

Thus, (3.13) gives (3.8).

Now, we specialize the choice of the cut-off function ϕ. Taking

$$\phi(\eta)=\Psi(\frac{|\xi|^4+|\tilde{\xi}|^4+\tau^2}{R^4}),\ \eta=(\xi,\tilde{\xi},\tau)\in H^n, R>0, \tag{3.14}$$

with $\Psi\in C_c^\infty(\mathbb{R}^+)$ is the standard cut-off function

$$\Psi(r)=\begin{cases}1, & \text{if } 0\leq r\leq 1,\\ \searrow, & \text{if } 1<r\leq 2,\\ 0, & \text{if } r>2.\end{cases}$$

We note that supp(ϕ) is a subset of

$$\tilde{\Omega}_R=\{(\xi,\tilde{\xi},\tau)\in H^n||\xi|^4+|\tilde{\xi}|^4+|\tau|^2\leq 2R^4\},$$

and supp($\nabla_H\phi$) are supported on

$$\Omega_R=\{(\xi,\tilde{\xi},\tau)\in H^n|R^4<|\xi|^4+|\tilde{\xi}|^4+|\tau|^2\leq 2R^4\}.$$

Let $r=\frac{|\xi|^4+|\tilde{\xi}|^4+\tau^2}{R^4}$, we calculate

$$\begin{aligned}
|\nabla_H\phi| &= \left(\sum_{i=1}^n |X_i\phi|^2+|Y_i\phi|^2\right)^{\frac{1}{2}}\\
&= 4R^{-4}|\Psi'(r)|[|\xi|^6+|\tilde{\xi}|^6+\tau^2(|\xi|^2+|\tilde{\xi}|^2)+2\tau\xi\cdot\tilde{\xi}(|\xi|^2-|\tilde{\xi}|^2)]^{\frac{1}{2}}\\
&\leq C_1R^{-1}.
\end{aligned} \tag{3.15}$$

Lemma 3.6. *Let u be a solution of (1.1), and $l = \frac{p+q}{2} > 1$, then,*

$$\left(\int_{D_R(0)} u^l(\xi)\phi^k(\xi)d\xi \right)^2 \le CR^\alpha, \tag{3.16}$$

for an opportune real exponent α given by

$$\alpha = \frac{(p+q)(Q-m+\beta) - 2(m-1)Q}{p+q-m+1},$$

and ϕ as a cut-off function in (3.14).

Proof. If $R > \rho$ is large enough, then we have the following:

$$(K * u^p)(\xi) \ge cR^{-\beta} \int_{D_{2R}(0)} u^p(\eta)d\eta \ge cR^{-\beta} \int_{H^n} u^p(\eta)\phi^k(\eta)d\eta. \tag{3.17}$$

Furthermore, for $l = \frac{p+q}{2} > 1$, by the Hölder inequality, we have

$$\begin{aligned}
&\left(\iint_{H^n \times H^n} u^p(\eta)\phi^k(\eta)u^q(\xi)\phi^k(\xi)d\xi d\eta \right)^2 \\
&= \left(\iint_{H^n \times H^n} u^p(\eta)\phi^k(\eta)u^q(\xi)\phi^k(\xi)d\xi d\eta \right) \\
&\quad \times \left(\iint_{H^n \times H^n} u^p(\eta)\phi^k(\eta)u^q(\xi)\phi^k(\xi)d\xi d\eta \right) \\
&\ge \left(\iint_{H^n \times H^n} u^{\frac{p+q}{2}}(\eta)\phi^k(\eta)u^{\frac{p+q}{2}}(\xi)\phi^k(\xi)d\xi d\eta \right)^2 \\
&= \left(\int_{H^n} u^l(\xi)\phi^k(\xi)d\xi \right)^4 = J^4.
\end{aligned}$$

Hence,

$$\left(\iint_{H^n \times H^n} u^p(\eta)\phi^k(\eta)u^q(\xi)\phi^k(\xi)d\xi d\eta \right) \ge J^2. \tag{3.18}$$

By (3.17) and (3.18), we have

$$\int_{H^n} (K * u^p)u^q\phi^k d\xi \ge cR^{-\beta} \iint_{H^n \times H^n} u^p(\eta)\phi^k(\eta)u^q(\xi)\phi^k d\xi d\eta \ge cR^{-\beta}J^2.$$

Thus,

$$\begin{aligned}
J^2 &\le \frac{1}{c}R^\beta \int_{H^n} (K * u^p)u^q\phi^k d\xi \\
&\le \frac{C_\epsilon}{c}R^\beta \left(\int_{H^n} u^l\phi^k d\xi \right)^{\frac{m-1}{l}} \left(\int_{H^n} \phi^{k-\frac{lm}{l-m+d+1}} |\nabla_H\phi|^{\frac{lm}{l-m+d+1}} d\xi \right)^{\frac{(l-m+d+1)(m-1)}{lm}}
\end{aligned}$$

$$\times\left(\int_{H^n}\phi^{k-\frac{lm}{l-m+1-d(m-1)}}|\nabla_H\phi|^{\frac{lm}{l-m+1-d(m-1)}}d\xi\right)^{\frac{l-m+1-d(m-1)}{lm}}. \tag{3.19}$$

We perform the change of variables $R\bar{\xi} = \xi, R\hat{\xi} = \tilde{\xi}$, and $R^2\tilde{\tau} = \tau$, from (3.15), we obtain the following estimates:

$$\begin{aligned}\int_{H^n}\phi^{k-\frac{lm}{l-m+d+1}}|\nabla_H\phi|^{\frac{lm}{l-m+d+1}}d\xi &\leq C_1R^{-\frac{lm}{l-m+d+1}}\int_{\Omega_R}d\xi\\ &\leq C_1R^{Q-\frac{lm}{l-m+d+1}},\end{aligned}$$

and

$$\begin{aligned}\int_{H^n}\phi^{k-\frac{lm}{l-m+1-d(m-1)}}|\nabla_H\phi|^{\frac{lm}{l-m+1-d(m-1)}}d\xi &\leq C_1R^{-\frac{lm}{l-m+1-d(m-1)}}\int_{\Omega_R}d\xi\\ &\leq C_1R^{Q-\frac{lm}{l-m+1-d(m-1)}}.\end{aligned} \tag{3.20}$$

Consequently, by (3.19), we obtain

$$\begin{aligned}J^2 &\leq \frac{C_\epsilon C_1}{c}R^\beta\left(\int_{H^n}u^l\phi^k d\xi\right)^{\frac{m-1}{l}}\left(R^{Q-\frac{lm}{l-m+d+1}}\right)^{\frac{(l-m+d+1)(m-1)}{lm}}\\ &\times\left(R^{Q-\frac{lm}{l-m+1-d(m-1)}}\right)^{\frac{l-m+1-d(m-1)}{lm}}\\ &\leq C(J^2)^{\frac{m-1}{2l}}R^{\frac{(l-m+1)Q}{l}-m+\beta}.\end{aligned}$$

Hence, we have

$$\left(\int_{D_R(0)}u^l(\xi)\phi^k(\xi)d\xi\right)^2 \leq CR^\alpha, \tag{3.21}$$

where $\alpha = (\frac{(l-m+1)Q}{l} - m + \beta)(\frac{2l}{2l-m+1})$. Recall that $l = \frac{p+q}{2} > 1$, thus, we have the following:

$$\begin{aligned}\alpha &= (\frac{(l-m+1)Q}{l} - m + \beta)(\frac{2l}{2l-m+1})\\ &= \frac{(p+q)(Q-m+\beta) - 2(m-1)Q}{p+q-m+1}.\end{aligned}$$

Proof of Theorem 3.2. Now, We claim that u is constant. For this aim, first suppose that (3.2) holds, this means that $\alpha < 0$ in (3.21). Hence, from (3.21), by letting $R \to \infty$, we get

$$J = \int_{H^n}u^l(\xi)\phi^k(\xi)d\xi = 0,$$

which contradicts our assumptions on u, thus completing the proof of the theorem.

Let us consider the following system

$$\begin{cases} L_{\mathcal{A}}u = -\text{div}_H(\mathcal{A}(\xi, u, \nabla_H u)) \geq (K(\xi) * v^{p_1})v^{q_1},\\ L_{\mathcal{B}}v = -\text{div}_H(\mathcal{B}(\xi, v, \nabla_H v)) \geq (I(\xi) * u^{p_2})u^{q_2},\end{cases} \tag{3.22}$$

where $\mathcal{A}, \mathcal{B}$ are Carathéodory functions with

$$\mathcal{A}(\xi, z, \eta) \cdot \eta \geq C_1 |\mathcal{A}(\xi, z, \eta)|^{m_1'}, \;\; \mathcal{B}(\xi, z, \eta) \cdot \eta \geq C_2 |\mathcal{B}(\xi, z, \eta)|^{m_2'}, \tag{3.23}$$

for some $m_1', m_2' > 1$ and m_1, m_2 are the corresponding conjugate exponents of m_1, m_2, respectively. The quality $K * v^{p_1}$ and $I * u^{p_2}$ are defined as standard convolution operators on the Heisenberg group by

$$K * v^{p_1}(\xi) = \int_{H^n} K(\eta^{-1} \circ \xi) v^{p_1}(\eta) d\eta, \text{ for all } \xi \in H^n,$$

and

$$I * u^{p_2}(\xi) = \int_{H^n} I(\eta^{-1} \circ \xi) u^{p_2}(\eta) d\eta, \text{ for all } \xi \in H^n,$$

respectively. We assume that there exist positive constants c_1, c_2, R_0 and real exponents $\beta_1, \beta_2 \in \mathbb{R}$ such that

$$K(\xi) \geq c_1 |\xi|^{-\beta_1}, I(\xi) \geq c_2 |\xi|^{-\beta_2}, \tag{3.24}$$

for all ξ with $|\xi| \geq R_0$.

Let us give a definition of the weak solution to (3.22) as follows.

Definition 3.7. *By an entire weak solution of* (3.22)*, we mean a couple* (u, v) *of nonnegative functions of class* $W^{1,1}_{loc}(H^n) \times W^{1,1}_{loc}(H^n)$ *such that* (u, v) *is a distribution solution of* (3.22)*, that is,*

$$\mathcal{A}(\xi, u, \nabla_H u) \in L^{m_1'}_{loc}(H^n)^{2n}, \; \mathcal{B}(\xi, v, \nabla_H v) \in L^{m_2'}_{loc}(H^n)^{2n},$$
$$(K * v^{p_1}) v^{q_1} \in L^1_{loc}(H^n), \; (I * u^{p_2}) u^{q_2} \in L^1_{loc}(H^n),$$
$$\int_{H^n} \mathcal{A}(\xi, u, \nabla_H u) \cdot \nabla_H \psi d\xi \geq \int_{H^n} (K * v^{p_1}) v^{q_1} \psi d\xi,$$
$$\int_{H^n} \mathcal{B}(\xi, v, \nabla_H v) \cdot \nabla_H \psi d\xi \geq \int_{H^n} (I * u^{p_2}) u^{q_2} \psi d\xi,$$

for all $\psi \in C^\infty_c(H^n)$ *with* $\psi \geq 0$.

Theorem 3.8. *Assume that the inequalities*

$$Q \leq \max\{K_1, K_2\}, \tag{3.25}$$

and

$$p_1 + q_1 > m_2 - 1, \; p_2 + q_2 > m_1 - 1, \tag{3.26}$$

hold with $Q = 2n + 2$*, which is the homogeneous dimension of* H^n*, then, the system of elliptic type inequalities (3.22) does not have an entire nontrivial weak solution, where*

$$K_1 = \frac{(p_2 + q_2)(p_1 + q_1)(m_2 - \beta_2) + (p_2 + q_2)(m_2 - 1)(m_1 - \beta_1)}{(m_2 - 1)(-p_2 - q_2 - 2m_1 + 2) + (p_1 + q_1)(p_2 + q_2)},$$
$$K_2 = \frac{(p_2 + q_2)(p_1 + q_1)(m_1 - \beta_1) + (p_1 + q_1)(m_1 - 1)(m_2 - \beta_2)}{(m_1 - 1)(-p_1 - q_1 - 2m_2 + 2) + (p_1 + q_1)(p_2 + q_2)}.$$

Proof of Theorem 3.8. Assume that there exists a entire weak solution to (3.22). Let $\tau > 0$, and define $u_\tau = u + \tau$ and $v_\tau = v + \tau$, so that $u_\tau, v_\tau > 0$ and $\nabla_H u = \nabla_H u_\tau, \nabla_H v = \nabla_H v_\tau$.
Step 1. The following inequalities hold:

$$\int_{H^n} (K * v^{p_1}) v^{q_1} u_\tau^{-\beta} \psi^k d\xi + \int_{H^n} u_\tau^{-\beta-1} \psi^k |\mathcal{A}(\xi, u, \nabla_H u)|^{m_1'} d\xi \\ \leq \hat{C} \int_{H^n} u_\tau^{-\beta+m_1-1} \psi^{k-m_1} |\nabla_H \psi|^{m_1} d\xi, \tag{3.27}$$

$$\int_{H^n} (I * u^{p_2}) u^{q_2} v_\tau^{-\alpha} \psi^k d\xi + \int_{H^n} v_\tau^{-\alpha-1} \psi^k |\mathcal{B}(\xi, v, \nabla_H v)|^{m_2'} d\xi \\ \leq \tilde{C} \int_{H^n} v_\tau^{-\alpha+m_2-1} \psi^{k-m_2} |\nabla_H \psi|^{m_2} d\xi,$$

where $\hat{C}, \tilde{C}$ are positive constants.

To prove Step 1, let $\psi \in C_c^\infty(H^n)$ be a standard nonnegative cut-off function. Let $\epsilon > 0$ be sufficiently small, and $(\varsigma_\epsilon)_{\epsilon>0}$ be a standard family of mollifiers, that is,

$$\varsigma_\epsilon \in C_0^\infty(H^n; \mathbb{R}_0^+), \quad \mathrm{supp}(\varsigma_\epsilon) \subset D_\epsilon(0), \quad \int \varsigma_\epsilon = 1.$$

For $\tau > 0$, define

$$\tilde{u}(\xi) = \tau + \int_{H^n} \varsigma_\epsilon(\eta^{-1} \circ \xi) u(\eta) d\eta, \quad \tilde{v}(\xi) = \tau + \int_{H^n} \varsigma_\epsilon(\eta^{-1} \circ \xi) v(\eta) d\eta.$$

In particular, since $u, v \in L^1_{loc}(H^n)$, then $\tilde{u}, \tilde{v} \in C^1(H^n; \mathbb{R}^+)$, which shows that both $\tilde{u}^{-\beta}\psi^k$ and $\tilde{v}^{-\alpha}\psi^k$ can be used as test functions in (3.22), $\alpha, \beta \in (0, 1)$, so that, using also (3.23),

$$\int_{H^n} (K * v^{p_1}) v^{q_1} \tilde{u}^{-\beta} \psi^k d\xi + C_1 \beta \int_{H^n} \tilde{u}^{-\beta-1} \psi^k |\mathcal{A}(\xi, u, \nabla_H u)|^{m_1'} d\xi \\ \leq k \int_{H^n} \tilde{u}^{-\beta+m_1-1} \psi^{k-m_1} |\nabla_H \psi|^{m_1} d\xi,$$

and

$$\int_{H^n} (I * u^{p_2}) u^{q_2} \tilde{v}^{-\alpha} \psi^k d\xi + C_2 \alpha \int_{H^n} \tilde{v}^{-\alpha-1} \psi^k |\mathcal{B}(\xi, v, \nabla_H v)|^{m_2'} d\xi \\ \leq k \int_{H^n} \tilde{v}^{-\alpha+m_2-1} \psi^{k-m_2} |\nabla_H \psi|^{m_2} d\xi.$$

Since $\tilde{u} \to u_\tau$ and $\tilde{v} \to v_\tau$ in $L^1_{loc}(H^n)$ as $\epsilon \to 0$, then, by the Lebesgue dominated convergence theorem, we arrive to

$$\int_{H^n} (K * v^{p_1}) v^{q_1} u_\tau^{-\beta} \psi^k d\xi + C_1 \beta \int_{H^n} u_\tau^{-\beta-1} \psi^k |\mathcal{A}(\xi, u, \nabla_H u)|^{m_1'} d\xi \\ \leq k \int_{H^n} u_\tau^{-\beta+m_1-1} \psi^{k-m_1} |\nabla_H \psi|^{m_1} d\xi,$$

and

$$\int_{H^n} (I * u^{p_2}) u^{q_2} v_\tau^{-\alpha} \psi^k d\xi + C_2 \alpha \int_{H^n} v_\tau^{-\alpha-1} \psi^k |\mathcal{B}(\xi, v, \nabla_H v)|^{m_2'} d\xi$$
$$\leq k \int_{H^n} v_\tau^{-\alpha+m_2-1} \psi^{k-m_2} |\nabla_H \psi|^{m_2} d\xi.$$

Step 2. The inequalities

$$\begin{aligned}
&\int_H (K * v^{p_1}) v^{q_1} \psi^k d\xi \\
&\leq k \hat{C}^{\frac{1}{m_1'}} \left(\int_{H^n} u^{l_1} \psi^k d\xi \right)^{\frac{1}{\lambda m_1'} + \frac{1}{\mu m_1}} \\
&\quad \times \left(\int_{H^n} \psi^{k-m_1\lambda'} |\nabla_H \psi|^{m_1\lambda'} d\xi \right)^{\frac{1}{\lambda' m_1'}} \left(\int_{H^n} \psi^{k-m_1\mu'} |\nabla_H \psi|^{m_1\mu'} d\xi \right)^{\frac{1}{m_1 \mu'}},
\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}
&\int_{H^n} (I * u^{p_2}) u^{q_2} \psi^k d\xi \\
&\leq k \tilde{C}^{\frac{1}{m_2'}} \left(\int_{H^n} v^{l_2} \psi^k d\xi \right)^{\frac{1}{\chi m_2'} + \frac{1}{\iota m_2}} \\
&\quad \times \left(\int_{H^n} \psi^{k-m_2\chi'} |\nabla_H \psi|^{m_2\chi'} d\xi \right)^{\frac{1}{\chi' m_2'}} \left(\int_{H^n} \psi^{k-m_2\iota'} |\nabla_H \psi|^{m_2\iota} d\xi \right)^{\frac{1}{m_2 \iota'}},
\end{aligned} \tag{3.29}$$

hold, where $\mu, \lambda, \chi, \iota > 1$, which will be given later, and $\psi \in C_c^1(H^n)$ is a standard nonnegative cut-off function to be specified later.

To show the validity of Step 2, we multiply the first inequality in (3.22) by ψ^k so that we obtain the following using the Hölder inequality with exponents $m_1, m_1' > 1$,

$$\begin{aligned}
\int_{H^n} (K * v^{p_1}) v^{q_1} \psi^k d\xi &\leq \int_{\mathrm{supp}(\nabla_H \psi)} |\mathcal{A}(\xi, u, \nabla_H u)| |\nabla_H \psi^k| d\xi \\
&\leq k \left(\int_{H^n} u_\tau^{-\beta-1} \psi^k |\mathcal{A}(\xi, u, \nabla_H u)|^{m_1'} d\xi \right)^{\frac{1}{m_1'}} \\
&\quad \times \left(\int_{H^n} u_\tau^{(\beta+1)(m_1-1)} \psi^{k-m_1} |\nabla_H \psi|^{m_1} d\xi \right)^{\frac{1}{m_1}}.
\end{aligned}$$

Hence, by inserting the estimation (3.27) in the right hand side of the above inequality, we obtain the following:

$$\int_{H^n} (K * v^{p_1}) v^{q_1} \psi^k d\xi$$

$$\leq k\hat{C}^{\frac{1}{m_1'}}\left(\int_{H^n} u_\tau^{-\beta+m_1-1}\psi^{k-m_1}|\nabla_H\psi|^{m_1}d\xi\right)^{\frac{1}{m_1'}}$$
$$\times\left(\int_{H^n} u_\tau^{(\beta+1)(m_1-1)}\psi^{k-m_1}|\nabla_H\psi|^{m_1}d\xi\right)^{\frac{1}{m_1}}. \tag{3.30}$$

Analogously, we arrive at

$$\int_{H^n}(I*u^{p_2})u^{q_2}\psi^k d\xi$$
$$\leq k\tilde{C}^{\frac{1}{m_2'}}\left(\int_{H^n} v_\tau^{-\alpha+m_2-1}\psi^{k-m_2}|\nabla_H\psi|^{m_2}d\xi\right)^{\frac{1}{m_2'}}$$
$$\times\left(\int_{H^n} v_\tau^{(\alpha+1)(m_2-1)}\psi^{k-m_2}|\nabla_H\psi|^{m_2}d\xi\right)^{\frac{1}{m_2}}. \tag{3.31}$$

Further applications of the Hölder inequality with exponents $\lambda,\mu>1$ give

$$\int_{H^n} u_\tau^{-\beta+m_1-1}\psi^{k-m_1}|\nabla_H\psi|^{m_1}d\xi$$
$$\leq\left(\int_{\mathrm{supp}(\nabla_H\psi)} u_\tau^{l_1}\psi^k d\xi\right)^{\frac{1}{\lambda}}\left(\int_{H^n}\psi^{k-m_1\lambda'}|\nabla_H\psi|^{m_1\lambda'}d\xi\right)^{\frac{1}{\lambda'}}, \tag{3.32}$$

and

$$\int_{H^n} u_\tau^{(\beta+1)(m_1-1)}\psi^{k-m_1}|\nabla_H\psi|^{m_1}d\xi$$
$$\leq\left(\int_{\mathrm{supp}(\nabla_H\psi)} u^{l_1}\psi^k d\xi\right)^{\frac{1}{\mu}}\left(\int_{H^n}\psi^{k-m_1\mu'}|\nabla_H\psi|^{m_1\mu'}d\xi\right)^{\frac{1}{\mu'}}, \tag{3.33}$$

where

$$\lambda=\frac{l_1}{m_1-\beta-1},\quad \lambda'=\frac{l_1}{l_1-m_1+\beta+1},$$
$$\mu=\frac{l_1}{(\beta+1)(m_1-1)},\quad \mu'=\frac{l_1}{l_1-m_1+1-\beta(m_1-1)}.$$

Hence, by inserting (3.32) and (3.33) in (3.30), we deduce the following:

$$\int_{H^n}(K*v^{p_1})v^{q_1}\psi^k d\xi$$
$$\leq k\hat{C}^{\frac{1}{m_1'}}\left(\int_{H^n} u_\tau^{l_1}\psi^k d\xi\right)^{\frac{1}{\lambda m_1'}+\frac{1}{\mu m_1}}$$
$$\times\left(\int_{H^n}\psi^{k-m_1\lambda'}|\nabla_H\psi|^{m_1\lambda'}d\xi\right)^{\frac{1}{\lambda' m_1'}}\left(\int_{H^n}\psi^{k-m_1\mu'}|\nabla_H\psi|^{m_1\mu'}d\xi\right)^{\frac{1}{m_1\mu'}}. \tag{3.34}$$

An application of Fatou's Lemma when $\tau\to 0$ in (3.34) immediately gives (3.28).

A similar reasoning allows us to prove

$$\begin{aligned}&\int_{H^n} v_\tau^{-\alpha+m_2-1}\psi^{k-m_2}|\nabla_H\psi|^{m_2}d\xi\\&\leq\left(\int_{\mathrm{supp}(\nabla_H\psi)} v_\tau^{l_2}\psi^k d\xi\right)^{\frac{1}{\chi}}\left(\int_{H^n}\psi^{k-m_2\chi'}|\nabla_H\psi|^{m_2\chi'}d\xi\right)^{\frac{1}{\chi'}},\end{aligned}\tag{3.35}$$

and

$$\begin{aligned}&\int_{H^n} v_\tau^{(\alpha+1)(m_2-1)}\psi^{k-m_2}|\nabla_H\psi|^{m_2}d\xi\\&\leq\left(\int_{\mathrm{supp}(\nabla_H\psi)} v^{l_2}\psi^k d\xi\right)^{\frac{1}{\iota}}\left(\int_{H^n}\psi^{k-m_2\iota'}|\nabla_H\psi|^{m_2\iota'}d\xi\right)^{\frac{1}{\iota'}},\end{aligned}\tag{3.36}$$

where

$$\chi=\frac{l_2}{m_2-\alpha-1},\ \chi'=\frac{l_2}{l_2-m_2+\alpha+1},$$
$$\iota=\frac{l_2}{(\alpha+1)(m_2-1)},\ \iota'=\frac{l_2}{l_2-m_2+1-\alpha(m_2-1)}.$$

Hence, by inserting (3.35) and (3.36) in (3.31), we have the following:

$$\begin{aligned}&\int_H (I*u^{p_2})u^{q_2}\psi^k d\xi\\&\leq k\tilde{C}^{\frac{1}{m_2'}}\left(\int_{H^n} v_\tau^{l_2}\psi^k d\xi\right)^{\frac{1}{\chi m_2'}+\frac{1}{\iota m_2}}\\&\times\left(\int_{H^n}\psi^{k-m_2\chi'}|\nabla_H\psi|^{m_2\chi'}d\xi\right)^{\frac{1}{\chi' m_2'}}\left(\int_{H^n}\psi^{k-m_2\iota'}|\nabla_H\psi|^{m_2\iota}d\xi\right)^{\frac{1}{m_2\iota'}}.\end{aligned}\tag{3.37}$$

An application of Fatou's Lemma when $\tau\to 0$ in (3.37) immediately gives (3.29).
Step 3. Let $l_1=\frac{p_2+q_2}{2}$ and $l_2=\frac{p_1+q_1}{2}$, if we define

$$J_1=\int_{D_R(0)} u^{l_2}\psi^k d\xi,\ \ J_2=\int_{D_R(0)} v^{l_1}\psi^k d\xi,\tag{3.38}$$

where D_R is the ball of H^n which has a center at $\xi=0$ and a radius $R>0$, then

$$\begin{cases} J_1^2\leq CR^{\sigma_1},\\ J_2^2\leq CR^{\sigma_2},\end{cases}\tag{3.39}$$

for some real exponents σ_1 and σ_2, which are given below.

To show Step 3, we first recall the proof of Lemma 3.6, which leads to

$$\int_{H^n}(K*v^{p_1})v^{q_1}\psi^k d\xi\geq c_1R^{-\beta_1}J_2^2,\tag{3.40}$$

and

$$\int_{H^n}(I*u^{p_2})u^{q_2}\psi^k d\xi \geq c_2R^{-\beta_2}J_1^2.$$

Therefore, from (3.28), we have the following:

$$\begin{aligned}
J_2^2 &\leq \frac{1}{c_1}R^{\beta_1}\int_{H^n}(K*v^{p_1})v^{q_1}\psi^k d\xi \\
&\leq \frac{k}{c_1}\hat{C}^{\frac{1}{m_1'}}R^{\beta_1}(J_1)^{\frac{1}{\lambda m_1'}+\frac{1}{\mu m_1}} \\
&\quad\times\left(\int_{H^n}\psi^{k-m_1\lambda'}|\nabla_H\psi|^{m_1\lambda'}d\xi\right)^{\frac{1}{\lambda' m_1'}}\left(\int_{H^n}\psi^{k-m_1\mu'}|\nabla_H\psi|^{m_1\mu'}d\xi\right)^{\frac{1}{m_1\mu'}}.
\end{aligned} \tag{3.41}$$

Similarly, we have

$$\begin{aligned}
J_1^2 &\leq \frac{1}{c_2}R^{\beta_2}\int_{H^n}(I*u^{p_2})u^{q_2}\psi^k d\xi \\
&\leq \frac{k}{c_2}\tilde{C}^{\frac{1}{m_2'}}R^{\beta_2}(J_2)^{\frac{1}{\chi m_2'}+\frac{1}{\iota m_2}} \\
&\quad\times\left(\int_{H^n}\psi^{k-m_2\chi'}|\nabla_H\psi|^{m_2\chi'}d\xi\right)^{\frac{1}{\chi' m_2'}}\left(\int_{H^n}\psi^{k-m_2\iota'}|\nabla_H\psi|^{m_2\iota'}d\xi\right)^{\frac{1}{m_1\iota'}}.
\end{aligned} \tag{3.42}$$

We use a combination of (3.41) and (3.42) , which leads to the following:

$$\begin{aligned}
J_1^2 &\leq \frac{k}{c_2}\tilde{C}^{\frac{1}{m_2'}}R^{\beta_2}(J_2)^{\frac{1}{\chi m_2'}+\frac{1}{\iota m_2}} \\
&\quad\times\left(\int_{H^n}\psi^{k-m_2\chi'}|\nabla_H\psi|^{m_2\chi'}d\xi\right)^{\frac{1}{\chi' m_2'}}\left(\int_{H^n}\psi^{k-m_2\iota'}|\nabla_H\psi|^{m_2\iota'}d\xi\right)^{\frac{1}{m_2\iota'}} \\
&\leq \frac{k}{c_2}\tilde{C}^{\frac{1}{m_2'}}R^{\beta_2}\Bigg[\frac{k}{c_1}\hat{C}^{\frac{1}{m_1'}}R^{\beta_1}(J_1)^{\frac{1}{\lambda m_1'}+\frac{1}{\mu m_1}}\left(\int_{H^n}\psi^{k-m_1\lambda'}|\nabla_H\psi|^{m_1\lambda'}d\xi\right)^{\frac{1}{\lambda' m_1'}} \\
&\quad\times\left(\int_{H^n}\psi^{k-m_1\mu'}|\nabla_H\psi|^{m_1\mu'}d\xi\right)^{\frac{1}{m_1\mu'}}\Bigg]^{\frac{1}{2\chi m_2'}+\frac{1}{2\iota m_2}} \\
&\quad\times\left(\int_{H^n}\psi^{k-m_2\chi'}|\nabla_H\psi|^{m_2\chi'}d\xi\right)^{\frac{1}{\chi' m_2'}}\left(\int_{H^n}\psi^{k-m_2\iota'}|\nabla_H\psi|^{m_2\iota'}d\xi\right)^{\frac{1}{m_2\iota'}} \\
&\leq CR^{\beta_2+\frac{\beta_1}{2\chi m_2'}+\frac{\beta_1}{2\iota m_2}}(J_1^2)^{(\frac{1}{2\lambda m_1'}+\frac{1}{2\mu m_1})(\frac{1}{2\chi m_2'}+\frac{1}{2\iota m_2})} \\
&\quad\times\left(\int_{H^n}\psi^{k-m_1\lambda'}|\nabla_H\psi|^{m_1\lambda'}d\xi\right)^{(\frac{1}{\lambda' m_1'})(\frac{1}{2\chi m_2'}+\frac{1}{2\iota m_2})} \\
&\quad\times\left(\int_{H^n}\psi^{k-m_1\mu'}|\nabla_H\psi|^{m_1\mu'}d\xi\right)^{(\frac{1}{m_1\mu'})(\frac{1}{2\chi m_2'}+\frac{1}{2\iota m_2})}
\end{aligned}$$

$$\times\left(\int_{H^n}\psi^{k-m_2\chi'}|\nabla_H\psi|^{m_2\chi'}d\xi\right)^{\frac{1}{\chi' m_2'}}\left(\int_{H^n}\psi^{k-m_2\iota'}|\nabla_H\psi|^{m_2\iota'}d\xi\right)^{\frac{1}{m_2\iota'}}. \tag{3.43}$$

In (3.43), we compute the following:

$$\beta_2+\frac{\beta_1}{2\chi m_2'}+\frac{\beta_1}{2\iota m_2}=\beta_2+\frac{(m_2-1)\beta_1}{2l_2},$$

$$\frac{1}{\lambda' m_1'}=\frac{(l_1-m_1+\beta+1)(m_1-1)}{l_1m_1},\ \frac{1}{m_1\mu'}=\frac{l_1-m_1+1-\beta(m_1-1)}{l_1m_1},$$
$$\frac{1}{\chi' m_2'}=\frac{(l_2-m_2+\alpha+1)(m_2-1)}{l_2m_2},\ \frac{1}{m_2\iota'}=\frac{l_2-m_2+1-\alpha(m_2-1)}{l_2m_2},$$
$$m_1\lambda'=\frac{l_1m_1}{l_1-m_1+\beta+1},\ m_1\mu'=\frac{l_1m_1}{l_1-m_1+1-\beta(m_1-1)},$$
$$m_2\chi'=\frac{l_2m_2}{l_2-m_2+\alpha+1},\ m_2\iota'=\frac{l_2m_2}{l_2-m_2+1-\alpha(m_2-1)}.$$

From (3.43), we deduce

$$\begin{aligned}
&(J_1^2)^{1-(\frac{1}{2\lambda m_1'}+\frac{1}{2\mu m_1})(\frac{1}{2\chi m_2'}+\frac{1}{2\iota m_2})}=(J_1^2)^{1-(\frac{m_1-1}{2l_1})(\frac{m_2-1}{2l_2})}\\
&\leq CR^{\beta_2+\frac{(m_2-1)\beta_1}{2l_2}}\left(\int_{H^n}\psi^{k-m_1\lambda'}|\nabla_H\psi|^{m_1\lambda'}d\xi\right)^{(\frac{(l_1-m_1+\beta+1)(m_1-1)}{l_1m_1})(\frac{m_2-1}{2l_2})}\\
&\times\left(\int_{H^n}\psi^{k-m_1\mu'}|\nabla_H\psi|^{m_1\mu'}d\xi\right)^{(\frac{l_1-m_1+1-\beta(m_1-1)}{l_1m_1})(\frac{m_2-1}{2l_2})}\\
&\times\left(\int_{H^n}\psi^{k-m_2\chi'}|\nabla_H\psi|^{m_2\chi'}d\xi\right)^{\frac{(l_2-m_2+\alpha+1)(m_2-1)}{l_2m_2}}\\
&\times\left(\int_{H^n}\psi^{k-m_2\iota'}|\nabla_H\psi|^{m_2\iota'}d\xi\right)^{\frac{l_2-m_2+1-\alpha(m_2-1)}{l_2m_2}},
\end{aligned} \tag{3.44}$$

where, by (3.26),

$$1-(\frac{m_1-1}{2l_1})(\frac{m_2-1}{2l_2})>0.$$

Now, we specialize the choice of the cut-off function ψ, by taking $\psi(\xi)=\psi_R(\xi)$, where $\psi\in C_0^\infty(H^n)$, with

$$\psi_R(\eta)=\Psi(\frac{|\xi|^4+|\tilde{\xi}|^4+\tau^2}{R^4}),\ \eta=(\xi,\tilde{\xi},\tau)\in H^n, R>0, \tag{3.45}$$

with $\Psi\in C^\infty(\mathbb{R}^+)$ is a the standard cut-off function

$$\Psi(r)=\begin{cases}1, & \text{if } 0\leq r\leq 1,\\ \searrow, & \text{if } 1<r\leq 2,\\ 0, & \text{if } r>2.\end{cases}$$

We note that supp(ψ) is subset of

$$\tilde{\Omega}_R = \{(\xi, \tilde{\xi}, \tau) \in H^n | |\xi|^4 + |\tilde{\xi}|^4 + |\tau|^2 \leq 2R^4\},$$

and supp($\nabla_H \psi$) are supported on

$$\Omega_R = \{(\xi, \tilde{\xi}, \tau) \in H^n | R^4 < |\xi|^4 + |\tilde{\xi}|^4 + |\tau|^2 \leq 2R^4\}.$$

It follows from (3.15) that

$$|\nabla_H \psi_R| \leq CR^{-1}. \tag{3.46}$$

Thus,

$$\begin{aligned}
&\left(\int_{H^n} \psi^{k-m_1\lambda'} |\nabla_H \psi|^{m_1\lambda'} d\xi\right)^{(\frac{(l_1-m_1+\beta+1)(m_1-1)}{l_1 m_1})(\frac{m_2-1}{2l_2})} \\
&\leq \left(CR^{-m_1\lambda'} \int_{\Omega_R} d\xi\right)^{(\frac{(l_1-m_1+\beta+1)(m_1-1)}{l_1 m_1})(\frac{m_2-1}{2l_2})} \\
&\leq C_1 R^{(m_1-1)(m_2-1)(\frac{Q(l_1-m_1+\beta+1)}{2l_1 l_2 m_1} - \frac{1}{2l_2})},
\end{aligned}$$

$$\begin{aligned}
&\left(\int_{H^n} \psi^{k-m_1\mu'} |\nabla_H \psi|^{m_1\mu'} d\xi\right)^{(\frac{l_1-m_1+1-\beta(m_1-1)}{l_1 m_1})(\frac{m_2-1}{2l_2})} \\
&\leq \left(CR^{-m_1\mu'} \int_{\Omega_R} d\xi\right)^{(\frac{l_1-m_1+1-\beta(m_1-1)}{l_1 m_1})(\frac{m_2-1}{2l_2})} \\
&\leq C_2 R^{(m_2-1)(\frac{Q(l_1-m_1+1-\beta(m_1-1))}{2l_1 l_2 m_1} - \frac{1}{2l_2})},
\end{aligned}$$

$$\begin{aligned}
&\left(\int_{H^n} \psi^{k-m_2\chi'} |\nabla_H \psi|^{m_2\chi'} d\xi\right)^{\frac{(l_2-m_2+\alpha+1)(m_2-1)}{l_2 m_2}} \\
&\leq \left(CR^{-m_2\chi'} \int_{\Omega_R} d\xi\right)^{\frac{(l_2-m_2+\alpha+1)(m_2-1)}{l_2 m_2}} \\
&\leq C_3 R^{\frac{Q(l_2-m_2+\alpha+1)(m_2-1)}{l_2 m_2} - m_2 + 1},
\end{aligned}$$

$$\begin{aligned}
&\left(\int_{H^n} \psi^{k-m_2\iota'} |\nabla_H \psi|^{m_2\iota'} d\xi\right)^{\frac{l_2-m_2+1-\alpha(m_2-1)}{l_2 m_2}} \\
&\leq \left(CR^{-m_2\iota'} \int_{\Omega_R} d\xi\right)^{\frac{l_2-m_2+1-\alpha(m_2-1)}{l_2 m_2}} \\
&\leq C_4 R^{\frac{Q(l_2-m_2+1-\alpha(m_2-1))}{l_2 m_2} - 1}.
\end{aligned}$$

Consequently, from (3.44), we obtain the following:

$$\begin{cases} J_1^2 \leq CR^{\sigma_1}, \\ J_2^2 \leq CR^{\sigma_2}, \end{cases}$$

where

$$\sigma_1 = \frac{4l_1l_2(\beta_2 - m_2) + 2l_1(m_2 - 1)(\beta_1 - m_1) + 2Q[(m_2 - 1)(-l_1 - m_1 + 1) + 2l_1l_2]}{4l_1l_2 - (m_2 - 1)(m_1 - 1)},$$

and

$$\sigma_2 = \frac{4l_2l_1(\beta_1 - m_1) + 2l_2(m_1 - 1)(\beta_2 - m_2) + 2Q[(m_2 - 1)(-l_2 - m_2 + 1) + 2l_1l_2]}{4l_1l_2 - (m_1 - 1)(m_2 - 1)}.$$

From the estimates of Step 3, it immediately follows that (u, v) is constant. We recall that $l_1 = \frac{p_2+q_2}{2}$, $l_2 = \frac{p_1+q_1}{2}$. Consequently, from (3.39), we note that either $\sigma_1 \leq 0$ or $\sigma_2 \leq 0$ if and only if (3.25) holds. In the case where $\sigma_1 \leq 0$, the integral J_1, which is increasing in R, is uniformly bounded with respect to R. Using the dominated convergence theorem, we obtain the following:

$$\lim_{R\to+\infty} J_1 = 0,$$

which implies that $u \equiv 0$ in H^n. The proof in the case where $\sigma_2 \leq 0$ is analogous.

4. Conclusions

We established the criteria for the nonexistence of nonnegative solutions to quasilinear elliptic inequalities and divergence-type systems on the Heisenberg group. Through contradiction arguments rooted in maximum principle-type inequalities, we demonstrate that specific critical exponents governed by the group's homogeneous dimension and the nonlinearities' interaction act as thresholds beyond which solutions cannot exist. Further investigations could explore applications to coupled systems with mixed nonlinearities, weighted function spaces, or time-dependent analogues, where similar geometric constraints may impose novel nonexistence regimes.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest.

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