AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 10(5): 12447–12463.
DOI: 10.3934/math.2025562
Received: 14 March 2025
Revised: 09 May 2025
Accepted: 22 May 2025
Published: 28 May 2025

Research article

Null controllability of Atangana-Baleanu fractional stochastic systems with Poisson jumps and fractional Brownian motion

Yazid Alhojilan[1,*] **and Hamdy M. Ahmed**[2]

[1] Department of Mathematics, College of Science, Qassim University, Saudi Arabia
[2] Department of Physics and Engineering Mathematics, Higher Institute of Engineering, El-Shorouk Academy, El-Shorouk City, Cairo, Egypt

* **Correspondence:** Email: yhjielan@qu.edu.sa.

Abstract: Null controllability is an essential concept in control theory, guaranteeing that the state of a system can be controlled to reach zero. We focused on investigating the sufficient conditions for the null controllability of Atangana-Baleanu (A-B) fractional stochastic differential equations (SDEs) involving Poisson jumps and fractional Brownian motion (fBm) within Hilbert space, a significant area of research in control theory and stochastic analysis. We employed a combination of tools including fractional analysis, compact semigroup theory, fixed point theorems, and stochastic analysis to derive the desired results. An example is included to illustrate the application of our findings.

Keywords: Atangana-Baleanu fractional derivative; stochastic systems; mild solutions; control theory; Poisson random measures
Mathematics Subject Classification: 93C10, 34K37, 60J65, 93B05

1. Introduction

Fractional calculus constitutes a realm within mathematical analysis wherein the traditional notions of differentiation and integration are expanded to encompass non-integer or fractional orders. Fractional stochastic differential equations (SDEs) serve as invaluable tools for modeling intricate systems amidst uncertainty, offering a more faithful depiction of the dynamic evolution of such systems over time. Theoretical foundations of fractional SDEs have been extensively developed [1–3], while their applications span a wide range of fields, including finance, engineering, and economics [4–6].

In recent years, a substantial body of research has focused on the existence, uniqueness, and stability of solutions to fractional SDEs, which serves as the foundational requirement for their mathematical and physical validity. Various analytical tools, including fixed-point theorems, stochastic semigroup theory, and measure-theoretic techniques, have been employed to establish these properties under different fractional operators and noise sources [7–9]. Another key area of development is the averaging principle, which is crucial for analyzing multi-scale stochastic systems. By separating fast and slow dynamics, the averaging technique simplifies the system while preserving essential stochastic and fractional features [10–12].

The concept of controllability lies at the heart of control theory, serving as a fundamental paradigm for understanding the ability to steer the dynamics of a system from one state to another through appropriate control inputs. While the controllability of deterministic systems has been extensively studied and well-understood, the advent of stochasticity introduces additional complexities and challenges, giving rise to a rich area of research focused on the controllability of SDEs. For example, in [13], the author established the sufficient conditions for controllability of fractional stochastic delay equations. The approximate controllability of impulsive neutral stochastic differential equations with fractional Brownian motion in a Hilbert space was examined in [14]. Kavitha et al. [15] studied the controllability of Hilfer fractional differential equations with infinite delay via measures of noncompactness. Luo and Huang [16] investigated the relative controllability for conformable impulsive delay differential equations. Haque, Ali, and Nieto [17] discussed the controllability of psi-Hilfer fractional differential equations with infinite delay via measure of noncompactness. Zhao et al. [18] explored the approximate controllability and optimal control for fractional systems characterized by multiple delays.

Null controllability is a fundamental concept in control theory, particularly in the context of partial differential equations (PDEs) and systems governed by dynamical laws. In other words, a system is said to be null controllable if, starting from any initial condition, there exists a control input that can drive the system to a state where all state variables are zero, or within an acceptable range of desired values, in a finite time [19–21].

In recent years, the introduction of the Atangana-Baleanu fractional derivative has sparked significant interest in the study of fractional DEs, offering a novel perspective on the dynamics of complex systems with memory and long-range dependence [22]. Several authors have investigated the fractional DEs containing A-B fractional derivatives, for example, Syam and Al-Refai [23] established existence and uniqueness results to the linear and nonlinear fractional differential equations with Atangana–Baleanu fractional derivative. Kaliraj et al. [24] studied the controllability for impulsive integro-differential equation via Atangana–Baleanu fractional derivative. Devi and Kumar [25] explored the existence and uniqueness results for integro fractional differential equations with Atangana-Baleanu fractional derivative. Ahmed et al. [26] established the approximate controllability of Sobolev-type Atangana-Baleanu fractional differential inclusions with noise effect and Poisson jumps. Balasubramaniam [27] derived the necessary and sufficient conditions for the controllability of Atangana-Baleanu-Caputo neutral fractional differential equations. However, there have been no documented studies in the literature concerning the null controllability of Atangana–Baleanu fractional SDEs incorporating fBm and Poisson jumps . Inspired by this gap in research, we aim to explore the null controllability of such Atangana–Baleanu fractional SDEs with fBm and Poisson jumps in Hilbert space, structured as follows:

$$\begin{cases} {}^{ABC}\mathscr{D}^{\aleph}_{0+}\varkappa(\mathfrak{h}) = \Pi\varkappa(\mathfrak{h}) + \mathscr{B}\psi(\mathfrak{h}) + \mathscr{N}(\mathfrak{h},\varkappa(\mathfrak{h})) + \mathscr{W}(\mathfrak{h},\varkappa(\mathfrak{h}))\frac{d\mathfrak{B}^{\mathscr{H}}(\mathfrak{h})}{d\mathfrak{h}} \\ \qquad\qquad + \int_{\mathscr{Z}}\mu(\mathfrak{h},\varkappa(\mathfrak{h}),\mathscr{G})\tilde{\mathfrak{N}}(d\mathfrak{h},d\mathscr{G}), \qquad \mathfrak{h}\in\mathscr{M} = (0,\mathscr{S}]. \\ \varkappa(0) = \varkappa_0, \end{cases} \tag{1.1}$$

The expression ${}^{ABC}\mathscr{D}_{0+}$ represents the A-B Caputo fractional derivative with order $\frac{1}{2} < \aleph < 1$. The function $\varkappa(\cdot)$ operates in a Hilbert space denoted as $\mathcal{K}$, equipped with an inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. The term $\mathfrak{B}^{\mathscr{H}}$ signifies a fBm on another separable and real Hilbert space $\mathscr{Y}$, characterized by a Hurst parameter $\frac{1}{2} < \mathscr{H} < 1$.

The operator $\Pi : \mathfrak{D}(\Pi) \subset \mathcal{K} \to \mathcal{K}$ acts as the infinitesimal generator of a family of $\aleph$-resolvent denoted as $(\mathfrak{S}_{\aleph}(\mathfrak{h}))_{\mathfrak{h}\geq 0}$ and $(\mathfrak{Q}_{\aleph}(\mathfrak{h}))_{\mathfrak{h}\geq 0}$, defined on a separable Hilbert space $\mathcal{K}$. The control function $\psi(\cdot)$ is specified within $\mathfrak{L}_2(\mathscr{M}, \mathfrak{U})$, where $\mathfrak{U}$ represents another separable Hilbert space of admissible control functions. Here, $\mathscr{B}$ denotes a bounded linear operator mapping from $\mathfrak{U}$ to $\mathcal{K}$.

Additionally, there are nonlinear functions represented by $\mathscr{N} : \mathscr{M}\times\mathcal{K} \to \mathcal{K}, \mu : \mathscr{M}\times\mathcal{K}\times\mathscr{Z} \to \mathcal{K}$, and $\mathscr{W} : \mathscr{M} \times \mathcal{K} \to \mathfrak{L}_2^0(\mathscr{Y}, \mathcal{K})$.

The remainder of the manuscript is structured as follows: Section 2 comprises a compilation of notations, definitions, and lemmas. In Section 3, we delve into investigating the precise null controllability of the system described by Eq (1.1). Section 4 depicts an example to exemplify the theoretical results derived.

2. Preliminaries

Definition 2.1. *(see [22]) Let* $\mathfrak{g} \in H^1(a, b)$*,* $a < b$ *and* $0 < \aleph < 1$*. The A-B fractional derivative of a function* $\mathfrak{g}$ *in Caputo sense is defined as:*

$$ {}^{ABC}\mathscr{D}_{a^+}^{\aleph}\mathfrak{g}(\mathfrak{h}) = \frac{\varpi(\aleph)}{1-\aleph}\int_a^{\mathfrak{h}} \mathfrak{g}'(\mathscr{E})\mathbb{M}_{\aleph}(-\theta(\mathfrak{h}-\mathscr{E})^{\aleph})d\mathscr{E},\ \theta = \frac{\aleph}{1-\aleph}, \tag{2.1}$$

where the function

$$\mathbb{M}_{\aleph}(\mathscr{G}) = \sum_{n=0}^{\infty} \frac{\mathscr{G}^n}{\Gamma(n\aleph + 1)}$$

denotes the Mittag-Leffler function.

Additionally, the normalization function, denoted by $\varpi(\aleph)$, is expressed as $(1 - \aleph) + \frac{\aleph}{\Gamma(\aleph)}$. It is defined in such a way that $\varpi(0) = \varpi(1) = 1$.

The expression for the fractional integral of A-B is given as:

$$ {}^{AB}I_{a^+}^{\aleph}\mathfrak{g}(\mathfrak{h}) = \frac{1-\aleph}{\varpi(\aleph)}\mathfrak{g}(\mathfrak{h}) + \frac{\aleph}{\varpi(\aleph)\Gamma(\aleph)}\int_a^{\mathfrak{h}} (\mathfrak{h}-\mathscr{E})^{\aleph-1}\mathfrak{g}(\mathscr{E})d\mathscr{E}. \tag{2.2}$$

Fix a time interval $[0, \mathscr{S}]$ and let $(\Omega, \xi, \mathscr{P})$ be a complete probability space equipped with a comprehensive collection of right-continuous increasing sub σ-algebras $\{\xi_{\mathfrak{h}} : \mathfrak{h} \in [0, \mathscr{S}]\}$ all nested within ξ. Assume $(\mathscr{Z}, \Psi, \mathfrak{b}(d\mathscr{G}))$ is a measurable space with σ-finite. A stationary Poisson point process $(\mathfrak{p}_{\mathfrak{h}})_{\mathfrak{h}\geq 0}$, defined on $(\Omega, \xi, \mathscr{P})$ with values in $\mathscr{Z}$ and with characteristic measure $\mathfrak{b}$. We represent as $\mathfrak{N}(\mathfrak{h}, d\mathscr{G})$ the counting measure of $\mathfrak{p}_{\mathfrak{h}}$ where $\tilde{\mathfrak{N}}(\mathfrak{h}, \Theta) := E(\mathfrak{N}(\mathfrak{h}, \Theta)) = \mathfrak{h}\mathfrak{b}(\Theta)$ for $\Theta \in \Psi$. We define $\tilde{\mathfrak{N}}(\mathfrak{h}, d\mathscr{G}) := \mathfrak{N}(\mathfrak{h}, d\mathscr{G}) - \mathfrak{h}\mathfrak{b}(d\mathscr{G})$, which represents the Poisson martingale measure generated by $\mathfrak{p}_{\mathfrak{h}}$.

Here, $\mathfrak{L}(\mathscr{Y}, \mathcal{K})$ represents the space of bounded linear operators from $\mathscr{Y}$ to $\mathcal{K}$. We assume $\mathfrak{Q} \in \mathfrak{L}(\mathscr{Y}, \mathscr{Y})$ be an operator defined by $\mathfrak{Q}\tau_n = \mathfrak{b}_n\tau_n$ where the trace of $\mathfrak{Q}$, denoted $tr\mathfrak{Q}$, is finite ($tr\mathfrak{Q} = \sum_{n=1}^{\infty} \mathfrak{b}_n < \infty$). Here, $\mathfrak{b}_n \geq 0$ $(n = 1, 2, ...)$ are non-negative real numbers and $\{\tau_n\}$ $(n = 1, 2, ...)$ forms a complete orthonormal basis in $\mathscr{Y}$. $\|\cdot\|$ constitutes the norm in $\mathfrak{L}(\mathscr{Y}, \mathcal{K})$, $\mathscr{Y}$ and $\mathcal{K}$.

We establish the fBm in $\mathscr{Y}$ as follows:

$$\mathfrak{B}^{\mathscr{H}}(\mathfrak{h}) = \mathfrak{B}^{\mathscr{H}}_{\mathfrak{Q}}(\mathfrak{h}) = \sum_{n=1}^{\infty} \sqrt{\mathfrak{b}_n}\tau_n\beta^{\mathscr{H}}_n(\mathfrak{h}).$$

The variables $\beta^{\mathscr{H}}_n$ represent real, independent fBms.

We introduce the space $\mathfrak{L}^0_2$, denoted as $\mathfrak{L}^0_2(\mathscr{Y},\mathcal{K})$, encompassing all $\mathfrak{Q}$-Hilbert Schmidt operators $\eta : \mathscr{Y} \to \mathcal{K}$. Recall that a $\eta \in \mathfrak{L}(\mathscr{Y},\mathcal{K})$ is termed a $\mathfrak{Q}$-Hilbert-Schmidt operator if the expression $\|\eta\|^2_{\mathfrak{L}^0_2} := \sum_{n=1}^{\infty} \| \sqrt{\mathfrak{b}_n}\eta\tau_n\|^2$ is finite. Additionally, the space $\mathfrak{L}^0_2$, endowed with $\langle\vartheta,\eta\rangle_{\mathfrak{L}_{2^0}} = \sum_{n=1}^{\infty}\langle\vartheta\tau_n,\eta\tau_n\rangle$, forms a separable Hilbert space.

Lemma 2.2. *(see [28]) If function $\eta : [0,\mathscr{S}] \to \mathfrak{L}^0_2(\mathscr{Y},\mathcal{K})$ meets the condition $\int_0^{\mathscr{S}} \|\eta(\mathscr{E})\|^2_{\mathfrak{L}^0_2} < \infty$, then we can conclude that*

$$E\left\|\int_0^{\mathfrak{h}} \eta(\mathscr{E})d\mathfrak{B}^{\mathscr{H}}(\mathscr{E})\right\|^2 \leq 2\mathscr{H}\mathfrak{h}^{2\mathscr{H}-1}\int_0^{\mathfrak{h}} \|\eta(\mathscr{E})\|^2_{\mathfrak{L}^0_2}d\mathscr{E}.$$

Definition 2.3. *(see [29]) The set of resolvents, denoted $\rho(\Pi)$, consists of complex numbers ζ for which the operator $(\zeta - \Pi) : \mathfrak{D}(\Pi) \to \mathcal{K}$ is a bijective mapping. According to the closed graph theorem, the operator $\mathfrak{R}(\zeta,\Pi) = (\zeta - \Pi)^{-1}$ is bounded for $\zeta \in \rho(\Pi)$ on $\mathcal{K}$, serving as the resolvent of Π at ζ. Consequently, for all $\zeta \in \rho(\Pi)$, the equation $\Pi\mathfrak{R}(\zeta,\Pi) = \zeta\mathfrak{R}(\zeta,\Pi) - I$ holds true.*

Definition 2.4. *(see [29]) If Π is a linear and closed sectorial operator, then $\exists\, \hbar > 0$, $\mathfrak{I}$ real, and Λ within the interval $[\frac{\pi}{2},\pi]$, such that (s.t.)*
(i) $\sum_{\Lambda,\mathfrak{I}} = \{\zeta \in \mathbb{C} : \zeta \neq \mathfrak{I}, |\arg(\zeta - \mathfrak{I})| < \Lambda\} \subset \rho(\Pi)$,
(ii) $\|\mathfrak{R}(\zeta,\Pi)\| \leq \frac{\hbar}{|\zeta-\mathfrak{I}|}$, $\zeta \in \sum_{\Lambda,\mathfrak{I}}$,
are verified.

Consider $\mathfrak{C}(\mathscr{M},\mathfrak{L}_2(\Omega,\mathcal{K}))$, the Banach space comprising all continuous mappings from $\mathscr{M}$ to $\mathfrak{L}_2(\Omega,\mathcal{K})$, where each function satisfies the condition $\sup_{\mathfrak{h}\in\mathscr{M}} E\|\varkappa(\mathfrak{h})\|^2 < \infty$.

Let $\bar{\mathfrak{C}}$ denote the set $\{\varkappa : \varkappa(\cdot) \in \mathfrak{C}(\mathscr{M},\mathfrak{L}_2(\Omega,\mathcal{K}))\}$, with its norm $\|\cdot\|_{\bar{\mathfrak{C}}}$ defined as

$$\|\cdot\|_{\bar{\mathfrak{C}}} = (\sup_{\mathfrak{h}\in\mathscr{M}} E\|\varkappa(\mathfrak{h})\|^2)^{\frac{1}{2}}.$$

Definition 2.5. *We define $\varkappa \in \bar{\mathfrak{C}}$ as a mild solution to (1.1) if it meets the condition:*

$$\begin{aligned}
\varkappa(\mathfrak{h}) \;=\; & F\mathfrak{S}_{\aleph}(\mathfrak{h})\varkappa_0 + \frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}[\mathscr{N}(\mathscr{E},\varkappa(\mathscr{E})) + \mathscr{B}\psi(\mathscr{E})]d\mathscr{E} \\
& + \frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\mathscr{W}(\mathscr{E},\varkappa(\mathscr{E}))d\mathfrak{B}^{\mathscr{H}}(\mathscr{E}) \\
& + \frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\int_{\mathscr{Z}}\mu(\mathscr{E},\varkappa(\mathscr{E}),\mathscr{G})\tilde{\mathfrak{N}}(d\mathscr{E},d\mathscr{G}) \\
& + \frac{\aleph F^2}{V(\aleph)}\int_0^{\mathfrak{h}}\mathfrak{Q}_{\aleph}(\mathfrak{h}-\mathscr{E})[\mathscr{N}(\mathscr{E},\varkappa(\mathscr{E}))+\mathscr{B}\psi(\mathscr{E})]d\mathscr{E} + \frac{\aleph F^2}{V(\aleph)}\int_0^{\mathfrak{h}}\mathfrak{Q}_{\aleph}(\mathfrak{h}-\mathscr{E})\mathscr{W}(\mathscr{E},\varkappa(\mathscr{E}))d\mathfrak{B}^{\mathscr{H}}(\mathscr{E}) \\
& + \frac{\aleph F^2}{V(\aleph)}\int_0^{\mathfrak{h}}\mathfrak{Q}_{\aleph}(\mathfrak{h}-\mathscr{E})\int_{\mathscr{Z}}\mu(\mathscr{E},\varkappa(\mathscr{E}),\mathscr{G})\tilde{\mathfrak{N}}(d\mathscr{E},d\mathscr{G}),
\end{aligned}$$

where $F = \vartheta^*(\vartheta^* I - \Pi)^{-1}$ and $\wp = -\delta^*\Pi(\vartheta^* I - \Pi)^{-1}$, with $\vartheta^* = \frac{V(\aleph)}{1-\aleph}$, $\quad \delta^* = \frac{\aleph}{1-\aleph}$,

$$\begin{aligned}\mathfrak{S}_\aleph(\mathfrak{h}) &= \mathbb{M}_\aleph(-\wp\mathfrak{h}^\aleph) = \frac{1}{2\pi i}\int_\Upsilon e^{\mathscr{E}t}\mathscr{E}^{\aleph-1}(\mathscr{E}^\aleph I - \wp)^{-1}d\mathscr{E},\\ \mathfrak{Q}_\aleph(\mathfrak{h}) &= \mathfrak{h}^{\aleph-1}\mathbb{M}_{\aleph,\aleph}(-\wp\mathfrak{h}^\aleph) = \frac{1}{2\pi i}\int_\Upsilon e^{\mathscr{E}t}(\mathscr{E}^\aleph I - \wp)^{-1}d\mathscr{E},\end{aligned}$$

and the path Υ is lying on $\sum_{\Lambda,\mathfrak{I}}$.

3. Null controllability investigation

Here, we examine the null controllability for (1.1).

If $\Pi \in \Pi^\aleph(\varrho_0, \mathfrak{I}_0)$ where $\Pi^\aleph(\varrho_0, \mathfrak{I}_0)$ denotes operators that generate fractional resolvent families, then for $C_1 > 0$ and $C_2 > 0$, the following holds:

$$\|\mathfrak{S}_\aleph(\mathfrak{h})\| \le C_1 e^{\mathfrak{I}\mathfrak{h}} \text{ and } \|\mathfrak{Q}_\aleph(\mathfrak{h})\| \le C_2 e^{\mathfrak{I}\mathfrak{h}}(1 + \mathfrak{h}^{\aleph-1}), \text{ for every } \mathfrak{h} > 0,\ \mathfrak{I} > \mathfrak{I}_0.$$

Let $C_3 = \sup_{\mathfrak{h}\ge 0}\|\mathfrak{S}_\aleph(\mathfrak{h})\|$, $C_4 = \sup_{\mathfrak{h}\ge 0} C_2 e^{\mathfrak{I}\mathfrak{h}}(1 + \mathfrak{h}^{\aleph-1})$. So we get

$$\|\mathfrak{S}_\aleph(\mathfrak{h})\| \le C_3, \quad \|\mathfrak{Q}_\aleph(\mathfrak{h})\| \le C_4\, \mathfrak{h}^{\aleph-1} \text{ [30]}.$$

To examine the null controllability of Eq (1.1), we analyze the fractional stochastic linear system

$$\begin{cases} {}^{ABC}\mathscr{D}^\aleph_{0+}\lambda(\mathfrak{h}) = \Pi\lambda(\mathfrak{h}) + \mathscr{N}(\mathfrak{h}) + \mathscr{B}\psi(\mathfrak{h}) + \mathscr{W}(\mathfrak{h})\frac{d\mathfrak{B}^{\mathscr{H}}(\mathfrak{h})}{d\mathfrak{h}}, & \mathfrak{h} \in \mathscr{M} = (0, \mathscr{S}],\\ \lambda(0) = \lambda_0, \end{cases} \tag{3.1}$$

associated with the system (1.1).

Consider

$$\mathfrak{L}_0^{\mathscr{S}}\psi = \frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathscr{S}}(\mathscr{S}-\mathscr{E})^{\aleph-1}\mathfrak{B}\psi(\mathscr{E})d\mathscr{E} + \frac{\aleph F^2}{V(\aleph)}\int_0^{\mathscr{S}}\mathfrak{Q}_\aleph(\mathscr{S}-\mathscr{E})\mathfrak{B}\psi(\mathscr{E})d\mathscr{E}\ : \mathfrak{L}_2(\mathscr{M}, \mathfrak{U}) \to \mathcal{K},$$

where $\mathfrak{L}_0^{\mathscr{S}}\psi$ possesses a bounded inverse operator denoted as $(\mathfrak{L}_0)^{-1}$, operating within the space $\mathfrak{L}_2(\mathscr{M}, \mathfrak{U})/\ker(\mathfrak{L}_0^{\mathscr{S}})$, and

$$\begin{aligned}\mathfrak{N}_0^{\mathscr{S}}(\lambda, \mathscr{N}, \mathscr{W}) &= F\mathfrak{S}_\aleph(\mathscr{S})\lambda + \frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathscr{S}}(\mathscr{S}-\mathscr{E})^{\aleph-1}\mathscr{N}(\mathscr{E})d\mathscr{E}\\ &\quad + \frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathscr{S}}(\mathscr{S}-\mathscr{E})^{\aleph-1}\mathscr{W}(\mathscr{E})d\mathfrak{B}^{\mu}(\mathscr{E})\\ &\quad + \frac{\aleph F^2}{V(\aleph)}\int_0^{\mathscr{S}}\mathfrak{Q}_\aleph(\mathscr{S}-\mathscr{E})\mathscr{N}(\mathscr{E})d\mathscr{E}\\ &\quad + \frac{\aleph F^2}{V(\aleph)}\int_0^{\mathscr{S}}\mathfrak{Q}_\aleph(\mathscr{S}-\mathscr{E})\mathscr{W}(\mathscr{E})d\mathfrak{B}^{\mathscr{H}}(\mathscr{E}) : \mathcal{K} \times \mathfrak{L}_2(\mathscr{M}, \mathcal{K}) \to \mathcal{K}.\end{aligned}$$

Definition 3.1. *(see [31]) The system described by Eq (3.1) is termed exact null controllable over $\mathscr{M}$ if $Im\mathfrak{L}_0^{\mathscr{S}} \supset Im\mathfrak{N}_0^{\mathscr{S}}$ or $\exists$ a $\kappa > 0$ s.t. $\|(\mathfrak{L}_0^{\mathscr{S}})^*\lambda\|^2 \ge \kappa\|(\mathfrak{N}_0^{\mathscr{S}})^*\lambda\|^2\ \forall\ \lambda \in \mathcal{K}$.*

Lemma 3.2. *(see [32]) Assume that (3.1) exhibits exact null controllability over the interval $\mathscr{M}$. Consequently, the operator $(\mathfrak{L}_0)^{-1}\mathfrak{N}_0^{\mathscr{S}} \times \mathfrak{L}_2(\mathscr{M}, \mathcal{K}) \to \mathfrak{L}_2(\mathscr{M}, \psi)$ is bounded, and the control*

$$\begin{aligned}\psi(\mathfrak{h}) \;=\; & -(\mathfrak{L}_0)^{-1}\Big[F\mathfrak{S}_{\aleph}(\mathscr{S})\lambda_0 + \frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathscr{S}}(\mathscr{S}-\mathscr{E})^{\aleph-1}\mathscr{N}(\mathscr{E})d\mathscr{E} \\ & + \frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathscr{S}}(\mathscr{S}-\mathscr{E})^{\aleph-1}\mathscr{W}(\mathscr{E})d\mathfrak{B}^{\mathscr{H}}(\mathscr{E}) \\ & + \frac{\aleph F^2}{V(\aleph)}\int_0^{\mathscr{S}}\mathfrak{Q}_{\aleph}(\mathscr{S}-\mathscr{E})\mathscr{N}(\mathscr{E})d\mathscr{E} + \frac{\aleph F^2}{V(\aleph)}\int_0^{\mathscr{S}}\mathfrak{Q}_{\aleph}(\mathscr{S}-\mathscr{E})\mathscr{W}(\mathscr{E})d\mathfrak{B}^{\mathscr{H}}(\mathscr{E})\Big](\mathfrak{h})\end{aligned}$$

drives the system described by Eq (3.1) from an initial state λ_0 to the zero state.

Here, $\mathfrak{L}_0$ represents the restriction of $\mathfrak{L}_0^{\mathscr{S}}$ to $[\ker \mathfrak{L}_0^{\mathscr{S}}]^{\perp}$, while $\mathscr{N}$ belongs to $\mathfrak{L}_2(\mathscr{M}, \mathcal{K})$ and $\mathscr{W}$ belongs to $\mathfrak{L}_2^0(\mathscr{M}, \mathfrak{L}(\lambda, \mathcal{K}))$.

Definition 3.3. *The system defined by Eq (1.1) is deemed exact null controllable over $\mathscr{M}$ if $\exists$ a stochastic control $\psi \in \mathfrak{L}_2(\mathscr{M}, \mathfrak{U})$ s.t. the solution $\varkappa(\mathfrak{h})$ of (1.1) meets the condition $\varkappa(\mathscr{S}) = 0$.*

Let us impose the following assumptions:
(H_0) $(\mathfrak{S}_{\aleph})(\mathfrak{h})_{\mathfrak{h}\geq 0}$ and $(\mathfrak{Q}_{\aleph})(\mathfrak{h})_{\mathfrak{h}\geq 0}$ are compact.
(H_1) The fractional linear system described by Eq (3.1) is exactly null controllable over $\mathscr{M}$.
(H_2) $\mathscr{N} : \mathscr{M} \times \mathcal{K} \to \mathcal{K}$ meets the following:
(i) $\mathscr{N}$ be continuous. Suppose $\mathscr{N} \in \bar{\mathfrak{C}}\ \forall\ \mathcal{K} \in \bar{\mathfrak{C}}$, which guarantees ${}^{ABC}\mathscr{D}_{0+}^{\aleph}\mathcal{K} \in \bar{\mathfrak{C}}$ exists.
(ii) $\forall\ \mathfrak{q} \in \mathfrak{N},\ \mathfrak{q} > 0$, $\exists$ a positive function $\mathscr{N}_{\mathfrak{q}}(\cdot) : \mathscr{M} \to \mathscr{R}^+$ s.t.

$$\sup_{\|\varkappa\|^2 \leq \mathfrak{q}} E\|\mathscr{N}(\mathfrak{h}, \varkappa)\|^2 \leq \mathscr{N}_{\mathfrak{q}}(\mathfrak{h}),$$

$s \to (\mathfrak{h} - \mathscr{E})^{\aleph-1}\mathscr{N}_{\mathfrak{q}}(\mathscr{E}) \in L^1([0, \mathfrak{h}], \mathfrak{R}^+)$, and $\exists$ a $\delta > 0$ s.t.

$$\liminf_{\mathfrak{q}\to\infty} \frac{\int_0^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\mathscr{N}_{\mathfrak{q}}(\mathscr{E})d\mathscr{E}}{q} = \delta < \infty,\ \mathfrak{h} \in \mathscr{M}.$$

(H_3) $\mathscr{W} : \mathscr{M} \times \mathcal{K} \to \mathfrak{L}_2^0(\mathfrak{K}, \mathcal{K})$ fulfills the following:
(i) $\mathscr{W} : J \times \mathcal{K} \to \mathfrak{L}_2^0(\mathfrak{K}, \mathcal{K})$ is continuous function.
(ii) $\forall\ \mathfrak{q} > 0$; $\mathfrak{q} \in \mathfrak{N}$, $\exists$ a positive function $\mathfrak{g}_{\mathfrak{q}}(\cdot) : \mathscr{M} \to \mathfrak{R}^+$ s.t.

$$\sup_{\|\varkappa\|^2 \leq q} E\|\mathscr{W}(\mathfrak{h}, x)\|^2_{\mathfrak{L}_{2^0}} \leq \mathfrak{g}_{\mathfrak{q}}(\mathfrak{h}),$$

$s \to (\mathfrak{h} - \mathscr{E})^{\aleph-1}\mathfrak{g}_{\mathfrak{q}}(\mathscr{E}) \in L^1([0, \mathfrak{h}], \mathfrak{R}^+)$, and $\exists$ a $\delta > 0$ s. t.

$$\liminf_{\mathfrak{q}\to\infty} \frac{\int_0^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\mathfrak{g}_{\mathfrak{q}}(\mathscr{E})d\mathscr{E}}{\mathfrak{q}} = \delta < \infty,\ \mathfrak{h} \in \mathscr{M}.$$

(H_4) $\mu : \mathscr{M} \times \mathcal{K} \times \mathscr{Z} \to \mathcal{K}$ verifies the following:
(i) $\mu : \mathscr{M} \times \mathcal{K} \times \mathscr{Z} \to \mathcal{K}$ is continuous function.

(ii) $\forall$ $\mathfrak{q} > 0$; $\mathfrak{q} \in \mathfrak{R}$, $\exists$ a positive function $\chi_{\mathfrak{q}}(\cdot) : \mathscr{M} \to \mathfrak{R}^+$ s.t.

$$\sup_{\|\varkappa\|^2 \leq \mathfrak{q}} \int_{\mathscr{Z}} E\|\mu(\mathfrak{h}, \varkappa(\mathfrak{h}), \mathscr{G})\|^2 \mathfrak{b}(d\mathscr{G}) \leq \chi_{\mathfrak{q}}(\mathfrak{h}),$$

$s \to (\mathfrak{h} - \mathscr{E})^{\aleph-1}\chi_{\mathfrak{q}}(\mathscr{E}) \in \mathfrak{L}^1([0, \mathfrak{h}], \mathfrak{R}^+)$, and $\exists$ a $\delta > 0$ s.t.

$$\liminf_{\mathfrak{q}\to\infty} \frac{\int_0^{\mathfrak{h}} (\mathfrak{h} - \mathscr{E})^{\aleph-1}\chi_{\mathfrak{q}}(\mathscr{E})d\mathscr{E}}{\mathfrak{q}} = \delta < \infty,\ \mathfrak{h} \in \mathscr{M}.$$

Theorem 3.4. *Let* $(H_0) - (H_4)$ *hold, then (1.1) is exactly null controllable over* $\mathscr{M}$ *s.t.*

$$\frac{10\delta\mathscr{S}^{\aleph} + 5\delta\mathscr{H}\mathscr{S}^{2\mathscr{H}+\aleph-1}}{\aleph}\Big[\frac{\|\wp\|^2\|F\|^2(1-\aleph)^2}{V^2(\aleph)\Gamma^2(\aleph)} + \frac{\aleph^2\|F\|^4C_4^2}{V^2(\aleph)}\Big]$$
$$\times\Big[1 + \frac{5\|\mathscr{B}\|^2\|\mathfrak{L}_0^{-1}\|^2\mathscr{S}^{2\aleph-1}}{2\aleph-1}\Big(\frac{\|\wp\|^2\|F\|^2(1-\aleph)^2}{V^2(\aleph)\Gamma^2(\aleph)} + \frac{\aleph^2\|F\|^4C_4^2}{V^2(\aleph)}\Big)\Big] < 1. \tag{3.2}$$

Proof. For any function $\varkappa(\cdot)$, the operator Φ on $\bar{\mathfrak{C}}$ is defined in the following manner

$$\begin{aligned}(\Phi\varkappa)(\mathfrak{h}) &= F\mathfrak{S}_{\aleph}(\mathfrak{h})\varkappa_0 + \frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}[\mathscr{N}(\mathscr{E},\varkappa(\mathscr{E})) + \mathscr{B}\psi(\mathscr{E})]d\mathscr{E}\\
&+\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\mathscr{W}(\mathscr{E},\varkappa(\mathscr{E}))d\mathfrak{B}^{\mathscr{H}}(\mathscr{E})\\
&+\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\int_{\mathscr{Z}}\mu(\mathscr{E},\varkappa(\mathscr{E}),\mathscr{G})\tilde{\mathfrak{N}}(d\mathscr{E},d\mathscr{G})\\
&+\frac{\aleph F^2}{V(\aleph)}\int_0^{\mathfrak{h}}\mathfrak{Q}_{\aleph}(\mathfrak{h}-\mathscr{E})[\mathscr{N}(\mathscr{E},\varkappa(\mathscr{E}))\\
&+\mathscr{B}\psi(\mathscr{E})]d\mathscr{E} + \frac{\aleph F^2}{V(\aleph)}\int_0^{\mathfrak{h}}\mathfrak{Q}_{\aleph}(\mathfrak{h}-\mathscr{E})\mathscr{W}(\mathscr{E},\varkappa(\mathscr{E}))d\mathfrak{B}^{\mathscr{H}}(\mathscr{E})\\
&+\frac{\aleph F^2}{V(\aleph)}\int_0^{\mathfrak{h}}\mathfrak{Q}_{\aleph}(\mathfrak{h}-\mathscr{E})\int_{\mathscr{Z}}\mu(\mathscr{E},\varkappa(\mathscr{E}),\mathscr{G})\tilde{\mathfrak{N}}(d\mathscr{E},d\mathscr{G}),\ \mathfrak{h}\in\mathscr{M},\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}\psi(\mathfrak{h}) &= -(\mathfrak{L}_0)^{-1}\Big[F\mathfrak{S}_{\aleph}(\mathscr{S})\varkappa_0 + \frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathscr{S}}(\mathscr{S}-\mathscr{E})^{\aleph-1}\mathscr{N}(\mathscr{E},\varkappa(\mathscr{E}))d\mathscr{E}\\
&+\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathscr{S}}(\mathscr{S}-\mathscr{E})^{\aleph-1}\mathscr{W}(\mathscr{E},\varkappa(\mathscr{E}))d\mathfrak{B}^{\mathscr{H}}(\mathscr{E})\\
&+\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathscr{S}}(\mathscr{S}-\mathscr{E})^{\aleph-1}\int_{\mathscr{Z}}\mu(\mathscr{E},\varkappa(\mathscr{E}),\mathscr{G})\tilde{\mathfrak{N}}(d\mathscr{E},d\mathscr{G})\\
&+\frac{\aleph F^2}{V(\aleph)}\int_0^{\mathscr{S}}\mathfrak{Q}_{\aleph}(\mathscr{S}-\mathscr{E})\mathscr{N}(\mathscr{E},\varkappa(\mathscr{E}))d\mathscr{E} + \frac{\aleph F^2}{V(\aleph)}\int_0^{\mathscr{S}}\mathfrak{Q}_{\aleph}(\mathscr{S}-\mathscr{E})\mathscr{W}(\mathscr{E},\varkappa(\mathscr{E}))d\mathfrak{B}^{\mathscr{H}}(\mathscr{E})\\
&+\frac{\aleph F^2}{V(\aleph)}\int_0^{\mathscr{S}}\mathfrak{Q}_{\aleph}(\mathscr{S}-\mathscr{E})\int_{\mathscr{Z}}\mu(\mathscr{E},\varkappa(\mathscr{E}),\mathscr{G})\tilde{\mathfrak{N}}(d\mathscr{E},d\mathscr{G})\Big].\end{aligned}$$

We will demonstrate that Φ, mapping from $\bar{\mathfrak{C}}$ to itself possesses a fixed point. For integer $\mathfrak{q} > 0$, put $\mathfrak{B}_{\mathfrak{q}} = \{\iota \in \bar{\mathfrak{C}}, \|\iota\|^2_{\bar{\mathfrak{C}}} \leq \mathfrak{q}\}$. We assume that, there exists $\mathfrak{q} > 0$ s.t. $\Phi(\mathfrak{B}_{\mathfrak{q}}) \subseteq \mathfrak{B}_{\mathfrak{q}}$. If it is not true, then, for $\mathfrak{q} > 0$, there exists a function $\varkappa_{\mathfrak{q}}(\cdot) \in \mathfrak{B}_{\mathfrak{q}}$, s.t. $\Phi(\varkappa_{\mathfrak{q}}) \notin \mathfrak{B}_{\mathfrak{q}}$. Specifically, there exists $\mathfrak{h} = \mathfrak{h}(\mathfrak{q}) \in \mathscr{M}$, where $\mathfrak{h}(\mathfrak{q})$ depends on $\mathfrak{q}$, s.t. $\|\Phi(\varkappa_{\mathfrak{q}}))(\mathfrak{h})\|^2_{\bar{\mathfrak{C}}} > \mathfrak{q}$.

From (H_2) in conjunction with the Hölder inequality, we derive

$$\begin{aligned}
&E\left\|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\mathscr{N}(\mathscr{E},\varkappa(\mathscr{E}))d\mathscr{E}+\frac{\aleph F^2}{V(\aleph)}\int_0^{\mathfrak{h}}\mathfrak{Q}_{\aleph}(\mathfrak{h}-\mathscr{E})\mathscr{N}(\mathscr{E},\varkappa(\mathscr{E}))d\mathscr{E}\right\|^2\\
\leq\ &\left\{\left[\frac{\|\wp\|\|F\|(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\right]^2+\left[\frac{\aleph\|F\|^2C_4}{V(\aleph)}\right]^2\right\}E\left[\int_0^{\mathfrak{h}}\|(\mathfrak{h}-\mathscr{E})^{\aleph-1}\mathscr{N}(\mathscr{E},\varkappa(\mathscr{E}))\|d\mathscr{E}\right]^2\\
\leq\ &\left\{\left[\frac{\|\wp\|\|F\|(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\right]^2+\left[\frac{\aleph\|F\|^2C_4}{V(\aleph)}\right]^2\right\}\int_0^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}d\mathscr{E}\int_0^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}E\|\mathscr{N}(\mathscr{E},\varkappa(\mathscr{E}))\|^2d\mathscr{E}\\
\leq\ &\frac{\mathscr{S}^{\aleph}}{\aleph}\left\{\left[\frac{\|\wp\|\|F\|(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\right]^2+\left[\frac{\aleph\|F\|^2C_4}{V(\aleph)}\right]^2\right\}\int_0^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\mathscr{N}_{\mathfrak{q}}(\mathscr{E})d\mathscr{E}. \qquad (3.4)
\end{aligned}$$

Also, from Burkholder-Gungy's inequality and Lemma 2.2 along with $(H3)$, yields

$$\begin{aligned}
&E\left\|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\mathscr{W}(\mathscr{E},\varkappa(\mathscr{E}))d\mathfrak{B}^{\mathscr{H}}(\mathscr{E})+\frac{\aleph F^2}{V(\aleph)}\int_0^{\mathfrak{h}}\mathfrak{Q}_{\aleph}(\mathfrak{h}-\mathscr{E})\mathscr{W}(\mathscr{E},\varkappa(\mathscr{E}))d\mathfrak{B}^{\mathscr{H}}(\mathscr{E})\right\|^2\\
\leq\ &2\mathscr{H}\mathscr{S}^{2\mathscr{H}-1}\left\{\left[\frac{\|\wp\|\|F\|(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\right]^2+\left[\frac{\aleph\|F\|^2C_4}{V(\aleph)}\right]^2\right\}E\left[\int_0^{\mathfrak{h}}\|(\mathfrak{h}-\mathscr{E})^{\aleph-1}\mathscr{W}(\mathscr{E},\varkappa(\mathscr{E}))\|_{\mathfrak{L}_2^0}d\mathscr{E}\right]^2\\
\leq\ &2\mathscr{H}\mathscr{S}^{2\mathscr{H}-1}\left\{\left[\frac{\|\wp\|\|F\|(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\right]^2+\left[\frac{\aleph\|F\|^2C_4}{V(\aleph)}\right]^2\right\}\int_0^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}d\mathscr{E}\int_0^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}E\|\mathscr{W}(\mathscr{E},\varkappa(\mathscr{E}))\|^2_{\mathfrak{L}_2^0}d\mathscr{E}\\
\leq\ &\frac{2\mathscr{H}\mathscr{S}^{2\mathscr{H}+\aleph-1}}{\aleph}\left\{\left[\frac{\|\wp\|\|F\|(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\right]^2+\left[\frac{\aleph\|F\|^2C_4}{V(\aleph)}\right]^2\right\}\int_0^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\mathfrak{g}_q(\mathscr{E})d\mathscr{E}. \qquad (3.5)
\end{aligned}$$

From Hölder inequality and $(H4)$, we obtain

$$\begin{aligned}
&E\left\|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\int_{\mathscr{Z}}\mu(\mathscr{E},\varkappa(\mathscr{E}),\mathscr{G})\tilde{\mathfrak{N}}(d\mathscr{E},d\mathscr{G})+\frac{\aleph F^2}{V(\aleph)}\int_0^{\mathfrak{h}}\mathfrak{Q}_{\aleph}(\mathfrak{h}-\mathscr{E})\int_{\mathscr{Z}}\mu(\mathscr{E},\varkappa(\mathscr{E}),\mathscr{G})\tilde{\mathfrak{N}}(d\mathscr{E},d\mathscr{G})\right\|^2\\
\leq\ &\left\{\left[\frac{\|\wp\|\|F\|(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\right]^2+\left[\frac{\aleph\|F\|^2C_4}{V(\aleph)}\right]^2\right\}E\left[\int_0^{\mathfrak{h}}\|(\mathfrak{h}-\mathscr{E})^{\aleph-1}\int_{\mathscr{Z}}\mu(\mathscr{E},\varkappa(\mathscr{E}),\mathscr{G})\tilde{\mathfrak{N}}(d\mathscr{E},d\mathscr{G})\|\right]^2\\
\leq\ &\left\{\left[\frac{\|\wp\|\|F\|(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\right]^2+\left[\frac{\aleph\|F\|^2C_4}{V(\aleph)}\right]^2\right\}\int_0^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}d\mathscr{E}\int_0^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\int_{\mathscr{Z}}E\|\mu(\mathscr{E},\varkappa(\mathscr{E}),\mathscr{G})\|^2\mathfrak{b}(d\mathscr{G})d\mathscr{E}\\
\leq\ &\frac{\mathscr{S}^{\aleph}}{\aleph}\left\{\left[\frac{\|\wp\|\|F\|(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\right]^2+\left[\frac{\aleph\|F\|^2C_4}{V(\aleph)}\right]^2\right\}\int_0^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\chi_{\mathfrak{q}}(\mathscr{E})d\mathscr{E}. \qquad (3.6)
\end{aligned}$$

However, from (3.4)–(3.6), we get

$$\begin{aligned}
&\sup_{\mathfrak{h}\in\mathscr{M}}E\left\|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\mathscr{B}\psi(\mathscr{E})d\mathscr{E}+\frac{\aleph F^2}{V(\aleph)}\int_0^{\mathfrak{h}}\mathfrak{Q}_{\aleph}(\mathfrak{h}-\mathscr{E})\mathscr{B}\psi(\mathscr{E})d\mathscr{E}\right\|^2\\
\leq\ &\frac{\|\mathscr{B}\|^2\|\mathfrak{L}_0^{-1}\|^2\mathscr{S}^{2\aleph-1}}{2\aleph-1}\left(\left[\frac{\|\wp\|\|F\|(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\right]^2+\left[\frac{\aleph\|F\|^2C_4}{V(\aleph)}\right]^2\right)\Big\{5\|F\|^2C_3^2E\|\varkappa_0\|^2\\
&+\frac{5\mathscr{S}^{\aleph}}{\aleph}\left(\left[\frac{\|\wp\|\|F\|(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\right]^2+\left[\frac{\aleph\|F\|^2C_4}{V(\aleph)}\right]^2\right)\int_0^{\mathscr{S}}(\mathscr{S}-\mathscr{E})^{\aleph-1}\mathscr{N}_{\mathfrak{q}}(\mathscr{E})d\mathscr{E}
\end{aligned}$$

$$+\frac{10H\mathscr{S}^{2H+\aleph-1}}{\aleph}\Big(\Big[\frac{\|\wp\|\|F\|(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\Big]^2+\Big[\frac{\aleph\|F\|^2C_4}{V(\aleph)}\Big]^2\Big)\int_0^{\mathscr{S}}(\mathscr{S}-\mathscr{E})^{\aleph-1}\mathfrak{g}_{\mathfrak{q}}(\mathscr{E})d\mathscr{E}$$
$$+\frac{5\mathscr{S}^{\aleph}}{\aleph}\Big(\Big[\frac{\|\wp\|\|F\|(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\Big]^2+\Big[\frac{\aleph\|F\|^2C_4}{V(\aleph)}\Big]^2\Big)\int_0^{\mathscr{S}}(\mathscr{S}-\mathscr{E})^{\aleph-1}\chi_{\mathfrak{q}}(\mathscr{E})d\mathscr{E}\Big\}. \tag{3.7}$$

$$\begin{aligned}
\mathfrak{q} \leq\ & \|\Phi(\varkappa_{\mathfrak{q}})(\mathfrak{h})\|^2_{\mathcal{C}}=\sup_{\mathfrak{h}\in\mathscr{M}}E\|\Phi(\varkappa_{\mathfrak{q}})(\mathfrak{h})\|^2\\
\leq\ & 5\sup_{\mathfrak{h}\in\mathscr{M}}E\|F\mathfrak{S}_{\aleph}(\mathfrak{h})\varkappa_0\|^2+5\sup_{\mathfrak{h}\in\mathscr{M}}E\Big\|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\mathscr{N}(\mathscr{E},\varkappa(\mathscr{E}))d\mathscr{E}\\
&+\frac{\aleph F^2}{V(\aleph)}\int_0^{\mathfrak{h}}\mathfrak{Q}_{\aleph}(\mathfrak{h}-\mathscr{E})\mathscr{N}(\mathscr{E},\varkappa(\mathscr{E}))d\mathscr{E}\Big\|^2\\
&+5\sup_{\mathfrak{h}\in\mathscr{M}}E\Big\|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\mathscr{B}\psi(\mathscr{E})d\mathscr{E}+\frac{\aleph F^2}{V(\aleph)}\int_0^{\mathfrak{h}}\mathfrak{Q}_{\aleph}(\mathfrak{h}-\mathscr{E})\mathscr{B}\psi(\mathscr{E})d\mathscr{E}\Big\|^2\\
&+5\sup_{\mathfrak{h}\in\mathscr{M}}E\Big\|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\mathscr{W}(\mathscr{E},\varkappa(\mathscr{E}))d\mathfrak{B}^{\mathscr{H}}(\mathscr{E}) \qquad (3.8)\\
&+\frac{\aleph F^2}{V(\aleph)}\int_0^{\mathfrak{h}}\mathfrak{Q}_{\aleph}(\mathfrak{h}-\mathscr{E})\mathscr{W}(\mathscr{E},\varkappa(\mathscr{E}))d\mathfrak{B}^{\mathscr{H}}(\mathscr{E})\Big\|^2\\
&+5\sup_{\mathfrak{h}\in\mathscr{M}}E\Big\|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\int_{\mathscr{Z}}\mu(\mathscr{E},\varkappa(\mathscr{E}),\mathscr{G})\tilde{\mathfrak{N}}(d\mathscr{E},d\mathscr{G})\\
&+\frac{\aleph F^2}{V(\aleph)}\int_0^{\mathfrak{h}}\mathfrak{Q}_{\aleph}(\mathfrak{h}-\mathscr{E})\int_{\mathscr{Z}}\mu(\mathscr{E},\varkappa(\mathscr{E}),\mathscr{G})\tilde{\mathfrak{N}}(d\mathscr{E},d\mathscr{G})\Big\|^2\\
\leq\ & 5\|F\|^2C_3^2E\|\varkappa_0\|^2+\frac{5\mathscr{S}^{\aleph}}{\aleph}\Big(\Big[\frac{\|\wp\|\|F\|(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\Big]^2+\Big[\frac{\aleph\|F\|^2C_4}{V(\aleph)}\Big]^2\Big)\int_0^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\mathscr{N}_{\mathfrak{q}}(\mathscr{E})d\mathscr{E}\\
&+\frac{10\mathscr{H}\mathscr{S}^{2\mathscr{H}+\aleph-1}}{\aleph}\Big(\Big[\frac{\|\wp\|\|F\|(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\Big]^2+\Big[\frac{\aleph\|F\|^2C_4}{V(\aleph)}\Big]^2\Big)\int_0^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\mathfrak{g}_q(\mathscr{E})d\mathscr{E}\\
&+\frac{5\mathscr{S}^{\aleph}}{\aleph}\Big(\Big[\frac{\|\wp\|\|F\|(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\Big]^2+\Big[\frac{\aleph\|F\|^2C_4}{V(\aleph)}\Big]^2\Big)\int_0^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\chi_{\mathfrak{q}}(\mathscr{E})d\mathscr{E}\\
&+\frac{5\|\mathscr{B}\|^2\|\mathfrak{L}_0^{-1}\|^2\mathscr{S}^{2\aleph-1}}{2\aleph-1}\Big(\Big[\frac{\|\wp\|\|F\|(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\Big]^2+\Big[\frac{\aleph\|F\|^2C_4}{V(\aleph)}\Big]^2\Big)\Big\{5\|F\|^2C_3^2E\|\varkappa_0\|^2\\
&+\frac{5\mathscr{S}^{\aleph}}{\aleph}\Big(\Big[\frac{\|\wp\|\|F\|(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\Big]^2+\Big[\frac{\aleph\|F\|^2C_4}{V(\aleph)}\Big]^2\Big)\int_0^{\mathscr{S}}(\mathscr{S}-\mathscr{E})^{\aleph-1}\mathscr{N}_{\mathfrak{q}}(\mathscr{E})d\mathscr{E}\\
&+\frac{10H\mathscr{S}^{2H+\aleph-1}}{\aleph}\Big(\Big[\frac{\|\wp\|\|F\|(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\Big]^2+\Big[\frac{\aleph\|F\|^2C_4}{V(\aleph)}\Big]^2\Big)\int_0^{\mathscr{S}}(\mathscr{S}-\mathscr{E})^{\aleph-1}\mathfrak{g}_{\mathfrak{q}}(\mathscr{E})d\mathscr{E}\\
&+\frac{5\mathscr{S}^{\aleph}}{\aleph}\Big(\Big[\frac{\|\wp\|\|F\|(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\Big]^2+\Big[\frac{\aleph\|F\|^2C_4}{V(\aleph)}\Big]^2\Big)\int_0^{\mathscr{S}}(\mathscr{S}-\mathscr{E})^{\aleph-1}\chi_{\mathfrak{q}}(\mathscr{E})d\mathscr{E}\Big\}. \qquad (3.9)
\end{aligned}$$

By dividing both sides of (3.8) by $\mathfrak{q}$ and letting $\mathfrak{q}\to+\infty$, we obtain

$$\frac{10\delta\mathscr{S}^{\aleph}+5\delta\mathscr{H}\mathscr{S}^{2H+\aleph-1}}{\aleph}\Big[\frac{\|\wp\|^2\|F\|^2(1-\aleph)^2}{V^2(\aleph)\Gamma^2(\aleph)}+\frac{\aleph^2\|F\|^4C_4^2}{V^2(\aleph)}\Big]$$
$$\times\Big[1+\frac{5\|\mathscr{B}\|^2\|\mathfrak{L}_0^{-1}\|^2\mathscr{S}^{2\aleph-1}}{2\aleph-1}\Big(\frac{\|\wp\|^2\|F\|^2(1-\aleph)^2}{V^2(\aleph)\Gamma^2(\aleph)}+\frac{\aleph^2\|F\|^4C_4^2}{V^2(\aleph)}\Big)\Big]\geq 1.$$

This contradicts (3.2). Therefore, $\Phi(\mathfrak{B}_{\mathfrak{q}})\subseteq\mathfrak{B}_{\mathfrak{q}}$, for $\mathfrak{q}>0$.

Indeed, Φ maps $\mathfrak{B}_{\mathfrak{q}}$ into a compact subset of $\mathfrak{B}_{\mathfrak{q}}$. In reality, Φ maps $\mathfrak{B}_{\mathfrak{q}}$ into a compact subset of itself. To establish this, we begin by demonstrating that $\mathfrak{B}_{\mathfrak{q}}(\mathfrak{h})=\{(\Phi\varkappa)(\mathfrak{h}):\varkappa\in\mathfrak{B}_{\mathfrak{q}}\}$ is precompact in $\mathcal{K}$, $\forall\ \mathfrak{h}\in\mathscr{M}$. This is trivial for $\mathfrak{h}=0$, because $\mathfrak{B}_{\mathfrak{q}}(0)=\{\varkappa_0\}$. Now, consider a fixed $\mathfrak{h}$, where $0<\mathfrak{h}\leq\mathscr{S}$. For $0<\epsilon<\mathfrak{h}$, take

$$\begin{aligned}(\Phi^{\epsilon}\varkappa)(\mathfrak{h}) &= F\mathfrak{S}_{\aleph}(\mathfrak{h})\varkappa_0+\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathfrak{h}-\epsilon}(\mathfrak{h}-\mathscr{E})^{\aleph-1}[\mathscr{N}(\mathscr{E},\varkappa(\mathscr{E}))+\mathscr{B}\psi(\mathscr{E})]d\mathscr{E}\\
&\quad+\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathfrak{h}-\epsilon}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\mathscr{W}(\mathscr{E},\varkappa(\mathscr{E}))d\mathfrak{B}^{\mathscr{H}}(\mathscr{E})\\
&\quad+\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathfrak{h}-\epsilon}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\int_{\mathscr{Z}}\mu(\mathscr{E},\varkappa(\mathscr{E}),\mathscr{G})\tilde{\mathfrak{N}}(d\mathscr{E},d\mathscr{G})\\
&\quad+\frac{\aleph F^2}{V(\aleph)}\int_0^{\mathfrak{h}-\epsilon}\mathfrak{Q}_{\aleph}(\mathfrak{h}-\mathscr{E})[\mathscr{N}(\mathscr{E},\varkappa(\mathscr{E}))+\mathscr{B}\psi(\mathscr{E})]d\mathscr{E}\\
&\quad+\frac{\aleph F^2}{V(\aleph)}\int_0^{\mathfrak{h}-\epsilon}\mathfrak{Q}_{\aleph}(\mathfrak{h}-\mathscr{E})\mathscr{W}(\mathscr{E},\varkappa(\mathscr{E}))d\mathfrak{B}^{\mathscr{H}}(\mathscr{E})\\
&\quad+\frac{\aleph F^2}{V(\aleph)}\int_0^{\mathfrak{h}-\epsilon}\mathfrak{Q}_{\aleph}(\mathfrak{h}-\mathscr{E})\int_{\mathscr{Z}}\mu(\mathscr{E},\varkappa(\mathscr{E}),\mathscr{G})\tilde{\mathfrak{N}}(d\mathscr{E},d\mathscr{G}).\end{aligned}$$

From H_0, the set $\mathfrak{B}^{\epsilon}(\mathfrak{h})=\{(\Phi^{\epsilon}\varkappa)(\mathfrak{h}):\varkappa\in\mathfrak{B}_{\mathfrak{q}}\}$ is a precompact set in $\mathcal{K}$ $\forall\ \epsilon$ where $0<\epsilon<\mathfrak{h}$.

Furthermore, for any $\varkappa\in\mathfrak{B}_{\mathfrak{q}}$, we have

$$\begin{aligned}&\|(\Phi\varkappa)(\mathfrak{h})-(\Phi^{\epsilon}\varkappa)(\mathfrak{h})\|_{\bar{C}}^2=\sup_{\mathfrak{h}\in\mathscr{M}}E\|(\Phi\varkappa)(\mathfrak{h})-(\Phi^{\epsilon}\varkappa)(\mathfrak{h})\|^2\\
\leq\ & 5\sup_{\mathfrak{h}\in\mathscr{M}}E\Big\|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_{\mathfrak{h}-\epsilon}^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\mathscr{N}(\mathscr{E},\varkappa(\mathscr{E}))d\mathscr{E}+\frac{\aleph F^2}{V(\aleph)}\int_{\mathfrak{h}-\epsilon}^{\mathfrak{h}}\mathfrak{Q}_{\aleph}(\mathfrak{h}-\mathscr{E})\mathscr{N}(\mathscr{E},\varkappa(\mathscr{E}))d\mathscr{E}\Big\|^2\\
&+5\sup_{\mathfrak{h}\in\mathscr{M}}E\Big\|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_{\mathfrak{h}-\epsilon}^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\mathscr{B}\psi(\mathscr{E})d\mathscr{E}+\frac{\aleph F^2}{V(\aleph)}\int_{\mathfrak{h}-\epsilon}^{\mathfrak{h}}\mathfrak{Q}_{\aleph}(\mathfrak{h}-\mathscr{E})\mathscr{B}\psi(\mathscr{E})d\mathscr{E}\Big\|^2\\
&+5\sup_{\mathfrak{h}\in\mathscr{M}}E\Big\|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_{\mathfrak{h}-\epsilon}^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\mathscr{W}(\mathscr{E},\varkappa(\mathscr{E}))d\mathfrak{B}^{\mathscr{H}}(\mathscr{E})\\
&+\frac{\aleph F^2}{V(\aleph)}\int_{\mathfrak{h}-\epsilon}^{\mathfrak{h}}\mathfrak{Q}_{\aleph}(\mathfrak{h}-\mathscr{E})\mathscr{W}(\mathscr{E},\varkappa(\mathscr{E}))d\mathfrak{B}^{\mathscr{H}}(\mathscr{E})\Big\|^2\\
&+5\sup_{\mathfrak{h}\in\mathscr{M}}E\Big\|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_{\mathfrak{h}-\epsilon}^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\int_{\mathscr{Z}}\mu(\mathscr{E},\varkappa(\mathscr{E}),\mathscr{G})\tilde{\mathfrak{N}}(d\mathscr{E},d\mathscr{G})\\
&+\frac{\aleph F^2}{V(\aleph)}\int_{\mathfrak{h}-\epsilon}^{\mathfrak{h}}\mathfrak{Q}_{\aleph}(\mathfrak{h}-\mathscr{E})\int_{\mathscr{Z}}\mu(\mathscr{E},\varkappa(\mathscr{E}),\mathscr{G})\tilde{\mathfrak{N}}(d\mathscr{E},d\mathscr{G})\Big\|^2\end{aligned}$$

$$
\begin{aligned}
\leq\ & \frac{5\epsilon^{\aleph}}{\aleph}\Big(\Big[\frac{\|\wp\|\|F\|(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\Big]^2+\Big[\frac{\aleph\|F\|^2C_4}{V(\aleph)}\Big]^2\Big)\int_{\mathfrak{h}-\epsilon}^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\mathscr{N}_{\mathfrak{q}}(\mathscr{E})d\mathscr{E}\\
&+\frac{10\mathscr{H}\epsilon^{2\mathscr{H}+\aleph-1}}{\aleph}\Big(\Big[\frac{\|\wp\|\|F\|(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\Big]^2+\Big[\frac{\aleph\|F\|^2C_4}{V(\aleph)}\Big]^2\Big)\int_{\mathfrak{h}-\epsilon}^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\mathfrak{g}_{\mathfrak{q}}(\mathscr{E})d\mathscr{E}\\
&+\frac{5\epsilon^{\aleph}}{\aleph}\Big(\Big[\frac{\|\wp\|\|F\|(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\Big]^2+\Big[\frac{\aleph\|F\|^2C_4}{V(\aleph)}\Big]^2\Big)\int_{\mathfrak{h}-\epsilon}^{\mathfrak{h}}(\mathfrak{h}-\mathscr{E})^{\aleph-1}\chi_{\mathfrak{q}}(\mathscr{E})d\mathscr{E}\\
&+\frac{\|\mathscr{B}\|^2\|\mathfrak{L}_0^{-1}\|^2\epsilon^{2\aleph-1}}{2\aleph-1}\Big(\Big[\frac{\|\wp\|\|F\|(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\Big]^2+\Big[\frac{\aleph\|F\|^2C_4}{V(\aleph)}\Big]^2\Big)\Big\{5\|F\|^2C_3^2E\|\varkappa_0\|^2\\
&+\frac{5\epsilon^{\aleph}}{\aleph}\Big(\Big[\frac{\|\wp\|\|F\|(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\Big]^2+\Big[\frac{\aleph\|F\|^2C_4}{V(\aleph)}\Big]^2\Big)\int_{\mathscr{S}-\epsilon}^{\mathscr{S}}(\mathscr{S}-\mathscr{E})^{\aleph-1}\mathscr{N}_{\mathfrak{q}}(\mathscr{E})d\mathscr{E}\\
&+\frac{10\mathscr{H}\epsilon^{2H+\aleph-1}}{\aleph}\Big(\Big[\frac{\|\wp\|\|F\|(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\Big]^2+\Big[\frac{\aleph\|F\|^2C_4}{V(\aleph)}\Big]^2\Big)\int_{\mathscr{S}-\epsilon}^{\mathscr{S}}(\mathscr{S}-\mathscr{E})^{\aleph-1}\mathfrak{g}_{\mathfrak{q}}(\mathscr{E})d\mathscr{E}\\
&+\frac{\epsilon^{\aleph}}{\aleph}\Big(\Big[\frac{\|\wp\|\|F\|(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\Big]^2+\Big[\frac{\aleph\|F\|^2C_4}{V(\aleph)}\Big]^2\Big)\int_{\mathscr{S}-\epsilon}^{\mathscr{S}}(\mathscr{S}-\mathscr{E})^{\aleph-1}\chi_{\mathfrak{q}}(\mathscr{E})d\mathscr{E}\Big\}.
\end{aligned}
$$

We observe that $\forall\ x\in\mathfrak{B}_{\mathfrak{q}}$, $\|(\Phi x)(\mathfrak{h})-(\Phi^{\epsilon}x)(\mathfrak{h})\|_{\bar{C}}^2\to 0$ as ϵ approaches 0^+. Thus, $\exists$ precompact sets arbitrarily close to the set $\mathfrak{B}_{\mathfrak{q}}(\mathfrak{h})$, indicating that $\mathfrak{B}_{\mathfrak{q}}(\mathfrak{h})$ itself is precompact in $\mathcal{K}$.

Next, we demonstrate that $\{\Phi\varkappa : \varkappa\in\mathfrak{B}_{\mathfrak{q}}\}$ is an equicontinuous family of functions. Let $\varkappa\in\mathfrak{B}_{\mathfrak{q}}$ and $\mathfrak{h}_1,\mathfrak{h}_2\ \in\mathscr{M}$ such that $0<\mathfrak{h}_1<\mathfrak{h}_2$, then

$$
\begin{aligned}
& \|(\Phi x)(\mathfrak{h}_2)-(\Phi x)(\mathfrak{h}_1)\|_{\bar{C}}^2\\
\leq\ & 5\|F\mathfrak{S}_{\aleph}(\mathfrak{h}_2)\varkappa_0-F\mathfrak{S}_{\aleph}(\mathfrak{h}_1)x_0\|_{\bar{C}}^2\\
&+5\|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathfrak{h}_1}[(\mathfrak{h}_2-\mathscr{E})^{\aleph-1}-(\mathfrak{h}_1-\mathscr{E})^{\aleph-1}][\mathscr{N}(\mathscr{E},\varkappa(\mathscr{E}))+\mathscr{B}\psi(\mathscr{E})]d\mathscr{E}\|_{\bar{C}}^2\\
&+5\|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_{\mathfrak{h}_1}^{\mathfrak{h}_2}[(\mathfrak{h}_2-\mathscr{E})^{\aleph-1}[\mathscr{N}(\mathscr{E},\varkappa(\mathscr{E}))+\mathscr{B}\psi(\mathscr{E})]d\mathscr{E}\|_{\bar{C}}^2\\
&+5\|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathfrak{h}_1}[(\mathfrak{h}_2-\mathscr{E})^{\aleph-1}-(\mathfrak{h}_1-\mathscr{E})^{\aleph-1}]\mathscr{W}(\mathscr{E},\varkappa(\mathscr{E}))d\mathfrak{B}^{\mathscr{H}}(\mathscr{E})\|_{\bar{C}}^2\\
&+5\|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_{\mathfrak{h}_1}^{\mathfrak{h}_2}(\mathfrak{h}_2-\mathscr{E})^{\aleph-1}\mathscr{W}(\mathscr{E},\varkappa(\mathscr{E}))d\mathfrak{B}^{\mathscr{H}}(\mathscr{E})\|_{\bar{C}}^2\\
&+5\|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathfrak{h}_1}[(\mathfrak{h}_2-\mathscr{E})^{\aleph-1}-(\mathfrak{h}_1-\mathscr{E})^{\aleph-1}]\int_{\mathscr{Z}}\mu(\mathscr{E},\varkappa(\mathscr{E}),\mathscr{G})\tilde{\mathfrak{N}}(d\mathscr{E},d\mathscr{G})\|_{\bar{C}}^2\\
&+5\|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_{\mathfrak{h}_1}^{\mathfrak{h}_2}(\mathfrak{h}_2-\mathscr{E})^{\aleph-1}\int_{\mathscr{Z}}\mu(\mathscr{E},\varkappa(\mathscr{E}),\mathscr{G})\tilde{\mathfrak{N}}(d\mathscr{E},d\mathscr{Z})\|_{\bar{C}}^2\\
&+5\|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathfrak{h}_1}[\mathfrak{Q}_{\aleph}(\mathfrak{h}_2-\mathscr{E})-\mathfrak{Q}_{\aleph}(\mathfrak{h}_1-\mathscr{E})][\mathscr{N}(\mathscr{E},\varkappa(\mathscr{E}))+\mathscr{B}\psi(\mathscr{E})]d\mathscr{E}\|_{\bar{C}}^2\\
&+5\|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_{\mathfrak{h}_2}^{\mathfrak{h}_1}\mathfrak{Q}_{\aleph}(\mathfrak{h}_2-\mathscr{E})\mathscr{N}(\mathscr{E},\varkappa(\mathscr{E}))+\mathscr{B}\psi(\mathscr{E})]d\mathscr{E}\|_{\bar{C}}^2\\
&+5\|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathfrak{h}_1}[\mathfrak{Q}_{\aleph}(\mathfrak{h}_2-\mathscr{E})-\mathfrak{Q}_{\aleph}(\mathfrak{h}_1-\mathscr{E})]\mathscr{W}(\mathscr{E},\varkappa(\mathscr{E}))d\mathfrak{B}^{\mathscr{H}}(\mathscr{E})\|_{\bar{C}}^2
\end{aligned}
$$

$$+5\|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_{\mathfrak{h}_1}^{\mathfrak{h}_2}\mathfrak{Q}_{\aleph}(\mathfrak{h}_2-\mathscr{E})\mathscr{W}(\mathscr{E},\varkappa(\mathscr{E}))d\mathfrak{B}^{\mathscr{H}}(\mathscr{E})\|_{\tilde{C}}^2$$
$$+5\|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_0^{\mathfrak{h}_1}[\mathfrak{Q}_{\aleph}(\mathfrak{h}_2-\mathscr{E})-\mathfrak{Q}_{\aleph}(\mathfrak{h}_1-\mathscr{E})]\int_{\mathscr{Z}}\mu(\mathscr{E},\varkappa(\mathscr{E}),\mathscr{G})\tilde{\mathfrak{N}}(d\mathscr{E},d\mathscr{G})\|_{\tilde{C}}^2$$
$$+5\|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\int_{\mathfrak{h}_1}^{\mathfrak{h}_2}\mathfrak{Q}_{\aleph}(\mathfrak{h}_2-\mathscr{E})\int_{\mathscr{Z}}\mu(\mathscr{E},\varkappa(\mathscr{E}),\mathscr{G})\tilde{\mathfrak{N}}(d\mathscr{E},d\mathscr{G})\|_{\tilde{C}}^2.$$

Based on the earlier observation, we note that $\|(\Phi\varkappa)(\mathfrak{h}_2)-(\Phi\varkappa)(\mathfrak{h}_1)\|_{\tilde{C}}^2\to 0$ independently of $\varkappa\in\mathfrak{B}_{\mathfrak{q}}$ as $\mathfrak{h}_2$ tends to $\mathfrak{h}_1$. The compactness of $\mathfrak{S}_{\aleph}(\mathfrak{h})$ and $\mathfrak{Q}_{\aleph}(\mathfrak{h})$ for $\mathfrak{h}>0$ ensures continuity in the uniform operator topology. Therefore, $\Phi(\mathfrak{B}_{\mathfrak{q}})$ exhibits both equicontinuity and boundedness. According to Arzela-Ascoli theorem, $\Phi(\mathfrak{B}_{\mathfrak{q}})$ is precompact in $\mathcal{K}$. Therefore, Φ is a completely continuous operator on $\mathcal{K}$. By the Schauder fixed point theorem, Φ possesses a fixed point in $\mathfrak{B}_{\mathfrak{q}}$. Any fixed point of Φ serves as a mild solution to (1.1) over $\mathscr{M}$. Consequently, (1.1) has exact null controllability on $\mathscr{M}$. □

4. Illustration

To validate the obtained results, we examine the A-B fractional stochastic PDE with fBm and Poisson jumps as follows:

$$\begin{cases} {}^{ABC}\mathscr{D}_{0+}^{\frac{3}{4}}\varkappa(\mathfrak{h},\mathfrak{f})=\frac{\partial^2}{\partial\mathfrak{f}^2}\varkappa(\mathfrak{h},\mathfrak{f})+\psi(\mathfrak{h},\mathfrak{f})+\mathscr{N}(\mathfrak{h},\varkappa(\mathfrak{h},\mathfrak{f}))+\mathscr{W}(\mathfrak{h},\varkappa(\mathfrak{h},\mathfrak{f}))\frac{d\mathfrak{B}^{\mathscr{H}}(\mathfrak{h})}{d\mathfrak{h}}\\ +\int_{\mathscr{Z}}\mu(\mathfrak{h},\varkappa(\mathfrak{h},\mathfrak{f}),\mathscr{G})\tilde{\mathfrak{N}}(d\mathfrak{h},d\mathscr{G}),\ \ \mathfrak{h}\in\mathscr{M},\ 0<\mathfrak{f}<1,\\ \varkappa(\mathfrak{h},0)=\varkappa(\mathfrak{h},1)=0,\ \mathfrak{h}\in\mathscr{M},\\ \varkappa(0,\mathfrak{f})=\varkappa_0,\ 0\le\mathfrak{f}\le 1, \end{cases} \tag{4.1}$$

where ${}^{ABC}\mathscr{D}_{0+}^{\frac{3}{4}}$ is the A-B derivative, of order $\frac{3}{4}$ and $\mathfrak{B}^{\mathscr{H}}$ is a fBm.

Let $\varkappa(\mathfrak{h})(\mathfrak{f})=\varkappa(\mathfrak{h},\mathfrak{f})$, $\mathscr{N}(\mathfrak{h},\varkappa(\mathfrak{h}))(\mathfrak{f})=\mathscr{N}(\mathfrak{h},\varkappa(\mathfrak{h},\mathfrak{f}))$, $\mathscr{W}(\mathfrak{h},\varkappa(\mathfrak{h}))(\mathfrak{f})=\mathscr{W}(\mathfrak{h},\varkappa(\mathfrak{h},\mathfrak{f}))$ and $\mu(\mathfrak{h},\varkappa(\mathfrak{h}),\mathscr{G})(\mathfrak{f})=\mu(\mathfrak{h},\varkappa(\mathfrak{h},\mathfrak{f}),\mathscr{G})$.

We define $\mathscr{B}\psi=\psi(\mathfrak{h},\mathfrak{f}),\ 0\le\mathfrak{f}\le 1,\ \psi\in\mathfrak{U}$.

Here, consider $\mathcal{K}=\mathscr{Y}=\mathfrak{U}=\mathfrak{L}^2([0,1])$ and the operator $\Pi:\mathfrak{D}(\Pi)\subset\mathcal{K}\to\mathcal{K}$ is defined by $\Pi=\frac{\partial^2}{\partial\mathfrak{f}^2}$, where $\mathfrak{D}(\Pi)=\{\varkappa\in\mathcal{K};\varkappa,\frac{\partial\varkappa}{\partial\mathfrak{f}}$ are absolutely continuous, $\frac{\partial^2\varkappa}{\partial\mathfrak{f}^2}\in\mathcal{K},\ \varkappa(0)=\varkappa(1)=0\}$.

Therefore,

$$\Pi\varkappa=\sum_{n=1}^{\infty}n^2(\varkappa,\varkappa_n)\varkappa_n,\quad \varkappa\in\mathfrak{D}(\Pi).$$

Here, $\varkappa_n(\mathfrak{f})=\sqrt{2}\,\sin(n\pi\mathfrak{f}),\ n\in N$ represents the orthogonal set of eigenvectors of Π. Operator Π is the generator of an analytical semigroup $\mathfrak{S}(\mathfrak{h}),\mathfrak{h}>0$, acting on $\mathcal{K}$, and is defined as

$$\mathfrak{S}(\mathfrak{h})x=\sum_{n=1}^{\infty}e^{-n^2}(\varkappa,\varkappa_n)\varkappa_n,\ \ \varkappa\in\varkappa,\ \ \|\mathfrak{S}(\mathfrak{h})\|\le 1.$$

Hence, $\mathfrak{S}(\mathfrak{h}),\mathfrak{h}>0$, forms a uniformly bounded compact semigroup, implying that $\mathfrak{R}(\zeta,\Pi)=(\zeta I-\Pi)^{-1}$ is a compact operator $\forall\ \mathfrak{b}\in\rho(\Pi)$.

If $\psi\in\mathfrak{L}_2(\mathscr{M},\mathfrak{U})$, then $\mathfrak{B}=I,\ \mathfrak{B}^*=I$.

Let the fractional linear system

$$\begin{cases} {}^{ABC}\mathscr{D}_{0+}^{\frac{3}{4}}\lambda(\mathfrak{h},\mathfrak{f}) = \frac{\partial^2}{\partial \mathfrak{f}^2}\lambda(\mathfrak{h},\mathfrak{f}) + \psi(\mathfrak{h},\mathfrak{f}) + \mathscr{N}(\mathfrak{h},\mathfrak{f}) + \mathscr{W}(\mathfrak{h},\mathfrak{f})d\omega(\mathfrak{h}), \mathfrak{h}\in\mathscr{M},\ 0<\mathfrak{f}<1, \\ \lambda(\mathfrak{h},0) = \lambda(\mathfrak{h},1) = 0, \mathfrak{h}\in\mathscr{M}, \\ \lambda(0,\mathfrak{f}) = \lambda_0,\ 0\le \mathfrak{f}\le 1, \end{cases} \tag{4.2}$$

is exact null controllability if $\exists$ a $\kappa > 0$, s.t.

$$\begin{aligned} &\|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\|^2 \int_0^{\mathscr{S}} \|(\mathscr{S}-\mathscr{E})^{\aleph-1}\mathscr{B}^* y\|^2 d\mathscr{E} + \|\frac{\aleph F^2}{V(\aleph)}\|^2 \int_0^{\mathscr{S}} \|\mathfrak{Q}^*_{\aleph}(\mathscr{S}-\mathscr{E})y\|^2 d\mathscr{E} \\ &\ge \kappa[\|F\mathfrak{S}^*_{\aleph}(\mathscr{S})\lambda\|^2 + \|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\|^2 \int_0^{\mathscr{S}} \|(\mathscr{S}-\mathscr{E})^{\aleph-1}\lambda\|^2 d\mathscr{E} + \|\frac{\aleph F^2}{V(\aleph)}\|^2 \int_0^{\mathscr{S}} \|\mathfrak{Q}^*_{\aleph}(\mathscr{S}-\mathscr{E})y\|^2 d\mathscr{E}]. \end{aligned}$$

Or equivalently

$$\begin{aligned} &\|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\|^2 \int_0^{\mathscr{S}} \|(\mathscr{S}-\mathscr{E})^{\aleph-1}\lambda\|^2 d\mathscr{E} + \|\frac{\aleph F^2}{V(\aleph)}\|^2 \int_0^{\mathscr{S}} \|\mathfrak{Q}_{\aleph}(\mathscr{S}-\mathscr{E})\lambda\|^2 d\mathscr{E} \\ &\ge \kappa[\|F\mathfrak{S}_{\aleph}(\mathscr{S})\lambda\|^2 + \|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\|^2 \int_0^{\mathscr{S}} \|(\mathscr{S}-\mathscr{E})^{\aleph-1}\lambda\|^2 d\mathscr{E} + \|\frac{\aleph F^2}{V(\aleph)}\|^2 \int_0^{\mathscr{S}} \|\mathfrak{Q}_{\aleph}(\mathscr{S}-\mathscr{E})\lambda\|^2 d\mathscr{E}]. \end{aligned}$$

If $\mathscr{N} = 0$ and $\mathscr{W} = 0$ in (4.2), then the fractional linear system achieves exact null controllability provided that

$$\|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\|^2 \int_0^{\mathscr{S}} \|(\mathscr{S}-\mathscr{E})^{\aleph-1}\lambda\|^2 d\mathscr{E} + \|\frac{\aleph F^2}{V(\aleph)}\|^2 \int_0^{\mathscr{S}} \|\mathfrak{Q}_{\aleph}(\mathscr{S}-\mathscr{E})\lambda\|^2 d\mathscr{E} \ge b\|F\mathfrak{S}_{\aleph}(\mathscr{S})\lambda\|^2.$$

Therefore,

$$\begin{aligned} &\|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\|^2 \int_0^{\mathscr{S}} \|(\mathscr{S}-\mathscr{E})^{\aleph-1}\lambda\|^2 d\mathscr{E} + \|\frac{\aleph F^2}{V(\aleph)}\|^2 \int_0^{\mathscr{S}} \|\mathfrak{Q}_{\aleph}(\mathscr{S}-\mathscr{E})\lambda\|^2 d\mathscr{E} \\ &\ge \frac{b}{1+\mathscr{S}}[\|F\mathfrak{S}_{\aleph}(\mathscr{S})\lambda\|^2 + \|\frac{\wp F(1-\aleph)}{V(\aleph)\Gamma(\aleph)}\|^2 \int_0^{\mathscr{S}} \|(\mathscr{S}-\mathscr{E})^{\aleph-1}\lambda\|^2 d\mathscr{E} + \|\frac{\aleph F^2}{V(\aleph)}\|^2 \int_0^{\mathscr{S}} \|\mathfrak{Q}_{\aleph}(\mathscr{S}-\mathscr{E})\lambda\|^2 d\mathscr{E}]. \end{aligned}$$

Therefore, (4.2) achieves exact null controllability over $\mathscr{M}$. Thus, hypothesis $(H1)$ is fulfilled.

Set $\aleph = \frac{3}{4},\ \mathscr{H} = 0.75,\ \|\wp\| = 1,\ \|F\| = 1,\ \|\mathscr{B}\| = 1,\ \|\mathfrak{L}_0^{-1}\| = 1,\ C_4 = 1,\ \mathscr{S} = 0.01,\ V(\aleph) = 1$. From the above choice, system (4.1) can be written in the abstract form of (1.1) and all conditions of Theorem 3.4 are satisfied, and

$$\begin{aligned} &\frac{10\mathscr{S}^{\aleph} + 5\mathscr{H}\mathscr{S}^{2\mathscr{H}+\aleph-1}}{\aleph}\Big[\frac{\|\wp\|^2\|F\|^2(1-\aleph)^2}{V^2(\aleph)\Gamma^2(\aleph)} + \frac{\aleph^2\|F\|^4C_4^2}{V^2(\aleph)}\Big] \\ &\qquad\times\Big[1 + \frac{\|\mathscr{B}\|^2\|\mathfrak{L}_0^{-1}\|^2\mathscr{S}^{2\aleph-1}}{2\aleph-1}\Big(\frac{\|\wp\|^2\|F\|^2(1-\aleph)^2}{V^2(\aleph)\Gamma^2(\aleph)} + \frac{\aleph^2\|F\|^4C_4^2}{V^2(\aleph)}\Big)\Big] < 1, \end{aligned}$$

therefore, (4.1) achieves exact null controllability over $\mathscr{M}$.

5. Conclusions

In this work, we established sufficient conditions for the null controllability of Atangana-Baleanu fractional stochastic differential equations in Hilbert space, which involves Poisson jumps and fractional Brownian motion. The use of a combination of mathematical tools, including fractional analysis, compact semigroup theory, fixed point theorems, and stochastic analysis, proved effective in deriving these conditions. This integrated approach demonstrated its potential applicability to a broader range of stochastic control problems involving fractional dynamics. Finally, an example was included to illustrate the applicability of the major results.

For future work, we can investigate the approximate controllability for Atangana-Baleanu fractional stochastic differential inclusions involving the Clarke subdifferential and the control function on the boundary.

Author contributions

Yazid Alhojilan: Formal analysis, Validation; Hamdy M. Ahmed: Methodology, Writing–review & editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

Acknowledgments

The researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University for financial support (QU-APC-2025).

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. H. M. Ahmed, Sobolev-type fractional stochastic integrodifferential equations with nonlocal conditions in Hilbert space, *J. Theor. Probab.*, **30** (2017), 771–783. https://doi.org/10.1007/s10959-016-0665-9

2. W. Wang, S. Cheng, Z. Guo, X. Yan, A note on the continuity for Caputo fractional stochastic differential equations, *Chaos*, **30** (2020), 073106. https://doi.org/10.1063/1.5141485

3. A. Ahmadova, N. I. Mahmudov, Existence and uniqueness results for a class of fractional stochastic neutral differential equations, *Chaos, Soliton. Fract.*, **139** (2020), 110253. https://doi.org/10.1016/j.chaos.2020.110253

4. I. Ali, S. U. Khan, A dynamic competition analysis of stochastic fractional differential equation arising in finance via pseudospectral method, *Mathematics*, **11** (2023), 1328. https://doi.org/10.3390/math11061328
5. O. A. M. Omar, R. A. Elbarkouky, H. M. Ahmed, Fractional stochastic modelling of COVID-19 under wide spread of vaccinations: Egyptian case study, *Alex. Eng. J.*, **61** (2022), 8595–8609. https://doi.org/10.1016/j.aej.2022.02.002
6. B. P. Moghaddam, A. Babaei, A. Dabiri, A. Galhano, Fractional stochastic partial differential equations: numerical advances and practical applications—A state of the art review, *Symmetry*, **16** (2024), 563. https://doi.org/10.3390/sym16050563
7. D. Chalishajar, D. Kasinathan, R. Kasinathan, R. Kasinathan, Viscoelastic Kelvin–Voigt model on Ulam–Hyer's stability and T-controllability for a coupled integro fractional stochastic systems with integral boundary conditions via integral contractors, *Chaos, Soliton. Fract.*, **191** (2025), 115785. https://doi.org/10.1016/j.chaos.2024.115785
8. D. Chalishajar, D. Kasinathan, R. Kasinathan, R. Kasinathan, Ulam–Hyers–Rassias stability of Hilfer fractional stochastic impulsive differential equations with non-local condition via time-changed Brownian motion followed by the currency options pricing model, *Chaos, Soliton. Fract.*, **197** (2025), 116468. https://doi.org/10.1016/j.chaos.2025.116468
9. M. Yang, Q. Wang, Existence of mild solutions for a class of Hilfer fractional evolution equations with nonlocal conditions, *FCAA*, **20** (2017), 679–705. https://doi.org/10.1515/fca-2017-0036
10. M. Yang, Q. Huan, H. Cui, Q. Wang, Hilfer fractional stochastic evolution equations on the positive semi-axis, *Alex. Eng. J.*, **104** (2024), 386–395. https://doi.org/10.1016/j.aej.2024.07.111
11. J. Huo, M. Yang, Averaging principle for Hilfer–Katugampola fractional stochastic differential equations, *Math. Methods Appl. Sci.*, **47** (2024), 14037–14053. https://doi.org/10.1002/mma.10254
12. M. Yang, T. Lv, Q. Wang, The averaging principle for Hilfer fractional stochastic evolution equations with Lévy noise, *Fractal Fract.*, **10** (2023), 701. https://doi.org/10.3390/fractalfract7100701
13. H. M. Ahmed, Controllability of fractional stochastic delay equations, *Lobachevskii J. Math.*, **30** (2009), 195–202. https://doi.org/10.1134/S1995080209030019
14. H. M. Ahmed, Approximate controllability of impulsive neutral stochastic differential equations with fractional Brownian motion in a Hilbert space, *Adv. Differ. Equ.*, **2014** (2014), 1–11. https://doi.org/10.1186/1687-1847-2014-113
15. K. Kavitha, V. Vijayakumar, R. Udhayakumar, C. Ravichandran, Results on controllability of Hilfer fractional differential equations with infinite delay via measures of noncompactness, *Asian J. Control*, **24** (2022), 1406–1415. https://doi.org/10.1002/asjc.2549
16. D. Luo, J. Huang, Relative controllability for conformable impulsive delay differential equations, *IMA J. Math. Control Inf.*, **41** (2024), 378–400. https://doi.org/10.1093/imamci/dnae013
17. I. Haque, J. Ali, J. J. Nieto, Controllability of psi-Hilfer fractional differential equations with infinite delay via measure of noncompactness, *Nonlinear Anal.: Model. Control*, **29** (2024), 379–399. https://doi.org/10.15388/namc.2024.29.34706

18. H. Zhao, J. Zhang, J. Lu, J. Hu, Approximate controllability and optimal control in fractional differential equations with multiple delay controls, fractional Brownian motion with Hurst parameter in $0 < H < \frac{1}{2}$, and Poisson jumps, *Commun. Nonlinear Sci. Numer. Simul.*, **128** (2024), 107636. https://doi.org/10.1016/j.cnsns.2023.107636

19. T. Sathiyaraj, M. Fečkan, J. Wang, Null controllability results for stochastic delay systems with delayed perturbation of matrices, *Chaos, Soliton. Fract.*, **138** (2020), 109927. https://doi.org/10.1016/j.chaos.2020.109927

20. J. P. Dauer, N. I. Mahmudov, Exact null controllability of semilinear integrodifferential systems in Hilbert spaces, *J. Math. Anal. Appl.*, **299** (2004), 322–332. https://doi.org/10.1016/j.jmaa.2004.01.050

21. J. Wang, H. M. Ahmed, Null controllability of nonlocal Hilfer fractional stochastic differential equations, *Miskolc Math. Notes*, **18** (2017), 1073–1083. https://doi.org/10.18514/MMN.2017.2396

22. A. Atangana, D. Baleanu, New fractional derivatives with non-local and non-singular kernel: theory and application to heat transfer model, *Therm. Sci.*, **20** (2016) 763–769. https://doi.org/10.2298/TSCI160111018A

23. M. I. Syam, M. Al-Refai, Fractional differential equations with Atangana–Baleanu fractional derivative: analysis and applications, *Chaos, Soliton. Fract.: X*, **2** (2019), 100013. https://doi.org/10.1016/j.csfx.2019.100013

24. K. Kaliraj, E. Thilakraj, C. Ravichandran, K. S. Nisar, Controllability analysis for impulsive integro-differential equation via Atangana–Baleanu fractional derivative, *Math. Methods Appl. Sci.*, 2021. https://doi.org/10.1002/mma.7693

25. A. Devi, A. Kumar, Existence and uniqueness results for integro fractional differential equations with Atangana-Baleanu fractional derivative, *J. Math. Ext.*, **15** (2022), 1–24. https://doi.org/10.30495/JME.SI.2021.2128

26. A. M. Sayed Ahmed, H. M. Ahmed, N. S. E. Abdalla, A. Abd-Elmonem, E. M. Mohamed, Approximate controllability of Sobolev-type Atangana-Baleanu fractional differential inclusions with noise effect and Poisson jumps, *AIMS Math.*, **8** (2023), 25288–25310. https://doi.org/10.3934/math.20231290

27. P. Balasubramaniam, Controllability of semilinear noninstantaneous impulsive ABC neutral fractional differential equations, *Chaos, Soliton. Fract.*, **152** (2021), 111276. https://doi.org/10.1016/j.chaos.2021.111276

28. B. Boufoussi, S. Hajji, Neutral stochastic functional differential equations driven by a fractional Brownian motion in a Hilbert space, *Stat. Probabil. Lett.*, **82** (2012), 1549–1558. https://doi.org/10.1016/j.spl.2012.04.013

29. A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, New York: Springer, 1983. https://doi.org/10.1007/978-1-4612-5561-1

30. X. B. Shu, Y. Lai, Y. Chen, The existence of mild solutions for impulsive fractional partial differential equations, *Nonlinear Anal.: Theory, Method. Appl.*, **74** (2011), 2003–2011. https://doi.org/10.1016/j.na.2010.11.007

31. X. Fu, Y. Zhang, Exact null controllability of non-autonomous functional evolution systems with nonlocal conditions, *Acta Math. Sci.*, **33** (2013), 747–757. https://doi.org/10.1016/S0252-9602(13)60035-1

32. J. Y. Park, P. Balasubramaniam, Exact null controllability of abstract semilinear functional integrodifferential stochastic evolution equations in Hilbert spaces, *Taiwanese J. Math.*, **13** (2009), 2093–2103. https://doi.org/10.11650/twjm/1500405659

AIMS Press