



*Research article***Three infinite families of Hamilton-connected convex polytopes and their detour index****Sakander Hayat¹, Bagus Imanda¹, Asad Khan^{2,*} and Mohammed J. F. Alenazi³**

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Abstract: A path in a graph encompassing its whole vertex set is called Hamiltonian. Such a path with sharing the same initial and terminal vertices is called a Hamiltonian cycle. A graph comprising a Hamiltonian path (resp. cycle) is said to be traceable (resp. Hamiltonian). Graphs possessing Hamiltonian paths between every pair of their vertices are said to be Hamilton-connected. The computational complexity of evaluating a graph to be Hamilton-connected is NP-complete. A detour is the longest path in a graph. The detour index is the sum of the length of detours between every unordered pair of vertices. Computing the detour index of a graph is an NP-complete problem as well. A finite subset $P \subset \mathbb{R}^e$ is called a convex polytope if P is a convex hull. In this paper, we devised two distinct methods to prove a graph to be Hamilton-connected and employed these methods to construct some infinite families of Hamilton-connected convex polytopes. The convex polytope B_e has been shown to be non-Hamilton-connected in the literature. We showed that the existing proof for B_e is false and showed that this family is, in fact, Hamilton-connected. The paper is concluded with study implications followed by some future directions.

Keywords: graph; Hamiltonian path; Hamiltonian cycle; Hamilton-connected graph; detour index; NP-complete problems; convex polytopes

Mathematics Subject Classification: 05C38, 05C40, 05C45, 05C90

1. Introduction

In this paper, all graphs are assumed to be simple, finite, connected, and without loops. Any terms not explicitly defined can be found in Section 2.

Ore in 1963 [24] was the first graph-theorist who put forward the notion of Hamilton-connected structures, sparking significant research into the topics of Hamilton-connectedness and Hamiltonicity. Computational complexity of evaluating a graph to be Hamiltonian-connected or even Hamiltonian is categorized as NP-complete [11]. A study of cubic Hamiltonian graphs was conducted by Frucht [9] who delivered their canonical representations. In 1993, Wei [34] provided valency restrictions, which ensures a graph to possess pair-wise Hamiltonian paths. Wei [34] also demonstrated that any 3-connected graph meeting these conditions have to satisfy Hamilton-connectedness property. Yang et al. [38] presented a proof for Hamiltonicity of generalized torus graphs. Moreover, Hamiltonicity of triangular grid graphs was explored by Gordon et al. [12]. Additionally, Hamiltonian and super-Eulerian graphs were explored by Yang et al. [37] who pinpointed forbidden subgraphs in these structures. For more on applications of Hamiltonian paths and detours in communication systems and network design, we refer to [6, 13, 31, 39].

The Hamilton-connectivity of block graph's square was proven by Chartrand et al. [7]. Hamiltonian-connected tournament structures were researched by Thomasson [30]. Adequate conditions for graphs to possess Hamilton-connectedness structure were studied by Kewen et al. [18]. Zhou et al. [40] established Hamilton-connectivity conditions based on signless Laplacian spectral radius, adjacency spectral radius, and edge count. The Harary and Wiener topological invariants of large-diametrical Hamiltonian-connected graphs were computed by Zhou et al. [42]. A spectral analogy of Erdős' theorems was explored by Wei et al. [36] for Hamiltonian-connected graphs. The alphabet-grid graph and its Hamilton-connectivity was investigated by Hung et al. [16]. An extension of Nikiforov and Fiedler's results was derived by Zhou et al. [41] who considered Hamiltonian-connected graphs to have sufficiently-large smallest valency and put forward a spectral condition with respect to the signless Laplacian matrix. Network coherence is studied by Liu et al. [23] for 2^p -tree networks.

Convex polytopes' graphs are constructed by retaining the vertex-vertex adjacency as well as vertex-edge incidence relations. Bača [2–4] is among the first mathematician to study these geometric graphs. Bača in [4] and [3] investigated convex polytopes for their magic labeling, as well as graceful and anti-graceful labeling. Later, in [2], his focus was shifted to anti-magic face labeling. Imran et al. [17] determined the exact value of the metric dimension for several classes of convex polytopes and classified it as constant. Raza et al. [27] and Raza et al. [28] explored the fault-tolerance and vertex-edge resolvability in convex polytopes, respectively. Raza et al. [26] and Simić et al. [29] computed exact values of the locating-domination number of some classes of convex polytopes and established tight bounds for some other families.

1.1. The detour index

In structure-property modeling, Lukovits [20] presented a strong applicability of the detour index in chemistry. Trinajstić et al. [33] expanded on the detour index's applications to the Wiener index by conducting a comparative test, assessing its usefulness in predicting the boiling points of hydrocarbons and organic compounds. Additionally, Rücker and Rücker [25] employed the detour index to forecast the boiling points in alkanes.

Lukovits et al. [22] introduced an algorithm for tracing detours between two vertices in a graph and applied it to compute its detour indices representing fused bicyclic skeletons. Furthermore, Rücker et al. [25] and Trinajstić et al. [32] developed computational methods for identifying detours and calculating the detour index of graphs. It is noted in [14] that determining a graph's detour index is an NP-complete problem. Trinajstić et al. [32] put forward a computational method for the detour matrix for graphs of manageable size, where the detour index is from the detour matrix. The computational complexity of the longest path problem has been investigated by Fomin et al. [10].

Determining whether a graph is Hamilton-connected or calculating its detour index are known to be NP-complete problems [11, 14]. Due to this complexity, it is natural to focus on these problems for specific graph families. Here, we investigate the Hamilton-connectivity of certain infinite families of convex polytopes. We also generate certain families of Hamilton-connected convex polytopes and employ two distinct proof techniques to establish our results. Notably, we correct a significant error made by Hayat et al. [15], who incorrectly claimed that the convex polytope B_ε is not Hamilton-connected, whereas we prove that the family B_ε is indeed Hamilton-connected.

Next, we remark on the computational complexity of the longest path problem above the diameter.

Theorem 1.1. *The problem of the Longest Path above Diameter is NP-complete.*

Proof. Let G be a graph with $n \geq 2$ vertices. We construct graph G' as follows (see Figure 1):

- Construct a copy of G .
- Construct a vertex u and make it adjacent to every vertex of the copy of G .
- Construct two vertices s and t , and then (s, u) and (u, t) paths P_s and P_t , respectively, of length $n \geq 1$.

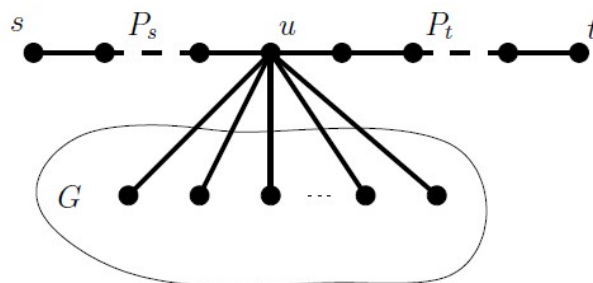


Figure 1. Construction of G' .

Notice that $\text{diam}(G) = \text{length}(P_s) + \text{length}(P_t) = 2n - 2$. It is easy to verify that G' has a path of length $2n - 1$ if and only if G has a path of length $n - 1$, that is, G is Hamiltonian. Because the Hamiltonian path is well-known to be NP-complete [11], we conclude that the Longest Path above Diameter is NP-complete. $\square$

2. Preliminaries

A graph Ω is defined as an ordered pair $\Omega = (V(\Omega), E(\Omega))$, where $V(\Omega)$ represents the vertex set (i.e., a set of points called vertices) and $E(\Omega) \subseteq \binom{V(\Omega)}{2}$ denotes the edge set (i.e., a set of lines connecting the points, known as edges). The number of vertices, denoted by $\varepsilon := |V(\Omega)|$, is referred to as the order

of Ω . For two vertices $y, z \in V(\Omega)$, the notation $y \sim z$ indicates that y and z are adjacent, meaning they are connected by an edge. For a subset $Y \subseteq V(\Omega)$ and vertices $w, x \in V(\Omega)$, if $Z = z_i : 1 \leq i \leq p$, then $w \sim z_i : 1 \leq i \leq p \sim x$ signifies that $w \sim z_1$ and $z_p \sim x$, while the adjacency between the intermediate vertices z_i ($2 \leq i \leq p$) remains unchanged. For a positive integer $\varepsilon \in \mathbb{Z}^+$, we use $\varepsilon \mid 2$ (respectively, $\varepsilon \nmid 2$) to indicate that ε is even (respectively, odd).

A cycle in a graph Ω is called Hamiltonian if it passes through all vertices of Ω exactly once. Similarly, a path in Ω is referred to as a Hamiltonian path if it visits every vertex in the graph. Not all graphs contain a Hamiltonian cycle; for instance, any tree is acyclic and therefore cannot have a Hamiltonian cycle, although it may contain a Hamiltonian path. A graph Ω is considered Hamiltonian if it possesses a Hamiltonian cycle. By definition, cycle graphs and clique graphs are Hamiltonian. A graph is said to be traceable if it has a Hamiltonian path. While every Hamiltonian graph is traceable, some traceable graphs are not Hamiltonian. A classic example is the Petersen graph, which contains a Hamiltonian path but lacks a Hamiltonian cycle. A graph is defined as Hamilton-connected if there is a Hamiltonian path between every pair of vertices in Ω .

Let Ω be a graph. A subset $S \subseteq V(\Omega)$ (resp. $T \subseteq E(\Omega)$) is called a vertex-cut (resp. edge-cut) if removing S (resp. T) from Ω results in a graph with more than one connected component. A vertex cut of size 1 corresponds to an articulation vertex. The vertex connectivity $\kappa(\Omega)$ (resp. edge connectivity $\lambda(\Omega)$) is defined as the minimum cardinality of a vertex-cut (resp. edge-cut) in Ω . A graph is said to be k -vertex-connected (resp. k -edge-connected) if its vertex-connectivity (resp. edge-connectivity) is greater than or equal to a fixed number k . In other words, a graph is k -vertex-connected or simply k -connected if there is no vertex-cut of cardinality $k - 1$ in Ω .

Let $\delta(\Omega)$ be the minimum degree of Ω . Whitney [35] in 1932 showed the following result:

Theorem 2.1. [35] *For any graph Ω , the following inequalities hold:*

$$\kappa(\Omega) \leq \lambda(\Omega) \leq \delta(\Omega).$$

The following well-known Kužel-Xiong Theorem was shown by Kužel in his Ph.D. thesis [19].

Theorem 2.2. [19] *Every 4-connected line graph is Hamiltonian if and only if it is Hamilton-connected.*

Beineke [5] independently gave the following characterization of line graphs.

Theorem 2.3. *The following statement are equivalent for a graph Ω :*

- (i) Ω is the line graph of some graph.
- (ii) The nine graphs in Figure 2 are forbidden in Ω , i.e., none of these graphs can be induced subgraphs of Ω .

See page 47 of Harary's book [14] for more information on this characterization.

For a graph Ω , let $\delta(x, y)$ represent the length of the longest path (i.e., detour) between vertices x and y in Ω . The detour index [21] is defined as the sum of the detours between all unordered pairs of vertices in Ω . The detour index of a graph Ω is typically denoted by $\omega(\Omega)$.

$$\omega(\Omega) = \sum_{\{y, z\} \subset V(\Omega)} \delta(y, z).$$

We end this section with an important and well-known result bounding the detour index in terms of graph parameters.

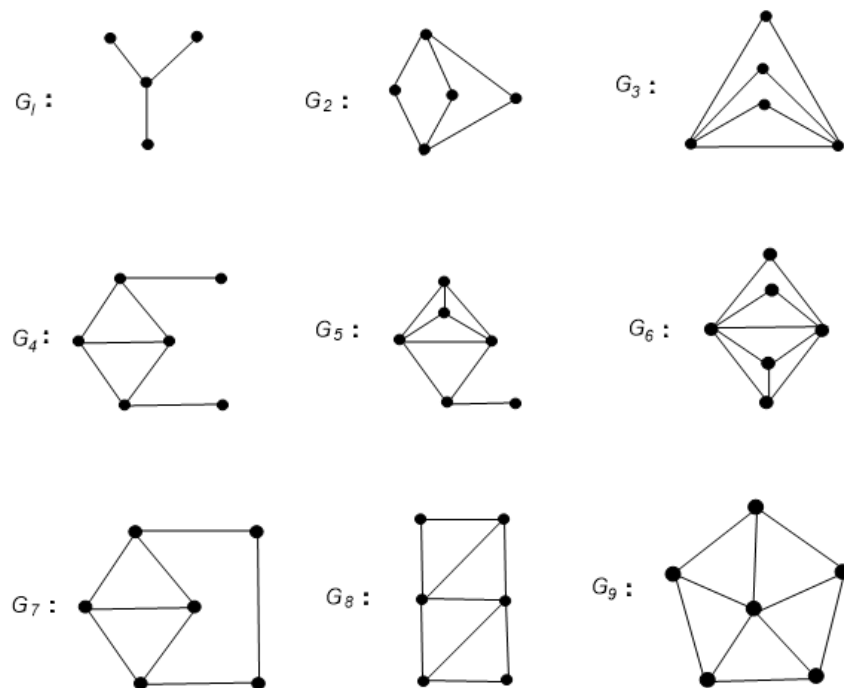


Figure 2. The nine forbidden subgraphs for a line graph.

Theorem 2.4. [8] Let Ω be an n -vertex graph with $n \geq 3$ and $\omega(\Omega)$ be its detour index. Then

$$(n-1)^2 \leq \omega(\Omega) \leq \frac{n(n-1)^2}{2}$$

with left equality holds if and only if Ω is Hamilton-connected, and right inequality holds if and only if $\Omega \cong S_n$.

3. Hamilton-connectivity and the detour index of E_ε

In this section, we introduce a novel family of convex polytopes called the convex polytope E_ε of dimension ε . Here, we aim to prove Hamilton-connectedness of E_ε and employ it to evaluate its detour index.

Assuming the subscript r to be modulo ε , the vertex/edge sets for E_ε are:

$$V(E_\varepsilon) = \{a_r, b_r, c_r : 1 \leq r \leq \varepsilon\},$$

$$E(E_\varepsilon) = \{a_r a_{r+1}, a_r b_r, a_r b_{r+1}, b_r b_{r+1}, b_r c_r, b_r c_{r+1}, c_r c_{r+1} : 1 \leq r \leq \varepsilon\}.$$

With its vertices' labeling, Figure 3 delivers the construction of E_ε . For our proofs, we define panels 1, 2, and 3 to comprise the vertices a_r , b_r , and c_r , respectively.

The next theorem delivers the main result of this section. The proof technique employs Theorem 2.2 by Kužel and Xiong [19], which asserts that a 4-connected Hamiltonian line graph is Hamilton-connected. Thus, we divide our proof for Ω in three steps: (1) Ω is a line graph; (2) Ω is 4-connected; and (3) Ω is Hamiltonian.

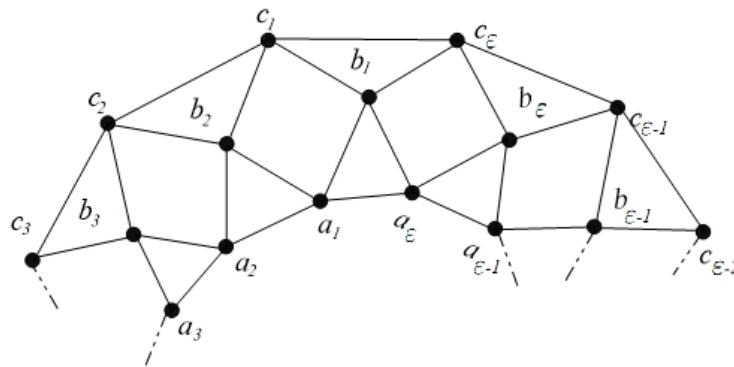


Figure 3. The convex polytope E_ε .

Theorem 3.1. *If $\varepsilon \geq 5$, then E_ε is Hamilton-connected.*

Proof. Let Ω be the ε -dimensional convex polytope E_ε . We divide the proof into a number of claims.

Claim 1: Ω is a line graph.

Note that Ω does not contain any of the nine forbidden subgraphs in Figure 2. Thus, by Theorem 2.3, Ω is a line graph. Moreover, it can be seen that E_ε is the line graph of prism graph $C_\varepsilon \times P_2$, which is also a family of convex polytopes (see Section 1).

Claim 2: For $\varepsilon \geq 5$, Ω is a 4-connected, i.e., $\kappa(\Omega) = 4$.

Note that Ω is a 4-regular graph. For $z \in V(\Omega)$, let $S = N_\Omega(z) = \{z_1, z_2, z_3, z_4\}$ represent the neighborhood of z in Ω . Observe that $V \setminus \Omega$ is disconnected, with z as an isolated connected component. This implies that $\kappa(\Omega) \leq 4$. To demonstrate that $\kappa(\Omega) \geq 4$, let $S = \{v, w, x\}$ be a subset of $V(\Omega)$, where v, w , and x are arbitrary vertices. We show that deleting S from Ω leaves the graph connected.

Here, we divide our discussion into a number of cases.

Case 1. Panels 1 or 3 of Ω comprise all three vertices of S .

In panels 1 or 3, the vertices v, w , and x may be arranged as either consecutive vertices, two adjacent vertices with one non-adjacent, or all three being non-adjacent. If v, w , and x are positioned in panel 1, their neighbors will be found in panels 1 and 2. Conversely, if they are in panel 3, the neighboring vertices will be in panels 2 and 3. In every situation, these neighboring vertices exhibit degrees of either two or three in the graph $\Omega - S$. This demonstrates that S does not constitute a vertex cut set for Ω , as the graph $\Omega - S$ continues to be connected.

Case 2. All three vertices of S are situated in panel 2 of Ω .

In panel 2, the vertices v, w , and x can be either consecutive or non-consecutive. Regardless of their arrangement, the neighboring vertices are in panels 1 and 3 if they are part of panel 2. All neighboring vertices in panels 1 or 3 have degrees of either 2 or 3 in $\Omega - S$. Examining the structure of Ω as shown in Figure 3, it is evident that S does not serve as a vertex cut set for Ω , since $\Omega - S$ remains connected.

Case 3. Two vertices of S are positioned in either panel 1 or panel 3, and the remaining vertex is in panel 2.

If all three vertices of S are adjacent, forming a triangle, then the neighbors of v, w , and x are found in panels 1, 2, or 3 of Ω , each having degrees of either 2 or 3 in $\Omega - S$. Additionally, if the two vertices in panel 1 or 3 are adjacent while the vertex in panel 2 is non-adjacent to the other two vertices of S , the neighbors of v, w , and x will also belong to panels 1, 2, or 3, maintaining degrees of either 2 or 3 in

$\Omega - S$. Last, if the two vertices in panel 1 or 3 are non-adjacent, the number of neighbors for v , w , and x increases, with these neighbors still having degrees of either 2 or 3 in $\Omega - S$. Therefore, as shown in Figure 3, S does not constitute a vertex cut set in this scenario either, since $\Omega - S$ remains connected.

Case 4. Two vertices of S are situated in panel 2, while the third vertex is located in either panel 1 or panel 3.

This scenario is quite straightforward since the two vertices in panel 2 cannot be adjacent. Consequently, all neighbors of v , w , and x are present in panels 1, 2, and 3, irrespective of whether the remaining vertex is in panel 1 or panel 3. Each of these neighbors has a degree of at least 2, indicating that S does not function as a vertex cut set for Ω , as $\Omega - S$ remains connected.

Case 5. Each vertex of S is located in a different panel in Ω .

Applying the same reasoning as in the previous cases, we conclude that $\Omega - S$ remains connected in this scenario as well.

By integrating our findings from Cases 1–5, we determine that no set of three vertices in Ω can serve as a vertex cut set. This establishes that $\kappa(\Omega) \geq 4$. When combined with the result $\kappa(\Omega) \leq 4$, this completes the proof of Claim 2.

Claim 3: Ω is a Hamiltonian.

The following is the Hamiltonian cycle in Ω :

$$x = c_\varepsilon \circ \{c_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 1\} \circ \{b_s a_s : 1 \leq s \leq \varepsilon - 1\} \circ a_\varepsilon b_\varepsilon c_\varepsilon = x.$$

Claims 1–3 show that Ω is a family of Hamiltonian, 4-connected line graphs. By using Theorem 2.2, Ω is Hamilton-connected. $\square$

Note that $|V(E_\varepsilon)| = 3\varepsilon$. Theorems 2.4 and 3.1, together with $|V(E_\varepsilon)| = 3\varepsilon$, deliver the following corollary:

Corollary 3.1. *Assuming $\varepsilon \geq 5$, we have*

$$\omega(E_\varepsilon) = \frac{3\varepsilon(3\varepsilon - 1)^2}{2}.$$

Proof. Since $|V(E_\varepsilon)| = 3\varepsilon$. Theorem 2.4 with $n = 3\varepsilon$ finishes the proof. $\square$

4. Hamilton-connectivity and the detour index of Q_ε

In this section, we explore Hamilton-connectivity of the convex polytope Q_ε of dimension ε . It was introduced by Bača [3] in 1988. We, in this section, aim to prove Hamilton-connectedness of Q_ε and employ it to evaluate its detour index.

Assuming the subscript r to be modulo ε , the vertex/edge sets for Q_ε are:

$$\begin{aligned} V(Q_\varepsilon) &= \{a_r, b_r, c_r, d_r : 1 \leq r \leq \varepsilon\}, \\ E(Q_\varepsilon) &= \{a_r a_{r+1}, a_r b_r, b_r b_{r+1}, b_r c_r, b_r c_{r+1}, c_r d_r, d_r d_{r+1} : 1 \leq r \leq \varepsilon\}. \end{aligned}$$

With its vertices' labeling, Figure 4 delivers the construction of Q_ε .

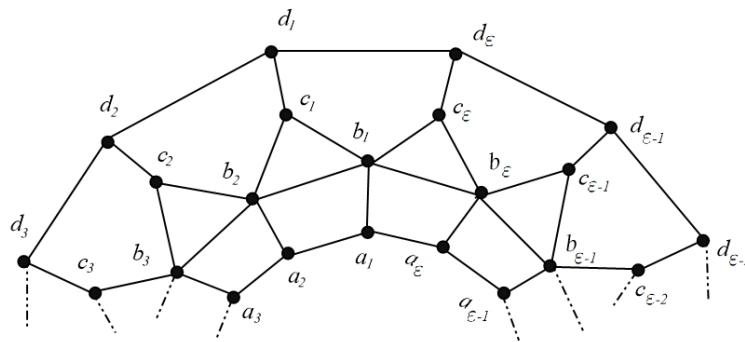


Figure 4. The ε -dimensional convex polytope Q_ε .

Here, we prove the main result, asserting that Q_ε is Hamilton-connected. The proof technique utilizes the methodology of Alspach and Liu [1] using the well-known Posa exchange in constructing Hamiltonian paths between every pair of vertices in a graph.

Theorem 4.1. Assuming $\varepsilon \geq 6$, the graph Q_ε is Hamilton-connected.

Proof. We show the result by constructing pairwise Hamiltonian paths between vertices of Q_ε . Assume $H_p(w, z)$ is a Hamiltonian path between $w, z \in V(Q_\varepsilon)$. Furthermore, assume $Q_\varepsilon = A \cup B \cup C \cup D$ such that $A = \{a_1, a_2, \dots, a_\varepsilon\}$, $B = \{b_1, b_2, \dots, b_\varepsilon\}$, $C = \{c_1, c_2, \dots, c_\varepsilon\}$ and $D = \{d_1, d_2, \dots, d_\varepsilon\}$.

Case 1. $w = a_1$ and $z = a_r$, $2 \leq r \leq \varepsilon$

Subcase 1.1. $r = 2$

$$H_p(w, z) : w = a_1 \circ \{b_j c_s : 1 \leq s \leq \varepsilon - 1\} \circ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 1\} \circ d_\varepsilon c_\varepsilon b_\varepsilon \circ \\ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} = z.$$

Subcase 1.2. $r \equiv 0 \pmod{2}$, $2 < r < \varepsilon$

$$H_p(w, z) : w = a_1 \circ \{a_s : 2 \leq s \leq r - 1\} \circ b_{r-1} \circ \{b_{2s} c_{2s} d_{2s} d_{2s-1} c_{2s-1} b_{2s-1} : \\ \max\left(1, \frac{r-2}{2}\right) \leq s \leq \min\left(1, \frac{r-2}{2}\right)\} \circ c_\varepsilon \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r + 1\} \circ c_{r-1} \circ \\ \{b_j c_s : r \leq s \leq \varepsilon - 1\} \circ b_\varepsilon \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} = z.$$

Subcase 1.3. $r \equiv 1 \pmod{2}$, $2 < r < \varepsilon - 1$

$$H_p(w, z) : w = a_1 \circ \{a_s : 2 \leq s \leq r - 1\} \circ \{b_{2s} c_{2s} d_{2s} d_{2s-1} c_{2s-1} b_{2s-1} : \max\left(1, \frac{r-1}{2}\right) \leq s \leq \min\left(1, \frac{r-1}{2}\right)\} \circ \\ c_\varepsilon \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} \circ c_i b_r \circ \{b_j c_s : r + 1 \leq s \leq \varepsilon - 1\} \circ b_\varepsilon \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} = z.$$

Subcase 1.4. $r = \varepsilon - 1$, and $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = a_1 \circ \{a_s : 2 \leq s \leq r - 1\} \circ \{b_{2s} c_{2s} d_{2s} d_{2s-1} c_{2s-1} b_{2s-1} : 1 \leq s \leq \frac{r-1}{2}\} \circ c_\varepsilon \circ \\ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} \circ c_i b_i b_\varepsilon a_\varepsilon a_{\varepsilon-1} = z.$$

Subcase 1.5. $r = \varepsilon$

$$H_p(w, z) : w = a_1 \circ \{a_s : 2 \leq s \leq \varepsilon - 1\} \circ b_{\varepsilon-1} \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : 2 \leq s \leq \varepsilon - 1\} \circ c_\varepsilon d_\varepsilon \circ \\ \{d_s : 1 \leq s \leq \varepsilon - 1\} \circ c_{\varepsilon-1} b_\varepsilon a_\varepsilon = z.$$

Case 2. $w = a_1$ and $z = b_r, 1 \leq r \leq \varepsilon$

Subcase 2.1. $r = 1$

$$H_p(w, z) : w = a_1 \circ \{a_s : 2 \leq s \leq \varepsilon\} \circ b_\varepsilon \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 2\} \circ c_1 \circ \{d_s : 1 \leq s \leq \varepsilon\} \circ c_\varepsilon b_1 = z.$$

Subcase 2.2. $r \equiv 0 \pmod{2}, 2 \leq r < \varepsilon$

$$H_p(w, z) : w = a_1 \circ \{a_s : 2 \leq s \leq \varepsilon\} \circ b_\varepsilon \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - r - 2\} \circ b_{r+1}c_{r+1} \circ \{d_s : r + 1 \leq s \leq \varepsilon\} \circ \\ c_\varepsilon \circ \{b_{2s-1}c_{2s-1}d_{2s-1}d_{2s}c_{2s}b_{2s} : 1 \leq s \leq \frac{r}{2}\} = z.$$

Subcase 2.3. $r \equiv 1 \pmod{2}, 2 < r < \varepsilon - 1$

$$H_p(w, z) : w = a_1 \circ \{a_s : 2 \leq s \leq \varepsilon\} \circ b_\varepsilon \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - r - 1\} \circ c_r \circ \{d_s : r \leq s \leq \varepsilon\} \circ \\ c_\varepsilon \circ \{b_{2s-1}c_{2s-1}d_{2s-1}d_{2s}c_{2s}b_{2s} : 1 \leq s \leq \frac{r-1}{2}\} \circ b_r = z.$$

Subcase 2.4. $r = \varepsilon - 1$, and $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = a_1 \circ \{a_s : 2 \leq s \leq \varepsilon\} \circ b_\varepsilon c_{\varepsilon-1} d_{\varepsilon-1} d_\varepsilon c_\varepsilon \circ \{b_{2s-1}c_{2s-1}d_{2s-1}d_{2s}c_{2s}b_{2s} : 1 \leq s \leq \frac{r-1}{2}\} \circ \\ b_r = z.$$

Subcase 2.5. $r = \varepsilon$

$$H_p(w, z) : w = a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_2 c_2 d_2 d_1 c_1 b_1 c_\varepsilon \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 3\} \circ c_3 b_3 \circ \\ \{b_j c_s : 4 \leq s \leq \varepsilon - 1\} \circ b_\varepsilon = z.$$

Case 3. $w = a_1$ and $z = c_r, 1 \leq r \leq \varepsilon$

Subcase 3.1. $r = 1$

$$H_p(w, z) : w = a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_j c_s : 2 \leq s \leq \varepsilon - 1\} \circ b_\varepsilon b_1 c_\varepsilon \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ \\ d_1 c_1 = z.$$

Subcase 3.2. $r = 2$

$$H_p(w, z) : w = a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_2 \circ \{b_j c_s : 3 \leq s \leq \varepsilon\} \circ b_1 c_1 d_1 \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \\ c_2 = z.$$

Subcase 3.3. $r = 3$

$$H_p(w, z) : w = a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_2 c_2 \circ \{d_s : 2 \leq s \leq \varepsilon\} \circ d_1 c_1 b_1 \circ$$

$$\{c_{\varepsilon-s}b_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 4\} \circ b_3c_3 = z.$$

Subcase 3.4. $r \equiv 0 \pmod{2}$, $3 < r < \varepsilon - 1$

$$\begin{aligned} H_p(w, z) : w = a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_{2s}c_{2s}d_{2s}d_{2s-1}c_{2s-1}b_{2s-1} : \\ \max\left(1, \frac{r-2}{2}\right) \leq s \leq \min\left(1, \frac{r-2}{2}\right)\} \circ b_r \circ \{b_jc_s : r+1 \leq s \leq \varepsilon\} \circ b_1c_1d_1 \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} \circ \\ c_r = z. \end{aligned}$$

Subcase 3.5. $r \equiv 1 \pmod{2}$, $3 < r < \varepsilon - 1$

$$\begin{aligned} H_p(w, z) : w = a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_{2s}c_{2s}d_{2s}d_{2s-1}c_{2s-1}b_{2s-1} : \\ \max\left(1, \frac{r-3}{2}\right) \leq s \leq \min\left(1, \frac{r-3}{2}\right)\} \circ b_{r-1}c_{r-1} \circ \{d_s : r-1 \leq s \leq \varepsilon\} \circ d_1c_1b_1 \circ \\ \{c_{\varepsilon-s}b_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r - 1\} \circ b_ic_r = z. \end{aligned}$$

Subcase 3.6. $r = \varepsilon - 1$

$$\begin{aligned} H_p(w, z) : w = a_1 \circ \{a_s : 2 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ d_1c_1b_1 \circ \\ \{b_jc_s : 2 \leq s \leq \varepsilon - 1\} = z. \end{aligned}$$

Subcase 3.7. $r = \varepsilon$

$$\begin{aligned} H_p(w, z) : w = a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_jc_s : 2 \leq s \leq \varepsilon - 1\} \circ b_\varepsilon b_1c_1 \circ \{d_s : 1 \leq s \leq \varepsilon\} \circ \\ c_\varepsilon = z. \end{aligned}$$

Case 4. $w = a_1$ and $z = d_r$, $1 \leq r \leq \varepsilon$

Subcase 4.1. $r = 1$

$$\begin{aligned} H_p(w, z) : w = a_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 1\} \circ c_\varepsilon \circ \\ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} = z. \end{aligned}$$

Subcase 4.2. $r = 2$

$$\begin{aligned} H_p(w, z) : w = a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_jc_s : r \leq s \leq \varepsilon\} \circ b_1c_1d_1 \circ \\ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} = z. \end{aligned}$$

Subcase 4.3. $r = 3$

$$\begin{aligned} H_p(w, z) : w = a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_2b_1c_1d_1d_2c_2 \circ \{b_jc_s : 3 \leq s \leq \varepsilon\} \circ \\ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 3\} = z. \end{aligned}$$

Subcase 4.4. $r \equiv 0 \pmod{2}$, $3 < r < \varepsilon$

$$H_p(w, z) : w = a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_{2s}c_{2s}d_{2s}d_{2s-1}c_{2s-1}b_{2s-1} :$$

$$\max\left(1, \frac{r-2}{2}\right) \leq s \leq \min\left(1, \frac{r-2}{2}\right) \circ \{b_j c_s : r \leq s \leq \varepsilon\} \circ b_1 c_1 d_1 \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} = z.$$

Subcase 4.5. $r \equiv 1 \pmod{2}$, $3 < r < \varepsilon$

$$H_p(w, z) : w = a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_2 b_1 c_1 d_1 d_2 c_2 \circ \{b_{2s-1} c_{2s-1} d_{2s-1} d_{2s} c_{2s} b_{2s} :$$

$$2 \leq s \leq \frac{r-1}{2}\} \circ \{b_j c_s : r \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} = z.$$

Subcase 4.6. $r = \varepsilon$

$$H_p(w, z) : w = a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_j c_s : 2 \leq s \leq \varepsilon\} \circ b_1 c_1 \circ \{d_s : 1 \leq s \leq \varepsilon\} = z.$$

Case 5. $w = b_1$ and $z = a_r$, $1 \leq r \leq \varepsilon$

Subcase 5.1. $r = 1$

$$H_p(w, z) : w = b_1 c_\varepsilon \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ c_1 \circ \{b_j c_s : 2 \leq s \leq \varepsilon - 1\} \circ b_\varepsilon \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} = z.$$

Subcase 5.2. $r = 2$

$$H_p(w, z) : w = b_1 \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 5\} \circ b_4 c_4 \circ \{d_s : 4 \leq s \leq \varepsilon\} \circ d_1 c_1 b_2 c_2 d_2 d_3 c_3 b_3 \circ$$

$$\{a_s : 3 \leq s \leq \varepsilon\} \circ a_1 a_2 = z.$$

Subcase 5.3. $r \equiv 0 \pmod{2}$, $2 < r < \varepsilon - 1$

$$H_p(w, z) : w = b_1 b_\varepsilon c_\varepsilon d_\varepsilon d_1 c_1 \circ \{b_{2s} c_{2s} d_{2s} d_{2s+1} c_{2s+1} b_{2s+1} : 1 \leq s \leq \frac{r-2}{2}\} \circ \{a_{\varepsilon-s} : r+1 \leq s \leq \varepsilon - 1\} \circ$$

$$\{a_{\varepsilon-s} : 0 \leq s \leq r-1\} \circ \{b_j c_s : r+1 \leq s \leq \varepsilon - 1\} \circ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - r\} \circ c_i b_i a_r = z.$$

Subcase 5.4. $r \equiv 1 \pmod{2}$, $2 < r < \varepsilon - 1$

$$H_p(w, z) : w = \{b_{2s-1} c_{2s-1} d_{2s-1} d_{2s} c_{2s} b_{2s} : 1 \leq s \leq \frac{r-1}{2}\} \circ \{a_{\varepsilon-s} : r+1 \leq s \leq \varepsilon - 1\} \circ$$

$$\{a_{\varepsilon-s} : 0 \leq s \leq r-1\} \circ \{b_j c_s : r+1 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} \circ c_i b_i a_r = z.$$

Subcase 5.5. $r = \varepsilon - 1$

$$H_p(w, z) : w = b_1 c_\varepsilon \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ c_1 \circ \{b_j c_s : 2 \leq s \leq \varepsilon - 1\} \circ b_\varepsilon a_\varepsilon \circ \{a_s : 1 \leq s \leq \varepsilon - 1\} = z.$$

Subcase 5.6. $r = \varepsilon$

$$H_p(w, z) : w = \{b_j c_s : 1 \leq s \leq \varepsilon - 3\} \circ \{d_{\varepsilon-s} : 3 \leq s \leq \varepsilon - 1\} \circ d_\varepsilon c_\varepsilon b_\varepsilon c_{\varepsilon-1} d_{\varepsilon-1} d_{\varepsilon-2} c_{\varepsilon-2} b_{\varepsilon-2} b_{\varepsilon-1} \circ$$

$$\{a_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 1\} \circ a_\varepsilon = z.$$

Case 6. $w = b_1$ and $z = b_r$, $2 \leq r \leq \varepsilon$

Subcase 6.1. $r = 2$, and $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ \{b_{2s} c_{2s} d_{2s} d_{2s-1} c_{2s-1} b_{2s-1} : \max\left(2, \frac{\varepsilon}{2}\right) \leq s \leq \min\left(2, \frac{\varepsilon}{2}\right)\} \circ$$

$$b_3c_2d_2d_1c_1b_2 = z.$$

Subcase 6.2. $r = 2$, and $\varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = b_1c_\varepsilon d_\varepsilon d_{\varepsilon-1}c_{\varepsilon-1}b_\varepsilon a_\varepsilon \circ \{a_s : 1 \leq s \leq \varepsilon - 3\} \circ b_{\varepsilon-3}c_{\varepsilon-3}b_{\varepsilon-2}a_{\varepsilon-2}a_{\varepsilon-1}b_{\varepsilon-1}c_{\varepsilon-2}d_{\varepsilon-2}d_{\varepsilon-3} \circ$$

$$\{d_{2s+1}c_{2s+1}b_{2s+1}b_{2s}c_{2s}d_{2s} : \max\left(2, \frac{\varepsilon-5}{2}\right) \leq s \leq \min\left(2, \frac{\varepsilon-5}{2}\right)\} \circ d_3c_3b_3c_2d_2d_1c_1b_2 = z.$$

Subcase 6.3. $r = 3$

$$H_p(w, z) : w = b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon d_\varepsilon d_1c_1b_2c_2 \circ \{d_s : 2 \leq s \leq \varepsilon - 1\} \circ$$

$$\{c_{\varepsilon-s}b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 3\} = z.$$

Subcase 6.4. $r \equiv 0 \pmod{2}$, $3 < r < \varepsilon$

$$H_p(w, z) : w = \{b_{2s-1}c_{2s-1}d_{2s-1}d_{2s}c_{2s}b_{2s} : 1 \leq s \leq \frac{r-2}{2}\} \circ \{a_{\varepsilon-s} : \varepsilon - r + 2 \leq s \leq \varepsilon - 1\} \circ$$

$$\{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r + 1\} \circ b_{r-1}c_{r-1} \circ \{d_s : r - 1 \leq s \leq \varepsilon\} \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} = z.$$

Subcase 6.5. $r \equiv 1 \pmod{2}$, $3 < r < \varepsilon$

$$H_p(w, z) : w = b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon d_\varepsilon d_1c_1 \circ \{b_{2s}c_{2s}d_{2s}d_{2s+1}c_{2s+1}b_{2s+1} : 1 \leq s \leq \frac{r-3}{2}\} \circ$$

$$b_{r-1}c_{r-1} \circ \{d_s : r \leq s \leq \varepsilon - 1\} \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - r\} = z.$$

Subcase 6.6. $r = \varepsilon$, and $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = b_1c_1d_1d_2c_2b_2a_2a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 3\} \circ \{b_{2s-1}c_{2s-1}d_{2s-1}d_{2s}c_{2s}b_{2s} :$$

$$2 \leq s \leq \frac{\varepsilon-1}{2}\} = z.$$

Subcase 6.7. $r = \varepsilon$, and $\varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = b_1c_1d_1d_2c_2b_2a_2a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 5\} \circ b_5c_4b_4a_4a_3b_3c_3d_3d_4d_5c_5 \circ$$

$$\{b_{2s}c_{2s}d_{2s}d_{2s+1}c_{2s+1}b_{2s+1} : 3 \leq s \leq \frac{\varepsilon-1}{2}\} = z.$$

Case 7. $w = b_1$ and $z = c_r$, $1 \leq r \leq \varepsilon$

Subcase 7.1. $r = 1$

$$H_p(w, z) : w = b_1a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_jc_s : 2 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ c_1 = z.$$

Subcase 7.2. $r = 2$

$$H_p(w, z) : w = b_1a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_2c_1 \circ \{d_s : 1 \leq s \leq \varepsilon\} \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 3\} \circ$$

$$c_2 = z.$$

Subcase 7.3. $r \equiv 0 \pmod{2}$, $2 < r < \varepsilon - 1$, and $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ \{b_{2s}c_{2s}d_{2s}d_{2s-1}c_{2s-1}b_{2s-1} : \\ \max\left(\frac{r+2}{2}, \frac{\varepsilon}{2}\right) \leq s \leq \min\left(\frac{r+2}{2}, \frac{\varepsilon}{2}\right)\} \circ b_r \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : \varepsilon - r + 1 \leq s \leq \varepsilon - 2\} \circ c_1 \circ \{d_s : 1 \leq s \leq r\} \circ \\ c_r = z.$$

Subcase 7.4. $r \equiv 1 \pmod{2}$, $2 < r < \varepsilon - 1$, and $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ \{b_{2s}c_{2s}d_{2s}d_{2s-1}c_{2s-1}b_{2s-1} : \max\left(\frac{r+3}{2}, \frac{\varepsilon}{2}\right) \leq s \leq \min\left(\frac{r+3}{2}, \frac{\varepsilon}{2}\right)\} \circ \\ b_{r+1}c_{r+1} \circ \{d_{\varepsilon-s} : \varepsilon - r - 1 \leq s \leq \varepsilon - 1\} \circ c_1 \circ \{b_jc_s : 2 \leq s \leq r\} = z.$$

Subcase 7.5. $r \equiv 0 \pmod{2}$, $2 < r < \varepsilon - 1$, and $\varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ \{b_{2s+1}c_{2s+1}d_{2s+1}d_{2s}c_{2s}b_{2s} : \\ \max\left(\frac{r+2}{2}, \frac{\varepsilon-1}{2}\right) \leq s \leq \min\left(\frac{r+2}{2}, \frac{\varepsilon-1}{2}\right)\} \circ b_{r+1}c_{r+1} \circ \{d_{\varepsilon-s} : \varepsilon - r - 1 \leq s \leq \varepsilon - 1\} \circ c_1 \circ \\ \{b_jc_s : 2 \leq s \leq r\} = z.$$

Subcase 7.6. $r \equiv 1 \pmod{2}$, $2 < r < \varepsilon - 1$, and $\varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ \{b_{2s+1}c_{2s+1}d_{2s+1}d_{2s}c_{2s}b_{2s} : \\ \max\left(\frac{r+1}{2}, \frac{\varepsilon-1}{2}\right) \leq s \leq \min\left(\frac{r+1}{2}, \frac{\varepsilon-1}{2}\right)\} \circ b_r \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : \varepsilon - r + 1 \leq s \leq \varepsilon - 2\} \circ c_1 \circ \\ \{d_s : 1 \leq s \leq r\} \circ c_r = z.$$

Subcase 7.7. $r = \varepsilon - 1$

$$H_p(w, z) : w = b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ c_1 \circ \{b_jc_s : 2 \leq s \leq \varepsilon - 1\} = z.$$

Subcase 7.8. $r = \varepsilon$

$$H_p(w, z) : w = b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 2\} \circ c_1 \circ \{d_s : 1 \leq s \leq \varepsilon\} \circ c_\varepsilon = z.$$

Case 8. $w = b_1$ and $z = d_r$, $1 \leq r \leq \varepsilon$

Subcase 8.1. $r = 1$, $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = b_1c_1 \circ \{b_{2s}c_{2s}d_{2s}d_{2s+1}c_{2s+1}b_{2s+1} : 1 \leq s \leq \frac{\varepsilon-2}{2}\} \circ \{a_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 1\} \circ \\ a_\varepsilon b_\varepsilon c_\varepsilon d_\varepsilon d_1 = z.$$

Subcase 8.2. $r = 1$, $\varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ \{b_{2s+1}c_{2s+1}d_{2s+1}d_{2s}c_{2s}b_{2s} : \max\left(1, \frac{\varepsilon-1}{2}\right) \leq s \leq \min\left(1, \frac{\varepsilon-1}{2}\right)\} \circ$$

$$c_1 d_1 = z.$$

Subcase 8.3. $r = 2$

$$H_p(w, z) : w = b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon d_\varepsilon d_1 c_1 \circ \{b_j c_s : 2 \leq s \leq \varepsilon - 1\} \circ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 2\} = z.$$

Subcase 8.4. $r \equiv 0 \pmod{2}$, $2 < r < \varepsilon$, and $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ \{b_{2s} c_{2s} d_{2s} d_{2s-1} c_{2s-1} b_{2s-1} : \max\left(\frac{r+2}{2}, \frac{\varepsilon}{2}\right) \leq s \leq \min\left(\frac{r+2}{2}, \frac{\varepsilon}{2}\right)\} \circ$$

$$\{c_{\varepsilon-s} b_{\varepsilon-s} : \varepsilon - r \leq s \leq \varepsilon - 2\} \circ c_1 \circ \{d_s : 1 \leq s \leq r\} = z.$$

Subcase 8.5. $r \equiv 1 \pmod{2}$, $2 < r < \varepsilon$, and $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = \{b_{2s-1} c_{2s-1} d_{2s-1} d_{2s} c_{2s} b_{2s} : 1 \leq s \leq \frac{r-1}{2}\} \circ \{a_{\varepsilon-s} : \varepsilon - r + 1 \leq s \leq \varepsilon - 1\} \circ$$

$$\{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} \circ \{b_j c_s : r \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} = z.$$

Subcase 8.6. $r \equiv 0 \pmod{2}$, $2 < r < \varepsilon$, and $\varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon d_\varepsilon d_1 c_1 \circ \{b_{2s} c_{2s} d_{2s} d_{2s+1} c_{2s+1} b_{2s+1} : 1 \leq s \leq \frac{r-2}{2}\} \circ$$

$$\{b_j c_s : r \leq s \leq \varepsilon - 1\} \circ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - r\} = z.$$

Subcase 8.7. $r \equiv 1 \pmod{2}$, $2 < r < \varepsilon$, and $\varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ \{b_{2s+1} c_{2s+1} d_{2s+1} d_{2s} c_{2s} b_{2s} :$$

$$\max\left(\frac{r+1}{2}, \frac{\varepsilon-1}{2}\right) \leq s \leq \min\left(\frac{r+1}{2}, \frac{\varepsilon-1}{2}\right)\} \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : \varepsilon - r \leq s \leq \varepsilon - 2\} \circ c_1 \circ \{d_s : 1 \leq s \leq r\} = z.$$

Subcase 8.8. $r = \varepsilon$, and $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = b_1 c_\varepsilon b_\varepsilon a_\varepsilon \circ \{a_s : 1 \leq s \leq \varepsilon - 1\} \circ \{b_{2s+1} c_{2s+1} d_{2s+1} d_{2s} c_{2s} b_{2s} :$$

$$\max\left(1, \frac{\varepsilon-2}{2}\right) \leq s \leq \min\left(1, \frac{\varepsilon-2}{2}\right)\} \circ c_1 d_1 d_\varepsilon = z.$$

Subcase 8.9. $r = \varepsilon$, and $\varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = b_1 c_1 d_1 d_2 c_2 b_2 a_2 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 3\} \circ \{b_{2s-1} c_{2s-1} d_{2s-1} d_{2s} c_{2s} b_{2s} :$$

$$2 \leq s \leq \frac{\varepsilon-1}{2}\} \circ b_\varepsilon c_\varepsilon d_\varepsilon = z.$$

Case 9. $w = c_1$ and $z = a_r$, $1 \leq r \leq \varepsilon$

Subcase 9.1. $r = 1$

$$H_p(w, z) : w = c_1 \circ \{d_s : 1 \leq s \leq \varepsilon\} \circ c_\varepsilon b_1 \circ \{b_j c_s : 2 \leq s \leq \varepsilon - 1\} \circ b_\varepsilon \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} = z.$$

Subcase 9.2. $r = 2$

$$H_p(w, z) : w = c_1 \circ \{d_s : 1 \leq s \leq \varepsilon\} \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_1a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} = z.$$

Subcase 9.3. $r = 3$

$$H_p(w, z) : w = c_1 \circ \{d_s : 1 \leq s \leq \varepsilon\} \circ c_\varepsilon b_1 b_\varepsilon \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 2\} \circ a_2 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 3\} = z.$$

Subcase 9.4. $r \equiv 0 \pmod{2}$, $3 < r < \varepsilon$, and $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = c_1 d_1 \circ \{d_{2s}c_{2s}b_{2s}a_{2s}a_{2s-1}b_{2s-1}c_{2s-1}d_{2s-1} : \max\left(\frac{r+2}{2}, \frac{\varepsilon}{2}\right) \leq s \leq \min\left(\frac{r+2}{2}, \frac{\varepsilon}{2}\right)\} \circ \{d_{2s}c_{2s}b_{2s}b_{2s-1}c_{2s-1}d_{2s-1} : \max\left(2, \frac{r}{2}\right) \leq s \leq \min\left(2, \frac{r}{2}\right)\} \circ d_2 c_2 b_2 b_1 \circ \{a_s : 1 \leq s \leq r\} = z.$$

Subcase 9.5. $r \equiv 1 \pmod{2}$, $3 < r < \varepsilon$, and $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = c_1 \circ \{d_s : 1 \leq s \leq r-1\} \circ c_{r-1} \circ \{b_{2s-1}c_{2s-1}d_{2s-1}d_{2s}c_{2s}b_{2s} : \frac{r+1}{2} \leq s \leq \frac{\varepsilon}{2}\} \circ b_1 \circ \{b_j c_s : 2 \leq s \leq r-2\} \circ b_{r-1} \circ \{a_{\varepsilon-s} : \varepsilon - r + 1 \leq s \leq \varepsilon - 1\} \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} = z.$$

Subcase 9.6. $r \equiv 0 \pmod{2}$, $3 < r < \varepsilon$, and $\varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = c_1 \circ \{d_s : 1 \leq s \leq r-1\} \circ c_{r-1} \circ \{b_{2s}c_{2s}d_{2s}d_{2s+1}c_{2s+1}b_{2s+1} : \frac{r}{2} \leq s \leq \frac{\varepsilon-1}{2}\} \circ b_1 \circ \{b_j c_s : 2 \leq s \leq r-2\} \circ b_{r-1} \circ \{a_{\varepsilon-s} : \varepsilon - r + 1 \leq s \leq \varepsilon - 1\} \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} = z.$$

Subcase 9.7. $r \equiv 1 \pmod{2}$, $3 < r < \varepsilon$, and $\varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = c_1 \circ \{d_s : 1 \leq s \leq r-2\} \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : \varepsilon - r + 2 \leq s \leq \varepsilon - 2\} \circ b_1 \circ \{b_{2s+1}c_{2s+1}d_{2s+1}d_{2s}c_{2s}b_{2s} : \max\left(\frac{r-1}{2}, \frac{\varepsilon-1}{2}\right) \leq s \leq \min\left(\frac{r-1}{2}, \frac{\varepsilon-1}{2}\right)\} \circ \{a_{\varepsilon-s} : \varepsilon - r + 1 \leq s \leq \varepsilon - 1\} \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} = z.$$

Subcase 9.8. $r = \varepsilon$

$$H_p(w, z) : w = c_1 \circ \{d_s : 1 \leq s \leq \varepsilon\} \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} = z.$$

Case 10. $w = c_1$ and $z = b_r$, $1 \leq r \leq \varepsilon$

Subcase 10.1. $r = 1$

$$H_p(w, z) : w = c_1 \circ \{d_s : 1 \leq s \leq \varepsilon\} \circ c_\varepsilon b_1 b_\varepsilon \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 2\} \circ \{a_s : 2 \leq s \leq \varepsilon\} \circ a_1 b_1 = z.$$

Subcase 10.2. $r = 2$

$$H_p(w, z) : w = c_1 d_1 \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ c_2 \circ \{b_j c_s : 3 \leq s \leq \varepsilon\} \circ b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_2 = z.$$

Subcase 10.3. $r \equiv 0 \pmod{2}$, $2 < r < \varepsilon$, and $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = c_1 \circ \{b_j c_s : 2 \leq s \leq r-1\} \circ \{d_{\varepsilon-s} : \varepsilon - r + 1 \leq s \leq \varepsilon - 1\} \circ d_\varepsilon c_\varepsilon b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ$$

$$b_\varepsilon \circ \{b_{2s+1} c_{2s+1} d_{2s+1} d_{2s} c_{2s} b_{2s} : \max\left(\frac{r}{2}, \frac{\varepsilon-2}{2}\right) \leq s \leq \min\left(\frac{r}{2}, \frac{\varepsilon-2}{2}\right)\} = z.$$

Subcase 10.4. $r \equiv 1 \pmod{2}$, $2 < r < \varepsilon$, and $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = c_1 \circ \{d_s : 1 \leq s \leq r-1\} \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : \varepsilon - r + 1 \leq s \leq \varepsilon - 2\} \circ b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ$$

$$\{b_{2s} c_{2s} d_{2s} d_{2s-1} c_{2s-1} b_{2s-1} : \max\left(\frac{r+1}{2}, \frac{\varepsilon}{2}\right) \leq s \leq \min\left(\frac{r+1}{2}, \frac{\varepsilon}{2}\right)\} = z.$$

Subcase 10.5. $r \equiv 0 \pmod{2}$, $2 < r < \varepsilon$, and $\varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = c_1 d_1 \circ \{d_{2s} c_{2s} b_{2s} b_{2s+1} c_{2s+1} d_{2s+1} : 1 \leq s \leq \frac{r-2}{2}\} \circ \{d_s : r \leq s \leq \varepsilon\} \circ c_\varepsilon b_1 \circ$$

$$\{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - r\} = z.$$

Subcase 10.6. $r \equiv 1 \pmod{2}$, $2 < r < \varepsilon$, and $\varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = c_1 \circ \{b_j c_s : 2 \leq s \leq r-1\} \circ \{d_{\varepsilon-s} : \varepsilon - r + 1 \leq s \leq \varepsilon - 1\} \circ d_\varepsilon c_\varepsilon b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ$$

$$b_\varepsilon \circ \{b_{2s} c_{2s} d_{2s} d_{2s-1} c_{2s-1} b_{2s-1} : \max\left(\frac{r+1}{2}, \frac{\varepsilon-1}{2}\right) \leq s \leq \min\left(\frac{r+1}{2}, \frac{\varepsilon-1}{2}\right)\} = z.$$

Subcase 10.7. $r = \varepsilon$

$$H_p(w, z) : w = c_1 \circ \{d_s : 1 \leq s \leq \varepsilon\} \circ c_\varepsilon b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_j c_s : 2 \leq s \leq \varepsilon - 1\} \circ b_\varepsilon = z.$$

Case 11. $w = c_1$ and $z = c_r$, $2 \leq r \leq \varepsilon$

Subcase 11.1. $r = 2$

$$H_p(w, z) : w = c_1 d_1 \circ \{d_s : 2 \leq s \leq \varepsilon\} \circ c_\varepsilon b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 3\} \circ$$

$$b_2 c_2 = z.$$

Subcase 11.2. $r = 3$

$$H_p(w, z) : w = c_1 d_1 d_2 c_2 b_2 b_3 \circ \{b_j c_s : 4 \leq s \leq \varepsilon - 1\} \circ b_\varepsilon \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ b_1 c_\varepsilon \circ$$

$$\{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} \circ c_r = z.$$

Subcase 11.3. $r \equiv 0 \pmod{2}$, $2 < r < \varepsilon$

$$H_p(w, z) : w = c_1 d_1 \circ \{d_{2s} c_{2s} b_{2s} b_{2s+1} c_{2s+1} d_{2s+1} : 1 \leq s \leq \frac{r-2}{2}\} \circ \{d_s : r \leq s \leq \varepsilon\} \circ c_\varepsilon b_1 \circ$$

$$\{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - r - 1\} \circ b_1 c_r = z.$$

Subcase 11.4. $r \equiv 1 \pmod{2}$, $3 < r < \varepsilon$

$$H_p(w, z) : w = c_1 d_1 \circ \{d_{2s} c_{2s} b_{2s} b_{2s+1} c_{2s+1} d_{2s+1} : 1 \leq s \leq \frac{r-3}{2}\} \circ d_{r-1} c_{r-1} b_{r-1} b_r \circ$$

$$\{b_j c_s : r+1 \leq s \leq \varepsilon-1\} \circ b_\varepsilon \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon-1\} \circ b_1 c_\varepsilon \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon-r\} \circ c_r = z.$$

Subcase 11.5. $r = \varepsilon - 1$, and $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = c_1 d_1 \circ \{d_{2s} c_{2s} b_{2s} b_{2s+1} c_{2s+1} d_{2s+1} : 1 \leq s \leq \frac{r-3}{2}\} \circ d_{r-1} c_{r-1} b_{r-1} b_i b_\varepsilon \circ$$

$$\{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon-1\} \circ b_1 c_\varepsilon \circ \{d_{\varepsilon-s} : 0 \leq s \leq r\} \circ c_r = z.$$

Subcase 11.6. $r = \varepsilon - 1$, and $\varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = c_1 d_1 \circ \{d_{2s} c_{2s} b_{2s} b_{2s+1} c_{2s+1} d_{2s+1} : 1 \leq s \leq \frac{r-2}{2}\} \circ \{d_s : r \leq s \leq \varepsilon\} \circ c_\varepsilon b_1 \circ$$

$$\{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon b_{\varepsilon-1} c_{\varepsilon-1} = z.$$

Subcase 11.7. $r = \varepsilon$

$$H_p(w, z) : w = c_1 d_1 \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon-2\} \circ c_2 b_2 \circ \{b_j c_s : 3 \leq s \leq \varepsilon-1\} \circ b_\varepsilon \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon-1\} \circ$$

$$b_1 c_\varepsilon = z.$$

Case 12. $w = c_1$ and $z = d_r$, $1 \leq r \leq \varepsilon$

Subcase 12.1. $r = 1$

$$H_p(w, z) : w = c_1 b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon-2\} \circ \{b_j c_s : 2 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon-1\} = z.$$

Subcase 12.2. $r = 2$

$$H_p(w, z) : w = c_1 d_1 d_\varepsilon c_\varepsilon b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon-2\} \circ \{b_j c_s : 2 \leq s \leq \varepsilon-2\} \circ b_{\varepsilon-1} b_\varepsilon c_{\varepsilon-1} \circ$$

$$\{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon-2\} = z.$$

Subcase 12.3. $r \equiv 0 \pmod{2}$, $2 < r < \varepsilon$, and $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = c_1 \circ \{d_s : 1 \leq s \leq r-1\} \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : \varepsilon-r+1 \leq s \leq \varepsilon-2\} \circ \{a_s : 2 \leq s \leq \varepsilon\} \circ$$

$$a_1 b_1 \circ \{b_{2s} c_{2s} d_{2s} d_{2s-1} c_{2s-1} b_{2s-1} : \max\left(\frac{r+2}{2}, \frac{\varepsilon}{2}\right) \leq s \leq \min\left(\frac{r+2}{2}, \frac{\varepsilon}{2}\right)\} \circ b_i c_i d_r = z.$$

Subcase 12.4. $r \equiv 1 \pmod{2}$, $2 < r < \varepsilon$, and $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = c_1 b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon-2\} \circ \{b_j c_s : 2 \leq s \leq r-1\} \circ \{d_{\varepsilon-s} : \varepsilon-r+1 \leq s \leq \varepsilon-1\} \circ$$

$$\{d_{2s} c_{2s} b_{2s} b_{2s-1} c_{2s-1} d_{2s-1} : \max\left(\frac{r+1}{2}, \frac{\varepsilon}{2}\right) \leq s \leq \min\left(\frac{r+1}{2}, \frac{\varepsilon}{2}\right)\} = z.$$

Subcase 12.5. $r \equiv 0 \pmod{2}$, $2 < r < \varepsilon$, and $\varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = c_1 b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon-2\} \circ \{b_j c_s : 2 \leq s \leq r-1\} \circ \{d_{\varepsilon-s} : \varepsilon-r+1 \leq s \leq \varepsilon-1\} \circ$$

$$\{d_{2s+1}c_{2s+1}b_{2s+1}b_{2s}c_{2s}d_{2s} : \max\left(\frac{r}{2}, \frac{\varepsilon-1}{2}\right) \leq s \leq \min\left(\frac{r}{2}, \frac{\varepsilon-1}{2}\right)\} = z.$$

Subcase 12.6. $r \equiv 1 \pmod{2}$, $2 < r < \varepsilon$, and $\varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = c_1 \circ \{d_s : 1 \leq s \leq r-1\} \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : \varepsilon-r+1 \leq s \leq \varepsilon-2\} \circ \{a_s : 2 \leq s \leq \varepsilon\} \circ$$

$$a_1b_1 \circ \{b_{2s+1}c_{2s+1}d_{2s+1}d_{2s}c_{2s}b_{2s} : \max\left(\frac{r+1}{2}, \frac{\varepsilon-1}{2}\right) \leq s \leq \min\left(\frac{r+1}{2}, \frac{\varepsilon-1}{2}\right)\} \circ b_i c_i d_r = z.$$

Subcase 12.7. $r = \varepsilon$

$$H_p(w, z) : w = c_1 \circ \{d_s : 1 \leq s \leq \varepsilon-1\} \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : 1 \leq s \leq \varepsilon-2\} \circ b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ$$

$$b_\varepsilon c_\varepsilon d_\varepsilon = z.$$

Case 13. $w = d_1$ and $z = a_r$, $1 \leq r \leq \varepsilon$

Subcase 13.1. $r = 1$

$$H_p(w, z) : w = d_1 \circ \{d_s : 2 \leq s \leq \varepsilon\} \circ c_\varepsilon \circ \{b_j c_s : 1 \leq s \leq \varepsilon-1\} \circ b_\varepsilon \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon-1\} = z.$$

Subcase 13.2. $r = 3$

$$H_p(w, z) : w = d_1 \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon-2\} \circ c_2 \circ \{b_j c_s : 3 \leq s \leq \varepsilon\} \circ b_1 c_1 b_2 a_2 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon-3\} = z.$$

Subcase 13.3. $r \equiv 0 \pmod{2}$

$$H_p(w, z) : w = \{d_{2s-1}c_{2s-1}b_{2s-1}b_{2s}c_{2s}d_{2s} : 1 \leq s \leq \frac{r}{2}\} \circ \{d_s : r+1 \leq s \leq \varepsilon\} \circ$$

$$\{c_{\varepsilon-s}b_{\varepsilon-s} : 0 \leq s \leq \varepsilon-r-1\} \circ \{a_s : r+1 \leq s \leq \varepsilon\} \circ \{a_s : 1 \leq s \leq r\} = z.$$

Subcase 13.4. $r \equiv 1 \pmod{2}$, $3 < r < \varepsilon$, and $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = d_1 c_1 b_1 \circ \{a_s : 1 \leq s \leq r-1\} \circ b_{r-1} \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : \varepsilon-r+2 \leq s \leq \varepsilon-3\} \circ b_2 c_2 \circ$$

$$\{d_s : 2 \leq s \leq r-1\} \circ c_{r-1} \circ \{b_{2s-1}c_{2s-1}d_{2s-1}d_{2s}c_{2s}b_{2s} : \frac{r+1}{2} \leq s \leq \frac{\varepsilon}{2}\} \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon-r\} = z.$$

Subcase 13.5. $r \equiv 1 \pmod{2}$, $3 < r < \varepsilon$, and $\varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = \{d_{2s-1}c_{2s-1}b_{2s-1}a_{2s-1}a_{2s}b_{2s}c_{2s}d_{2s} : 1 \leq s \leq \frac{r-1}{2}\} \circ \{d_{2s-1}c_{2s-1}b_{2s-1}b_{2s}c_{2s}d_{2s} :$$

$$\frac{r+1}{2} \leq s \leq \frac{\varepsilon-1}{2}\} \circ d_\varepsilon c_\varepsilon b_\varepsilon \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon-r\} = z.$$

Subcase 13.6. $r = \varepsilon$

$$H_p(w, z) : w = d_1 \circ \{d_s : 2 \leq s \leq \varepsilon\} \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : 0 \leq s \leq \varepsilon-1\} \circ \{a_s : 1 \leq s \leq \varepsilon\} = z.$$

Case 14. $w = d_1$ and $z = b_r$, $1 \leq r \leq \varepsilon$

Subcase 14.1. $r = 1$, and $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = d_1 d_\varepsilon c_\varepsilon b_\varepsilon a_\varepsilon \circ \{a_s : 1 \leq s \leq \varepsilon - 1\} \circ \{b_{2s+1} c_{2s+1} d_{2s+1} d_{2s} c_{2s} b_{2s} : \\ \max\left(1, \frac{\varepsilon - 2}{2}\right) \leq s \leq \min\left(1, \frac{\varepsilon - 2}{2}\right)\} \circ c_1 b_1 = z.$$

Subcase 14.2. $r = 1$, and $\varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = d_1 c_1 \circ \{b_{2s} c_{2s} d_{2s} d_{2s+1} c_{2s+1} b_{2s+1} : 1 \leq s \leq \frac{\varepsilon - 1}{2}\} \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ b_1 = z.$$

Subcase 14.3. $r = 2$, and $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = d_1 d_2 c_2 \circ \{b_{2s-1} c_{2s-1} d_{2s-1} d_{2s} c_{2s} b_{2s} : 2 \leq s \leq \frac{\varepsilon}{2}\} \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ b_1 c_1 b_2 = z.$$

Subcase 14.4. $r = 2$, and $\varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = d_1 c_1 b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ \{b_{2s+1} c_{2s+1} d_{2s+1} d_{2s} c_{2s} b_{2s} : \\ \max\left(1, \frac{\varepsilon - 1}{2}\right) \leq s \leq \min\left(1, \frac{\varepsilon - 1}{2}\right)\} = z.$$

Subcase 14.5. $r \equiv 0 \pmod{2}$, $2 < r < \varepsilon$, and $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = d_1 \circ \{d_s : 2 \leq s \leq r - 1\} \circ d_i c_r \circ \{b_{2s-1} c_{2s-1} d_{2s-1} d_{2s} c_{2s} b_{2s} : \frac{r+2}{2} \leq s \leq \frac{\varepsilon}{2}\} \circ \\ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ \{b_j c_s : 1 \leq s \leq r - 1\} \circ b_r = z.$$

Subcase 14.6. $r \equiv 1 \pmod{2}$, $2 < r < \varepsilon$, and $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = d_1 \circ \{d_s : 2 \leq s \leq r - 1\} \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : \varepsilon - r + 1 \leq s \leq \varepsilon - 1\} \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ \\ \{b_{2s} c_{2s} d_{2s} d_{2s-1} c_{2s-1} b_{2s-1} : \max\left(\frac{r+1}{2}, \frac{\varepsilon}{2}\right) \leq s \leq \min\left(\frac{r+1}{2}, \frac{\varepsilon}{2}\right)\} = z.$$

Subcase 14.7. $r \equiv 0 \pmod{2}$, $2 < r < \varepsilon$, and $\varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = d_1 \circ \{d_s : 2 \leq s \leq r - 1\} \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : \varepsilon - r + 1 \leq s \leq \varepsilon - 1\} \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ \\ \{b_{2s+1} c_{2s+1} d_{2s+1} d_{2s} c_{2s} b_{2s} : \max\left(\frac{r}{2}, \frac{\varepsilon - 1}{2}\right) \leq s \leq \min\left(\frac{r}{2}, \frac{\varepsilon - 1}{2}\right)\} = z.$$

Subcase 14.8. $r \equiv 1 \pmod{2}$, $2 < r < \varepsilon$, and $\varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = d_1 \circ \{d_s : 2 \leq s \leq r\} \circ c_r \circ \{b_{2s} c_{2s} d_{2s} d_{2s+1} c_{2s+1} b_{2s+1} : \frac{r+1}{2} \leq s \leq \frac{\varepsilon - 1}{2}\} \circ \\ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ \{b_j c_s : 1 \leq s \leq r - 1\} \circ b_r = z.$$

Subcase 14.9. $r = \varepsilon$

$$H_p(w, z) : w = d_1 c_1 b_2 \circ \{a_s : 2 \leq s \leq \varepsilon\} \circ a_1 b_1 c_\varepsilon \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ c_2 \circ \{b_j c_s : 3 \leq s \leq \varepsilon - 1\} \circ b_\varepsilon = z.$$

Case 15. $w = d_1$ and $z = c_r$, $1 \leq r \leq \varepsilon$

Subcase 15.1. $r = 1$

$$H_p(w, z) : w = d_1 \circ \{d_s : 2 \leq s \leq \varepsilon\} \circ c_\varepsilon b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 2\} \circ c_1 = z.$$

Subcase 15.2. $r = 2$

$$H_p(w, z) : w = d_1 c_1 b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_2 \circ \{b_j c_s : 3 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ c_2 = z.$$

Subcase 15.3. $r = 3$

$$H_p(w, z) : w = d_1 c_1 b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_2 c_2 \circ \{d_s : 2 \leq s \leq \varepsilon\} \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 4\} \circ b_3 c_3 = z.$$

Subcase 15.4. $r \equiv 0 \pmod{2}$, $3 < r < \varepsilon$

$$H_p(w, z) : w = d_1 c_1 b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_{2s} c_{2s} d_{2s} d_{2s+1} c_{2s+1} b_{2s+1} : 1 \leq s \leq \frac{r-2}{2}\} \circ b_r \circ \{b_j c_s : r+1 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} \circ c_r = z.$$

Subcase 15.5. $r \equiv 1 \pmod{2}$, $3 < r < \varepsilon$

$$H_p(w, z) : w = d_1 \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r + 1\} \circ c_{r-1} b_r \circ \{a_s : r \leq s \leq \varepsilon\} \circ \{a_s : 1 \leq s \leq r - 1\} \circ b_{r-1} \circ \{b_{2s+1} c_{2s+1} d_{2s+1} d_{2s} c_{2s} b_{2s} : \max\left(1, \frac{r-3}{2}\right) \leq s \leq \min\left(1, \frac{r-3}{2}\right)\} \circ c_1 b_1 \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r - 1\} \circ c_r = z.$$

Subcase 15.6. $r = \varepsilon$

$$H_p(w, z) : w = d_1 c_1 b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 3\} \circ b_2 c_2 \circ \{d_s : 2 \leq s \leq \varepsilon\} \circ c_\varepsilon = z.$$

Case 16. $w = d_1$ and $z = d_r$, $2 \leq r \leq \varepsilon$

Subcase 16.1. $r = 2$

$$H_p(w, z) : w = d_1 \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 3\} \circ c_3 b_3 a_3 a_2 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 4\} \circ \{b_j c_s : 4 \leq s \leq \varepsilon\} \circ b_1 c_1 b_2 c_2 d_2 = z.$$

Subcase 16.2. $r \equiv 0 \pmod{2}$, $2 < r < \varepsilon - 1$

$$H_p(w, z) : w = d_1 \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r - 1\} \circ c_{r+1} b_{r+1} \circ \{b_j c_s : r+2 \leq s \leq \varepsilon\} \circ b_1 c_1 \circ \{b_{2s} c_{2s} d_{2s} d_{2s+1} c_{2s+1} b_{2s+1} : 1 \leq s \leq \frac{r-2}{2}\} \circ \{a_{\varepsilon-s} : \varepsilon - r + 1 \leq s \leq \varepsilon - 1\} \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} \circ b_i c_i d_r = z.$$

Subcase 16.3. $r \equiv 1 \pmod{2}$, $2 < r < \varepsilon - 1$, and $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = d_1 \circ \{d_s : 2 \leq s \leq r-1\} \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : \varepsilon - r + 1 \leq s \leq \varepsilon - 1\} \circ \{a_s : 1 \leq s \leq r\} \circ b_i c_i b_{r+1} \circ \{a_s : r+1 \leq s \leq \varepsilon\} \circ \{b_{2s}c_{2s}d_{2s}d_{2s-1}c_{2s-1}b_{2s-1} : \max\left(\frac{r+3}{2}, \frac{\varepsilon}{2}\right) \leq s \leq \min\left(\frac{r+3}{2}, \frac{\varepsilon}{2}\right)\} \circ c_{r+1}d_{r+1}d_r = z.$$

Subcase 16.4. $r \equiv 1 \pmod{2}$, $2 < r < \varepsilon - 1$, and $\varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = d_1 \circ \{d_s : 2 \leq s \leq r-1\} \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : \varepsilon - r + 1 \leq s \leq \varepsilon - 1\} \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ \{b_{2s+1}c_{2s+1}d_{2s+1}d_{2s}c_{2s}b_{2s} : \max\left(\frac{r+1}{2}, \frac{\varepsilon-1}{2}\right) \leq s \leq \min\left(\frac{r+1}{2}, \frac{\varepsilon-1}{2}\right)\} \circ b_i c_i d_r = z.$$

Subcase 16.5. $r = \varepsilon - 1$, and $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = d_1 d_\varepsilon c_\varepsilon b_\varepsilon \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 4\} \circ b_4 c_3 b_3 a_3 a_2 a_1 b_1 c_1 b_2 c_2 d_2 d_3 d_4 c_4 \circ \{b_{2s-1}c_{2s-1}d_{2s-1}d_{2s}c_{2s}b_{2s} : \varepsilon > 6, 3 \leq s \leq \frac{\varepsilon-2}{2}\} \circ b_{\varepsilon-1}c_{\varepsilon-1}d_{\varepsilon-1} = z.$$

Subcase 16.6. $r = \varepsilon - 1$, and $\varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = \{d_{2s-1}c_{2s-1}b_{2s-1}b_{2s}c_{2s}d_{2s} : 1 \leq s \leq \frac{\varepsilon-3}{2}\} \circ d_{\varepsilon-2}c_{\varepsilon-2}b_{\varepsilon-2} \circ \{a_{\varepsilon-s} : 2 \leq s \leq \varepsilon - 1\} \circ a_\varepsilon a_{\varepsilon-1} b_{\varepsilon-1} c_{\varepsilon-1} b_\varepsilon c_\varepsilon d_\varepsilon d_{\varepsilon-1} = z.$$

Subcase 16.7. $r = \varepsilon$

$$H_p(w, z) : w = d_1 \circ \{d_s : 2 \leq s \leq \varepsilon - 1\} \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 1\} \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon d_\varepsilon = z.$$

Existence of Hamiltonian path between any two vertices of the Q_ε completes the proof. $\square$

Having $|V(Q_\varepsilon)| = 4\varepsilon$, Theorems 2.4 and 4.1 deliver the following immediate result:

Corollary 4.1. *Assuming $\varepsilon \geq 6$, we have*

$$\omega(Q_\varepsilon) = \frac{4\varepsilon(4\varepsilon - 1)^2}{2}.$$

Proof. Since $|V(Q_\varepsilon)| = 4\varepsilon$. Theorem 2.4 with $n = 4\varepsilon$ delivers the formula in the corollary. $\square$

5. Hamilton-connectivity and the detour index of B_ε

In this section, we explore Hamilton-connectivity of the convex polytope B_ε of dimension ε , which was introduced by Imran et al. [17] in 2016. We, in this section, aim to prove Hamilton-connectedness of B_ε and employ it to evaluate its detour index.

Assuming the subscript r to be modulo ε , the vertex/edge sets for B_ε are:

$$V(B_\varepsilon) = \{a_r, b_r, c_r, d_r, e_r : 1 \leq r \leq \varepsilon\},$$

$$E(B_\varepsilon) = \{a_r a_{r+1}, a_r b_r, a_r b_{r+1}, b_r b_{r+1}, b_r c_r, c_r b_{r+1}, c_r d_r, d_r d_{r+1}, d_r e_r, e_r e_{r+1} : 1 \leq r \leq \varepsilon\}.$$

With its vertices' labeling, Figure 5 delivers the construction of B_ε .

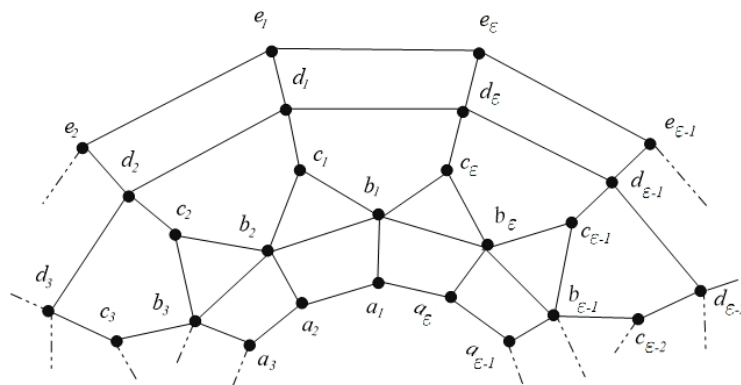


Figure 5. The graph B_ε .

Note that Hayat et al. [15, Theorem 5] showed the following result:

Theorem 5.1. [15, Theorem 5] *Considering $\varepsilon \geq 4$, the graph B_ε is not Hamilton-connected.*

They show in the proof that there is no Hamiltonian path between e_r and e_{r+1} if ε is odd. We show that their proof was wrong and there are multiple Hamiltonian paths between e_r and e_{r+1} if ε is odd. Correcting their proof, we show that the graph B_ε is, in fact, Hamilton-connected.

Next, we show this section's major results, delivering the proof for Hamilton-connectivity of B_ε . The proof technique utilizes the methodology of Alspach and Liu [1] using the well-known Posa exchange in constructing Hamiltonian paths between every pair of vertices in a graph.

Theorem 5.2. *If $\varepsilon \geq 6$, then B_ε is, in fact, Hamilton-connected.*

Proof. We show the result by constructing pairwise Hamiltonian paths between vertices of Q_ε . Assume $H_p(w, z)$ is a Hamiltonian path between $w, z \in V(Q_\varepsilon)$. Let $B_\varepsilon = A \cup B \cup C \cup D \cup E$ where $A = \{a_1, a_2, \dots, a_\varepsilon\}$, $B = \{b_1, b_2, \dots, b_\varepsilon\}$, $C = \{c_1, c_2, \dots, c_\varepsilon\}$, $D = \{d_1, d_2, \dots, d_\varepsilon\}$, and $E = \{e_1, e_2, \dots, e_\varepsilon\}$.

Case 1. $w = a_1$ and $z = a_r$, $2 \leq r \leq \varepsilon$

Subcase 1.1. $r = 2$

$$H_p(w, z) : w = a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 3\} \circ \{b_j c_s : 3 \leq s \leq \varepsilon\} \circ b_1 c_1 d_1 \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 3\} \circ \{e_s : 3 \leq s \leq \varepsilon\} \circ e_1 e_2 d_2 c_2 b_2 a_2 = z.$$

Subcase 1.2. $r \equiv 0 \pmod{2}$, and $2 < r < \varepsilon - 1$

$$H_p(w, z) : w = a_1 \circ \{a_s : 2 \leq s \leq r - 1\} \circ b_{r-1} \circ \{b_{2s} c_{2s} d_{2s} d_{2s-1} c_{2s-1} b_{2s-1} : \max(1, \frac{r-2}{2}) \leq s \leq \min(1, \frac{\varepsilon-2}{2})\} \circ c_\varepsilon d_\varepsilon e_\varepsilon \circ \{e_s : 1 \leq s \leq \varepsilon - 1\} \circ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - r + 1\} \circ c_{r-1} \circ \{b_j c_s : r \leq s \leq \varepsilon - 1\} \circ b_\varepsilon \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} = z.$$

Subcase 1.3. $r \equiv 1 \pmod{2}$, and $2 < r < \varepsilon - 1$

$$H_p(w, z) : w = a_1 \circ \{a_s : 2 \leq s \leq r - 1\} \circ \{b_{2s} c_{2s} d_{2s} d_{2s-1} c_{2s-1} b_{2s-1} :$$

$$\max(1, \frac{r-1}{2}) \leq s \leq \min\left(1, \frac{r-1}{2}\right) \circ c_\varepsilon d_\varepsilon e_\varepsilon \circ \{e_s : 1 \leq s \leq \varepsilon - 1\} \circ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - r\} \circ \\ c_i b_r \circ \{b_j c_s : r + 1 \leq s \leq \varepsilon - 1\} \circ b_\varepsilon \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} = z.$$

Subcase 1.4. $r = \varepsilon - 1$

$$H_p(w, z) : w = a_1 a_\varepsilon b_\varepsilon b_{\varepsilon-1} c_{\varepsilon-1} d_{\varepsilon-1} \circ \{e_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 1\} \circ e_\varepsilon d_\varepsilon c_\varepsilon b_1 c_1 \circ \{d_s : 1 \leq s \leq \varepsilon - 2\} \circ \\ \{c_{\varepsilon-s} b_{\varepsilon-s} : 2 \leq s \leq \varepsilon - 2\} \circ \{a_s : 2 \leq s \leq \varepsilon - 1\} = z.$$

Subcase 1.5. $r = \varepsilon$

$$H_p(w, z) : w = a_1 b_1 c_1 d_1 e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{d_s : 2 \leq s \leq \varepsilon\} \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \\ \{a_s : 2 \leq s \leq \varepsilon\} = z.$$

Case 2. $w = a_1$ and $z = b_r$, $1 \leq r \leq \varepsilon$

Subcase 2.1. $r = 1$

$$H_p(w, z) : w = a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_j c_s : 2 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \\ \{e_s : 2 \leq s \leq \varepsilon\} \circ e_1 d_1 c_1 b_1 = z.$$

Subcase 2.2. $r = 2$

$$H_p(w, z) : w = a_1 \circ \{a_s : 2 \leq s \leq \varepsilon\} \circ b_\varepsilon b_{\varepsilon-1} c_{\varepsilon-1} d_{\varepsilon-1} \circ \{e_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 1\} \circ e_\varepsilon d_\varepsilon c_\varepsilon b_1 c_1 \circ \\ \{d_s : 1 \leq s \leq \varepsilon - 2\} \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 2 \leq s \leq \varepsilon - 2\} = z.$$

Subcase 2.3. $r \equiv 0 \pmod{2}$, and $2 \leq r < \varepsilon - 1$

$$H_p(w, z) : w = a_1 \circ \{a_s : 2 \leq s \leq \varepsilon\} \circ b_\varepsilon b_{\varepsilon-1} c_{\varepsilon-1} d_{\varepsilon-1} \circ \{e_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 1\} \circ e_\varepsilon d_\varepsilon c_\varepsilon \circ \\ \{b_{2s-1} c_{2s-1} d_{2s-1} d_{2s} c_{2s} b_{2s} : 1 \leq s \leq \frac{r-2}{2}\} \circ b_{r-1} c_{r-1} \circ \{d_s : r - 1 \leq s \leq \varepsilon - 2\} \circ \\ \{c_{\varepsilon-s} b_{\varepsilon-s} : 2 \leq s \leq \varepsilon - r\} = z.$$

Subcase 2.4. $r \equiv 1 \pmod{2}$, and $2 < r < \varepsilon - 1$

$$H_p(w, z) : w = a_1 \circ \{a_s : 2 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon d_\varepsilon \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ \{d_{2s-1} c_{2s-1} b_{2s-1} b_{2s} c_{2s} d_{2s} : \\ 1 \leq s \leq \frac{r-1}{2}\} \circ \{d_s : r \leq s \leq \varepsilon - 1\} \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - r\} = z.$$

Subcase 2.5. $r = \varepsilon - 1$

$$H_p(w, z) : w = a_1 \circ \{a_s : 2 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon \circ \{b_j c_s : 1 \leq s \leq \varepsilon - 2\} \circ \{d_{\varepsilon-s} : 2 \leq s \leq \varepsilon - 1\} \circ d_\varepsilon e_\varepsilon \circ \\ \{e_s : 1 \leq s \leq \varepsilon - 1\} \circ d_{\varepsilon-1} c_{\varepsilon-1} b_{\varepsilon-1} = z.$$

Subcase 2.6. $r = \varepsilon$

$$H_p(w, z) : w = a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_2 b_1 c_1 d_1 e_1 e_2 d_2 c_2 \circ \{b_j c_s : 3 \leq s \leq \varepsilon - 1\} \circ$$

$$\{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 3\} \circ \{e_s : 3 \leq s \leq \varepsilon\} \circ d_\varepsilon c_\varepsilon b_\varepsilon = z.$$

Case 3. $w = a_1$ and $z = c_r$, $1 \leq r \leq \varepsilon$

Subcase 3.1. $r = 1$

$$H_p(w, z) : w = a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_j c_s : 2 \leq s \leq \varepsilon - 1\} \circ b_\varepsilon b_1 c_\varepsilon d_\varepsilon e_\varepsilon \circ$$

$$\{e_s : 1 \leq s \leq \varepsilon - 1\} \circ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 1\} \circ c_1 = z.$$

Subcase 3.2. $r = 2$

$$H_p(w, z) : w = a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_2 b_1 c_1 d_1 e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{d_s : 2 \leq s \leq \varepsilon\} \circ$$

$$\{c_{\varepsilon-s} b_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 3\} \circ c_2 = z.$$

Subcase 3.3. $r \equiv 0 \pmod{2}$, and $2 < r < \varepsilon - 1$

$$H_p(w, z) : w = a_1 \circ \{a_s : 2 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon d_\varepsilon \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ d_1 c_1 b_1 \circ \{b_{2s} c_{2s} d_{2s} d_{2s+1} c_{2s+1} b_{2s+1} :$$

$$1 \leq s \leq \frac{r-2}{2}\} \circ b_r \circ \{b_j c_s : r+1 \leq s \leq \varepsilon - 1\} \circ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - r\} \circ c_r = z.$$

Subcase 3.4. $r \equiv 1 \pmod{2}$, and $2 < r < \varepsilon - 1$

$$H_p(w, z) : w = a_1 \circ \{a_s : 2 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon d_\varepsilon \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ \{d_{2s-1} c_{2s-1} b_{2s-1} b_{2s} c_{2s} d_{2s} :$$

$$1 \leq s \leq \frac{r-1}{2}\} \circ \{d_s : r \leq s \leq \varepsilon - 1\} \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r - 1\} \circ b_i c_r = z.$$

Subcase 3.5. $r = \varepsilon - 1$

$$H_p(w, z) : w = a_1 \circ \{a_s : 2 \leq s \leq \varepsilon\} \circ b_\varepsilon b_{\varepsilon-1} \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 2 \leq s \leq \varepsilon - 1\} \circ c_\varepsilon d_\varepsilon \circ$$

$$\{d_s : 1 \leq s \leq \varepsilon - 2\} \circ \{e_{\varepsilon-s} : 2 \leq s \leq \varepsilon - 1\} \circ e_\varepsilon e_{\varepsilon-1} d_{\varepsilon-1} c_{\varepsilon-1} = z.$$

Subcase 3.6. $r = \varepsilon$

$$H_p(w, z) : w = a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_j c_s : 2 \leq s \leq \varepsilon - 1\} \circ b_\varepsilon b_1 c_1 d_1 e_1 \circ$$

$$\{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{d_s : 2 \leq s \leq \varepsilon\} \circ c_\varepsilon = z.$$

Case 4. $w = a_1$ and $z = d_r$, $1 \leq r \leq \varepsilon$

Subcase 4.1. $r = 1$

$$H_p(w, z) : w = a_1 \circ \{a_s : 2 \leq s \leq \varepsilon\} \circ b_\varepsilon \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 1\} \circ c_\varepsilon \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ$$

$$\{e_s : 2 \leq s \leq \varepsilon\} \circ e_1 d_1 = z.$$

Subcase 4.2. $r = 2$

$$H_p(w, z) : w = a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_j c_s : 2 \leq s \leq \varepsilon\} \circ b_1 c_1 d_1 \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ$$

$$\{e_s : 2 \leq s \leq \varepsilon\} \circ e_1 e_2 d_2 = z.$$

Subcase 4.3. $r \equiv 0 \pmod{2}$, and $2 < r < \varepsilon$

$$H_p(w, z) : w = a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_j c_s : 2 \leq k \leq \varepsilon\} \circ b_1 c_1 d_1 e_1 \circ \{e_{2s} d_{2s} d_{2s+1} e_{2s+1} : \\ 1 \leq s \leq \frac{r-2}{2}\} \circ \{e_s : r \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} = z.$$

Subcase 4.4. $r \equiv 1 \pmod{2}$, and $3 < r < \varepsilon$

$$H_p(w, z) : w = a_1 \circ \{a_s : 2 \leq s \leq \varepsilon\} \circ b_\varepsilon \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 1\} \circ c_\varepsilon d_\varepsilon e_\varepsilon \circ \{e_{2s-1} d_{2s-1} d_{2s} e_{2s} : \\ 1 \leq s \leq \frac{r-1}{2}\} \circ \{e_s : r \leq s \leq \varepsilon - 1\} \circ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - r\} = z.$$

Subcase 4.5. $r = \varepsilon$

$$H_p(w, z) : w = a_1 \circ \{a_s : 2 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon \circ \{b_j c_s : 1 \leq s \leq \varepsilon - 1\} \circ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 1\} \circ \\ \{e_s : 1 \leq s \leq \varepsilon\} \circ d_\varepsilon = z.$$

Case 5. $w = a_1$ and $z = e_r$, $1 \leq r \leq \varepsilon$

Subcase 5.1. $r = 1$

$$H_p(w, z) : w = a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_j c_s : 2 \leq s \leq \varepsilon\} \circ b_1 c_1 \circ \{d_s : 1 \leq s \leq \varepsilon\} \circ \\ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} = z.$$

Subcase 5.2. $r = 2$

$$H_p(w, z) : w = a_1 \circ \{a_s : 2 \leq s \leq \varepsilon\} \circ b_\varepsilon \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 1\} \circ c_\varepsilon d_\varepsilon e_\varepsilon e_1 \circ \\ \{d_s : 1 \leq s \leq \varepsilon - 1\} \circ \{e_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 2\} = z.$$

Subcase 5.3. $r \equiv 0 \pmod{2}$, and $2 < r < \varepsilon$

$$H_p(w, z) : w = a_1 \circ \{a_s : 2 \leq s \leq \varepsilon\} \circ b_\varepsilon \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 1\} \circ c_\varepsilon d_\varepsilon e_\varepsilon e_1 d_1 \circ \\ \{d_{2s} e_{2s} e_{2s+1} d_{2s+1} : 1 \leq s \leq \frac{r-2}{2}\} \circ \{d_s : r \leq s \leq \varepsilon - 1\} \circ \{e_{\varepsilon-s} : 1 \leq s \leq \varepsilon - r\} = z.$$

Subcase 5.4. $r \equiv 1 \pmod{2}$, and $2 < r < \varepsilon$

$$H_p(w, z) : w = a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_j c_s : 2 \leq s \leq \varepsilon\} \circ b_1 c_1 \circ \{d_{2s-1} e_{2s-1} e_{2s} d_{2s} : \\ 1 \leq s \leq \frac{r-1}{2}\} \circ \{d_s : r \leq s \leq \varepsilon\} \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} = z.$$

Subcase 5.5. $r = \varepsilon$

$$H_p(w, z) : w = a_1 \circ \{a_s : 2 \leq s \leq \varepsilon\} \circ b_\varepsilon \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 1\} \circ c_\varepsilon \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ \\ \{e_s : 1 \leq s \leq \varepsilon\} = z.$$

Case 6. $w = b_1$ and $z = a_r$, $1 \leq r \leq \varepsilon$

Subcase 6.1. $r = 1$

$$H_p(w, z) : w = b_1 c_1 \circ \{b_j c_s : 2 \leq s \leq \varepsilon - 1\} \circ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 1\} \circ \{e_s : 1 \leq s \leq \varepsilon\} \circ d_\varepsilon c_\varepsilon b_\varepsilon \circ \\ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} = z.$$

Subcase 6.2. $r = 2$

$$H_p(w, z) : w = b_1 b_\varepsilon c_\varepsilon d_\varepsilon d_1 c_1 b_2 c_2 d_2 e_2 e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 3\} \circ \{d_s : 3 \leq s \leq \varepsilon - 1\} \circ \\ \{c_{\varepsilon-s} b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 3\} \circ \{a_s : 3 \leq s \leq \varepsilon\} \circ a_1 a_2 = z.$$

Subcase 6.3. $r \equiv 0 \pmod{2}$, and $2 < r < \varepsilon - 1$

$$H_p(w, z) : w = b_1 b_\varepsilon c_\varepsilon d_\varepsilon \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ d_1 c_1 \circ \{b_{2s} c_{2s} d_{2s} d_{2s+1} c_{2s+1} b_{2s+1} : \\ 1 \leq s \leq \frac{r-2}{2}\} \circ \{a_{\varepsilon-s} : \varepsilon - r + 1 \leq s \leq \varepsilon - 1\} \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r - 1\} \circ \\ \{b_j c_s : r + 1 \leq s \leq \varepsilon - 1\} \circ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - r\} \circ c_i b_i a_r = z.$$

Subcase 6.4. $r \equiv 1 \pmod{2}$, and $2 < r < \varepsilon - 1$

$$H_p(w, z) : w = b_1 c_1 d_1 e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{d_{2s} c_{2s} b_{2s} b_{2s+1} c_{2s+1} d_{2s+1} : 1 \leq s \leq \frac{r-1}{2}\} \circ \\ \{d_s : r + 1 \leq s \leq \varepsilon\} \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r - 1\} \circ \{a_s : r + 1 \leq s \leq \varepsilon\} \circ \{a_s : 1 \leq s \leq r\} = z.$$

Subcase 6.5. $r = \varepsilon - 1$

$$H_p(w, z) : w = b_1 c_1 \circ \{b_j c_s : 2 \leq s \leq \varepsilon - 1\} \circ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 1\} \circ \{e_s : 1 \leq s \leq \varepsilon\} \circ d_\varepsilon c_\varepsilon b_\varepsilon a_\varepsilon \circ \\ \{a_s : 1 \leq s \leq \varepsilon - 1\} = z.$$

Subcase 6.6. $r = \varepsilon$

$$H_p(w, z) : w = b_1 c_1 d_1 e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{d_s : 2 \leq s \leq \varepsilon\} \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \\ \{a_s : 2 \leq s \leq \varepsilon\} = z.$$

Case 7. $w = b_1$ and $z = b_r, 2 \leq r \leq \varepsilon$

Subcase 7.1. $r = 2$

$$H_p(w, z) : w = b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon d_\varepsilon e_\varepsilon e_{\varepsilon-1} d_{\varepsilon-1} \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 3\} \circ c_2 \circ \\ \{d_s : 2 \leq s \leq \varepsilon\} \circ \{e_{\varepsilon-s} : 2 \leq s \leq \varepsilon - 1\} \circ d_1 c_1 b_2 = z.$$

Subcase 7.2. $r = 3$

$$H_p(w, z) : w = b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon d_\varepsilon \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ d_1 c_1 b_2 c_2 \circ \\ \{d_s : 3 \leq s \leq \varepsilon - 1\} \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 3\} = z.$$

Subcase 7.3. $r \equiv 0 \pmod{2}$, and $3 < r < \varepsilon$

$$\begin{aligned} H_p(w, z) : w = \{b_{2s-1}c_{2s-1}d_{2s-1}d_{2s}c_{2s}b_{2s} : 1 \leq k \leq \frac{r-2}{2}\} \circ \{a_{\varepsilon-s} : \varepsilon - r + 2 \leq s \leq \varepsilon - 1\} \circ \\ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r + 1\} \circ b_{r-1}c_{r-1}d_{r-1} \circ \{e_{\varepsilon-s} : \varepsilon - r + 1 \leq s \leq \varepsilon - 1\} \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} \circ \\ \{d_s : r \leq s \leq \varepsilon\} \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} = z. \end{aligned}$$

Subcase 7.4. $r \equiv 1 \pmod{2}$, and $3 < r < \varepsilon$

$$\begin{aligned} H_p(w, z) : w = b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon d_\varepsilon \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ d_1 c_1 \circ \{b_{2s}c_{2s}d_{2s}d_{2s+1}c_{2s+1}b_{2s+1} : \\ 1 \leq s \leq \frac{r-3}{2}\} \circ b_{r-1}c_{r-1} \circ \{d_s : r - 1 \leq s \leq \varepsilon - 1\} \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - r\} = z. \end{aligned}$$

Subcase 7.5. $r = \varepsilon$

$$\begin{aligned} H_p(w, z) : w = b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_2 c_1 d_1 e_1 e_2 d_2 c_2 \circ \{b_j c_s : 3 \leq s \leq \varepsilon - 1\} \circ \\ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 3\} \circ \{e_s : 3 \leq s \leq \varepsilon\} \circ d_\varepsilon c_\varepsilon b_\varepsilon = z. \end{aligned}$$

Case 8. $w = b_1$ and $z = c_r$, $1 \leq r \leq \varepsilon$

Subcase 8.1. $r = 1$

$$\begin{aligned} H_p(w, z) : w = b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_j c_s : 2 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \\ \{e_s : 2 \leq s \leq \varepsilon\} \circ e_1 d_1 c_1 = z. \end{aligned}$$

Subcase 8.2. $r = 2$

$$\begin{aligned} H_p(w, z) : w = b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_2 b_1 c_1 d_1 e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{d_s : 2 \leq s \leq \varepsilon\} \circ \\ \{c_{\varepsilon-s}b_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 3\} \circ c_2 = z. \end{aligned}$$

Subcase 8.3. $r = 3$

$$\begin{aligned} H_p(w, z) : w = b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_2 c_2 \circ \{d_s : 2 \leq s \leq \varepsilon\} \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ \\ d_1 c_1 b_1 \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 4\} \circ b_3 c_3 = z. \end{aligned}$$

Subcase 8.4. $r \equiv 0 \pmod{2}$, $3 < r < \varepsilon - 1$

$$\begin{aligned} H_p(w, z) : w = b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon d_\varepsilon \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ d_1 c_1 \circ \\ \{b_{2s}c_{2s}d_{2s}d_{2s+1}c_{2s+1}b_{2s+1} : 1 \leq s \leq \frac{r-2}{2}\} \circ b_r \circ \{b_j c_s : r + 1 \leq s \leq \varepsilon - 1\} \circ \\ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - r\} \circ c_r = z. \end{aligned}$$

Subcase 8.5. $r \equiv 1 \pmod{2}$, $3 < r < \varepsilon - 1$

$$H_p(w, z) : w = \{b_{2s-1}c_{2s-1}d_{2s-1}d_{2s}c_{2s}b_{2s} : 1 \leq s \leq \frac{r-1}{2}\} \circ \{a_{\varepsilon-s} : \varepsilon - r + 1 \leq s \leq \varepsilon - 1\} \circ$$

$$\{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} \circ b_r \circ \{b_j c_s : r + 1 \leq s \leq \varepsilon\} \circ d_\varepsilon e_\varepsilon \circ \{e_s : 1 \leq s \leq \varepsilon - 1\} \circ \\ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - r\} \circ c_r = z.$$

Subcase 8.6. $r = \varepsilon - 1$

$$H_p(w, z) : w = b_1 c_1 \circ \{b_j c_s : 2 \leq s \leq \varepsilon - 2\} \circ b_{\varepsilon-1} \circ \{a_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 1\} \circ a_\varepsilon b_\varepsilon c_\varepsilon d_\varepsilon \circ \\ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ \{d_s : 1 \leq s \leq \varepsilon - 1\} \circ c_{\varepsilon-1} = z.$$

Subcase 8.7. $r = \varepsilon$

$$H_p(w, z) : w = b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_2 c_1 d_1 \circ \{e_s : 1 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \\ c_2 \circ \{b_j c_s : 3 \leq s \leq \varepsilon\} = z.$$

Case 9. $w = b_1$ and $z = d_r, 1 \leq r \leq \varepsilon$

Subcase 9.1. $r = 1, \varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = b_1 c_1 \circ \{b_{2s} c_{2s} d_{2s} d_{2s+1} c_{2s+1} b_{2s+1} : 1 \leq s \leq \frac{\varepsilon-2}{2}\} \circ \{a_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 1\} \circ \\ a_\varepsilon b_\varepsilon c_\varepsilon d_\varepsilon \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ d_1 = z.$$

Subcase 9.2. $r = 1, \varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon d_\varepsilon e_\varepsilon \circ \{e_s : 1 \leq s \leq \varepsilon - 1\} \circ \{d_{2s} c_{2s} b_{2s} b_{2s-1} c_{2s-1} d_{2s-1} : \\ \max(2, \frac{\varepsilon-1}{2}) \leq s \leq \min(2, \frac{\varepsilon-1}{2})\} \circ d_2 c_2 b_2 c_1 d_1 = z.$$

Subcase 9.3. $r = 2$

$$H_p(w, z) : w = b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon d_\varepsilon \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ d_1 c_1 \circ \{b_j c_s : 2 \leq s \leq \varepsilon - 1\} \\ \circ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 2\} = z.$$

Subcase 9.4. $r = 3$

$$H_p(w, z) : w = b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_2 c_1 d_1 e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ d_2 c_2 \circ \\ \{b_j c_s : 3 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 3\} = z.$$

Subcase 9.5. $r \equiv 0 \pmod{2}, 2 < r < \varepsilon$

$$H_p(w, z) : w = b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon d_\varepsilon \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ d_1 c_1 \circ \\ \{b_{2s} c_{2s} d_{2s} d_{2s+1} c_{2s+1} b_{2s+1} : 1 \leq s \leq \frac{r-2}{2}\} \circ \{b_j c_s : r \leq s \leq \varepsilon - 1\} \circ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - r\} = z.$$

Subcase 9.6. $r \equiv 1 \pmod{2}, 2 < r < \varepsilon$

$$H_p(w, z) : w = b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_2 c_1 d_1 e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ d_2 c_2 \circ$$

$$\{b_{2s-1}c_{2s-1}d_{2s-1}d_{2s}c_{2s}b_{2s} : 2 \leq s \leq \frac{r-1}{2}\} \circ \{b_jc_s : r \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} = z.$$

Subcase 9.7. $r = \varepsilon$, and $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = b_1c_\varepsilon b_\varepsilon a_\varepsilon \circ \{a_s : 1 \leq s \leq \varepsilon - 1\} \circ \{b_{2s+1}c_{2s+1}d_{2s+1}d_{2s}c_{2s}b_{2s} : \\ \max(1, \frac{\varepsilon-2}{2}) \leq s \leq \min(1, \frac{\varepsilon-2}{2})\} \circ c_1d_1 \circ \{e_s : 1 \leq s \leq \varepsilon\} \circ d_\varepsilon = z.$$

Subcase 9.8. $r = \varepsilon$, and $\varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = b_1a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_2c_1d_1e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ d_2c_2 \circ \\ \{b_{2s-1}c_{2s-1}d_{2s-1}d_{2s}c_{2s}b_{2s} : 2 \leq s \leq \frac{\varepsilon-1}{2}\} \circ b_\varepsilon c_\varepsilon d_\varepsilon = z.$$

Case 10. $w = b_1$ and $z = e_r$, $1 \leq r \leq \varepsilon$

Subcase 10.1. $r = 1$

$$H_p(w, z) : w = b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon d_\varepsilon \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{d_s : 2 \leq s \leq \varepsilon - 1\} \circ \\ \{c_{\varepsilon-s}b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 2\} \circ c_1d_1e_1 = z.$$

Subcase 10.2. $r = 2$

$$H_p(w, z) : w = b_1a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_2c_1d_1e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 3\} \circ \{d_s : 3 \leq s \leq \varepsilon\} \circ \\ \{c_{\varepsilon-s}b_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 3\} \circ c_2d_2e_2 = z.$$

Subcase 10.3. $r \equiv 0 \pmod{2}$, $2 < r < \varepsilon - 1$

$$H_p(w, z) : w = b_1a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_2c_1d_1e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r - 1\} \circ \{d_s : r + 1 \leq s \leq \varepsilon\} \circ \\ \{c_{\varepsilon-s}b_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 3\} \circ c_2 \circ \{d_{2s}e_{2s}e_{2s+1}d_{2s+1} : 1 \leq s \leq \frac{r-2}{2}\} \circ d_ie_r = z.$$

Subcase 10.4. $r \equiv 1 \pmod{2}$, $2 < r < \varepsilon - 1$

$$H_p(w, z) : w = b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon d_\varepsilon \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r - 1\} \circ \{d_s : r + 1 \leq s \leq \varepsilon - 1\} \circ \\ \{c_{\varepsilon-s}b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 2\} \circ c_1 \circ \{d_{2s-1}e_{2s-1}e_{2s}d_{2s} : 1 \leq s \leq \frac{r-1}{2}\} \circ d_ie_r = z.$$

Subcase 10.5. $r = \varepsilon - 1$

$$H_p(w, z) : w = b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon \circ \{b_jc_s : 1 \leq s \leq \varepsilon - 1\} \circ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 1\} \circ \\ d_\varepsilon e_\varepsilon \circ \{e_s : 1 \leq s \leq \varepsilon - 1\} = z.$$

Subcase 10.6. $r = \varepsilon$ and $\varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = b_1a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_2c_1d_1 \circ \{e_s : 1 \leq s \leq \varepsilon - 1\} \circ \\ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 2\} \circ c_2 \circ \{b_jc_s : 3 \leq s \leq \varepsilon\} \circ d_\varepsilon e_\varepsilon = z.$$

Case 11. $w = c_1$ and $z = a_r, 1 \leq r \leq \varepsilon$

Subcase 11.1. $r = 1$

$$H_p(w, z) : w = c_1 d_1 \circ \{e_s : 1 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ c_2 \circ \{b_j c_s : 3 \leq s \leq \varepsilon\} \circ b_1 b_2 \circ \\ \{a_s : 2 \leq s \leq \varepsilon\} \circ a_1 = z.$$

Subcase 11.2. $r = 2$

$$H_p(w, z) : w = c_1 d_1 \circ \{e_s : 1 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ c_2 b_2 \circ \{b_j c_s : 3 \leq s \leq \varepsilon\} \circ \\ b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} = z.$$

Subcase 11.3. $r = 3$

$$H_p(w, z) : w = c_1 d_1 \circ \{e_s : 1 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ c_2 \circ \{b_j c_s : 3 \leq s \leq \varepsilon\} \circ \\ b_1 b_2 a_2 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 3\} = z.$$

Subcase 11.4. $r \equiv 0 \pmod{2}, 3 < r < \varepsilon - 1$

$$H_p(w, z) : w = c_1 d_1 \circ \{e_s : 1 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} \circ c_i b_r \circ \{b_j c_s : r + 1 \leq s \leq \varepsilon\} \circ b_1 \circ \\ \{b_{2s} c_{2s} d_{2s} d_{2s+1} c_{2s+1} b_{2s+1} : 1 \leq s \leq \frac{r-2}{2}\} \circ \{a_{\varepsilon-s} : \varepsilon - r + 1 \leq s \leq \varepsilon - 1\} \circ \\ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} = z.$$

Subcase 11.5. $r \equiv 1 \pmod{2}, 3 < r < \varepsilon - 1$

$$H_p(w, z) : w = c_1 d_1 e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{d_{2s} c_{2s} b_{2s} b_{2s+1} c_{2s+1} d_{2s+1} : 1 \leq s \leq \frac{r-1}{2}\} \circ \\ \{d_s : r + 1 \leq s \leq \varepsilon\} \circ c_\varepsilon b_1 b_\varepsilon \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - r - 1\} \circ \{a_s : r + 1 \leq s \leq \varepsilon\} \circ \\ \{a_s : 1 \leq s \leq r\} = z.$$

Subcase 11.6. $r = \varepsilon - 1$

$$H_p(w, z) : w = c_1 b_1 \circ \{b_j c_s : 2 \leq s \leq \varepsilon - 1\} \circ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 1\} \circ \{e_s : 1 \leq s \leq \varepsilon\} \circ \\ d_\varepsilon c_\varepsilon b_\varepsilon a_\varepsilon \circ \{a_s : 1 \leq s \leq \varepsilon - 1\} = z.$$

Subcase 11.7. $r = \varepsilon$

$$H_p(w, z) : w = c_1 d_1 \circ \{e_s : 1 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ c_2 b_2 \circ \{b_j c_s : 3 \leq s \leq \varepsilon\} \circ b_1 \circ \\ \{a_s : 1 \leq s \leq \varepsilon\} = z.$$

Case 12. $w = c_1$ and $z = b_r, 1 \leq r \leq \varepsilon$

Subcase 12.1. $r = 1$

$$H_p(w, z) : w = c_1 d_1 e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{d_s : 2 \leq s \leq \varepsilon\} \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ$$

$$\{a_s : 2 \leq s \leq \varepsilon\} \circ a_1 b_1 = z.$$

Subcase 12.2. $r = 2$

$$H_p(w, z) : w = c_1 d_1 \circ \{e_s : 1 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ c_2 \circ \{b_j c_s : 3 \leq s \leq \varepsilon\} \circ$$

$$b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_2 = z.$$

Subcase 12.3. $r = 3$

$$H_p(w, z) : w = c_1 d_1 \circ \{e_s : 1 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ c_2 b_2 \circ \{a_s : 2 \leq s \leq \varepsilon\} \circ a_1 b_1 \circ$$

$$\{c_{\varepsilon-s} b_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 3\} = z.$$

Subcase 12.4. $r \equiv 0 \pmod{2}$, $3 < r < \varepsilon$

$$H_p(w, z) : w = c_1 d_1 \circ \{e_s : 1 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} \circ c_r \circ \{b_j c_s : r + 1 \leq s \leq \varepsilon\} \circ$$

$$b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_{2s} c_{2s} d_{2s} d_{2s+1} c_{2s+1} b_{2s+1} : 1 \leq s \leq \frac{r-2}{2}\} \circ b_r = z.$$

Subcase 12.5. $r \equiv 1 \pmod{2}$, $3 < r < \varepsilon$

$$H_p(w, z) : w = c_1 d_1 \circ \{e_s : 1 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} \circ \{d_{2s} c_{2s} b_{2s} b_{2s-1} c_{2s-1} d_{2s-1} :$$

$$\max(2, \frac{r-1}{2}) \leq s \leq \min(2, \frac{r-1}{2})\} \circ d_2 c_2 b_2 \circ \{a_s : 2 \leq s \leq \varepsilon\} \circ a_1 b_1 \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} = z.$$

Subcase 12.6. $r = \varepsilon$

$$H_p(w, z) : w = c_1 d_1 e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{d_s : 2 \leq s \leq \varepsilon\} \circ c_\varepsilon b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ$$

$$\{b_j c_s : 2 \leq s \leq \varepsilon - 1\} \circ b_\varepsilon = z.$$

Case 13. $w = c_1$ and $z = c_r$, $2 \leq r \leq \varepsilon$

Subcase 13.1. $r = 2$

$$H_p(w, z) : w = c_1 d_1 e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{d_s : 2 \leq s \leq \varepsilon\} \circ c_\varepsilon b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon \circ$$

$$\{c_{\varepsilon-s} b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 3\} \circ b_2 c_2 = z.$$

Subcase 13.2. $r = 3$

$$H_p(w, z) : w = c_1 d_1 \circ \{e_s : 1 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ c_2 b_2 \circ \{a_s : 2 \leq s \leq \varepsilon\} \circ a_1 b_1 \circ$$

$$\{c_{\varepsilon-s} b_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 4\} \circ b_3 c_3 = z.$$

Subcase 13.3. $r \equiv 0 \pmod{2}$, and $3 < r < \varepsilon$

$$H_p(w, z) : w = c_1 d_1 \circ \{e_s : 1 \leq s \leq \varepsilon\} \circ d_\varepsilon c_\varepsilon b_\varepsilon \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ b_1 \circ \{b_{2s} c_{2s} d_{2s} d_{2s+1} c_{2s+1} b_{2s+1} :$$

$$1 \leq s \leq \frac{r-2}{2}\} \circ b_r \circ \{b_j c_s : r + 1 \leq s \leq \varepsilon - 1\} \circ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - r\} \circ c_r = z.$$

Subcase 13.4. $r \equiv 1 \pmod{2}$, and $3 < r < \varepsilon$

$$H_p(w, z) : w = c_1 d_1 e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ d_2 c_2 b_2 \circ \{b_{2s-1} c_{2s-1} d_{2s-1} d_{2s} c_{2s} b_{2s} : \\ 2 \leq k \leq \frac{r-1}{2}\} \circ b_r \circ \{b_j c_s : r+1 \leq s \leq \varepsilon - 1\} \circ b_\varepsilon \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ b_1 c_\varepsilon \circ \\ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} \circ c_r = z.$$

Subcase 13.5. $r = \varepsilon - 1$

$$H_p(w, z) : w = c_1 b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_j c_s : 2 \leq s \leq \varepsilon - 2\} \circ b_{\varepsilon-1} b_\varepsilon c_\varepsilon d_\varepsilon \circ \\ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ \{d_s : 1 \leq s \leq \varepsilon - 1\} \circ c_{\varepsilon-1} = z.$$

Subcase 13.6. $r = \varepsilon$

$$H_p(w, z) : w = c_1 d_1 \circ \{e_s : 1 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ c_2 b_2 \circ \{b_j c_s : 3 \leq s \leq \varepsilon - 1\} \circ b_\varepsilon \circ \\ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ b_1 c_\varepsilon = z.$$

Case 14. $w = c_1$ and $z = d_r$, $1 \leq r \leq \varepsilon$

Subcase 14.1. $r = 1$

$$H_p(w, z) : w = c_1 b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_j c_s : 2 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \\ \{e_s : 2 \leq s \leq \varepsilon\} \circ e_1 d_1 = z.$$

Subcase 14.2. $r = 2$

$$H_p(w, z) : w = c_1 d_1 \circ \{e_s : 1 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 3\} \circ c_3 b_3 \circ \{b_j c_s : 4 \leq s \leq \varepsilon\} \circ b_1 a_1 \circ \\ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_2 c_2 d_2 = z.$$

Subcase 14.3. $r = 3$

$$H_p(w, z) : w = c_1 d_1 e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ d_2 c_2 b_2 \circ \{b_j c_s : 3 \leq s \leq \varepsilon - 1\} \circ b_\varepsilon \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ \\ b_1 c_\varepsilon \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 3\} = z.$$

Subcase 14.4. $r \equiv 0 \pmod{2}$, $3 < r < \varepsilon - 1$

$$H_p(w, z) : w = c_1 d_1 \circ \{e_s : 1 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r - 1\} \circ c_{r+1} b_{r+1} \circ \{b_j c_s : r+2 \leq s \leq \varepsilon\} \circ \\ b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_{2s} c_{2s} d_{2s} d_{2s-1} c_{2s-1} b_{2s-1} : \max(1, \frac{r-2}{2}) \leq k \leq \min(1, \frac{r-2}{2})\} \circ \\ b_i c_i d_r = z.$$

Subcase 14.5. $r \equiv 1 \pmod{2}$, $3 < r < \varepsilon - 1$

$$H_p(w, z) : w = c_1 d_1 e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ d_2 c_2 b_2 \circ \{b_{2s-1} c_{2s-1} d_{2s-1} d_{2s} c_{2s} b_{2s} : 2 \leq k \leq \frac{r-1}{2}\} \circ$$

$$\{b_j c_s : r \leq s \leq \varepsilon - 1\} \circ b_\varepsilon \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ b_1 c_\varepsilon \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} = z.$$

Subcase 14.6. $r = \varepsilon - 1$

$$H_p(w, z) : w = c_1 b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_j c_s : 2 \leq s \leq \varepsilon\} \circ d_\varepsilon \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ \{d_s : 1 \leq s \leq \varepsilon - 1\} = z.$$

Subcase 14.7. $r = \varepsilon$

$$H_p(w, z) : w = c_1 d_1 e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{d_s : 2 \leq s \leq \varepsilon - 1\} \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 2\} \circ b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon d_\varepsilon = z.$$

Case 15. $w = c_1$ and $z = e_r, 1 \leq r \leq \varepsilon$

Subcase 15.1. $r = 1$

$$H_p(w, z) : w = c_1 \circ \{d_s : 1 \leq s \leq \varepsilon - 1\} \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 2\} \circ b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon d_\varepsilon \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} = z.$$

Subcase 15.2. $r \equiv 0 \pmod{2}$

$$H_p(w, z) : w = c_1 b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_j c_s : 2 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r - 1\} \circ \{e_s : r + 1 \leq s \leq \varepsilon\} \circ \{e_{2s-1} d_{2s-1} d_{2s} e_{2s} : 1 \leq s \leq \frac{r}{2}\} = z.$$

Subcase 15.3. $r \equiv 1 \pmod{2}, 2 < r < \varepsilon$

$$H_p(w, z) : w = c_1 \circ \{d_{2s-1} e_{2s-1} e_{2s} d_{2s} : 1 \leq s \leq \frac{r-1}{2}\} \circ \{d_s : r \leq s \leq \varepsilon - 1\} \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 2\} \circ b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon d_\varepsilon \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} = z.$$

Subcase 15.4. $r = \varepsilon$

$$H_p(w, z) : w = c_1 b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_j c_s : 2 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ \{e_s : 1 \leq s \leq \varepsilon\} = z.$$

Case 16. $w = d_1$ and $z = a_r, 1 \leq r \leq \varepsilon$

Subcase 16.1. $r = 1$

$$H_p(w, z) : w = d_1 \circ \{e_s : 1 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ c_2 \circ \{b_j c_s : 3 \leq s \leq \varepsilon\} \circ b_1 c_1 b_2 \circ \{a_s : 2 \leq s \leq \varepsilon\} \circ a_1 = z.$$

Subcase 16.2. $r \equiv 0 \pmod{2}, 1 < r < \varepsilon - 1$

$$H_p(w, z) : w = \{d_{2s-1} c_{2s-1} b_{2s-1} b_{2s} c_{2s} d_{2s} : 1 \leq s \leq \frac{r}{2}\} \circ \{e_{\varepsilon-s} : \varepsilon - r \leq s \leq \varepsilon - 1\} \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r - 1\} \circ \{d_s : r + 1 \leq s \leq \varepsilon\} \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r - 1\} \circ$$

$$\{a_s : r+1 \leq s \leq \varepsilon\} \circ \{a_s : 1 \leq s \leq r\} = z.$$

Subcase 16.3. $r \equiv 1 \pmod{2}$, $1 < r < \varepsilon - 1$

$$H_p(w, z) : w = d_1 d_\varepsilon e_\varepsilon \circ \{e_s : 1 \leq s \leq \varepsilon - 1\} \circ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - r - 1\} \circ c_{r+1} \circ \{b_j c_s : r+2 \leq s \leq \varepsilon\} \circ$$

$$b_1 c_1 \circ \{b_{2s} c_{2s} d_{2s} d_{2s+1} c_{2s+1} b_{2s+1} : 1 \leq s \leq \frac{r-1}{2}\} \circ b_{r+1} \circ \{a_s : r+1 \leq s \leq \varepsilon\} \circ \{a_s : 1 \leq s \leq r\} = z.$$

Subcase 16.4. $r = \varepsilon - 1$

$$H_p(w, z) : w = d_1 e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{d_s : 2 \leq s \leq \varepsilon\} \circ c_\varepsilon \circ \{b_j c_s : 1 \leq s \leq \varepsilon - 1\} \circ b_\varepsilon a_\varepsilon \circ$$

$$\{a_s : 1 \leq s \leq \varepsilon - 1\} = z.$$

Subcase 16.5. $r = \varepsilon$

$$H_p(w, z) : w = d_1 e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{d_s : 2 \leq s \leq \varepsilon\} \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ$$

$$\{a_s : 1 \leq s \leq \varepsilon\} = z.$$

Case 17. $w = d_1$ and $z = b_r$, $1 \leq r \leq \varepsilon$

Subcase 17.1. $r = 1$, $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = d_1 \circ \{e_s : 1 \leq s \leq \varepsilon\} \circ d_\varepsilon c_\varepsilon b_\varepsilon a_\varepsilon \circ \{a_s : 1 \leq s \leq \varepsilon - 1\} \circ \{b_{2s+1} c_{2s+1} d_{2s+1} d_{2s} c_{2s} b_{2s} :$$

$$\max(1, \frac{\varepsilon-2}{2}) \leq k \leq \min(1, \frac{\varepsilon-2}{2})\} \circ c_1 b_1 = z.$$

Subcase 17.2. $r = 1$, $\varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = d_1 c_1 b_2 a_2 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 3\} \circ b_3 c_2 d_2 e_2 e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 3\} \circ d_3 c_3 \circ$$

$$\{b_{2s} c_{2s} d_{2s} d_{2s+1} c_{2s+1} b_{2s+1} : 2 \leq k \leq \frac{\varepsilon-1}{2}\} \circ b_1 = z.$$

Subcase 17.3. $r = 2$, $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = d_1 e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ d_2 c_2 b_3 a_3 a_2 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 4\} \circ b_4 c_3 d_3 d_4 c_4 \circ$$

$$\{b_{2s-1} c_{2s-1} d_{2s-1} d_{2s} c_{2s} b_{2s} : 3 \leq s \leq \frac{\varepsilon}{2}\} \circ b_1 c_1 b_2 = z.$$

Subcase 17.4. $r = 2$, $\varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = d_1 c_1 b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon d_\varepsilon e_\varepsilon \circ \{e_s : 1 \leq s \leq \varepsilon - 1\} \circ \{d_{2s} c_{2s} b_{2s} b_{2s-1} c_{2s-1} d_{2s-1} :$$

$$\max(2, \frac{\varepsilon-1}{2}) \leq s \leq \min(2, \frac{\varepsilon-1}{2})\} \circ d_2 c_2 b_2 = z.$$

Subcase 17.5. $r \equiv 0 \pmod{2}$, $2 < r < \varepsilon$

$$H_p(w, z) : w = d_1 \circ \{e_s : 1 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} \circ c_r \circ \{b_j c_s : r+1 \leq s \leq \varepsilon\} \circ b_1 c_1 \circ$$

$$\{b_{2s}c_{2s}d_{2s}d_{2s+1}c_{2s+1}b_{2s+1} : 1 \leq s \leq \frac{r-2}{2}\} \circ \{a_{\varepsilon-s} : \varepsilon - r + 1 \leq s \leq \varepsilon - 1\} \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} \circ b_r = z.$$

Subcase 17.6. $r \equiv 1 \pmod{2}$, $2 < r < \varepsilon$

$$H_p(w, z) : w = d_1c_1b_1a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_2c_2d_2 \circ \{d_{2s-1}c_{2s-1}b_{2s-1}b_{2s}c_{2s}d_{2s} : r > 3, 2 \leq s \leq \frac{r-1}{2}\} \circ \{e_{\varepsilon-s} : \varepsilon - r + 1 \leq s \leq \varepsilon - 1\} \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} \circ \{d_s : r \leq s \leq \varepsilon\} \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} = z.$$

Subcase 17.7. $r = \varepsilon$

$$H_p(w, z) : w = d_1e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ d_2c_2b_3 \circ \{a_s : 3 \leq s \leq \varepsilon\} \circ a_1a_2b_2c_1b_1c_\varepsilon \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 3\} \circ c_3 \circ \{b_jc_s : 4 \leq s \leq \varepsilon - 1\} \circ b_\varepsilon = z.$$

Case 18. $w = d_1$ and $z = c_r$, $1 \leq r \leq \varepsilon$

Subcase 18.1. $r = 1$

$$H_p(w, z) : w = d_1 \circ \{e_s : 1 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ c_2 \circ \{b_jc_s : 3 \leq s \leq \varepsilon\} \circ b_1a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_2c_1 = z.$$

Subcase 18.2. $r = 2$

$$H_p(w, z) : w = d_1c_1b_1a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_2 \circ \{b_jc_s : 3 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 3\} \circ \{e_s : 3 \leq s \leq \varepsilon\} \circ e_1e_2d_2c_2 = z.$$

Subcase 18.3. $r \equiv 0 \pmod{2}$, $2 < r < \varepsilon$

$$H_p(w, z) : w = d_1c_1b_1a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_{2s}c_{2s}d_{2s}d_{2s+1}c_{2s+1}b_{2s+1} : 1 \leq k \leq \frac{r-2}{2}\} \circ b_r \circ \{b_jc_s : r + 1 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r - 1\} \circ \{e_s : r + 1 \leq s \leq \varepsilon\} \circ \{e_s : 1 \leq s \leq r\} \circ d_1c_r = z.$$

Subcase 18.4. $r \equiv 1 \pmod{2}$, $2 < r < \varepsilon$

$$H_p(w, z) : w = d_1c_1b_1a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_jc_s : 2 \leq s \leq r - 1\} \circ \{d_{2s}e_{2s}e_{2s-1}d_{2s-1} : r > 3, \max(2, \frac{r-1}{2}) \leq s \leq \min(2, \frac{r-1}{2})\} \circ d_2e_2e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} \circ \{d_s : r \leq s \leq \varepsilon\} \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r - 1\} \circ b_1c_r = z.$$

Subcase 18.5. $r = \varepsilon$

$$H_p(w, z) : w = d_1c_1b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 3\} \circ b_2c_2 \circ \{d_s : 2 \leq s \leq \varepsilon - 1\} \circ \{e_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 1\} \circ e_\varepsilon d_\varepsilon c_\varepsilon = z.$$

Case 19. $w = d_1$ and $z = d_r, 2 \leq r \leq \varepsilon$

Subcase 19.1. $r = 2$

$$H_p(w, z) : w = d_1 \circ \{e_s : 1 \leq s \leq \varepsilon\} \circ d_\varepsilon c_\varepsilon b_\varepsilon \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ \{b_j c_s : 1 \leq s \leq \varepsilon - 1\} \circ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 2\} = z.$$

Subcase 19.2. $r \equiv 0 \pmod{2}, 2 < r < \varepsilon$

$$H_p(w, z) : w = d_1 c_1 b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_j c_s : 2 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r - 1\} \circ \{e_s : r + 1 \leq s \leq \varepsilon\} \circ e_1 \circ \{e_{2s} d_{2s} d_{2s+1} e_{2s+1} : 1 \leq s \leq \frac{r-2}{2}\} \circ e_i d_r = z.$$

Subcase 19.3. $r \equiv 1 \pmod{2}, \varepsilon \equiv 0 \pmod{2}, 2 < r < \varepsilon - 1$

$$H_p(w, z) : w = d_1 \circ \{e_s : 1 \leq s \leq \varepsilon\} \circ d_\varepsilon c_\varepsilon b_\varepsilon a_\varepsilon a_1 b_1 c_1 b_2 \circ \{a_s : 2 \leq s \leq \varepsilon - 1\} \circ \{b_{2s+1} c_{2s+1} d_{2s+1} d_{2s} c_{2s} b_{2s} : \max(\frac{r+1}{2}, \frac{\varepsilon-2}{2}) \leq s \leq \min(\frac{r+1}{2}, \frac{\varepsilon-2}{2})\} \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : \varepsilon - r \leq s \leq \varepsilon - 3\} \circ c_2 \circ \{d_s : 2 \leq s \leq r\} = z.$$

Subcase 19.4. $r \equiv 1 \pmod{2}, \varepsilon \equiv 1 \pmod{2}, 2 < r < \varepsilon - 1$

$$H_p(w, z) : w = d_1 c_1 b_1 \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ \{b_{2s+1} c_{2s+1} d_{2s+1} d_{2s} c_{2s} b_{2s} : \frac{r+1}{2} \leq s \leq \frac{\varepsilon-1}{2}\} \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : \varepsilon - r \leq s \leq \varepsilon - 3\} \circ c_2 \circ \{d_s : 2 \leq s \leq r - 1\} \circ \{e_{\varepsilon-s} : \varepsilon - r + 1 \leq s \leq \varepsilon - 1\} \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} \circ d_r = z.$$

Subcase 19.5. $r = \varepsilon - 1, \varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = d_1 \circ \{e_s : 1 \leq s \leq \varepsilon\} \circ d_\varepsilon c_\varepsilon b_\varepsilon \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 4\} \circ b_4 c_3 b_3 a_3 a_2 a_1 b_1 c_1 b_2 c_2 d_2 d_3 d_4 c_4 \circ \{b_{2s-1} c_{2s-1} d_{2s-1} d_{2s} c_{2s} b_{2s} : \varepsilon > 6, 3 \leq s \leq \frac{\varepsilon-2}{2}\} \circ b_{\varepsilon-1} c_{\varepsilon-1} d_{\varepsilon-1} = z.$$

Subcase 19.6. $r = \varepsilon$

$$H_p(w, z) : w = d_1 e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{d_s : 2 \leq s \leq \varepsilon - 1\} \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 1\} \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon d_\varepsilon = z.$$

Case 20. $w = d_1$ and $z = e_r, 1 \leq r \leq \varepsilon$

Subcase 20.1. $r = 1$

$$H_p(w, z) : w = d_1 c_1 b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_j c_s : 2 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{e_s : 2 \leq s \leq \varepsilon\} \circ e_1 = z.$$

Subcase 20.2. $r = 2$

$$H_p(w, z) : w = d_1 e_1 e_\varepsilon d_\varepsilon c_\varepsilon b_\varepsilon a_\varepsilon a_1 b_1 c_1 b_2 \circ \{a_s : 2 \leq s \leq \varepsilon - 1\} \circ b_{\varepsilon-1} c_{\varepsilon-1} d_{\varepsilon-1} \circ \{e_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 3\} \circ$$

$$\{d_s : 3 \leq s \leq \varepsilon - 2\} \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : 2 \leq s \leq \varepsilon - 3\} \circ c_2d_2e_2 = z.$$

Subcase 20.3. $r \equiv 0 \pmod{2}$, $2 < r < \varepsilon - 1$

$$\begin{aligned} H_p(w, z) : w = d_1d_\varepsilon c_\varepsilon b_\varepsilon a_\varepsilon a_1b_1c_1b_2 \circ \{a_s : 2 \leq s \leq \varepsilon - 1\} \circ b_{\varepsilon-1}c_{\varepsilon-1}d_{\varepsilon-1}e_{\varepsilon-1}e_\varepsilon \circ \{e_s : 1 \leq s \leq r - 1\} \circ \\ \{d_{\varepsilon-s} : \varepsilon - r + 1 \leq s \leq \varepsilon - 2\} \circ c_2 \circ \{b_jc_s : 3 \leq s \leq r - 1\} \circ \{b_{2s}c_{2s}d_{2s}d_{2s+1}c_{2s+1}b_{2s+1} : \varepsilon > 6, \frac{r}{2} \leq s \leq \frac{\varepsilon - r}{2}\} \circ \\ b_{\varepsilon-2}c_{\varepsilon-2}d_{\varepsilon-2} \circ \{e_{\varepsilon-s} : 2 \leq s \leq \varepsilon - r\} = z. \end{aligned}$$

Subcase 20.4. $r \equiv 1 \pmod{2}$, $2 < r < \varepsilon - 1$

$$\begin{aligned} H_p(w, z) : w = d_1c_1b_1 \circ \{b_{2s}c_{2s}d_{2s}d_{2s+1}c_{2s+1}b_{2s+1} : 1 \leq s \leq \frac{r-1}{2}\} \circ \{a_{\varepsilon-s} : \varepsilon - r \leq s \leq \varepsilon - 1\} \circ \\ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r - 1\} \circ \{b_jc_s : r + 1 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r - 1\} \circ \{e_s : r + 1 \leq s \leq \varepsilon\} \circ \\ \{e_s : 1 \leq s \leq r\} = z. \end{aligned}$$

Subcase 20.5. $r = \varepsilon - 1$

$$\begin{aligned} H_p(w, z) : w = d_1 \circ \{d_s : 2 \leq s \leq \varepsilon - 1\} \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 1\} \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon d_\varepsilon e_\varepsilon \circ \\ \{e_s : 1 \leq s \leq \varepsilon - 1\} = z. \end{aligned}$$

Subcase 20.6. $r = \varepsilon$

$$\begin{aligned} H_p(w, z) : w = d_1d_\varepsilon c_\varepsilon b_\varepsilon a_\varepsilon a_1b_1c_1b_2 \circ \{a_s : 2 \leq s \leq \varepsilon - 1\} \circ b_{\varepsilon-1}c_{\varepsilon-1}d_{\varepsilon-1} \circ \{e_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 3\} \circ \\ \{d_s : 3 \leq s \leq \varepsilon - 2\} \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : 2 \leq s \leq \varepsilon - 3\} \circ c_2d_2e_2e_1e_\varepsilon = z. \end{aligned}$$

Case 21. $w = e_1$ and $z = a_r$, $1 \leq r \leq \varepsilon$

Subcase 21.1. $r = 1$

$$\begin{aligned} H_p(w, z) : w = e_1 \circ \{e_s : 2 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ c_1b_1 \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \\ \{a_s : 2 \leq s \leq \varepsilon\} \circ a_1 = z. \end{aligned}$$

Subcase 21.2. $r \equiv 0 \pmod{2}$, $1 < r < \varepsilon$

$$\begin{aligned} H_p(w, z) : w = e_1 \circ \{e_s : 2 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r - 1\} \circ c_{r+1} \circ \{b_jc_s : r + 2 \leq s \leq \varepsilon\} \circ \\ \{b_{2s-1}c_{2s-1}d_{2s-1}d_{2s}c_{2s}b_{2s} : 1 \leq s \leq \frac{r}{2}\} \circ b_{r+1} \circ \{a_s : r + 1 \leq s \leq \varepsilon\} \circ \{a_s : 1 \leq s \leq r\} = z. \end{aligned}$$

Subcase 21.3. $r \equiv 1 \pmod{2}$, $1 < r < \varepsilon$

$$\begin{aligned} H_p(w, z) : w = e_1 \circ \{e_s : 2 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} \circ c_ib_r \circ \{b_jc_s : r + 1 \leq s \leq \varepsilon\} \circ \\ \{b_{2s-1}c_{2s-1}d_{2s-1}d_{2s}c_{2s}b_{2s} : 1 \leq s \leq \frac{r-1}{2}\} \circ \{a_{\varepsilon-s} : \varepsilon - r + 1 \leq s \leq \varepsilon - 1\} \circ \\ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} = z. \end{aligned}$$

Subcase 21.4. $r = \varepsilon - 1$

$$H_p(w, z) : w = e_1 d_1 d_\varepsilon \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{d_s : 2 \leq s \leq \varepsilon - 1\} \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 1\} \circ \\ c_\varepsilon b_\varepsilon a_\varepsilon \circ \{a_s : 1 \leq s \leq \varepsilon - 1\} = z.$$

Subcase 21.5. $r = \varepsilon$

$$H_p(w, z) : w = e_1 \circ \{e_s : 2 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ c_1 \circ \{b_j c_s : 2 \leq s \leq \varepsilon\} \circ b_1 \circ \\ \{a_s : 1 \leq s \leq \varepsilon\} = z.$$

Case 22. $w = e_1$ and $z = b_r, 1 \leq r \leq \varepsilon$

Subcase 22.1. $r = 1$

$$H_p(w, z) : w = e_1 d_1 c_1 b_2 a_2 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 3\} \circ b_3 c_2 d_2 \circ \{e_s : 2 \leq s \leq \varepsilon\} \circ \\ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 3\} \circ c_3 \circ \{b_j c_s : 4 \leq s \leq \varepsilon\} \circ b_1 = z.$$

Subcase 22.2. $r \equiv 0 \pmod{2}, 1 < r < \varepsilon$

$$H_p(w, z) : w = e_1 \circ \{e_s : 2 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r - 1\} \circ c_{r+1} b_{r+1} \circ \{a_{\varepsilon-s} : \varepsilon - r - 1 \leq s \leq \varepsilon - 1\} \circ \\ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r - 2\} \circ \{b_j c_s : r + 2 \leq s \leq \varepsilon\} \circ \{b_{2s-1} c_{2s-1} d_{2s-1} d_{2s} c_{2s} b_{2s} : 1 \leq s \leq \frac{r}{2}\} = z.$$

Subcase 22.3. $r \equiv 1 \pmod{2}, 1 < r < \varepsilon$

$$H_p(w, z) : w = e_1 \circ \{e_s : 2 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} \circ c_r \circ \{b_j c_s : r + 1 \leq s \leq \varepsilon\} \circ \\ \{b_{2s-1} c_{2s-1} d_{2s-1} d_{2s} c_{2s} b_{2s} : 1 \leq s \leq \frac{r-1}{2}\} \circ \{a_{\varepsilon-s} : \varepsilon - r + 1 \leq s \leq \varepsilon - 1\} \circ \\ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - r\} \circ b_r = z.$$

Subcase 22.4. $r = \varepsilon - 1$

$$H_p(w, z) : w = e_1 \circ \{e_s : 2 \leq s \leq \varepsilon\} \circ d_\varepsilon c_\varepsilon b_\varepsilon c_{\varepsilon-1} \circ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 1\} \circ c_1 b_1 a_1 \circ \\ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_j c_s : 2 \leq s \leq \varepsilon - 2\} \circ b_{\varepsilon-1} = z.$$

Subcase 22.5. $r = \varepsilon$

$$H_p(w, z) : w = e_1 d_1 c_1 b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ \{b_j c_s : 2 \leq s \leq \varepsilon - 1\} \circ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 2\} \circ \\ \{e_s : 2 \leq s \leq \varepsilon\} \circ d_\varepsilon c_\varepsilon b_\varepsilon = z.$$

Case 23. $w = e_1$ and $z = c_r, 1 \leq r \leq \varepsilon$

Subcase 23.1. $r = 1$

$$H_p(w, z) : w = e_1 \circ \{e_s : 2 \leq s \leq \varepsilon\} \circ d_\varepsilon c_\varepsilon b_\varepsilon \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ b_1 \circ \{b_j c_s : 2 \leq s \leq \varepsilon - 1\} \circ \\ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 1\} \circ c_1 = z.$$

Subcase 23.2. $r = 2$

$$H_p(w, z) : w = e_1 \circ \{e_s : 2 \leq s \leq \varepsilon\} \circ d_\varepsilon c_\varepsilon b_\varepsilon \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ b_1 c_1 \circ \{d_s : 1 \leq s \leq \varepsilon - 1\} \circ \\ \{c_{\varepsilon-s} b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - 3\} \circ b_2 c_2 = z.$$

Subcase 23.3. $r \equiv 0 \pmod{2}$, $2 < r < \varepsilon - 1$

$$H_p(w, z) : w = e_1 \circ \{e_s : 2 \leq s \leq \varepsilon\} \circ d_\varepsilon c_\varepsilon b_\varepsilon \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ b_1 c_1 d_1 \circ \\ \{d_{2s} c_{2s} b_{2s} b_{2s+1} c_{2s+1} d_{2s+1} : 1 \leq s \leq \frac{r-2}{2}\} \circ \{d_s : r \leq s \leq \varepsilon - 1\} \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 1 \leq s \leq \varepsilon - r - 1\} \circ \\ b_i c_r = z.$$

Subcase 23.4. $r \equiv 1 \pmod{2}$, $2 < r < \varepsilon - 1$

$$H_p(w, z) : w = e_1 \circ \{e_s : 2 \leq s \leq \varepsilon\} \circ d_\varepsilon c_\varepsilon b_\varepsilon \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ \{b_{2s-1} c_{2s-1} d_{2s-1} d_{2s} c_{2s} b_{2s} : \\ 1 \leq s \leq \frac{r-1}{2}\} \circ b_r \circ \{b_j c_s : r+1 \leq s \leq \varepsilon - 1\} \circ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon - r\} \circ c_r = z.$$

Subcase 23.5. $r = \varepsilon - 1$

$$H_p(w, z) : w = e_1 \circ \{e_s : 2 \leq s \leq \varepsilon\} \circ d_\varepsilon c_\varepsilon b_\varepsilon a_\varepsilon \circ \{a_s : 1 \leq s \leq \varepsilon - 1\} \circ b_{\varepsilon-1} \circ \{c_{\varepsilon-s} b_{\varepsilon-s} : 2 \leq s \leq \varepsilon - 2\} \circ \\ b_1 c_1 \circ \{d_s : 1 \leq s \leq \varepsilon - 1\} \circ c_{\varepsilon-1} = z.$$

Subcase 23.6. $r = \varepsilon$

$$H_p(w, z) : w = e_1 \circ \{e_s : 2 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ c_1 \circ \{b_j c_s : 2 \leq s \leq \varepsilon - 1\} \circ b_\varepsilon \circ \\ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 1\} \circ b_1 c_\varepsilon = z.$$

Case 24. $w = e_1$ and $z = d_r$, $1 \leq r \leq \varepsilon$

Subcase 24.1. $r = 1$

$$H_p(w, z) : w = e_1 \circ \{e_s : 2 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ c_2 \circ \{b_j c_s : 3 \leq s \leq \varepsilon\} \circ b_1 a_1 \circ \\ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon - 2\} \circ b_2 c_1 d_1 = z.$$

Subcase 24.2. $r = 2$, $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = e_1 \circ \{e_s : 2 \leq s \leq \varepsilon\} \circ d_\varepsilon c_\varepsilon b_1 \circ \{a_s : 1 \leq s \leq \varepsilon - 2\} \circ b_{\varepsilon-2} c_{\varepsilon-2} b_{\varepsilon-1} a_{\varepsilon-1} a_\varepsilon b_\varepsilon c_{\varepsilon-1} \\ d_{\varepsilon-1} d_{\varepsilon-2} d_{\varepsilon-3} c_{\varepsilon-3} b_{\varepsilon-3} \circ \{b_{2s} c_{2s} d_{2s} d_{2s-1} c_{2s-1} b_{2s-1} : \varepsilon > 6, \max(2, \frac{\varepsilon-4}{2}) \leq s \leq \min(2, \frac{\varepsilon-4}{2})\} \circ \\ c_2 b_2 c_1 d_1 d_2 = z.$$

Subcase 24.3. $r = 2$, $\varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = e_1 \circ \{e_s : 2 \leq s \leq \varepsilon\} \circ d_\varepsilon d_1 c_1 b_2 \circ \{a_s : 2 \leq s \leq \varepsilon\} \circ a_1 b_1 c_\varepsilon b_\varepsilon \circ$$

$$\{b_{2s}c_{2s}d_{2s}d_{2s-1}c_{2s-1}b_{2s-1} : \max(2, \frac{\varepsilon-1}{2}) \leq s \leq \min(2, \frac{\varepsilon-1}{2})\} \circ c_2d_2 = z.$$

Subcase 24.4. $r \equiv 0 \pmod{2}$, $2 < r < \varepsilon$

$$H_p(w, z) : w = e_1 \circ \{e_s : 2 \leq s \leq \varepsilon\} \circ d_\varepsilon c_\varepsilon b_\varepsilon \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon-3\} \circ b_3c_2b_2a_2a_1b_1c_1d_1d_2d_3c_3 \circ$$

$$\{b_{2s}c_{2s}d_{2s}d_{2s+1}c_{2s+1}b_{2s+1} : r > 4, \varepsilon > 6, 2 \leq s \leq \frac{r-2}{2}\} \circ \{b_jc_s : r \leq s \leq \varepsilon-1\} \circ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon-r\} = z.$$

Subcase 24.5. $r \equiv 1 \pmod{2}$, $2 < r < \varepsilon$

$$H_p(w, z) : w = e_1 \circ \{e_s : 2 \leq s \leq \varepsilon\} \circ d_\varepsilon c_\varepsilon b_\varepsilon \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon-1\} \circ \{b_{2s-1}c_{2s-1}d_{2s-1}d_{2s}c_{2s}b_{2s} :$$

$$1 \leq s \leq \frac{r-1}{2}\} \circ \{b_jc_s : r \leq s \leq \varepsilon-1\} \circ \{d_{\varepsilon-s} : 1 \leq s \leq \varepsilon-r\} = z.$$

Subcase 24.6. $r = \varepsilon$, $\varepsilon \equiv 0 \pmod{2}$

$$H_p(w, z) : w = e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon-2\} \circ d_2d_1c_1b_2c_2 \circ \{b_{2s-1}c_{2s-1}d_{2s-1}d_{2s}c_{2s}b_{2s} : 2 \leq s \leq \frac{\varepsilon-2}{2}\} \circ$$

$$\{a_{\varepsilon-s} : 2 \leq s \leq \varepsilon-1\} \circ b_1c_\varepsilon b_\varepsilon a_\varepsilon a_{\varepsilon-1} b_{\varepsilon-1} c_{\varepsilon-1} d_{\varepsilon-1} d_\varepsilon = z.$$

Subcase 24.7. $r = \varepsilon$, $\varepsilon \equiv 1 \pmod{2}$

$$H_p(w, z) : w = e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon-2\} \circ d_2d_1c_1b_1a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon-2\} \circ b_2c_2 \circ$$

$$\{b_{2s-1}c_{2s-1}d_{2s-1}d_{2s}c_{2s}b_{2s} : 2 \leq s \leq \frac{\varepsilon-1}{2}\} \circ b_\varepsilon c_\varepsilon d_\varepsilon = z.$$

Case 25. $w = e_1$ and $z = e_r$, $1 \leq r \leq \varepsilon$

Subcase 25.1. $r = 2$

$$H_p(w, z) : w = e_1 \circ \{d_s : 1 \leq s \leq \varepsilon-1\} \circ \{c_{\varepsilon-s}b_{\varepsilon-s} : 1 \leq s \leq \varepsilon-1\} \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon d_\varepsilon \circ$$

$$\{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon-2\} = z.$$

Subcase 25.2. $r \equiv 0 \pmod{2}$, $2 < r < \varepsilon-1$

$$H_p(w, z) : w = e_1d_1 \circ \{d_{2s}e_{2s}e_{2s+1}d_{2s+1} : 1 \leq s \leq \frac{r-2}{2}\} \circ \{d_s : r \leq s \leq \varepsilon-1\} \circ$$

$$\{c_{\varepsilon-s}b_{\varepsilon-s} : 1 \leq s \leq \varepsilon-1\} \circ \{a_s : 1 \leq s \leq \varepsilon\} \circ b_\varepsilon c_\varepsilon d_\varepsilon \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon-r\} = z.$$

Subcase 25.3. $r \equiv 1 \pmod{2}$, $\varepsilon \equiv 0 \pmod{2}$, $2 < r < \varepsilon-1$

$$H_p(w, z) : w = e_1 \circ \{e_{\varepsilon-s} : 0 \leq s \leq \varepsilon-r-1\} \circ d_{r+1}c_{r+1}b_{r+1} \circ \{a_{\varepsilon-s} : \varepsilon-r-1 \leq s \leq \varepsilon-1\} \circ b_1c_\varepsilon b_\varepsilon \circ$$

$$\{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon-r-2\} \circ \{b_{2s-1}c_{2s-1}d_{2s-1}d_{2s}c_{2s}b_{2s} : \varepsilon > 6, r < \varepsilon-4, \frac{r+3}{2} \leq s \leq \frac{\varepsilon-r+1}{2}\} \circ$$

$$b_{\varepsilon-1}c_{\varepsilon-1}d_{\varepsilon-1}d_\varepsilon d_1c_1 \circ \{b_jc_s : 2 \leq s \leq r\} \circ \{d_{\varepsilon-s} : \varepsilon-r \leq s \leq \varepsilon-2\} \circ \{e_s : 2 \leq s \leq r\} = z.$$

Subcase 25.4. $r \equiv 1 \pmod{2}$, $\varepsilon \equiv 1 \pmod{2}$, $2 < r < \varepsilon - 1$

$$H_p(w, z) : w = e_1 \circ \{e_s : 2 \leq s \leq r-1\} \circ \{d_{\varepsilon-s} : \varepsilon-r+1 \leq s \leq \varepsilon-1\} \circ c_1 b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon-2\} \circ \\ \{b_j c_s : 2 \leq s \leq \varepsilon\} \circ \{d_{2s+1} e_{2s+1} e_{2s} d_{2s} : \max(\frac{r+1}{2}, \frac{\varepsilon-1}{2}) \leq s \leq \min(\frac{r+1}{2}, \frac{\varepsilon-1}{2})\} \circ d_i e_r = z.$$

Subcase 25.5. $r = \varepsilon - 1$

$$H_p(w, z) : w = e_1 d_1 c_1 b_2 a_2 a_1 b_1 c_\varepsilon b_\varepsilon \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon-3\} \circ b_3 c_2 d_2 \circ \{e_s : 2 \leq s \leq \varepsilon-2\} \circ \\ \{d_{\varepsilon-s} : 2 \leq s \leq \varepsilon-3\} \circ c_3 \circ \{b_j c_s : 4 \leq s \leq \varepsilon-1\} \circ d_{\varepsilon-1} d_\varepsilon e_\varepsilon e_{\varepsilon-1} = z.$$

Subcase 25.6. $r = \varepsilon$

$$H_p(w, z) : w = e_1 d_1 c_1 b_1 a_1 \circ \{a_{\varepsilon-s} : 0 \leq s \leq \varepsilon-2\} \circ \{b_j c_s : 2 \leq s \leq \varepsilon\} \circ \{d_{\varepsilon-s} : 0 \leq s \leq \varepsilon-2\} \circ \\ \{e_s : 2 \leq s \leq \varepsilon\} = z.$$

Existence of Hamiltonian path between any two vertices of the B_ε completes the proof. $\square$

Note that $|V(B_\varepsilon)| = 4\varepsilon$. Using it with Theorems 2.4 and 5.2 deliver the following immediate result:

Corollary 5.1. *Assuming $\varepsilon \geq 6$, we have*

$$\omega(B_\varepsilon) = \frac{4\varepsilon(4\varepsilon-1)^2}{2}.$$

Proof. Since $|V(B_\varepsilon)| = 4\varepsilon$. Theorem 2.4 with $n = 4\varepsilon$ delivers the result. $\square$

6. Conclusions

Contributions

- Introduced two distinct methods for proving Hamilton-connectedness of convex polytopes.
- Constructed three infinite families of Hamilton-connected convex polytopes.
- Corrected a misconception regarding the Hamilton-connectivity of the convex polytope B_ε , proving it is Hamilton-connected.
- Provided explicit computation of the detour index for these polytopes, establishing their structural properties.
- Extended previous results on Hamilton-connected graphs to new classes of geometric graphs derived from polytopes.

Implications

- Enhances understanding of Hamilton-connected structures within convex polytopes, contributing to graph theory and computational geometry.
- Provides new techniques for proving Hamilton-connectedness that can be applied to other combinatorial structures.
- Supports applications in network topology, where Hamilton-connected properties ensure robust and efficient routing.

- Offers insights into the relationship between geometric polytopes and their corresponding graph-theoretic properties.
- Strengthens theoretical foundations for detour index computations, aiding applications in cheminformatics and molecular modeling.

Limitations

- We focus on specific families of convex polytopes, leaving open the question of Hamilton-connectedness for other polytope classes.
- Computational complexity of verifying Hamilton-connectedness for arbitrary convex polytopes remains a challenge.
- The results rely on combinatorial proof techniques, which may not be directly extendable to all polyhedral structures.
- No empirical validation or algorithmic implementation is provided for the practical computation of detour indices in large graphs.

Future study

- Extend the proposed techniques to analyze Hamilton-connectedness in other polyhedral graphs beyond the studied families.
- Investigate algorithmic methods for efficiently computing detour indices in large-scale applications.
- Explore potential connections between Hamilton-connected convex polytopes and spectral graph theory.
- Develop computational tools to automate the detection of Hamilton-connected properties in graph representations of polytopes.
- Study the implications of Hamilton-connectedness in higher-dimensional convex polytopes and their applications in optimization and network design.

Author contributions

Sakander Hayat and Bagus Imanda: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Writing—original draft preparation, Writing—review and editing, Visualization; Asad Khan: Formal analysis, Investigation, Writing—original draft preparation, Editing; Mohammed J. F. Alenazi: Methodology, Validation, Resources, Writing—original draft preparation. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no known competing financial interests.

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