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Research article

Improved estimation of population parameter of in the existence of nonresponse using auxiliary information

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Abstract: The issue of survey nonresponse causes substantial effects on the reliability, together with validity of study findings. This research develops precise estimation methods for two distinct nonresponse situations: when nonresponse happens to primary survey variables like the number of students, and when nonresponse includes survey variables together with the accompanying auxiliary variables. The proposed estimators receive full a performance assessment through derived equations for bias and mean square error (MSE) based on first-order approximation. The accuracy and reliability assessment depends heavily on MSE calculations, since this method effectively merges systematic error measurements with random error measurements. The MSE values of different estimators receive numerical evaluation through a comparative analysis under equivalent operational conditions for assessing proposed estimator performance outcomes. The research seeks to find the estimator with the minimal MSE because this selection results in the most trustworthy estimates under nonresponse conditions. All the findings from this study create important guidelines for building educational survey designs.

Keywords: auxiliary variables; bias; efficiency; estimation of mean; mean squared error; nonresponse
Mathematics Subject Classification: 03H10, 37N40, 62P20, 68T07, 68T09, 91G15, 91G30

1. Introduction

A well-constructed sample is important in sampling theory because it serves as the foundation for obtaining accurate and relevant insights into a larger population. When a sample is carefully chosen, it represents the diversity and features of the entire population, allowing researchers to draw findings that are representative of larger patterns. Population parameters receive reliable statistical estimates through the application of estimators as important analytical tools. Well-informed decisions, initiative steering, and expanding our understanding of a population demand a representative sample and trustworthy estimators.

The analysis of nonresponse in sample collection plays an essential role according to sampling theory because it significantly influences validity and reliability. The act of study participants refusing to participate in research produces gaps in data that alter the research findings. The traits of survey participants who fail to respond differ strongly from those who complete the survey, creating distorted results. The health survey statistics will portray inaccurate health behaviors of the population, since they mostly collect data from people who possess good health understanding. Researchers must handle nonresponse correctly to sustain the representativeness of the study sample. Research teams should employ diverse methods, including follow-up reminders and incentives, together with substitute data collection methods to address nonresponse problems. These methods increase participation and guarantee proper representation of the demographic diversity within the population.

The growing problem of non responsiveness drives the need for establishing better statistical estimators. Population parameters can be obtained through sample data with the help of statistical tools called estimators. The reliability of these tools depends on the accurate representation of the sample. The outcome of traditional estimation methods can be inaccurate because these methods depend on an adequate representation of the population through the sample. To address this issue, researchers commonly employ approaches such as imputation, which guesses missing data on the basis of responses from like individuals, or weighting, which modifies the influence of specific responses to better align with known population characteristics.

In recent years, the conventional ratio, regression, and exponential processes have become widely used for calculating population means due to their easy computation and straightforward structure. Researchers have been able to more accurately estimate the unknown population's mean for the study variable by incorporating the population mean into the standard ratio and exponential estimators, as well as a number of auxiliary variable-related characteristics like skewness and kurtosis and coefficients of variation. The ratio, exponential, and regression estimation procedures also require prior knowledge of the population characteristics of the auxiliary variable. Assuming that auxiliary information was taken into account, [37] offered best linear unbiased (BLU) estimates, a popular regression estimator of the population mean of the studied variable. Using simple random sampling (SRS), Cochran proposed the traditional ratio estimate of the population mean [36]. It is generally known that the proper use of an auxiliary variable in survey sampling improves the estimators' precision both in the design and estimate phases. The survey sampling technique takes into account a number of different auxiliary variable integration strategies, including ratio, product, and regression-type procedures. The number of possible combinations for these techniques is unlimited. Numerous estimators are available, some of which combine regression, product, or ratio estimators with the simultaneous inclusion of a new variable. A large number of scientific researchers have created different estimators which effectively implement or replace auxiliary variables for optimizing estimation efficiency. Semary et al. [33] introduced a wide-ranging estimator framework for population proportion estimation with auxiliary attribute data. To enhance statistical modeling with auxiliary info about distributions, [28] along with [7], established generalized families which they then analyzed. Researchers, led by [6], expanded the Fréchet distribution and evaluated its performance in real-world data applications using simulation models. A new comprehensive class of estimators for population proportion using auxiliary attribute is simulation. An application [9] recently used supplementary information to create an enhanced estimate. Adamo [5] discussed the exchangeability property in causal models. Raja and Maqbool [25] used an auxiliary variable to examine a log-type estimator. Many additional researchers have generated estimators which improve the efficiency of population parameter estimation by combining auxiliary knowledge in strategic ways. Research by [24] developed Ln-type estimators which specifically deal with sensitive study variables to enable safer data collection. Ullah et al. [35] built two types of estimation methods that employed auxiliary distribution ranks and squares for simple sampling methods as well as stratified random sampling methods to achieve higher accuracy levels. Kurbah and Khongji [20] developed an exponential ratio-type estimator by incorporating both the mean and median values of two auxiliary variables in double sampling designs. Almulhim et al. [3] developed estimation methods which efficiently address nonresponse cases by using two auxiliary sources in simple random sampling. Rueda et al. [26] increased the application of auxiliary data for population quantile estimation to reach past traditional mean estimation methods. Koyuncu and Kadilsr [21] studied how ratio and product estimators behave in stratified sampling while demonstrating their most efficient use under ideal auxiliary data matches. The modified correlated measurement error model developed by [31] uses auxiliary data integration to improve mean estimation processes. Dyab et al. [10] worked with econometric panel data models while researching parameter estimation methods for temporal contexts through the use of high-frequency regressors. Adichwal et al. [4] developed flexible parameter estimation methods which operate well under random sampling without replacement conditions. Lone et al. [22] developed superior population variance estimation methods utilizing supplementary data to improve variability measurement. Following the development of a generalized class of exponential ratio estimators, [13] explored linear transformations of auxiliary variables to address non-linear patterns. Rao [27] provided fundamental methods to optimize ratio and regression estimators, followed by [32], who demonstrated how neutrosophic fuzzy sets can advance uncertain estimation in geographic information system (GIS) based applications.

1.1. Contribution of the proposed work

Numerous nonresponse scenarios demonstrate that the proposed estimators deliver superior performance in terms of the minimum mean squared error and maximum percent relative efficiency compared to those presented in references [13,14,22]. The proposed estimators surpass those presented in the references (a dual use of auxiliary variable generates improved mean estimation efficiency according to [14, 29, 30]). The research work of [34] introduced new exponential-type estimation methods that use transformed auxiliary variables to improve performances under certain preconditions. Grover and Kaur [13] developed a generalized system of exponential ratio-type estimators which use linear transformations of auxiliary variables for handling nonlinear features. Gemeay et al. [12] developed measurement error-based regression-type estimators, yet their methodology used only a single auxiliary variable. Yildirim et al. [38] improved the accuracy of stratified sampling techniques through modified ratio estimation methods that function without adding additional auxiliary data sources. Rao [27] developed essential theoretical research about optimizing ratio and regression estimators, yet his work lacked dual auxiliary variables or weighting strategies that distinguish modern advancements because they efficiently leverage dual auxiliary variables with optimized weightings to achieve higher precision and reliability. Results reveal that the proposed estimators deliver superior precision and reliability because they produce decreased mean square errors (MSEs) while generating elevated percentages relative efficiency (PREs) when employed with different sampling designs at varying correlation levels.

1.2. Novelty and motivation

This research introduces a primary innovation through its application of five different nonresponse rates at 35%, 30%, 25%, 20%, 15% for statistical estimation. We have identified this method as a new approach which previous research has not presented. The study shows how diverse levels of nonresponse influence estimator statistics, including MSE and PRE. The results obtained from the analysis validate the strength and reliability of our estimators in real survey settings because PRE and MSE improve steadily during decreasing nonresponse rate periods. This approach extends practical insight into the field by filling a gap that exists in current research. Therefore, it demonstrates both originality and practical benefits.

1.3. Practical application

These improved statistical methods from the study possess real-world applications including in the following situations.

- A significant number of people fail to participate in government survey and census work.
- Education surveys face limited participation because several students and graduates do not respond to requests.
- Respondents avoid answering questions in health surveys due to their personal nature.
- Surveys at national and local levels use additional institutional records to handle un responsive participants.

To improve estimators' efficiency and decrease nonresponse bias, several authors have focused on estimating the population mean. Mail surveys are more likely to have a high rate of nonresponse in their sample than special interview surveys. Originally, the issue of incomplete samples in surveys conducted by mail or over the phone was initially addressed in [11,15]. In relevant work, [1,2] worked and developed a restructured Searls family of estimators of the population mean in the presence of nonresponse, whereas [8,18,19] present a new estimator class to determine the value of the population mean when random nonresponse exists in two successive sampling phases. Hussain et al. [14] proposed two families of population distribution functions with dual use of auxiliary information under nonresponse, whereas [16, 17, 23] and the refrences therein proposed a generalized exponential type estimator of the population mean in the presence of nonresponse.

This study aims to improve the class of estimators used to determine the population mean from nonresponse under simple random sampling. We numerically compare the efficiency of the existing estimators with that of the proposed class of estimators. To demonstrate the suggested class of estimators, we offer two real-world datasets and simulated populations.

The following sections elaborate on the present paper's design. The methodology and terminology are defined in Section 2. Section 3 discusses some existing estimators and provides details on their bias and mean square error (MSE), and formulations for the minimal MSE. In Section 4, we present a new class of estimators for the population mean for nonresponse under simple random sampling (SRS) together with the bias and minimum MSE expressions. In Sections 5 and 6, we present the findings of the numerical analysis using real-life dataset and simulation. The results are discussed in Section 7. Finally the conclusion and some possible extensions of this work are discussed in Section 8.

2. Methodology

Take into consideration that $\Omega=\omega_1, \omega_2, , ..., \omega_N$ denotes a population of N units split into two categories having sizes of N_1 and N_2, where $N=N_1+N_2$. Thus, we denote the group that responded as $\Omega_1=\{\omega_1, \omega_2,, ..., \omega_1\}$ and the group that did not respond as $\Omega_2=\{\omega_1, \omega_2,, ..., N_2\}$. In order to estimate the population mean, simple random sampling with out replacement (SRSWOR) takes a sample of n from the underlying population, where n_1 represents responding units and $n_2=n-n_1$ is not. Furthermore, the sample sizes n_1 and n_2 are likely drawn from the groups Ω_2 (nonresponse) and Ω_1 (response), respectively. In addition, a sample of size $r=\frac{n_2}{k}$ units is created using a simple random sampling method without replacement from n_2, where k > 1.

Let *Y, X,* be the study and the auxiliary variable, respectively, while R_x is the ranks of *X*.

$\overline{Y}= \sum_{i=1}^{N} \frac{Y_i}{N}$, $\widehat{\overline{y}}= \sum_{i=1}^{n} \frac{Y_i}{n}$, $\overline{X}= \sum_{i=1}^{N} \frac{X_i}{N}$, $\widehat{\overline{x}}= \sum_{i=1}^{n} \frac{X_i}{n}$, $\overline{R}_x = \sum_{i=1}^{N} \frac{R_{xi}}{N}$, and $\widehat{\overline{R}}_x = \sum_{i=1}^{n} \frac{R_{xi}}{n}$ are the population and sample mean of *Y*, *X*, and R_x.

$\overline{Y}_2= \sum_{i=1}^{N_2} \frac{Y_i}{N_2}$, $\overline{X}_2= \sum_{i=1}^{N_2} \frac{X_i}{N_2}$, and $\overline{R}_{x_2}= \sum_{i=1}^{N_2} \frac{R_{xi}}{N_2}$ are the population mean of *Y*, *X*, and R_x for nonresponse group.

$\widehat{\overline{y}}_1= \sum_{i=1}^{n_1} \frac{Y_i}{n_1}$ $\widehat{\overline{x}}_1= \sum_{i=1}^{n_1} \frac{X_i}{n_1}$, and $\widehat{\overline{R}}_{x_1} = \sum_{i=1}^{n_1} \frac{R_{xi}}{n_1}$ denote the sample mean based on n_1 responding units out of n units.

$\widehat{\overline{y}}_{2r}= \sum_{i=1}^{r} \frac{Y_i}{r}$, $\widehat{\overline{x}}_{2r}=\sum_{i=1}^{r} \frac{X_i}{r}$, and $\widehat{\overline{R}}_{x_{2r}}=\sum_{i=1}^{r} \frac{R_{xi}}{r}$ be the sample mean based on r reacting units out of n_2 nonresponse units.

$S_y^2= \sum_{i=1}^{N} \frac{(Y_i-\overline{Y})^2}{N-1}$, $S_x^2= \sum_{i=1}^{N} \frac{(X_i-\overline{X})^2}{N-1}$, and $S_{R_x}^2= \sum_{i=1}^{N} \frac{(R_{xi}-\overline{R}_x)^2}{N-1}$ are the population variance of *Y*, *X*, and R_x respectively.

$S_{y_2}^2= \sum_{i=1}^{N_2} \frac{(Y_i-\overline{Y}_2)^2}{N_2-1}$, $S_x^2= \sum_{i=1}^{N_2} \frac{(X_i-\overline{X}_2)^2}{N_2-1}$, and $S_{R_{x2}}^2= \sum_{i=1}^{N_2} \frac{(R_{xi}-\overline{R}_{x2})^2}{N_2-1}$ are the population variance of *Y*, *X*, and R_x for the nonresponse group.

Let $C_Y = \frac{S_Y}{\overline{Y}}$, $C_X = \frac{S_X}{\overline{X}}$, and $C_{R_x} = \frac{S_{R_x}}{\overline{R_x}}$ be the population coefficient of variation of *Y*, *X*, and R_x.

Let $C_{Y_2}= \frac{S_{y2}}{\overline{Y}_2}$, $C_{X_2}= \frac{S_{x2}}{\overline{X}_2}$, and $C_{R_{x2}}= \frac{S_{R_{x2}}}{\overline{R}_{x2}}$ be the population coefficients of variation of *Y*, *X*, and R_x for the nonresponse group respectively.

Let $\rho_{yx}=S_{yx}/\left(S_yS_x\right)$, $\rho_{yR_x}=S_{yR_x}/\left(S_yS_{R_x}\right)$, and $\rho_{xR_x}=S_{xR_x}/\left(S_xS_{R_x}\right)$ be the population correlation coefficients of *Y*, *X*, and R_x. Let $\rho_{y_2x_2}=S_{y_2x_2}/\left(S_{y_2}S_{x_2}\right)$, $\rho_{y_2R_{x2}}=S_{y_2R_{x2}}/\left(S_{y_2}S_{R_{x2}}\right)$, and $\rho_{x_2R_{x2}}=S_{x_2R_{x2}}/\left(S_{x_2}S_{R_{x2}}\right)$ be the population correlation coefficients of *Y*, *X*, and R_x for the nonresponse group.

The populations mean *Y* may be written as follows:

$$\overline{Y} = W_1\widehat{\overline{Y}}_1 + W_2\widehat{\overline{Y}}_2, \tag{2.1}$$

$$\overline{X} = W_1\widehat{\overline{X}}_1 + W_2\widehat{\overline{X}}_2, \tag{2.2}$$

where $W_j=N_j$/N, $\overline{Y}_j= \sum_{i=1}^{N_j} \frac{Y_i}{N_j}$ for j=1, 2, $\overline{X}_j= \sum_{i=1}^{N_j} \frac{X_i}{N_j}$.

Following [1], we have suggested an unbiased estimator of $\overline{Y}$ under nonresponse, which is given by

$$\widehat{\overline{T}} = w_1\widehat{\overline{Y}}_{(1)} + w_2\widehat{\overline{Y}}_{(2r\ )}.$$

and

$$Var(\widehat{\overline{T}}) = \vartheta_1 S_{y1}^2 + \vartheta_2 S_y^2, \tag{2.3}$$

where $w_j = \frac{n_j}{n}$ for j= 1, 2. To get expression for the bias and MSEs of an estimators, we used the following error terms:

$$e_0^* = \frac{\widehat{\overline{y}}^{**} - \overline{Y}}{\overline{Y}}, e_1^* = \frac{\widehat{\overline{x}}^{**} - \overline{X}}{\overline{X}}, e_2^* = \frac{\widehat{\overline{R}}_x^{**} - \overline{R}_x}{\overline{R}_x}, e_0^* = \frac{\widehat{\overline{y}}^{*} - \overline{Y}}{\overline{Y}}, e_1 = \frac{\widehat{\overline{x}}^{*} - \overline{X}}{\overline{X}}, e_2 = \frac{\widehat{\overline{R}}_x^{*} - \overline{R}_x}{\overline{R}_x},$$

such that $E(e_i^*) = E(e_i) = 0$ for $i^* = 0, 1, 2$, and $i = 2, 3$, where $E(\cdot)$ stands for the mathematical expectation of $(\cdot)$. Assume

$$V_{rst} = E\left[e_0^r\, e_1^s\, e_2^t\right],$$

for nonresponse on both the study and auxiliary variable, and

$$\Lambda_{st} = E\left[e_1^{*r}\, e_2^{*s}\right],$$

with nonresponse on only the auxiliary variable.

$$E\left[e_0^{*2}\right] = \frac{\vartheta_1 S_y^2 + \vartheta_2 S_{y2}^2}{\overline{Y}^2} = V_{200}, E\left[e_1^{*2}\right] = \frac{\vartheta_1 S_x^2 + \vartheta_2 S_{x2}^2}{\overline{X}^2} = V_{020}, E\left[e_2^{*2}\right] = \frac{\vartheta_1 S_{rx}^2 + \vartheta_2 S_{rx2}^2}{\overline{R}_x^2} = V_{002},$$

$$E\left[e_0^* e_1^*\right] = \frac{\left(\vartheta_1 S_y S_x \rho_{yx}\right) + \left(\vartheta_2 S_{y(2)} S_{x(2)} \rho_{yx(2)}\right)}{\overline{Y}\overline{X}} = V_{110}, E\left[e_0^* e_2^*\right] = \frac{\left(\vartheta_1 S_y S_r x \rho_{yrx}\right) + \left(\vartheta_2 S_{y(2)} S_{rx(2)} \rho_{yrx(2)}\right)}{\overline{Y}\overline{R}_x} = V_{101},$$

$$E\left[e_1^* e_2^*\right] = \frac{\left(\vartheta_1 S_x S_r x \rho_{xrx}\right) + \left(\vartheta_2 S_{x(2)} S_{rx(2)} \rho_{xrx(2)}\right)}{\overline{X}\overline{R}_x} = V_{011}, E\left[e_1^2\right] = \frac{\vartheta_1 S_x^2}{\overline{X}^2} = \Lambda_{020}, E\left[e_2^2\right] = \frac{\vartheta_1 S_{rx}^2}{\overline{R}_x^2} = \Lambda_{002},$$

$$E\left[e_0^* e_1\right] = \frac{\left(\vartheta_1 S_y S_x \rho_{yx}\right)}{\overline{Y}\overline{X}} = \Lambda_{110}, E\left[e_0^* e_2\right] = \frac{\left(\vartheta_1 S_y S_{rx} \rho_{yrx}\right)}{\overline{Y}\overline{R}_x} = \Lambda_{101}, E\left[e_1 e_2\right] = \frac{\left(\vartheta_1 S_x S_{rx} \rho_{xrx}\right)}{\overline{X}\overline{R}_x} = \Lambda_{011},$$

$$\vartheta_1 = \left(\frac{1}{n} - \frac{1}{N}\right), \vartheta_2 = \frac{W_2.(K-1)}{n}.$$

3. Existing estimators

Some of the best-known estimators are discussed here under nonresponse. The explanation that follows includes their bias and MSE/minimum MSE expressions.

3.1. Situation I: when nonresponse occurs only in the study variable

(1) The ratio estimator with bias and MSEs are given by

$$\widehat{\overline{T}}^*_{Ratio} = \widehat{\overline{y}}^* \left(\frac{\overline{X}}{\widehat{\overline{x}}^*}\right), \tag{3.1}$$

$$Bias\left(\widehat{\overline{T}}^*_{Ratio}\right) = \overline{Y}\left(\Lambda_{020} - \Lambda_{110}\right),$$

and

$$MSE\left(\widehat{\overline{T}}^*_{Ratio}\right) = \overline{Y}^2\left[V_{200} + \Lambda_{020} - 2\Lambda_{110}\right]. \tag{3.2}$$

(2) The product estimator with bias and MSEs are given by

$$\widehat{\overline{T}}^{*}_{Product} = \widehat{\overline{y}}^{*}\left(\frac{\widehat{\overline{x}}^{*}}{\overline{X}}\right), \tag{3.3}$$

$$Bias\left(\widehat{\overline{T}}^{*}_{Product}\right) = \overline{Y}\Lambda_{110},$$

and

$$MSE\left(\widehat{\overline{T}}^{*}_{Product}\right) = \overline{Y}^{2}\left[V_{200} + \Lambda_{020} + 2\Lambda_{110}\right]. \tag{3.4}$$

(3) The difference estimator is given by

$$\widehat{\overline{T}}^{*}_{Difference} = \widehat{\overline{y}}^{*} + D\left(\overline{X} - \widehat{\overline{x}}^{*}\right), \tag{3.5}$$

where $D=\frac{\overline{Y}\Lambda_{110}}{\overline{X}\Lambda_{020}}$, and the variance of $\widehat{\overline{T}}^{**}_{Difference}$ at the optimum value of D is given by

$$Var\left(\widehat{\overline{T}}^{*}_{Difference}\right) = \frac{\overline{Y}^{2}\left(V_{200}\Lambda_{020} - \Lambda^{2}_{110}\right)}{\Lambda_{020}}. \tag{3.6}$$

(4) The suggested difference in the difference estimator of [13], which is given by

$$\widehat{\overline{T}}^{*}_{RD} = D_{1}\widehat{\overline{y}}^{*} + D_{2}\left(\overline{X} - \widehat{\overline{x}}^{*}\right). \tag{3.7}$$

D_1 and D_2 are given by

$$D_{1(opt)} = \frac{\Lambda_{020}}{\left[\Lambda_{020}\left\{1 + V_{200}\right\} - \Lambda^{2}_{110}\right]},\ D_{1(opt)} = \frac{\overline{Y}\Lambda_{110}}{\overline{X}\left[\Lambda_{020}\left\{1 + V_{200}\right\} - \Lambda^{2}_{110}\right]}.$$

The minimum MSE at $D_{1(opt)}$ and $D_{2(opt)}$ are given by

$$MSE\left(\widehat{\overline{T}}^{*}_{RD}\right) = \frac{\overline{Y}^{2}\left(V_{200}\Lambda_{020} - V^{2}_{110}\right)}{\Lambda_{020}\left\{1 + V_{200}\right\} - V^{2}_{110}}. \tag{3.8}$$

(5) The suggested estimator of [22] is given as

$$\widehat{\overline{T}}^{*}_{Grover} = \left\{k_{1}\widehat{\overline{y}}^{*} + k_{2}\left(\overline{X} - \widehat{\overline{x}}^{*}\right)\right\} exp\left(\frac{a\left(\overline{X} - \widehat{\overline{x}}^{*}\right)}{a\left(\overline{X} + \widehat{\overline{x}}^{*}\right) + 2b}\right). \tag{3.9}$$

The bias and MSE of $\widehat{\overline{T}}^{*}_{Grover}$ are given as

$$Bias\left(\widehat{\overline{T}}^{*}_{Grover}\right) \cong \overline{Y}\left(k_{1} - 1\right) + \frac{3}{8}\,\theta^{2}k_{1}\overline{Y}V_{200} + \frac{1}{2}\,\theta k_{2}\overline{X}\,\Lambda_{020} - \frac{1}{2}\,\theta\overline{Y}\,\Lambda_{110},$$

$$k_{1(opt)} = \frac{\Lambda_{020}\left(\Lambda_{020}-8\right)}{8\left(-V_{200}\Lambda_{020}-\Lambda_{110}{}^{2}-\Lambda_{020}\right)},$$

$$k_{2(opt)} = \frac{\overline{Y}\left(V_{020}^{2}-\Lambda_{020}\Lambda_{020}+4V_{200}\Lambda_{020}-4\Lambda_{110}{}^{2}\ -\ 4\Lambda_{020}+8\Lambda_{110}\right)}{8\overline{X}\left(V_{200}\Lambda_{020}-\Lambda_{110}{}^{2}+\Lambda_{020}\right)}.$$

The MSE of $\widehat{\overline{T}}^{*}_{Grover}$ is given by

$$MSE_{min}\left(\widehat{\overline{T}}^{*}{}_{Grover}\right) \cong Var_{min}\left(\widehat{\overline{T}}^{*}{}_{Difference}\right) - \frac{\overline{Y}^{2}\left(\theta^{2}\Lambda_{020}{}^{2}-8\Lambda_{110}{}^{2}+8\Lambda_{020}V_{200}\right)^{2}}{64\Lambda_{020}{}^{2}\left\{1+V_{200}\left(1-\rho_{yx}^{2}\right)\right\}}. \quad (3.10)$$

3.2. Situation II: when nonresponse occur in both the study and auxiliary variable

(1) The ratio estimator with bias and MSEs are given by

$$\widehat{\overline{T}}^{**}_{Ratio} = \widehat{\overline{y}}^{**}\left(\frac{\overline{X}}{\widehat{\overline{x}}^{**}}\right), \quad (3.11)$$

$$Bias\left(\widehat{\overline{T}}^{**}_{Ratio}\right) = \overline{Y}\left(V_{020}-V_{110}\right),$$

and

$$MSE\left(\widehat{\overline{T}}^{**}_{Ratio}\right) = \overline{Y}^{2}\left[V_{200}+V_{020}-2V_{110}\right]. \quad (3.12)$$

(2) The product estimator with bias and MSEs are given by

$$\widehat{\overline{T}}^{**}_{Product} = \widehat{\overline{y}}^{**}\left(\frac{\widehat{\overline{x}}^{**}}{\overline{X}}\right), \quad (3.13)$$

$$Bias\left(\widehat{\overline{T}}^{**}_{Product}\right) = \overline{Y}V_{110},$$

and

$$MSE\left(\widehat{\overline{T}}^{**}_{Product}\right) = \overline{Y}^{2}\left[V_{200}+V_{020}+2V_{110}\right]. \quad (3.14)$$

(3) The difference estimator is given by

$$\widehat{\overline{T}}^{**}_{Difference} = \widehat{\overline{y}}^{**} + D\left(\overline{X}-\widehat{\overline{x}}^{**}\right), \quad (3.15)$$

where D is a constant, and its value is given by $D=\frac{\overline{Y}V_{110}}{\overline{X}V_{020}}$, and the variance of $\widehat{\overline{T}}^{**}_{Difference}$ at the optimum value of D is given by

$$Var\left(\widehat{\overline{T}}^{**}_{Difference}\right) = \frac{\overline{Y}^{2}\left(V_{200}V_{020}-V_{110}^{2}\right)}{V_{020}}. \quad (3.16)$$

(4) The suggested difference in the difference estimator of [13], which is given by

$$\widehat{\overline{T}}^{**}_{RD} = D_1\widehat{\overline{y}}^{**} + D_2\left(\overline{X} - \widehat{\overline{x}}^{**}\right). \tag{3.17}$$

D_1 and D_2 are given by

$$D_{1(opt)} = \frac{V_{020}}{\left[V_{020}\left\{1 + V_{200}\right\} - V_{110}^2\right]},\ D_{2(opt)} = \frac{\overline{Y}V_{110}}{\overline{X}\left[V_{020}\left\{1 + V_{200}\right\} - V_{110}^2\right]}.$$

The minimum MSE at $D_{1(opt)}$ and $D_{2(opt)}$ are given by

$$MSE\left(\widehat{\overline{T}}^{**}_{RD}\right) = \frac{\overline{Y}^2\left(V_{200}V_{020} - V_{110}^2\right)}{V_{020}\left\{1 + V_{200}\right\} - V_{110}^2}. \tag{3.18}$$

(5) Following [22], the estimator of $\overline{Y}$ is given by

$$\widehat{\overline{T}}^{**}{}_{Grover} = \left\{k_1\widehat{\overline{y}} + k_2\left(\overline{X} - \widehat{\overline{x}}^{**}\right)\right\} exp\left(\frac{a\left(\overline{X} - \widehat{\overline{x}}^{**}\right)}{a\left(\overline{X} + \widehat{\overline{x}}^{**}\right) + 2b}\right), \tag{3.19}$$

where k_1 and k_2 are unknown constants. The bias and MSE of $\widehat{\overline{T}}^{**}{}_{Grover}$ are given as

$$Bias\left(\widehat{\overline{T}}^{**}{}_{Grover}\right) = \overline{Y}\left(k_1 - 1\right) + \frac{3}{8}\,\theta^2 k_1\overline{Y}V_{200} + \frac{1}{2}\,\theta k_2\overline{X}\,V_{020} - \frac{1}{2}\,\theta\overline{Y}\,V_{110},$$

$$k_{1(opt)} = \frac{V_{020}\left(V_{020} - 8\right)}{8\left(V_{020}(1 + V_{200}) - V_{110}{}^2\right)},$$

and

$$k_{2(opt)} = \frac{\overline{Y}\left(V_{020}^2 - V_{020}V_{020} + 4V_{200}V_{020} - 4V_{110}{}^2\ -\ 4V_{110} + 8V_{200}\right)}{8\overline{X}\left(V_{200}V_{020} - V_{110}{}^2 + V_{020}\right)}.$$

The MSE of $\widehat{\overline{T}}^{**}{}_{Grover}$ at the optimal values of k_1 and k_2 is given by

$$MSE_{min}\left(\widehat{\overline{T}}^{**}{}_{Grover}\right) \cong Var_{min}\left(\widehat{\overline{T}}^{**}{}_{Difference}\right) - \frac{\overline{Y}^2\left(\theta^2V_{020}{}^2 - 8V_{110}{}^2 + 8V_{020}V_{200}\right)^2}{64V_{020}{}^2\left\{1 + V_{200}\left(1 - \rho^2_{yx(2)}\right)\right\}}. \tag{3.20}$$

4. Proposed estimator

When auxiliary variables are addressed in both the design and estimation stages, estimators become more efficient. Inspired by [14], we provide a new family of estimators that exploit nonresponse under simple random sampling and rely on auxiliary variables. Up to the first order of approximation, mathematical properties such as bias and mean square error are obtained. The major feature of our generalized class of estimators is their better efficiency and adaptability compared with the existing estimators, as shown below.

4.1. Situation I: nonresponse occurs only in the study variable

When nonresponse occurs only in Y, then the proposed estimators under stratified random sampling for estimation of the mean are given by

$$\widehat{\overline{T}}^{*}{}_{Prop\ 1} = \widehat{\overline{y}}^{*} + Ø_1\left(\overline{X} - \widehat{\overline{x}}^{*}\right) + Ø_2\left(\overline{R}_x - \widehat{\overline{R}}_x^{*}\right), \tag{4.1}$$

$$\widehat{\overline{T}}^{*}{}_{Prop\ 2} = \widehat{\overline{y}}^{*}\left(\frac{\overline{X}}{\widehat{\overline{x}}^{*}}\right)^{Ø_1}\left(\frac{\overline{R}_x}{\widehat{\overline{R}}_x^{*}}\right)^{Ø_2}, \tag{4.2}$$

$$\widehat{\overline{T}}^{*}{}_{Prop\ 3} = Ł_1\widehat{\overline{y}}^{*} + Ł_2\left(\overline{X} - \widehat{\overline{x}}^{*}\right) + Ł_3\left(\overline{R}_x - \widehat{\overline{R}}_x^{*}\right), \tag{4.3}$$

where $(\hbar_1,\ \hbar_2)$, $(Ø_1,\ Ø_2)$, and $(Ł_1,\ Ł_2,\ Ł_3)$ are suitable constants.

Expressing Eqs (4.1)–(4.3), respectively, leads to

$$\widehat{\overline{T}}^{*}{}_{Prop\ 1} = \overline{Y}\left(1 + e_0^{*}\right) - Ø_1\overline{X}e_1 - Ø_2\overline{R}_x e_2, \tag{4.4}$$

$$\widehat{\overline{T}}^{*}{}_{Prop\ 2} = \overline{Y}\left(1 + e_0^{*}\right)\left(1 + e_1\right)^{-Ø_1} + \left(1 + e_2\right)^{-Ø_2}, \tag{4.5}$$

$$\widehat{\overline{T}}^{*}{}_{Prop\ 3} = Ł_1\overline{Y}\left(1 + e_0^{*}\right) - Ł_2\overline{X}e_1 - Ł_3\overline{R}_x e_2. \tag{4.6}$$

Rewriting Eqs (4.4)–(4.6) leads to

$$\widehat{\overline{T}}^{*}{}_{Prop\ 1} - \overline{Y} = \overline{Y}e_0^{*} - Ø_1\overline{X}e_1 - Ø_2\overline{R}_x e_2, \tag{4.7}$$

$$\widehat{\overline{T}}^{*}{}_{Prop\ 2} - \overline{Y} = \overline{Y}\left[e_0^{*} - Ø_1 e_1 - Ø_2 e_2 + \frac{Ø_1\left(Ø_1 + 1\right)}{2}e_1^2 + \frac{Ø_2\left(Ø_2 + 1\right)}{2}e_2^2 + Ø_1Ø_2 e_1 e_2 - Ø_1 e_0^{*} e_1\right], \tag{4.8}$$

$$\widehat{\overline{T}}^{*}{}_{Prop\ 3} = \overline{Y}\left(Ł_1 - 1\right) + \left(\widehat{\overline{T}}^{**}_{Prop\ 1} - \overline{Y}\right). \tag{4.9}$$

By taking expectations on both sides of Eqs (4.7)–(4.9), we get the biases

$$Bias\left(\widehat{\overline{T}}^{*}{}_{Prop\ 1}\right) = 0,$$

$$Bias\left(\widehat{\overline{T}}^{*}{}_{Prop\ 2}\right) = \overline{Y}\left[\frac{Ø_1\left(Ø_1 + 1\right)}{2}\Lambda_{020}^2 + \frac{Ø_2\left(Ø_2 + 1\right)}{2}\Lambda_{002}^2 + Ø_1Ø_2\Lambda_{011} - Ø_1\Lambda_{110} - Ø_2\Lambda_{101}\right],$$

$$Bias\left(\widehat{\overline{T}}^{*}{}_{Prop\ 2}\right) = \overline{Y}\left(Ł_1 - 1\right).$$

By taking the square of Eqs (4.7)–(4.9), we have

$$\left(\widehat{\overline{T}}^{*}{}_{Prop\ 1} - \overline{Y}\right)^2 = \overline{Y}^2\left(e_0^{*2} + \hbar_1^2 R_1^2 e_1^2 + \hbar_2^2 R_2^2 e_2^2 + 2\hbar_1\hbar_2 R_1 R_2 e_1 e_2 - 2\hbar_1 R_1 e_{*0} e_1 - 2\hbar_2 R_2 e_0^{*} e_2\right), \tag{4.10}$$

$$\left(\widehat{\overline{T}}^{*}{}_{Prop\ 2} - \overline{Y}\right)^2 = \overline{Y}^2\left(e_0^2 + Ø_1^2 e_1^2 + Ø_2^2 e_2^2 + 2Ø_1Ø_2 e_1 e_2 - 2Ø_1 R_1 e_0^{*} e_1 - 2Ø_2 e_0^{*} e_2\right), \tag{4.11}$$

$$\left(\widehat{\overline{\overline{T}}}^{*}{}_{Prop\ 2}-\overline{Y}\right)^{2} = \overline{Y}^{2}\left[\begin{array}{l} Ł_1^2\left(1+e_0^{*2}\right)-2Ł_1\left\{1+Ł_2R_1e_0^{*}e_1+Ł_3R_2e_0^{*}e_2\right\} \\ +2\left\{1+Ł_2^2R_1^2e_1^2+Ł_3^2R_2^2e_2^2-2Ł_2Ł_3R_1R_2e_1e_2\right\}\end{array}\right]. \tag{4.12}$$

Taking the expectation of Eqs (4.10)–(4.12), we have

$$MSE\left(\widehat{\overline{\overline{T}}}^{*}_{Prop\ 1}\right)=\overline{Y}^{2}\left(V_{200}^2+\hbar_1^2\,R_1^2\Lambda_{020}^2+\hbar_2^2\,R_2^2\Lambda_{002}^2+2\hbar_1\hbar_2R_1R_2\Lambda_{011}-2\hbar_1R_1\Lambda_{110}-2\hbar_2R_2\Lambda_{101}\right), \tag{4.13}$$

$$MSE\left(\widehat{\overline{\overline{T}}}^{*}{}_{Prop\ 2}\right)=\overline{Y}^{2}\left(V_{200}^2+Ø_1^2\,\Lambda_{020}^2+Ø_2^2\,\Lambda_{002}^2+2Ø_1Ø_2\Lambda_{011}-2Ø_1R_1\Lambda_{110}-2Ø_2\Lambda_{101}\right), \tag{4.14}$$

$$MSE\left(\widehat{\overline{\overline{T}}}^{*}{}_{Prop\ 3}\right) = \overline{Y}^{2}\left[\begin{array}{l} Ł_1^2\left(1+V_{200}^2\right)-2Ł_1\left\{1+Ł_2R_1\Lambda_{110}+Ł_3R_2\Lambda_{101}\right\} \\ +2\left\{1+Ł_2^2R_1^2\Lambda_{020}^2+Ł_3^2R_2^2\Lambda_{002}^2-2Ł_2Ł_3R_1R_2\Lambda_{011}\right\}\end{array}\right]. \tag{4.15}$$

Minimizing Eqs (4.13)–(4.15), we get the optimum values of ($\hbar_1,\ \hbar_2$), ($Ø_1,\ Ø_2$), and ($Ł_1,\ Ł_2,\ Ł_3$), we have

$$\hbar_1=\frac{\overline{Y}}{\overline{\overline{X}}}\frac{\Lambda_{020}^2\Lambda_{110}-\Lambda_{011}\Lambda_{101}}{\Lambda_{020}^2\Lambda_{002-}^2\Lambda_{011}^2},$$

$$\hbar_2=\frac{\overline{Y}}{\overline{R}_x}\frac{\Lambda_{020}^2\Lambda_{101}-\Lambda_{011}\Lambda_{110}}{\Lambda_{020}^2\Lambda_{002-}^2\Lambda_{011}^2},$$

$$Ø_1=R_1\hbar_1, Ø_2=R_2\hbar_2,$$

$$Ł_1=\frac{1+Ø_1R_1\Lambda_{110}+Ø_2R_2\Lambda_{101}}{1+V_{200}^2},$$

$$Ł_2=\frac{\overline{Y}}{\overline{\overline{X}}}\frac{\Lambda_{020}^2\Lambda_{110}-\Lambda_{011}\Lambda_{101}}{\Lambda_{020}^2\Lambda_{002-}^2\Lambda_{011}^2},$$

$$Ł_3=\frac{\overline{Y}}{\overline{R}_x}\frac{\Lambda_{020}^2\Lambda_{101}-\Lambda_{011}\Lambda_{110}}{\Lambda_{020}^2\Lambda_{002-}^2\Lambda_{011}^2}.$$

Putting the optimum values of ($\hbar_1,\ \hbar_2$), ($Ø_1,\ Ø_2$), and ($Ł_1,\ Ł_2,\ Ł_3$) in Eqs (4.13)–(4.15), we get the minimum MSEs of $\widehat{\overline{\overline{T}}}^{*}{}_{Prop\ 1}$, $\widehat{\overline{\overline{T}}}^{**}_{Prop\ 2}$, and $\widehat{\overline{\overline{T}}}^{*}{}_{Prop\ 3}$, which are given by

$$MSE\left(\widehat{\overline{\overline{T}}}^{*}{}_{Prop\ 1}\right)_{min}=\overline{Y}^{2}V_{200}^2\left[1-\mathbb{Q}_{y.xrx}\right]. \tag{4.16}$$

Similarly

$$MSE\left(\widehat{\overline{\overline{T}}}^{*}{}_{Prop\ 3}\right)_{min}=\overline{Y}^{2}V_{200}^2\left[1-\mathbb{Q}_{y.xrx}\right], \tag{4.17}$$

where

$$\mathbb{Q}_{y.xrx}=\frac{\Lambda_{020}^2\Lambda_{101}-\Lambda_{002}^2\Lambda_{110}-2\Lambda_{110}\Lambda_{011}\Lambda_{101}}{V_{200}^2\ \left\{\Lambda_{020}^2\Lambda_{002}^2-\Lambda_{110}\right\}},$$

$$MSE\left(\widehat{\overline{\overline{T}}}^{*}{}_{Prop\ 2}\right)=\frac{\overline{Y}^{2}\ MSE\left(\widehat{\overline{\overline{T}}}^{*}{}_{Prop\ 1}\right)_{min}}{\overline{Y}^{2}\ +MSE\left(\widehat{\overline{\overline{T}}}^{*}{}_{Prop\ 1}\right)_{min}}. \tag{4.18}$$

4.2. Situation II: nonresponse in the study and auxiliary variables

When nonresponse occurs in the study variable Y, auxiliary variable, and rank of the auxiliary variable, then the proposed estimators under stratified random sampling for estimation of the mean are given by

$$\widehat{\overline{\overline{T}}}^{**}_{Prop\ 1} = \widehat{\overline{y}}^{**} + Ø_1\left(\overline{X} - \widehat{\overline{x}}^{**}\right) + Ø_2\left(\overline{R}_x - \widehat{\overline{R}}^{**}_x\right), \tag{4.19}$$

$$\widehat{\overline{\overline{T}}}^{**}_{Prop\ 2} = \widehat{\overline{y}}^{**}\left(\frac{\overline{X}}{\widehat{\overline{x}}^{**}}\right)^{Ø_1}\left(\frac{\overline{R}_x}{\widehat{\overline{R}}^{**}_x}\right)^{Ø_2}, \tag{4.20}$$

$$\widehat{\overline{\overline{T}}}^{**}_{Prop\ 3} = Ł_1\widehat{\overline{y}}^{**} + Ł_2\left(\overline{X} - \widehat{\overline{x}}^{**}\right) + Ł_3\left(\overline{R}_x - \widehat{\overline{R}}^{**}_x\right), \tag{4.21}$$

where ($\hbar_1,\ \hbar_2$), ($Ø_1,\ Ø_2$), and ($Ł_1,\ Ł_2,\ Ł_3$) are suitable constants.

Expressing Eqs (4.19)–(4.21), respectively, leads to

$$\widehat{\overline{\overline{T}}}^{**}_{Prop\ 1} = \overline{Y}\left(1 + e^*_0\right) - Ø_1\overline{X}e^*_1 - Ø_2\overline{R}_x e^*_2, \tag{4.22}$$

$$\widehat{\overline{\overline{T}}}^{**}_{Prop\ 2} = \overline{Y}\left(1 + e^*_0\right)\left(1 + e^*_1\right)^{-Ø_1} + \left(1 + e^*_2\right)^{-Ø_2}, \tag{4.23}$$

$$\widehat{\overline{\overline{T}}}^{**}_{Prop\ 3} = Ł_1\overline{Y}\left(1 + e^*_0\right) - Ł_2\overline{X}e^*_1 - Ł_3\overline{R}_x e^*_2. \tag{4.24}$$

Rewriting Eqs (4.22)–(4.24), we have

$$\widehat{\overline{\overline{T}}}^{**}_{Prop\ 1} - \overline{Y} = \overline{Y}e^*_0 - Ø_1\overline{X}e^*_1 - Ø_2\overline{R}_x e^*_2, \tag{4.25}$$

$$\widehat{\overline{\overline{T}}}^{**}_{Prop\ 2} - \overline{Y} = \overline{Y}\left[e^*_0 - Ø_1 e^*_1 - Ø_2 e^*_2 + \frac{Ø_1\left(Ø_1 + 1\right)}{2}e^{*2}_1 + \frac{Ø_2\left(Ø_2 + 1\right)}{2}e^{*2}_2 + Ø_1Ø_2 e^*_1 e^*_2 - Ø_1 e^*_0 e^*_1\right], \tag{4.26}$$

$$\widehat{\overline{\overline{T}}}^{**}_{Prop\ 3} = \overline{Y}\left(Ł_1 - 1\right) + \left(\widehat{\overline{\overline{T}}}^{**}_{Prop\ 1} - \overline{Y}\right). \tag{4.27}$$

By taking expectations on both sides of Eqs (4.25)–(4.27), we get the biases

$$Bias\left(\widehat{\overline{\overline{T}}}^{**}_{Prop\ 1}\right) = 0,$$

$$Bias\left(\widehat{\overline{\overline{T}}}^{**}_{Prop\ 2}\right) = \overline{Y}\left[\frac{Ø_1\left(Ø_1 + 1\right)}{2}V^2_{020} + \frac{Ø_2\left(Ø_2 + 1\right)}{2}V^2_{002} + Ø_1Ø_2 V_{011} - Ø_1 V_{110} - Ø_2 V_{101}\right],$$

$$Bias\left(\widehat{\overline{\overline{T}}}^{**}_{Prop\ 2}\right) = \overline{Y}\left(Ł_1 - 1\right).$$

By taking the square of (4.25)–(4.27), we have

$$\left(\widehat{\overline{\overline{T}}}^{**}_{Prop\ 1} - \overline{Y}\right)^2 = \overline{Y}^2\left(e^{*2}_0 + \hbar^2_1 R^2_1 e^{*2}_1 + \hbar^2_2 R^2_2 e^{*2}_2 + 2\hbar_1\hbar_2 R_1 R_2 e^*_1 e^*_2 - 2\hbar_1 R_1 e^*_0 e^*_1 - 2\hbar_2 R_2 e^*_0 e^*_2\right), \tag{4.28}$$

$$\left(\widehat{\overline{\overline{T}}}^{**}_{Prop\ 2} - \overline{Y}\right)^2 = \overline{Y}^2\left(e^{*2}_0 + Ø^2_1\ e^{*2}_1 + Ø^2_2\ e^{*2}_0 + 2Ø_1\ Ø_2 e^*_1 e^*_2 - 2Ø_1\ R_1 e^*_0 e^*_1 - 2Ø_2 e^*_0 e^*_2\right), \tag{4.29}$$

$$\left(\widehat{\overline{\overline{T}}}^{**}_{Prop\ 2} - \overline{Y}\right)^2 = \overline{Y}^2 \begin{bmatrix} Ł_1^2\left(1 + e_0^{*2}\right) - 2Ł_1\left\{1 + Ł_2 R_1 e_0^* e_1^* + Ł_3 R_2 e_0^* e_2^*\right\} \\ + 2\left\{1 + Ł_2^2 R_1^2 e_1^{*2} + Ł_3^2 R_2^2 e_2^{*2} - 2Ł_2 Ł_3 R_1 R_2 e_1^* e_2^*\right\} \end{bmatrix}. \tag{4.30}$$

If we take the expectation of Eqs (4.28)–(4.30), the MSEs are

$$MSE\left(\widehat{\overline{\overline{T}}}^{**}_{Prop\ 1}\right) = \overline{Y}^2\left(V_{200}^2 + \hbar_1^2 R_1^2 V_{020}^2 + \hbar_2^2 R_2^2 V_{002}^2 + 2\hbar_1\hbar_2 R_1 R_2 V_{011} - 2\hbar_1 R_1 V_{110} - 2\hbar_2 R_2 V_{101}\right), \tag{4.31}$$

$$MSE\left(\widehat{\overline{\overline{T}}}^{**}_{Prop\ 2}\right) = \overline{Y}^2\left(V_{200}^2 + \text{Ø}_1^2 V_{020}^2 + \text{Ø}_2^2 V_{002}^2 + 2\text{Ø}_1\text{Ø}_2 V_{011} - 2\text{Ø}_1 R_1 V_{110} - 2\text{Ø}_2 V_{101}\right), \tag{4.32}$$

$$\left(\widehat{\overline{\overline{T}}}^{**}_{Prop\ 3}\right) \cong \overline{Y}^2 \begin{bmatrix} Ł_1^2\left(1 + V_{200}^2\right) - 2Ł_1\left\{1 + Ł_2 R_1 V_{110} + Ł_3 R_2 V_{101}\right\} \\ + 2\left\{1 + Ł_2^2 R_1^2 V_{020}^2 + Ł_3^2 R_2^2 V_{002}^2 - 2Ł_2 Ł_3 R_1 R_2 V_{011}\right\} \end{bmatrix}. \tag{4.33}$$

Minimizing Eqs (4.31)–(4.33), we get the optimum values of ($\hbar_1,\ \hbar_2$), (Ø$_1$, Ø$_2$), and (Ł$_1$, Ł$_2$, Ł$_3$), and we have

$$\hbar_1 = \frac{\overline{Y}}{\overline{\overline{X}}}\frac{V_{020}^2 V_{110} - V_{011} V_{101}}{V_{020}^2 V_{002-}^2 V_{011}^2},$$

$$\hbar_2 = \frac{\overline{Y}}{\overline{R}_x}\frac{V_{020}^2 V_{101} - V_{011} V_{110}}{V_{020}^2 V_{002-}^2 V_{011}^2},$$

$$\text{Ø}_1 = R_1\hbar_1, \text{Ø}_2 = R_2\hbar_2,$$

$$Ł_1 = \frac{1 + \text{Ø}_1 R_1 V_{110} + \text{Ø}_2 R_2 V_{101}}{1 + V_{200}^2},$$

$$Ł_2 = \frac{\overline{Y}}{\overline{\overline{X}}}\frac{V_{020}^2 V_{110} - V_{011} V_{101}}{V_{020}^2 V_{002-}^2 V_{011}^2},$$

$$Ł_3 = \frac{\overline{Y}}{\overline{R}_x}\frac{V_{020}^2 V_{101} - V_{011} V_{110}}{V_{020}^{2*} V_{002-}^2 V_{011}^2}.$$

Putting the optimum values of ($\hbar_1,\ \hbar_2$), (Ø$_1$, Ø$_2$), and (Ł$_1$, Ł$_2$, Ł$_3$) in Eqs (4.31)–(4.33), we get the minimum MSEs of $\widehat{\overline{\overline{T}}}^{**}_{Prop\ 1}$, $\widehat{\overline{\overline{T}}}^{**}_{Prop\ 2}$, and $\widehat{\overline{\overline{T}}}^{**}_{Prop\ 3}$, which are given by

$$MSE\left(\widehat{\overline{\overline{T}}}^{**}_{Prop\ 1}\right)_{min} = \overline{Y}^2 V_{200}^2\left[1 - \mathbb{Q}_{y.xrx(2)}\right]. \tag{4.34}$$

Similarly

$$MSE\left(\widehat{\overline{\overline{T}}}^{**}_{Prop\ 2}\right)_{min} = \overline{Y}^2 V_{200}^2\left[1 - \mathbb{Q}_{y.xrx(2)}\right], \tag{4.35}$$

where

$$\mathbb{Q}_{y.xrx(2)} = \frac{V_{020}^2 V_{101} - V_{002}^2 V_{110} - 2V_{110} V_{011} V_{101}}{V_{200}^2\left\{V_{020}^2 V_{002}^2 - V_{110}\right\}},$$

$$MSE\left(\widehat{\overline{\overline{T}}}^{**}_{Prop\ 3}\right) = \frac{\overline{Y}^2\ MSE\left(\widehat{\overline{\overline{T}}}^{**}_{Prop\ 1}\right)_{min}}{\overline{Y}^2 + MSE\left(\widehat{\overline{\overline{T}}}^{**}_{Prop\ 1}\right)_{min}}. \tag{4.36}$$

5. Numerical study

This section presents the mathematical results of all considered estimators. We used four different datasets. Here is a numerical representation of PRE:

$$\text{PRE}\left(\widehat{\overline{T}}_i^{**}, \widehat{\overline{T}}_U^{**}\right) = \frac{\text{Var}\left(\widehat{\overline{T}}_U^{**}\right)}{\text{MSE}\left(\widehat{\overline{T}}_i^{**}\right)} \times 100.$$

Population I [21] Turkey carried out a nationwide survey in 2007 including all 923 districts in six regions to determine the number of students enrolled in elementary and secondary education. According to the technique used, the final 35%, 30%, 25%, 20%, and 15% of units that remained silent were taken into account as a proxy for the population of non respondents (NR). This methodology was selected to guarantee a precise portrayal of the educational terrain throughout the country. The summary statistics of the real-life dataset are presented in Table 1.

Table 1. Summary statistics for real-life data.

Parameter	Value	Parameter	Value
N	923	S_y	749.9395
n	180	S_x	21331.13
λ	0.004472132	S_{r_x}	266.5914
$\bar{Y}$	436.4345	ρ_{yx}	0.9543029
$\bar{X}$	11440.5	ρ_{yr_x}	0.6444158
$\bar{R}_x$	462	ρ_{xr_x}	0.6306615
Nonresponse			
Parameter	Value	Parameter	Value
N_2	323	w_2	0.35000
N_2	277	w_2	0.30000
N_2	231	w_2	0.25000
N_2	185	w_2	0.20000
N_2	139	w_2	0.15000
λ_2 (when k=2)	0.001944444	$\rho_{yx(2)}$	0.9389602
λ_2 (when k=3)	0.003888889	$\rho_{yr_x(2)}$	0.7610628
λ_2 (when k=4)	0.005833333	$\rho_{xr_x(2)}$	0.7037817
λ_2 (when k=5)	0.007777778	$\beta_{2(x)}$	14.37321
$S_{y(2)}$	763.3681	$C^2_{y(2)}$	1.444186
$S_{x(2)}$	24870.81	$C^2_{x(2)}$	1.708471
$S_{r_x(2)}$	66.82812	$C^2_{r_x(2)}$	0.5761045

Custom nonresponse levels were observed during this study because the researchers excluded portions of 35%, 30%, 25%, 20%, and 15% of non responding sample units. The research utilized un responsive units as substitute values to represent varied degrees of nonresponse (NR). Performance changes of the proposed estimators were studied on the basis of modifications to silent unit proportions to determine nonresponse levels. By applying this method, researchers can effectively check both the robustness and efficiency levels in practical surveys dealing with nonresponse challenges as a main operational hurdle. The results are presented in Tables 2–21 below.

Table 2. MSEs of all estimators using real-life data under Situation I with a 35% nonresponse rate.

Estimators	MSEs	MSEs	MSEs	MSEs
	k=2	**k=3**	**k=4**	**k=5**
$\widehat{\overline{T}}^{**}_{SRS}$	3177.3679	4217.9671	4785.5666	5353.1661
$\widehat{\overline{T}}^{**}_{Ratio}$	929.8348	1970.434	2538.0335	3105.633
$\widehat{\overline{T}}^{**}_{Product}$	11347.6123	12388.2114	12955.8109	13523.4104
$\widehat{\overline{T}}^{**}_{Difference}$	886.8189	1927.418	2495.0175	3062.617
$\widehat{\overline{T}}^{**}_{RD}$	882.7091	1908.1098	2462.758	3014.1529
$\widehat{\overline{T}}^{**}_{Grover}$	878.5621	1899.9812	2452.4757	3001.7295
$\widehat{\overline{T}}^{**}_{\text{Prop 1}}$	**680.7385**	**1721.3376**	**2288.9371**	**2856.5367**
$\widehat{\overline{T}}^{**}_{\text{Prop 2}}$	**680.7385**	**1721.3376**	**2288.9371**	**2856.5367**
$\widehat{\overline{T}}^{**}_{\text{Prop 3}}$	**678.3143**	**1705.9211**	**2261.7576**	**2814.3304**

Table 3. MSEs of all estimators using real-life data under Situation II with a 35% nonresponse rate.

Estimators	MSEs	MSEs	MSEs	MSEs
	k=2	**k=3**	**k=4**	**k=5**
$\widehat{\overline{T}}^{**}_{SRS}$	3650.36754	4217.96705	4785.56656	5353.16608
$\widehat{\overline{T}}^{**}_{Ratio}$	281.13907	287.89089	294.64272	301.39454
$\widehat{\overline{T}}^{**}_{Product}$	15734.95928	18259.73251	20784.50574	23309.27897
$\widehat{\overline{T}}^{**}_{Difference}$	225.08635	225.22309	225.32665	225.4078
$\widehat{\overline{T}}^{**}_{RD}$	224.82067	224.95709	225.06041	225.14137
$\widehat{\overline{T}}^{**}_{Grover}$	221.97892	221.36993	220.64797	219.82377
$\widehat{\overline{T}}^{**}_{\text{Prop 1}}$	**56.46193**	**68.6282**	**78.66441**	**87.28453**
$\widehat{\overline{T}}^{**}_{\text{Prop 2}}$	**56.46193**	**68.6282**	**78.66441**	**87.28453**
$\widehat{\overline{T}}^{**}_{\text{Prop 3}}$	**56.4452**	**68.60348**	**78.63193**	**87.24455**

Table 4. PREs of all estimators using real-life data under Situation I with a 35% nonresponse rate.

Estimators	PREs	PREs	PREs	PREs
	k=2	**k=3**	**k=4**	**k=5**
$\widehat{\overline{T}}^{**}_{SRS}$	100	100	100	100
$\widehat{\overline{T}}^{**}_{Ratio}$	341.71315	214.06285	188.55412	172.36957
$\widehat{\overline{T}}^{**}_{Product}$	28.00032	34.04823	36.93761	39.58444
$\widehat{\overline{T}}^{**}_{Difference}$	358.28827	218.84029	191.80493	174.79058
$\widehat{\overline{T}}^{**}_{RD}$	359.9564	221.05474	194.31737	177.60101
$\widehat{\overline{T}}^{**}_{Grover}$	361.65546	222.00047	195.13207	178.33606
$\widehat{\overline{T}}^{**}_{\text{Prop 1}}$	**466.75307**	**245.04008**	**209.07374**	**187.40057**
$\widehat{\overline{T}}^{**}_{\text{Prop 2}}$	**466.75307**	**245.04008**	**209.07374**	**187.40057**
$\widehat{\overline{T}}^{**}_{\text{Prop 3}}$	**466.75307**	**245.04008**	**209.07374**	**187.40057**

Table 5. PREs of all estimators using real-life data under Situation II with a 35% nonresponse rate.

Estimators	PREs	PREs	PREs	PREs
	k=2	**k=3**	**k=4**	**k=5**
$\widehat{\overline{T}}^{**}_{SRS}$	100	100	100	100
$\widehat{\overline{T}}^{**}_{Ratio}$	1153.25708	1465.12695	1624.19305	1776.13238
$\widehat{\overline{T}}^{**}_{Product}$	23.3099	23.09983	23.02468	22.96582
$\widehat{\overline{T}}^{**}_{Difference}$	1412.57937	1872.79512	2123.83513	2374.88059
$\widehat{\overline{T}}^{**}_{RD}$	1414.2475	1875.00957	2126.34757	2377.69102
$\widehat{\overline{T}}^{**}_{Grover}$	1428.75709	1905.39293	2168.86956	2435.20807
$\widehat{\overline{T}}^{**}_{\text{Prop 1}}$	**7264.38928**	**6146.1138**	**6083.5219**	**6133.0064**
$\widehat{\overline{T}}^{**}_{\text{Prop 2}}$	**7264.38928**	**6146.1138**	**6083.5219**	**6133.0064**
$\widehat{\overline{T}}^{**}_{\text{Prop 3}}$	**7266.0574**	**6148.32824**	**6086.03434**	**6135.81683**

Table 6. MSEs of all estimators using real-life data under Situation I with a 30% nonresponse rate.

Estimators	MSEs	MSEs	MSEs	MSEs
	k=2	**k=3**	**k=4**	**k=5**
$\widehat{\overline{T}}^{**}_{SRS}$	4611.87	4024.793	4528.002	5031.21
$\widehat{\overline{T}}^{**}_{Ratio}$	2364.337	1777.26	2280.469	2783.677
$\widehat{\overline{T}}^{**}_{Product}$	12782.114	12195.038	12698.246	13201.454
$\widehat{\overline{T}}^{**}_{Difference}$	2321.321	1734.244	2237.453	2740.661
$\widehat{\overline{T}}^{**}_{RD}$	2293.371	1718.597	2211.475	2701.786
$\widehat{\overline{T}}^{**}_{Grover}$	2283.747	1711.204	2202.169	2690.576
$\widehat{\overline{T}}^{**}_{\text{Prop 1}}$	**2115.24**	**1528.164**	**2031.372**	**2534.581**
$\widehat{\overline{T}}^{**}_{\text{Prop 2}}$	**2115.24**	**1528.164**	**2031.372**	**2534.581**
$\widehat{\overline{T}}^{**}_{\text{Prop 3}}$	**2092.008**	**1516.001**	**2009.937**	**2501.297**

Table 7. MSEs of all estimators using real-life data under Situation II with a 30% nonresponse rate.

Estimators	MSEs	MSEs	MSEs	MSEs
	k=2	**k=3**	**k=4**	**k=5**
$\widehat{\overline{T}}^{**}_{SRS}$	3521.58514	4024.79345	4528.00177	5031.21008
$\widehat{\overline{T}}^{**}_{Ratio}$	316.76957	341.33664	365.90371	390.47079
$\widehat{\overline{T}}^{**}_{Product}$	15649.70254	18131.8474	20613.99225	23096.13711
$\widehat{\overline{T}}^{**}_{Difference}$	228.25776	229.29142	230.06496	230.6656
$\widehat{\overline{T}}^{**}_{RD}$	227.98456	229.01573	229.78741	230.3866
$\widehat{\overline{T}}^{**}_{Grover}$	225.01849	225.22361	225.07702	224.66573
$\widehat{\overline{T}}^{**}_{\text{Prop 1}}$	**38.59867**	**45.6848**	**51.90598**	**57.56118**
$\widehat{\overline{T}}^{**}_{\text{Prop 2}}$	**38.59867**	**45.6848**	**51.90598**	**57.56118**
$\widehat{\overline{T}}^{**}_{\text{Prop 3}}$	**38.59085**	**45.67385**	**51.89184**	**57.54379**

Table 8. PREs of all estimators using real-life data under Situation I with a 30% nonresponse rate.

Estimators	PREs	PREs	PREs	PREs
	k=2	**k=3**	**k=4**	**k=5**
$\widehat{\overline{T}}^{**}_{SRS}$	100	100	100	100
$\widehat{\overline{T}}^{**}_{Ratio}$	195.05977	226.46054	198.55575	180.73972
$\widehat{\overline{T}}^{**}_{Product}$	36.08065	33.00353	35.65848	38.11103
$\widehat{\overline{T}}^{**}_{Difference}$	198.67439	232.07764	202.37307	183.57652
$\widehat{\overline{T}}^{**}_{RD}$	201.09564	234.19067	204.75029	186.21792
$\widehat{\overline{T}}^{**}_{Grover}$	201.94313	235.20242	205.61559	186.99381
$\widehat{\overline{T}}^{**}_{\text{Prop 1}}$	**218.03053**	**263.37444**	**222.90358**	**198.50266**
$\widehat{\overline{T}}^{**}_{\text{Prop 2}}$	**218.03053**	**263.37444**	**222.90358**	**198.50266**
$\widehat{\overline{T}}^{**}_{\text{Prop 3}}$	**220.45177**	**265.48747**	**225.2808**	**201.14406**

Table 9. PREs of all estimators using real-life data under Situation II with a 30% nonresponse rate.

Estimators	PREs	PREs	PREs	PREs
	k=2	**k=3**	**k=4**	**k=5**
$\widehat{\overline{T}}^{**}_{SRS}$	100	100	100	100
$\widehat{\overline{T}}^{**}_{Ratio}$	1246.45727	1179.12728	1237.4845	1288.49845
$\widehat{\overline{T}}^{**}_{Product}$	21.93237	22.19737	21.96567	21.78377
$\widehat{\overline{T}}^{**}_{Difference}$	2003.63327	1755.31796	1968.14056	2181.17055
$\widehat{\overline{T}}^{**}_{RD}$	2006.05452	1757.43099	1970.51777	2183.81195
$\widehat{\overline{T}}^{**}_{Grover}$	2049.49104	1787.02113	2011.75662	2239.42039
$\widehat{\overline{T}}^{**}_{\text{Prop 1}}$	**8721.02011**	**8809.91745**	**8723.46807**	**8740.6309**
$\widehat{\overline{T}}^{**}_{\text{Prop 2}}$	**8721.02011**	**8809.91745**	**8723.46807**	**8740.6309**
$\widehat{\overline{T}}^{**}_{\text{Prop 3}}$	**8723.44135**	**8812.03048**	**8725.84528**	**8743.27231**

Table 10. MSEs of all estimators using real-life data under Situation I with a 25% nonresponse rate.

Estimators	**MSEs**	**MSEs**	**MSEs**	**MSEs**
	k=2	**k=3**	**k=4**	**k=5**
$\widehat{\overline{T}}^{**}_{SRS}$	2919.84271	3729.19109	4133.86528	4538.53947
$\widehat{\overline{T}}^{**}_{Ratio}$	291.50788	339.2528	363.12527	386.99773
$\widehat{\overline{T}}^{**}_{Product}$	12721.13483	16792.57885	18828.30086	20864.02287
$\widehat{\overline{T}}^{**}_{Difference}$	227.51478	231.06022	232.22426	233.14921
$\widehat{\overline{T}}^{**}_{RD}$	227.24334	230.78026	231.94148	232.86418
$\widehat{\overline{T}}^{**}_{Grover}$	225.11974	227.3985	227.83458	227.96812
$\widehat{\overline{T}}^{**}_{\text{Prop 1}}$	**27.03603**	**39.5819**	**44.78243**	**49.57974**
$\widehat{\overline{T}}^{**}_{\text{Prop 2}}$	**27.03603**	**39.5819**	**44.78243**	**49.57974**
$\widehat{\overline{T}}^{**}_{\text{Prop 3}}$	**27.0322**	**39.57368**	**44.77191**	**49.56684**

Table 11. MSEs of all estimators using real-life data under Situation II with a 25% nonresponse rate.

Estimators	**MSEs**	**MSEs**	**MSEs**	**MSEs**
	k=2	**k=3**	**k=4**	**k=5**
$\widehat{\overline{T}}^{**}_{SRS}$	2919.8427	3729.191	4133.865	4538.539
$\widehat{\overline{T}}^{**}_{Ratio}$	672.3096	1481.658	1886.332	2291.006
$\widehat{\overline{T}}^{**}_{Product}$	11090.087	11899.435	12304.11	12708.784
$\widehat{\overline{T}}^{**}_{Difference}$	629.2936	1438.642	1843.316	2247.99
$\widehat{\overline{T}}^{**}_{RD}$	627.2214	1427.858	1825.649	2221.769
$\widehat{\overline{T}}^{**}_{Grover}$	624.0665	1421.594	1817.84	2212.422
$\widehat{\overline{T}}^{**}_{\text{Prop 1}}$	**423.2133**	**1232.562**	**1637.236**	**2041.91**
$\widehat{\overline{T}}^{**}_{\text{Prop 2}}$	**423.2133**	**1232.562**	**1637.236**	**2041.91**
$\widehat{\overline{T}}^{**}_{\text{Prop 3}}$	**422.275**	**1224.637**	**1623.283**	**2020.253**

Table 12. PREs of all estimators using real-life data under Situation I with a 25% nonresponse rate.

Estimators	**PREs**	**PREs**	**PREs**	**PREs**
	k=2	**k=3**	**k=4**	**k=5**
$\widehat{\overline{\overline{T}}}^{**}_{SRS}$	100	100	100	100
$\widehat{\overline{\overline{T}}}^{**}_{Ratio}$	1001.63423	1099.23663	1138.41302	1172.75611
$\widehat{\overline{\overline{T}}}^{**}_{Product}$	22.95269	22.20738	21.95559	21.75295
$\widehat{\overline{\overline{T}}}^{**}_{Difference}$	1283.36399	1613.94772	1780.11776	1946.62442
$\widehat{\overline{\overline{T}}}^{**}_{RD}$	1284.89692	1615.90555	1782.28805	1949.00717
$\widehat{\overline{\overline{T}}}^{**}_{Grover}$	1297.01761	1639.93656	1814.41519	1990.86588
$\widehat{\overline{\overline{T}}}^{**}_{\text{Prop 1}}$	**10799.81965**	**9421.45513**	**9230.99712**	**9154.02028**
$\widehat{\overline{\overline{T}}}^{**}_{\text{Prop 2}}$	**10799.81965**	**9421.45513**	**9230.99712**	**9154.02028**
$\widehat{\overline{\overline{T}}}^{**}_{\text{Prop 3}}$	**10801.35258**	**9423.41297**	**9233.16741**	**9156.40303**

Table 13. PREs of all estimators using real-life data under Situation II with a 25% nonresponse rate.

Estimators	**PREs**	**PREs**	**PREs**	**PREs**
	k=2	**k=3**	**k=4**	**k=5**
$\widehat{\overline{\overline{T}}}^{**}_{SRS}$	100	100	100	100
$\widehat{\overline{\overline{T}}}^{**}_{Ratio}$	434.3003	251.69041	219.14832	198.10244
$\widehat{\overline{\overline{T}}}^{**}_{Product}$	26.3284	31.33923	33.59744	35.71183
$\widehat{\overline{\overline{T}}}^{**}_{Difference}$	463.9873	259.21606	224.26241	201.89319
$\widehat{\overline{\overline{T}}}^{**}_{RD}$	465.5203	261.1739	226.4327	204.27594
$\widehat{\overline{\overline{T}}}^{**}_{Grover}$	467.8737	262.32467	227.40533	205.13892
$\widehat{\overline{\overline{T}}}^{**}_{\text{Prop 1}}$	**689.9223**	**302.55615**	**252.49052**	**222.26931**
$\widehat{\overline{\overline{T}}}^{**}_{\text{Prop 2}}$	**689.9223**	**302.55615**	**252.49052**	**222.26931**
$\widehat{\overline{\overline{T}}}^{**}_{\text{Prop 3}}$	**691.4552**	**304.51398**	**254.66081**	**224.65206**

Table 14. MSEs of all estimators using real-life data under Situation I with a 20% nonresponse rate.

Estimators	MSEs	MSEs	MSEs	MSEs
	k=2	**k=3**	**k=4**	**k=5**
$\widehat{\overline{T}}^{**}_{SRS}$	2870.2485	4113.0284	4645.6483	5178.2683
$\widehat{\overline{T}}^{**}_{Ratio}$	622.7154	1865.4953	2398.1152	2930.7352
$\widehat{\overline{T}}^{**}_{Product}$	11040.4928	12283.2727	12815.8926	13348.5126
$\widehat{\overline{T}}^{**}_{Difference}$	579.6994	1822.4793	2355.0992	2887.7192
$\widehat{\overline{T}}^{**}_{RD}$	577.9405	1805.2069	2326.3356	2844.5934
$\widehat{\overline{T}}^{**}_{Grover}$	574.9769	1797.4779	2316.583	2832.8284
$\widehat{\overline{T}}^{**}_{\text{Prop 1}}$	**373.6191**	**1616.3989**	**2149.0189**	**2681.6388**
$\widehat{\overline{T}}^{**}_{\text{Prop 2}}$	**373.6191**	**1616.3989**	**2149.0189**	**2681.6388**
$\widehat{\overline{T}}^{**}_{\text{Prop 3}}$	**372.8876**	**1602.7974**	**2125.0433**	**2644.409**

Table 15. MSEs of all estimators using real-life data under Situation II with a 20% nonresponse rate.

Estimators	MSEs	MSEs	MSEs	MSEs
	k=2	**k=3**	**k=4**	**k=5**
$\widehat{\overline{T}}^{**}_{SRS}$	3580.40842	4113.02837	4645.64832	5178.26827
$\widehat{\overline{T}}^{**}_{Ratio}$	348.28658	388.61216	428.93775	469.26333
$\widehat{\overline{T}}^{**}_{Product}$	16199.45928	18956.4825	21713.50573	24470.52895
$\widehat{\overline{T}}^{**}_{Difference}$	234.53679	237.1781	239.10741	240.57841
$\widehat{\overline{T}}^{**}_{RD}$	234.24836	236.88314	238.80763	240.27494
$\widehat{\overline{T}}^{**}_{Grover}$	231.00051	232.62232	233.41094	233.61948
$\widehat{\overline{T}}^{**}_{\text{Prop 1}}$	**35.33021**	**41.63041**	**47.36476**	**52.73398**
$\widehat{\overline{T}}^{**}_{\text{Prop 2}}$	**35.33021**	**41.63041**	**47.36476**	**52.73398**
$\widehat{\overline{T}}^{**}_{\text{Prop 3}}$	**35.32366**	**41.62131**	**47.35299**	**52.71938**

Table 16. PREs of all estimators using real-life data under Situation I with a 20% nonresponse rate.

Estimators	PREs	PREs	PREs	PREs
	k=2	**k=3**	**k=4**	**k=5**
$\widehat{\overline{T}}^{**}_{SRS}$	100	100	100	100
$\widehat{\overline{T}}^{**}_{Ratio}$	460.92462	220.47916	193.72081	176.68837
$\widehat{\overline{T}}^{**}_{Product}$	25.99747	33.48479	36.24912	38.79285
$\widehat{\overline{T}}^{**}_{Difference}$	495.12705	225.68313	197.25913	179.32035
$\widehat{\overline{T}}^{**}_{RD}$	496.63394	227.84249	199.69811	182.03896
$\widehat{\overline{T}}^{**}_{Grover}$	499.19369	228.8222	200.53882	182.79499
$\widehat{\overline{T}}^{**}_{\text{Prop 1}}$	**768.2286**	**254.45626**	**216.17531**	**193.10088**
$\widehat{\overline{T}}^{**}_{\text{Prop 2}}$	**768.2286**	**254.45626**	**216.17531**	**193.10088**
$\widehat{\overline{T}}^{**}_{\text{Prop 3}}$	**769.73548**	**256.61562**	**218.61429**	**195.81949**

Table 17. PREs of all estimators using real-life data under Situation II with a 20% nonresponse rate.

Estimators	PREs	PREs	PREs	PREs
	k=2	**k=3**	**k=4**	**k=5**
$\widehat{\overline{T}}^{**}_{SRS}$	100	100	100	100
$\widehat{\overline{T}}^{**}_{Ratio}$	974.55415	1058.38899	1083.05887	1103.48879
$\widehat{\overline{T}}^{**}_{Product}$	22.91903	21.69721	21.3952	21.16124
$\widehat{\overline{T}}^{**}_{Difference}$	1253.36204	1734.15181	1942.91274	2152.42431
$\widehat{\overline{T}}^{**}_{RD}$	1254.86892	1736.31117	1945.35172	2155.14292
$\widehat{\overline{T}}^{**}_{Grover}$	1266.43187	1768.11426	1990.33018	2216.5396
$\widehat{\overline{T}}^{**}_{\text{Prop 1}}$	**11385.78054**	**9879.86495**	**9808.2369**	**9819.6049**
$\widehat{\overline{T}}^{**}_{\text{Prop 2}}$	**11385.78054**	**9879.86495**	**9808.2369**	**9819.6049**
$\widehat{\overline{T}}^{**}_{\text{Prop 3}}$	**11387.28743**	**9882.0243**	**9810.67588**	**9822.32351**

Table 18. MSEs of all estimators using real-life data under Situation I with a 15% nonresponse rate.

Estimators	**MSEs**	**MSEs**	**MSEs**	**MSEs**
	k=2	**k=3**	**k=4**	**k=5**
$\widehat{\overline{T}}^{**}_{SRS}$	2828.4667	4394.9577	5021.5541	5648.1505
$\widehat{\overline{T}}^{**}_{Ratio}$	580.9336	2147.4246	2774.021	3400.6174
$\widehat{\overline{T}}^{**}_{Product}$	10998.711	12565.202	13191.7984	13818.3948
$\widehat{\overline{T}}^{**}_{Difference}$	537.9176	2104.4086	2731.005	3357.6014
$\widehat{\overline{T}}^{**}_{RD}$	536.4028	2081.4127	2692.4017	3299.4405
$\widehat{\overline{T}}^{**}_{Grover}$	533.6005	2072.6112	2681.2277	3285.9093
$\widehat{\overline{T}}^{**}_{\text{Prop 1}}$	**331.8373**	**1898.3283**	**2524.9247**	**3151.5211**
$\widehat{\overline{T}}^{**}_{\text{Prop 2}}$	**331.8373**	**1898.3283**	**2524.9247**	**3151.5211**
$\widehat{\overline{T}}^{**}_{\text{Prop 3}}$	**331.2602**	**1879.5957**	**2491.8923**	**3100.226**

Table 19. MSEs of all estimators using real-life data under Situation II with a 15% nonresponse rate.

Estimators	**MSEs**	**MSEs**	**MSEs**	**MSEs**
	k=2	**k=3**	**k=4**	**k=5**
$\widehat{\overline{T}}^{**}_{SRS}$	3768.36131	4394.95771	5021.55411	5648.15051
$\widehat{\overline{T}}^{**}_{Ratio}$	385.452390	444.36088	503.26937	562.17786
$\widehat{\overline{T}}^{**}_{Product}$	17352.9975	20686.7898	24020.58213	27354.37445
$\widehat{\overline{T}}^{**}_{Difference}$	240.801970	244.68507	247.42073	249.45207
$\widehat{\overline{T}}^{**}_{RD}$	240.497930	244.37115	247.09975	249.1258
$\widehat{\overline{T}}^{**}_{Grover}$	236.756140	239.27251	240.45678	240.75101
$\widehat{\overline{T}}^{**}_{\text{Prop 1}}$	**37.1251500**	**44.31468**	**50.99742**	**57.35474**
$\widehat{\overline{T}}^{**}_{\text{Prop 2}}$	**37.1251500**	**44.31468**	**50.99742**	**57.35474**
$\widehat{\overline{T}}^{**}_{\text{Prop 3}}$	**37.1179100**	**44.30438**	**50.98377**	**57.33748**

Table 20. PREs of all estimators using real-life data under Situation I with a 15% nonresponse rate.

Estimators	PREs	PREs	PREs	PREs
	k=2	**k=3**	**k=4**	**k=5**
$\widehat{\overline{T}}^{**}_{SRS}$	100	100	100	100
$\widehat{\overline{T}}^{**}_{Ratio}$	486.8830	204.66179	181.02077	166.09191
$\widehat{\overline{T}}^{**}_{Product}$	25.71635	34.97721	38.06573	40.87414
$\widehat{\overline{T}}^{**}_{Difference}$	525.81782	208.84526	183.87202	168.2198
$\widehat{\overline{T}}^{**}_{RD}$	527.30277	211.15263	186.50835	171.1851
$\widehat{\overline{T}}^{**}_{Grover}$	530.0720	212.04931	187.28563	171.8900
$\widehat{\overline{T}}^{**}_{\text{Prop 1}}$	**852.36553**	**231.51726**	**198.87936**	**179.21982**
$\widehat{\overline{T}}^{**}_{\text{Prop 2}}$	**852.36553**	**231.51726**	**198.87936**	**179.21982**
$\widehat{\overline{T}}^{**}_{\text{Prop 3}}$	**853.85048**	**233.82463**	**201.51569**	**182.18512**

Table 21. PREs of all estimators using real-life data under Situation II with a 15% nonresponse rate.

Estimators	PREs	PREs	PREs	PREs
	k=2	**k=3**	**k=4**	**k=5**
$\widehat{\overline{T}}^{**}_{SRS}$	100	100	100	100
$\widehat{\overline{T}}^{**}_{Ratio}$	952.058280	989.05144	997.78655	1004.69103
$\widehat{\overline{T}}^{**}_{Product}$	22.8982800	21.24524	20.90521	20.64807
$\widehat{\overline{T}}^{**}_{Difference}$	1226.984230	1796.16914	2029.56082	2264.22278
$\widehat{\overline{T}}^{**}_{RD}$	1228.469190	1798.47651	2032.19715	2267.18808
$\widehat{\overline{T}}^{**}_{Grover}$	1239.549080	1836.800	2088.340	2346.05478
$\widehat{\overline{T}}^{**}_{Prop\ 1}$	**11671.26686**	**9917.61054**	**9846.6829**	**9847.74772**
$\widehat{\overline{T}}^{**}_{Prop\ 2}$	**11671.26686**	**9917.61054**	**9846.6829**	**9847.74772**
$\widehat{\overline{T}}^{**}_{Prop\ 3}$	**11672.75181**	**9919.91791**	**9849.31923**	**9850.71301**

6. Simulation study

The simulation research determines how different nonresponse (NR) levels impact mean estimates when using simple random sampling (SRS) through 10,000 replicate simulations. The method generates a total population of 600 to 1200 units during each iteration to extract 100 to 200 units through simple random sampling. The application of NR rates from 10% to 35% determines the observed sample population through reductions in the initial sizes.

Mixing uniform distribution sampling produces three variables Y, X, and Z that span values ranging across 200–800, 5000–10000, and 200–600, respectively. The data follow the specified joint distribution. The study uses different estimators for population mean estimation under NR, while the mean square error (MSE) along with the percentage relative efficiency (PRE) measure their performance. The research investigates two settings in Tables 22–25 through an analysis of MSE (in bold) and PRE (in bold) analysis. Scenario I contains NR applied to the study variable, and Scenario II involves NR applied to both the study and auxiliary variables.

Table 22. MSEs and PREs of estimators using simulation under Situation I with 10%, 15%, and 20% nonresponse rates.

NR	k	$\widehat{\overline{T}}^*_{SRS}$	$\widehat{\overline{T}}^*_{Ratio}$	$\widehat{\overline{T}}^*_{Product}$	$\widehat{\overline{T}}^*_{Difference}$	$\widehat{\overline{T}}^*_{RD}$	$\widehat{\overline{T}}^*_{Grover}$	$\widehat{\overline{T}}^*_{Prop\ 1,2}$	$\widehat{\overline{T}}^*_{Prop\ 3}$
10%	2	(2523.34, 100.00)	(387.49, 690.99)	(9925.65, 26.98)	(364.00, 735.58)	(363.35, 736.90)	(5566.18, 48.11)	**(361.64, 740.40)**	**(353.41, 757.63)**
	3	(2681.19, 100.00)	(545.39, 521.76)	(10083.75, 28.22)	(522.87, 545.22)	(521.49, 546.62)	(5724.91, 49.71)	**(519.29, 548.84)**	**(511.11, 557.80)**
	4	(2839.04, 100.00)	(703.17, 428.42)	(10241.85, 29.42)	(680.75, 443.21)	(678.45, 444.73)	(5883.64, 51.22)	**(675.93, 446.51)**	**(667.52, 452.01)**
	5	(2996.89, 100.00)	(860.86, 369.36)	(10400.00, 30.58)	(838.44, 379.70)	(835.01, 381.31)	(6042.37, 52.67)	**(831.87, 382.82)**	**(823.47, 386.55)**
15%	2	(2602.43, 100.00)	(466.43, 591.07)	(9997.14, 27.60)	(443.94, 623.41)	(442.98, 624.71)	(5645.00, 48.92)	**(440.99, 627.49)**	**(432.76, 639.54)**
	3	(2839.04, 100.00)	(703.17, 428.42)	(10241.85, 29.42)	(680.75, 443.21)	(678.45, 444.73)	(5883.64, 51.22)	**(675.93, 446.51)**	**(667.52, 452.01)**
	4	(3075.98, 100.00)	(940.05, 347.24)	(10484.87, 31.15)	(917.62, 356.15)	(913.42, 357.90)	(6122.27, 53.36)	**(909.94, 359.22)**	**(901.63, 362.39)**
	5	(3312.93, 100.00)	(1177.79, 298.72)	(10727.88, 32.80)	(1155.38, 304.81)	(1148.78, 306.60)	(6360.91, 55.31)	**(1144.51, 307.75)**	**(1136.17, 309.88)**
20%	2	(2681.19, 100.00)	(545.39, 521.76)	(10083.75, 28.22)	(522.87, 545.22)	(521.49, 546.62)	(5724.91, 49.71)	**(519.29, 548.84)**	**(511.11, 557.80)**
	3	(2996.89, 100.00)	(860.86, 369.36)	(10400.00, 30.58)	(838.44, 379.70)	(835.01, 381.31)	(6042.37, 52.67)	**(831.87, 382.82)**	**(823.47, 386.55)**
	4	(3312.93, 100.00)	(1177.79, 298.72)	(10727.88, 32.80)	(1155.38, 304.81)	(1148.78, 306.60)	(6360.91, 55.31)	**(1144.51, 307.75)**	**(1136.17, 309.88)**
	5	(3628.98, 100.00)	(1494.72, 258.99)	(11055.87, 34.90)	(1472.31, 263.15)	(1461.40, 265.06)	(6679.45, 57.71)	**(1455.25, 266.00)**	**(1446.94, 267.45)**

Table 23. MSEs and PREs of estimators using simulation under Situation I with 25%, 30%, and 35% nonresponse rates.

NR	k	$\widehat{\overline{T}}^*_{SRS}$	$\widehat{\overline{T}}^*_{Ratio}$	$\widehat{\overline{T}}^*_{Product}$	$\widehat{\overline{T}}^*_{Difference}$	$\widehat{\overline{T}}^*_{RD}$	$\widehat{\overline{T}}^*_{Grover}$	$\widehat{\overline{T}}^*_{Prop\ 1,2}$	$\widehat{\overline{T}}^*_{Prop\ 3}$
25%	2	(2651.42, 100.00)	(407.14, 690.99)	(10427.87, 26.98)	(382.43, 735.58)	(381.76, 736.90)	(5848.19, 48.11)	**(379.95, 740.40)**	**(371.31, 757.63)**
	3	(2816.99, 100.00)	(572.00, 521.76)	(10596.93, 28.22)	(549.43, 545.22)	(547.99, 546.62)	(6015.38, 49.71)	**(545.74, 548.84)**	**(537.99, 557.80)**
	4	(2982.57, 100.00)	(738.76, 428.42)	(10766.06, 29.42)	(716.21, 443.21)	(713.79, 444.73)	(6182.57, 51.22)	**(711.16, 446.51)**	**(702.34, 452.01)**
	5	(3148.15, 100.00)	(905.53, 369.36)	(10935.33, 30.58)	(883.00, 379.70)	(879.45, 381.31)	(6349.76, 52.67)	**(875.79, 382.82)**	**(866.96, 386.55)**
30%	2	(2733.04, 100.00)	(490.10, 591.07)	(10504.76, 27.60)	(466.41, 623.41)	(465.42, 624.71)	(5930.49, 48.92)	**(463.37, 627.49)**	**(454.30, 639.54)**
	3	(2982.57, 100.00)	(738.76, 428.42)	(10766.06, 29.42)	(716.21, 443.21)	(713.79, 444.73)	(6182.57, 51.22)	**(711.16, 446.51)**	**(702.34, 452.01)**
	4	(3232.10, 100.00)	(987.59, 347.24)	(11017.77, 31.15)	(964.07, 356.15)	(959.55, 357.90)	(6435.13, 53.36)	**(955.93, 359.22)**	**(946.26, 362.39)**
	5	(3481.62, 100.00)	(1237.39, 298.72)	(11261.01, 32.80)	(1214.87, 304.81)	(1207.85, 306.60)	(6687.69, 55.31)	**(1203.43, 307.75)**	**(1194.63, 309.88)**
35%	2	(2816.99, 100.00)	(572.00, 521.76)	(10596.93, 28.22)	(549.43, 545.22)	(547.99, 546.62)	(6015.38, 49.71)	**(545.74, 548.84)**	**(537.99, 557.80)**
	3	(3148.15, 100.00)	(905.53, 369.36)	(10935.33, 30.58)	(883.00, 379.70)	(879.45, 381.31)	(6349.76, 52.67)	**(875.79, 382.82)**	**(866.96, 386.55)**
	4	(3481.62, 100.00)	(1237.39, 298.72)	(11261.01, 32.80)	(1214.87, 304.81)	(1207.85, 306.60)	(6687.69, 55.31)	**(1203.43, 307.75)**	**(1194.63, 309.88)**
	5	(3815.10, 100.00)	(1569.26, 258.99)	(11591.26, 34.90)	(1546.75, 263.15)	(1535.52, 265.06)	(7025.62, 57.71)	**(1528.80, 266.00)**	**(1519.13, 267.45)**

Table 24. MSEs and PREs of estimators using simulation under Situation II with 10%, 15%, and 20% nonresponse rates.

NR	k	$\widehat{\overline{T}}^{**}_{SRS}$	$\widehat{\overline{T}}^{**}_{Ratio}$	$\widehat{\overline{T}}^{**}_{Product}$	$\widehat{\overline{T}}^{**}_{Difference}$	$\widehat{\overline{T}}^{**}_{RD}$	$\widehat{\overline{T}}^{**}_{Grover}$	$\widehat{\overline{T}}^{**}_{Prop\ 1,2}$	$\widehat{\overline{T}}^{**}_{Prop\ 3}$
10%	2	(2757.64, 100.00)	(422.87, 690.99)	(10818.46, 26.98)	(396.79, 735.58)	(395.87, 736.90)	(6073.83, 48.11)	**(393.09, 740.40)**	**(385.44, 757.63)**
	3	(2921.95, 100.00)	(624.13, 521.76)	(11504.13, 28.22)	(600.91, 545.22)	(599.46, 546.62)	(6540.10, 49.71)	**(599.04, 548.84)**	**(591.09, 557.80)**
	4	(3255.47, 100.00)	(805.88, 428.42)	(11763.77, 29.42)	(782.68, 443.21)	(780.36, 444.73)	(6762.80, 51.22)	**(779.41, 446.51)**	**(772.22, 452.01)**
	5	(3255.47, 100.00)	(991.15, 369.36)	(11956.16, 30.58)	(966.52, 379.70)	(962.68, 381.31)	(6925.38, 52.67)	**(961.00, 382.82)**	**(949.45, 386.55)**
15%	2	(2847.02, 100.00)	(509.01, 591.07)	(10995.79, 27.60)	(484.66, 623.41)	(483.26, 624.71)	(6163.09, 48.92)	**(480.73, 627.49)**	**(472.29, 639.54)**
	3	(3255.47, 100.00)	(805.88, 428.42)	(11763.77, 29.42)	(782.68, 443.21)	(780.36, 444.73)	(6762.80, 51.22)	**(779.41, 446.51)**	**(772.22, 452.01)**
	4	(3288.91, 100.00)	(1077.21, 347.24)	(11983.23, 31.15)	(1053.21, 356.15)	(1047.33, 357.90)	(6795.86, 53.36)	**(1047.84, 359.22)**	**(1037.27, 362.39)**
	5	(3791.85, 100.00)	(1347.63, 298.72)	(12259.23, 32.80)	(1318.43, 304.81)	(1308.65, 306.60)	(6945.73, 55.31)	**(1304.60, 307.75)**	**(1294.80, 309.88)**
20%	2	(2921.95, 100.00)	(624.13, 521.76)	(11504.13, 28.22)	(600.91, 545.22)	(599.46, 546.62)	(6540.10, 49.71)	**(599.04, 548.84)**	**(591.09, 557.80)**
	3	(3255.47, 100.00)	(991.15, 369.36)	(11956.16, 30.58)	(966.52, 379.70)	(962.68, 381.31)	(6925.38, 52.67)	**(961.00, 382.82)**	**(949.45, 386.55)**
	4	(3791.85, 100.00)	(1347.63, 298.72)	(12259.23, 32.80)	(1318.43, 304.81)	(1308.65, 306.60)	(6945.73, 55.31)	**(1304.60, 307.75)**	**(1294.80, 309.88)**
	5	(3943.81, 100.00)	(1702.16, 258.99)	(12536.80, 34.90)	(1670.32, 263.15)	(1658.18, 265.06)	(7302.71, 57.71)	**(1646.63, 266.00)**	**(1636.10, 267.45)**

Table 25. MSEs and PREs of estimators using simulation under Situation II with 25%, 30%, and 35% nonresponse rates.

NR	k	$\widehat{\overline{T}}^{**}_{SRS}$	$\widehat{\overline{T}}^{**}_{Ratio}$	$\widehat{\overline{T}}^{**}_{Product}$	$\widehat{\overline{T}}^{**}_{Difference}$	$\widehat{\overline{T}}^{**}_{RD}$	$\widehat{\overline{T}}^{**}_{Grover}$	$\widehat{\overline{T}}^{**}_{Prop\ 1,2}$	$\widehat{\overline{T}}^{**}_{Prop\ 3}$
25%	2	(2978.25, 100.00)	(457.89, 745.79)	(11684.55, 29.12)	(428.74, 794.02)	(426.84, 796.27)	(6560.14, 51.93)	**(424.74, 799.23)**	**(415.07, 816.23)**
	3	(3155.71, 100.00)	(674.67, 563.90)	(12444.47, 30.37)	(648.98, 589.24)	(647.21, 590.57)	(7053.71, 53.72)	**(646.12, 592.38)**	**(638.77, 602.42)**
	4	(3515.91, 100.00)	(869.34, 462.81)	(12773.86, 31.84)	(844.10, 478.67)	(842.39, 481.21)	(7301.62, 55.32)	**(841.81, 484.43)**	**(834.80, 487.17)**
	5	(3515.91, 100.00)	(1072.24, 398.11)	(12979.66, 32.97)	(1042.65, 410.50)	(1039.29, 412.62)	(7473.41, 57.51)	**(1037.08, 414.85)**	**(1024.72, 417.79)**
30%	2	(3078.99, 100.00)	(548.35, 637.96)	(11875.63, 29.74)	(523.03, 673.68)	(521.32, 674.92)	(6653.34, 52.83)	**(520.79, 678.84)**	**(509.87, 690.42)**
	3	(3515.91, 100.00)	(869.34, 462.81)	(12773.86, 31.84)	(844.10, 478.67)	(842.39, 481.21)	(7301.62, 55.32)	**(841.81, 484.43)**	**(834.80, 487.17)**
	4	(3552.24, 100.00)	(1163.79, 375.22)	(12962.09, 33.69)	(1137.47, 364.64)	(1130.73, 365.53)	(7340.33, 57.83)	**(1130.85, 367.97)**	**(1120.35, 368.99)**
	5	(4094.80, 100.00)	(1450.43, 323.32)	(13200.97, 35.46)	(1424.31, 331.19)	(1413.34, 333.13)	(7492.40, 59.91)	**(1407.97, 335.37)**	**(1393.38, 337.81)**
35%	2	(3155.71, 100.00)	(674.67, 563.90)	(12444.47, 30.37)	(648.98, 589.24)	(647.21, 590.57)	(7053.71, 53.72)	**(646.12, 592.38)**	**(638.77, 602.42)**
	3	(3515.91, 100.00)	(1072.24, 398.11)	(12979.66, 32.97)	(1042.65, 410.50)	(1039.29, 412.62)	(7473.41, 57.51)	**(1037.08, 414.85)**	**(1024.72, 417.79)**
	4	(4094.80, 100.00)	(1450.43, 323.32)	(13200.97, 35.46)	(1424.31, 331.19)	(1413.34, 333.13)	(7492.40, 59.91)	**(1407.97, 335.37)**	**(1393.38, 337.81)**
	5	(4263.31, 100.00)	(1838.33, 279.71)	(13551.83, 37.66)	(1804.54, 285.00)	(1781.43, 286.85)	(7996.93, 61.67)	**(1770.38, 287.73)**	**(1757.79, 289.22)**

The MSEs and PREs from existing and proposed estimators are visually presented in Figures 1–4. The MSE and PRE information for Situation I is presented in Figures 1 and 2, and Figures 3 and 4 display the same data for Situation II. These visual examples enable an assessment of performance comparing between existing and proposed estimators, which primarily uses actual and simulated data for precision and accuracy tests under different circumstances.

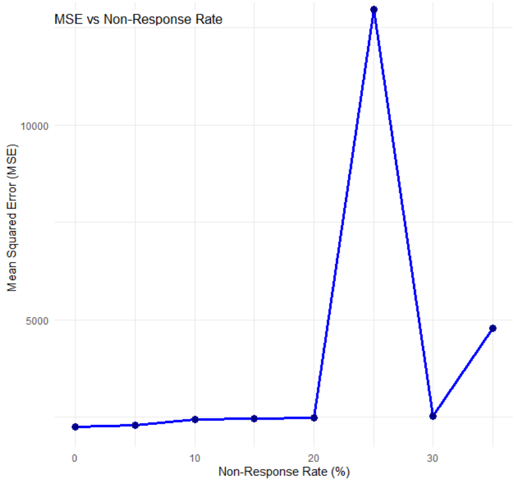

Figure 1. MSEs of the suggested and existing estimators using Situation I.

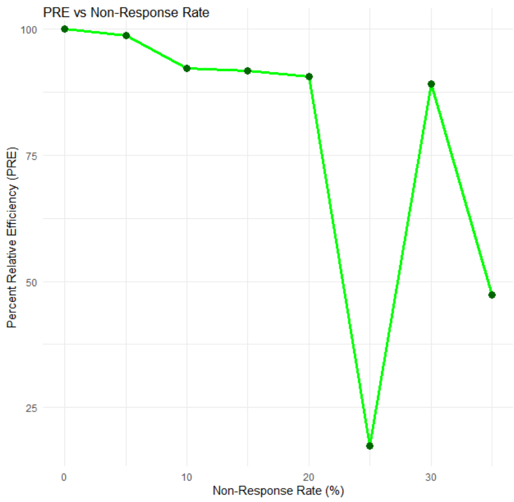

Figure 2. PREs of the suggested and existing estimators using Situation I.

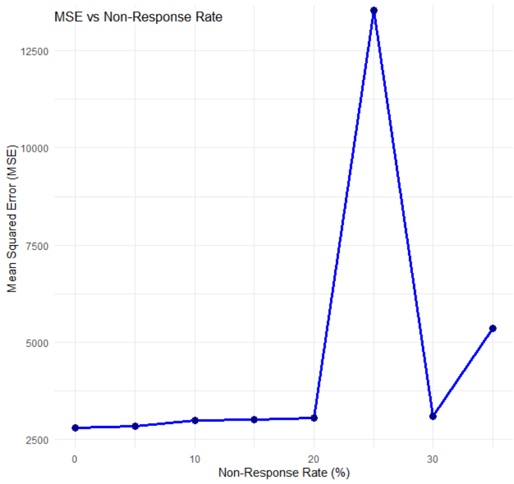

Figure 3. MSEs of the suggested and existing estimators using Situation II.

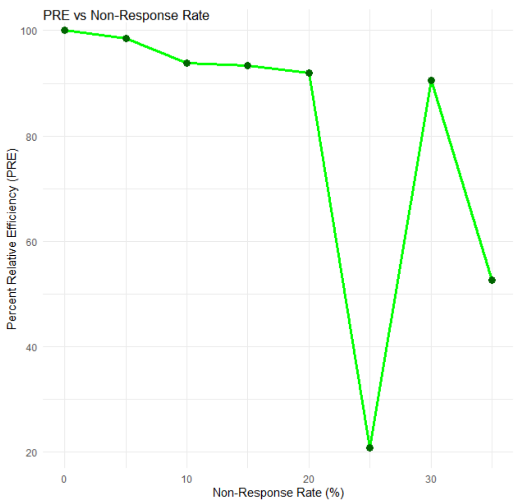

Figure 4. PREs of the suggested and existing estimators using Situation II.

7. Discussion and effect on MSE and PRE (based on all tables)

The mean square error from all tables show that the proposed estimators outperform existing or conventional estimators. The results demonstrate that the population mean estimation achieves better precision together with improved accuracy when conducted under nonresponse situations. The percentage of relative efficiency scores achieve better performance levels with the new estimator group than they do with standard methods. The value of PRE indicates how efficiently new estimators operate so that they need fewer sample observations to deliver precise results. The proposed estimators exhibit this behavior in the real-world data analysis, as well as when used in artificially generated information, showing their universal applicability. The MSE decreases while PRE increases consistently within all examined scenarios because the inclusion of auxiliary information strengthens the improved estimator structure's ability to deal with nonresponse effects.

The performance examination of the proposed class of estimators utilized the same real dataset when tested under Situations I and II. Mobile sample sizes were tested to determine how resilient the estimators were. The research evaluated the MSE and PRE performance of the proposed generalized estimators in comparison with traditional estimators. The real dataset produced the MSE and PRE values which are published in Tables 2 through 21. Numerical analysis reveals that the proposed estimators produce improved results compared with established ones since they generate lower MSE values and higher PRE levels effectively.

8. Conclusions and future recommendation

8.1. Conclusions

A new advanced estimator group provides better approaches to estimate population mean values when sampling occurs under nonresponse conditions. The proposed class contains 10 new estimators as members. Theoretical estimations for bias together with mean square error (MSE) first-order approximations exist. Testing with real datasets along with simulated results shows that the new estimators demonstrate better performance than traditional methods because they create lower MSE and higher PRE measurements. The implemented estimators prove their effectiveness in dealing with situations involving nonresponse. The effectiveness of auxiliary information delivery achieves maximum performance benefits in these situations. The methodology delivers a versatile structure for additional research development. The method presents scope for the inclusion of specific auxiliary variables that demonstrate measurement errors or display nonresponse distribution patterns or exhibit known population distribution values. The extensions provide basic yet effective tools to boost estimators' functionality in actual practical settings. The proposed estimators show strong potential for application in experimental surveys due to their ability to handle incomplete data and nonresponse errors effectively. Practitioners in the field of official statistics along with the health sector and educational institutions and other domains requiring accurate population estimation should adopt the proposed estimator class for its reliable and efficient performance.

8.2. Future recommendation

The current research examines nonresponse issues when using simple random sampling (SRS) for estimating the finite population mean. This method delivers important findings, though it creates space for multiple useful enhancements. Research evaluating this framework should integrate additional sampling approaches which include stratified random sampling, cluster sampling, systematic sampling, and probability proportional to size (PPS) sampling. These sampling methods present different strengths regarding efficiency and representativeness, especially during analysis of heterogeneous populations. The established methodology extends its applicability to estimating additional vital population measures that surpass the mean. The proposed method allows estimation of the cumulative distribution function (CDF) as well as the variance, median, quartiles, and other quantiles. The proposed estimators should be developed to calculate various population parameters including cumulative distribution functions and quantiles under different sampling methods to increase their practical value and make them suitable for social science research as well as public health studies and agricultural and official statistical applications.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in creating this article.

Conflict of interest

The author declares no conflicts of interest.

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