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Research article

Timelike hyperelastic strips

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Abstract: We explore certain extremals of the Sadowsky-type functional in Minkowski 3-space $\mathbb{R}^3_1$. Focusing on timelike rectifying strips, we provide a detailed characterization of the hyperelastic configurations. Specifically, we introduce timelike hyperelastic strips as surfaces derived from the solution curves of the minimization of this variational problem. Our investigation extends to the timelike planar critical points of the modified Sadowsky-type functional, where we demonstrate that these correspond to timelike hyperelastic curves situated on a timelike plane. Our research on timelike hyperelastic strips not only uncovers the fundamental Euler-Lagrange equations governing these structures but also highlights the geometric conservation laws that dictate their behavior. By deriving conserved quantities specific to these strips, we formulate the first and second conservation laws. This allows us to present significant insights into how these strips behave under both translational and rotational symmetries. We anticipate that these findings will pave the way for future studies exploring novel types of hyperelastic surfaces and their broader applications in geometric and physical contexts, particularly in relation to de Sitter and pseudo-hyperbolic spaces.

Keywords: p-elastic strips; Sadowsky-type functional; timelike hyperelastic strips; variational calculus
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1. Introduction

In the study of developable ruled surfaces, rectifying strips are those whose modified Darboux vector is aligned with the tangent and binormal vectors of a reference curve. These surfaces exhibit elastic behavior when minimizing a functional based on Sadowsky's 1930 work [17] (see [13] for a translation) which was designed to describe elastic properties of inextensible strips. The Wunderlich's 1962 study of the developable Möbius strip offered a key insight by reducing its elastic deformation problem to a variational framework, significantly advancing the understanding of inextensible surfaces in geometry [32] (see [23] for a translation). A necessary condition for a rectifying strip to be elastic is that its reference curve serves as a stationary point of the Sadowsky functional. This functional has a proportional relationship with the Willmore functional, which is also known for its connection to surfaces with minimal bending energy [25, 28].

Hangan [14] analyzed the extremals of the Sadowsky functional by deriving two Euler-Lagrange equations that offer insights into the geometric properties of elastic strips. In [19], the authors tackled the more complex problem of minimizing the energy of developable strips with finite width, resulting in six equilibrium equations for elastic strips, which they solved numerically. Chubelaschwili and Pinkall advanced the study by finding integrable solutions for elastic strips, deriving conservation laws using Euclidean symmetries. Additionally, they introduced two new integrable systems of elastic strips, offering a deeper understanding of these surfaces' physical and geometric behavior [4]. The concept of elastic strips has also been investigated into Minkowski 3-space $\mathbb{R}^3_1$, where the differential geometry of these surfaces is studied in a relativistic context. Two conservation laws were derived in $\mathbb{R}^3_1$ by incorporating Lorentzian translations and rotations (see [24–26]). The authors in [33] derived the equilibrium equations that belong to elastic strips and obtained the conservation laws along isotropic curves in a three-dimensional complex space. Recent research has focused on refining the computational methods used in these models, employing more advanced numerical techniques to solve for complex strip configurations (see [5–7, 18], etc.). Especially, Starostin and van der Heijden demonstrated that all non-force critical points correspond to spherical curves only when they are critical points of the Sadowsky-type functional $\int_\gamma C\kappa^n(1+\lambda^2)^n ds$,where C is constant and $n \neq 0, 1$ [20]. Recently, Tükel has showed that the Sadowsky-type functional (for $C = 1$) is proportional to the p-Willmore functional $\int_M H^n d\sigma$ (see [9–12]) using the similar methodology of Wunderlich [28]. In [29], the authors illustrate that the Sadowsky-type functional is proportional to the p-Willmore functional for non-degenerate developable rectifying surfaces, whether the base curves (directrices) are spacelike or timelike. Minkowski 3-space forms the geometric foundation of the theory of special relativity. If the surface under consideration changes parametrically with respect to time, classical Euclidean space becomes inadequate to describe such dynamic systems, and it becomes necessary to transition to Lorentz-Minkowski space. Minkowski 3-space possesses a dynamic structure and a metric that includes time. With these properties, it provides a crucial geometric framework for transitioning to high-energy physics and relativistic theories.

In this paper, we establish a connection between the Sadowsky-type functional

$$\int_\alpha \kappa^p (1+\lambda^2)^p \, ds, \tag{1.1}$$

and the p-Willmore functional $\int_M H^n d\sigma$ for a timelike ruled surface in three-dimensional Minkowski space $\mathbb{R}^3_1$ by using Wunderlich's approach. Then, we study and develop hyperelastic strips in $\mathbb{R}^3_1$ considering one of the causal characters of the space, namely timelike base curves. This approach aims to address geometric and physical problems that arise in space-time models, allowing for an analysis of the elastic behavior of strips through timelike base curves. We characterize the hyperelastic strips with two Euler-Lagrange equations. The timelike rectifying developable surfaces, which we refer to as timelike hyperelastic strips (or p-elastic strips), exhibit various geometric and elastic properties, making them significant in the study of elastic structures and energy minimization problems. Then, we show that the timelike planar critical points of the modified Sadowsky-type functional are nothing but timelike hyperelastic curves on a timelike plane. We demonstrate the practical application of the theoretical framework with an example. Furthermore, we calculate the internal force W_0 and the torque W_1 which are conserved quantities by using new variations involving Lorentzian translational and rotational motions. Finally, we find the first and second conservation laws of timelike hyperelastic strips.

2. Geometrical setup

Minkowski 3-space $\mathbb{R}_1^3$ is a metric space with a symmetric, bilinear and non-degenerate metric $\langle , \rangle$ with signature $(-,+,+)$. Any tangent vector $x \in \mathbb{R}_1^3$ is known spacelike, timelike or null (lightlike) if $< x, x >> 0$ or $x = 0$, $< x, x >< 0$, or $< x, x >= 0$ and $x \neq 0$, respectively. A curve α in $\mathbb{R}_1^3$ is called spacelike, timelike or null (lightlike) at t in I if its velocity vector $\alpha'(t)$ is spacelike, timelike or null, respectively [15, 16].

Let $\alpha : [0, \ell] \subset \mathbb{R} \to \mathbb{R}_1^3$ be a timelike curve with velocity $\upsilon = \|\gamma'\|$ in $\mathbb{R}_1^3$. At a point $\alpha(t)$ of α, let $T(t)$ denote the unit tangent vector field, $N(t)$ the unit principal normal vector field and $B(t) = T(t) \times N(t)$ the unit binormal vector field of α. $\{T, N, B\}$ is the Frenet frame along α. The derivative equations of Frenet frame $\{T, N, B\}$, for all $t \in I \subset \mathbb{R}$ are given by

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \upsilon\kappa & 0 \\ \upsilon\kappa & 0 & \upsilon\lambda\kappa \\ 0 & -\upsilon\lambda\kappa & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{2.1}$$

where $\lambda = \dfrac{\tau}{\kappa}$ is the modified torsion, and $\kappa > 0$, τ are the curvature and the torsion of α respectively, which are defined by

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3} \text{ and } \tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \times \alpha''\|^2}, \text{ [15, 27].}$$

Given that a developable strip is a specific instance of a ruled surface, the strip can be parameterized by a timelike directrix α as follows:

$$\begin{aligned} F_\alpha : [0, \ell] \times [-\omega, \omega] &\to \mathbb{R}_1^3 \\ (t, \delta) &\to F_\alpha(t, \delta) = \alpha(t) + \delta(\lambda T(t) + B(t)). \end{aligned} \tag{2.2}$$

In light of this, the following definition clarifies the concept of a timelike rectifying strip.

Definition 1. *[24] Any developable ruled surface defined as in (2.2) is called a timelike rectifying strip.*

We know that the mean curvature H of a timelike surface can be calculated as the formula $\dfrac{\kappa_1^2 + \kappa_2^2}{2}$, where κ_1 and κ_2 are principal curvatures [31]. The area element and the mean curvature of the surface are given by

$$d\sigma = (1 - \delta\lambda')\, d\delta\, dt \quad \text{and} \quad H = \frac{\kappa(1 + \lambda^2)^2}{2(1 - \delta\lambda')}, \text{ [29].}$$

The Gaussian curvature of the strip $F_\alpha(t,\delta)$ vanishes and so, one of the principal curvatures is zero. The energy functional can be written as

$$\frac{D}{2}\iint \kappa_1^p d\sigma',$$

where $D = \frac{2Yh^3}{3(1-\nu^2)}$ is the flexural rigidity, h is the thickness, ν is the Poisson's ratio, Y is Young's modulus, and $\kappa_1 = 2H$, $p \geq 2$ since F_α is developable and the strip is planar when relaxed. Thus, we arrive at [28, 29]

$$\frac{D}{2}\int_0^\ell\int_{-\omega}^{\omega}\frac{\kappa^p\left(1+\lambda^2\right)^p}{(1-\delta\lambda')^{p-1}}d\delta dt = D\omega\int_0^\ell h(\kappa,\lambda,\lambda')\,dt, \tag{2.3}$$

where

$$\begin{aligned} h(\kappa,\lambda,\lambda') &= \kappa^p\left(1+\lambda^2\right)^p V(\omega\lambda'), \\ V(\omega\lambda') &= \frac{(1-\omega\lambda')^{2-p}-(1+\omega\lambda')^{2-p}}{2(n-2)\,\omega\lambda'}. \end{aligned}$$

In the limit of narrow strips $\omega\lambda' \to 0$, we have $V(\omega\lambda') \to 1$ and no derivative enters the integrand in Eq (2.3) (see [21, 29]). This shows that the p-Willmore functional $\int_M H^p d\sigma$ is proportional to the Sadowsky-type functional $\int_{F_\alpha}\kappa^p(1+\lambda^p)\,dt$. Observe that for $\lambda = 0$ this is nothing but a functional of free hyperelastic curve (or p-elastic curve's) functional (see [1]).

We next study infinitely narrow timelike rectifying strips constructed by using critical points of the Sadowsky-type functional (1.1) within all timelike curves with fixed end points and

$$\dot{\ell} := \frac{\partial}{\partial\delta}\ell(\alpha_\delta)\Big|_{\delta=0} = 0, \tag{2.4}$$

where $\alpha(t)$ is a directrix of the timelike rectifying surface.

Lemma 1. *[24] If $\alpha_0 : [0,\ell] \to \mathbb{R}_1^3$ is an arc length parametrized timelike curve and*

$$\begin{aligned} \alpha : [0,\ell]\times[-\varepsilon,\varepsilon] &\to \mathbb{R}_1^3 \\ (s,\delta) &\to \alpha(s,\delta) = \alpha_\delta(s) = \alpha_0(s) + \delta\dot{\alpha}(s) \end{aligned}$$

is a variation of α_0 with variational vector

$$\dot{\alpha}(s) = \frac{\partial}{\partial\delta}\Big|_{\delta=0}\alpha_\delta(s) = u_1(s)\,T(s) + u_2(s)\,N(s) + u_3(s)\,B(s), \tag{2.5}$$

where $u_1(s) = -<\dot{\alpha}(s), T(s)>, u_2 =<\dot{\alpha}(s), N(s)>$ and $u_3 =<\dot{\alpha}(s), B(s)>$, then we have

$$\dot{\upsilon} = u_1' + u_2\kappa, \tag{2.6}$$

$$\dot{\kappa} = u_1\kappa' - u_2\kappa^2\left(1+\lambda^2\right) - 2u_3'\lambda\kappa - u_3(\lambda\kappa)' + u_2'', \tag{2.7}$$

$$\begin{aligned} \dot{\lambda} = {} & u_1\lambda' + u_2\left(\frac{(\lambda\kappa)''}{\kappa^2} - \frac{(\lambda\kappa)'\kappa'}{\kappa^3} + \lambda^3\kappa - \lambda\kappa\right) + u_2'\left(2\frac{\lambda'}{\kappa} + \frac{(\lambda\kappa)'}{\kappa^2}\right) \\ & + u_2''\frac{\lambda}{\kappa} - u_3\lambda\lambda' - u_3'\left(1-\lambda^2\right) - u_3''\frac{\kappa'}{\kappa^3} + u_3'''\frac{1}{\kappa^2}. \end{aligned} \tag{2.8}$$

3. Timelike hyperelastic strips

We analyze an inextensible strip that, when flattened onto a plane, forms a strip bounded by two parallel straight lines. Owing to its developable nature, the strip can always be reconstructed from any chosen reference curve on its surface, regardless of its deformation. For this reference curve, the strip's centerline is naturally chosen as it allows for reconstruction based on its curvature and torsion, providing a precise geometric representation of the strip's shape (see [5, 19, 22, 30]).

Now we define the modified Sadowsky-type functional for a timelike rectifying strip with regard to a Lagrange multiplier.

Definition 2. *Let F_α be a timelike rectifying strip in $\mathbb{R}^3_1$. If the timelike directrix α of F_α is a critical point for the modified Sadowsky functional*

$$S_\mu(\alpha) = \int_0^\ell \left(\kappa^p \left(1+\lambda^2\right)^p - \mu\right) \upsilon dt, \tag{3.1}$$

then F_α is a timelike hyperelastic strip. Here μ is a Lagrange multiplier, standing for the length constraint.

Suppose that a reparametrized timelike curve $\alpha : [0,\ell] \to \mathbb{R}^3_1$ is the directrix of the timelike hyperelastic strip. Then, we consider a variation of γ with the variational vector field (2.5). If we compute the first variation of the functional (3.1), we arrive at

$$S_\mu(\alpha_\delta) = \int_0^\ell \left(\kappa_\delta^p \left(1+\lambda_\delta^2\right)^p - \mu\right) \upsilon_\delta ds,$$

as

$$\frac{\partial}{\partial\delta} S_\mu(\alpha_\delta)\bigg|_{\delta=0} = \frac{\partial}{\partial\delta}\ell(\alpha_\delta)\bigg|_{\delta=0} \left(\left(\kappa_\delta^p \left(1+\lambda_\delta^2\right)^p - \mu\right)\upsilon_\delta\right) + \int_0^\ell \frac{\partial}{\partial\delta}\left(\kappa_\delta^p \left(1+\lambda_\delta^2\right)^p - \mu\right)\upsilon_\delta\bigg|_{\delta=0} ds,$$

taking into consideration (2.4), we get

$$\frac{\partial}{\partial\delta} S_\mu(\alpha_\delta)\bigg|_{\delta=0} = \int_0^\ell \frac{\partial}{\partial\delta}\left(\kappa_\delta^p \left(1+\lambda_\delta^2\right)^p - \mu\right)\upsilon_\delta\bigg|_{\delta=0} ds.$$

Now from the equalities (2.6)–(2.8), we conclude that

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial\delta} S_\mu(\alpha_\delta)\bigg|_{\delta=0} = {} & \frac{1}{2}\left(\kappa^p\left(1+\lambda^2\right)^p - \mu\right)(u_1' + \kappa u_2) + \frac{p}{2}\kappa^{p-1}\left(1+\lambda^2\right)^p (u_1\kappa' - u_2\kappa^2\left(1+\lambda^2\right) \\
& -2u_3'\lambda\kappa - u_3(\lambda\kappa)' + u_2'') + p\kappa^p\lambda\left(1+\lambda^2\right)^{p-1}(u_1\lambda' \\
& +u_2\left(\frac{(\lambda\kappa)''}{\kappa^2} - \frac{(\lambda\kappa)'\kappa'}{\kappa^3} + \lambda^3\kappa - \lambda\kappa\right) + u_2'\left(2\frac{\lambda'}{\kappa} + \frac{(\lambda\kappa)'}{\kappa^2}\right) + u_2''\frac{\lambda}{\kappa} \\
& -u_3\lambda\lambda' - u_3'\left(1-\lambda^2\right) - u_3''\frac{\kappa'}{\kappa^3} + u_3'''\frac{1}{\kappa^2}).
\end{aligned}$$

As these processes progress, the integrand can be expressed as follows:

$$\frac{1}{2}\left.\frac{\partial}{\partial\delta}\right|_{\delta=0}\left(\kappa_\delta^p\left(1+\lambda_\delta^2\right)^p-\mu\right)\upsilon_\delta=u_2f_1+u_3f_2+b', \tag{3.2}$$

where

$$\begin{aligned}
f_1 :=& \left(\tfrac{p(p-1)}{2}\kappa^{p-2}\kappa'\left(1+\lambda^2\right)^p+p\left(p-1\right)\kappa^{p-1}\left(1+\lambda^2\right)^{p-1}\lambda\lambda'\right)' \\
&-\tfrac{\kappa}{2}\left(\left((p-1)\left(1+\lambda^2\right)+4p\lambda^2\right)\kappa^p\left(1+\lambda^2\right)^{p-1}+\mu\right)+\lambda\kappa\left(\tfrac{p}{2}\kappa^p\left(1+\lambda^2\right)^p\lambda\right. \\
&\left.+\left(p\kappa^{p-3}\kappa'\left(1+\lambda^2\right)^{p-1}\lambda\right)'+\left(p\kappa^{p-2}\left(1+\lambda^2\right)^{p-1}\lambda\right)''\right), \\
f_2 :=& -\left(\tfrac{p}{2}\kappa^p\left(1+\lambda^2\right)^p\lambda-\left(2p\kappa^p\left(1+\lambda^2\right)^{p-1}\lambda\right)+\left(p\kappa^{p-3}\kappa'\left(1+\lambda^2\right)^{p-1}\lambda\right)'\right. \\
&\left.+\left(p\kappa^{p-2}\left(1+\lambda^2\right)^{p-1}\lambda\right)''\right)'+\lambda\kappa\left(\tfrac{p(p-1)}{2}\kappa'\kappa^{p-2}\left(1+\lambda^2\right)^p\right. \\
&\left.+\ p\left(p-1\right)\kappa^{p-1}\left(1+\lambda^2\right)^{p-1}\lambda\lambda'\right),
\end{aligned}$$

and

$$\begin{aligned}
b :=& \ u_1\tfrac{1}{2}\left(\kappa^p\left(1+\lambda^2\right)^p-\mu\right) \\
&+u_2\left(\left(3p\lambda\lambda'\kappa^{p-1}+p\kappa^{p-2}\lambda^2\kappa'\right)\left(1+\lambda^2\right)^{p-1}-\left(\tfrac{p}{2}\kappa^{p-1}\left(3\lambda^2+1\right)\left(1+\lambda^2\right)^{p-1}\right)'\right) \\
&+u_2'\left(\tfrac{p}{2}\kappa^{p-1}\left(3\lambda^2+1\right)\left(1+\lambda^2\right)^{p-1}\right) \\
&+u_3\left(\left(-2p\kappa^p\left(1+\lambda^2\right)^{p-1}\lambda\right)+\left(p\kappa^{p-3}\kappa'\lambda\left(1+\lambda^2\right)^{p-1}\right)'+\left(p\lambda\kappa^{p-2}\left(1+\lambda^2\right)^{p-1}\right)''\right) \\
&-u_3'\left(p\kappa^{p-3}\kappa'\lambda\left(1+\lambda^2\right)^{p-1}+\left(p\kappa^{p-2}\lambda\left(1+\lambda^2\right)^{p-1}\right)'\right)+u_3''\left(p\kappa^{p-2}\lambda\left(1+\lambda^2\right)^{p-1}\right).
\end{aligned} \tag{3.3}$$

Theorem 1. *Let F_α be a timelike rectifying strip. If α is a critical point of the modified Sadowsky functional S_μ, then it satisfies the following Euler-Lagrange equations:*

$$f_1=f_2=0. \tag{3.4}$$

In addition, we find that $b'=0$ when α is a critical point of S_μ under every variation of α, which implies that the integrand of S_μ remains invariant.

Proof. Suppose that a reparametrized timelike curve α is an extremal of S_μ. From (3.2), we obtain

$$\left.\frac{\partial}{\partial\delta}S_\mu\left(\alpha_\delta\right)\right|_{\delta=0}=\int_0^\ell\left(u_2\left(s\right)f_1\left(s\right)+u_3\left(s\right)f_2\left(s\right)\right)ds+b\left(\ell\right)-b\left(0\right)=0.$$

Because $b\left(\ell\right)=b\left(0\right)=0$ for a suitable variation, the Euler-Lagrange equations are satisfied. These boundary conditions guarantee that variations at the endpoints do not affect the variation of the functional. This highlights how the geometric features, such as the positions of the endpoints, influence the system's stability and configuration. In addition, when α is a stationary point of the functional (3.1), it satisfies (3.4) in such a way that

$$\frac{1}{2}\left.\frac{\partial}{\partial\delta}\right|_{\delta=0}\left(\kappa_\delta^p\left(1+\lambda_\delta^2\right)^p-\mu\right)\upsilon_\delta=u_2f_1+u_3f_2+b'=b'=0.$$

□

Based on the conditions outlined, we now define the notion of timelike hyperelastic strips as follows.

Definition 3. *A timelike rectifying strip F_α is called a timelike hyperelastic strip if a timelike curve α serving as the directrix of F_α satisfies the Euler-Lagrange equation (3.4).*

One can see from the Eq (3.4) that planar critical points of S_μ are just timelike hyperelastic curves on a non-degenerate plane (see for torsion-free results of the hyperelastic curves from the general equations for generalized elastic curves in [1–3, 8]). Furthermore, circular helices solve the Eq (3.4).

We next provide an example of timelike hyperelastic strips.

Example 1. *Let $\psi(s) = \left(s, \sqrt{2}\cosh s, \sqrt{2}\sinh s\right)$ be a unit speed timelike helix with the curvature and the modified torsion*

$$\kappa_\psi = \sqrt{2}, \text{ and } \lambda_\psi = \frac{1}{\sqrt{2}}, \text{ [16]}. \tag{3.5}$$

The timelike rectifying strip with directrix ψ is parametrized as follows:

$$F_\psi(s,\delta) = \left(s - \frac{\sqrt{2}}{2}\delta, \sqrt{2}\cosh s, \sqrt{2}\sinh s\right). \tag{3.6}$$

Taking into consideration (3.5) in Eq (3.4), we can easily see that (3.6) is a timelike hyperelastic strip with $\mu = \left(\frac{3}{\sqrt{2}}\right)^p\left(1 - 11\frac{p}{6}\right)$ (see Figure 1).

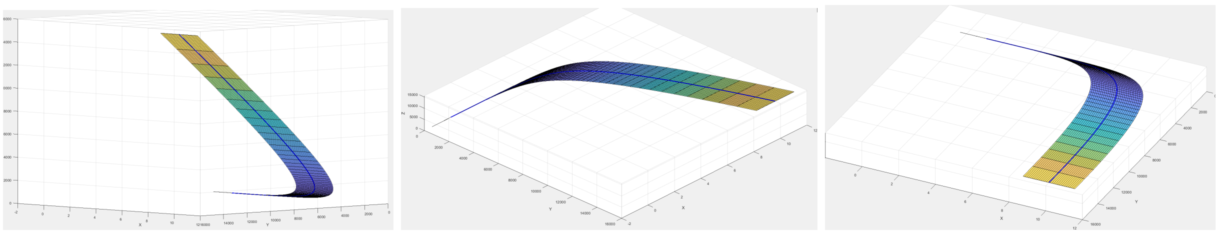

Figure 1. The timelike hyperelastic strip and its directrix corresponding to Example 1.

In Example 1, the hyperelasticity of a rectifying ruled surface is examined using a timelike curve $\psi(s)$. First, the rectifying strip, for which ψ is the base curve, is parametrized. The curvature and modified torsion data of $\psi(s)$ are applied to the Euler-Lagrange equation (3.4), and it is observed that the surface becomes a timelike hyperelastic strip when the Lagrange multiplier is chosen as $\mu = \left(\frac{3}{\sqrt{2}}\right)^p\left(1 - 11\frac{p}{6}\right)$. Moreover, it is also observed that the choice of the Lagrange multiplier μ depends on p.

4. Conservation laws of timelike hyperelastic strips

The conserved quantities derived under Lorentz symmetries naturally emerge within the context of Noether's Theorem. These quantities are essential for preserving the structural integrity and balance of the strip under deformation. The internal forces derived through Lorentz symmetries ensure the linear stability of the strip, while the moments govern its equilibrium dynamics. Furthermore, these quantities lead to conservation laws that offer a new perspective on the Euler-Lagrange equations and play a critical role in finding their solutions. In this section, we consider two different variations including only Lorentzian translations or rotations. We first find conserved quantities and then state the first and second conservation laws of timelike hyperelastic strips in $\mathbb{R}_1^3$.

Now, we consider translations in the following variation:

$$\alpha_\delta(s) = \alpha(s) + \delta\Gamma,$$

with the variation vector field

$$\dot{\alpha}_\delta(s) = \Gamma = u_1 T + u_2 N + u_3 B,$$

where Γ is an arbitrary vector in $\mathbb{R}_1^3$, $u_1 = - < \Gamma, T >$, $u_2 =< \Gamma, N >$, and $u_3 =< \Gamma, B >$. The derivatives, u_2', u_3' and u_3''. The derivatives u_2', u_3', u_3'' are calculated as

$$u_2' = \kappa < \Gamma, T > + \lambda\kappa < \Gamma, B >, \tag{4.1}$$
$$u_3' = -\lambda\kappa < \Gamma, N >, \tag{4.2}$$

and

$$u_3'' = -(\lambda\kappa)' < \Gamma, N > -\lambda\kappa^2 < \Gamma, T > -\lambda^2\kappa^2 < \Gamma, B >, \text{ [24]}. \tag{4.3}$$

Combining Eq (3.3) with the derivatives (4.1)–(4.3), we obtain

$$b =< \Gamma, W_0 >,$$

where

$$\begin{aligned} W_0 := & -\tfrac{1}{2}\left((p-1)\kappa^p\left(1+\lambda^2\right)^p + \mu\right)T \\ & + \left(\tfrac{p(p-1)}{2}\kappa^{p-2}\kappa'\left(1+\lambda^2\right)^p + p(p-1)\kappa^{p-1}\lambda\lambda'\left(1+\lambda^2\right)^{p-1}\right)N \\ & - \left(\tfrac{p}{2}\lambda\kappa^p\left(1+\lambda^2\right)^p - \left(2p\kappa^p\left(1+\lambda^2\right)^{p-1}\lambda\right)\right. \\ & \left. + \left(p\kappa^{p-3}\kappa'\lambda\left(1+\lambda^2\right)^{p-1}\right)' + \left(p\kappa^{p-2}\lambda\left(1+\lambda^2\right)^{p-1}\right)''\right)B. \end{aligned}$$

It is known that b remains constant when the timelike curve α is an extremal of the Sadowsky-type functional. Thus, W_0 is a constant for any $\Gamma \in \mathbb{R}_1^3$.

Now, we choose a variation that consists only of rotations. For $\tilde{\Gamma} \in \mathbb{R}_1^3$, $A_\delta \in SO_1(3)$, $u_1 = - < \tilde{\Gamma} \times \alpha, T >$, $u_2 =< \tilde{\Gamma} \times \alpha, N >$ and $u_3 =< \tilde{\Gamma} \times \alpha, B >$, we have

$$\left.\frac{\partial}{\partial\delta}\right|_{\delta=0} A_\delta\alpha(s) = \tilde{\Gamma} \times \alpha(s) = u_1(s)T(s) + u_2(s)N(s) + u_3(s)B(s).$$

Using a similar approach, we obtain

$$b =< \tilde{\Gamma}, W_1 >,$$

where

$$W_1 := -p\lambda\kappa^{p-1}\left(1+\lambda^2\right)^{p-1}T + \frac{1}{\kappa}\left(p\kappa^{p-1}\lambda\left(1+\lambda^2\right)^{p-1}\right)'N + \frac{p}{2}\kappa^{p-1}\left(1+\lambda^2\right)^{p-1}\left(1-\lambda^2\right)B - \alpha \times W_0$$

is a constant for timelike hyperelastic strips.

The following theorems, which describe the first and second conservation laws of timelike hyperelastic strips, demonstrate thatW_0 and W_1 determine timelike hyperelastic strips.

Theorem 2. *A rectifying strip F_α is a timelike hyperelastic strip if and only if the force vector $W_0 = a_1T + a_2N + a_3B$ is constant, where*

$$\begin{aligned} a_1 &:= -\frac{1}{2}\left((p-1)\,\kappa^p\left(1+\lambda^2\right)^p+\mu\right), && (4.4)\\ a_2 &:= \left(\frac{p\,(p-1)}{2}\kappa^{p-2}\kappa'\left(1+\lambda^2\right)^p + p\,(p-1)\,\kappa^{p-1}\lambda\lambda'\left(1+\lambda^2\right)^{p-1}\right), && (4.5)\end{aligned}$$

and

$$a_3 := -\left(\frac{p}{2}\lambda\kappa^p\left(1+\lambda^2\right)^p - \left(2p\kappa^p\left(1+\lambda^2\right)^{p-1}\lambda\right) + \left(p\kappa^{p-3}\kappa'\lambda\left(1+\lambda^2\right)^{p-1}\right)' + \left(p\kappa^{p-2}\lambda\left(1+\lambda^2\right)^{p-1}\right)''\right). \quad (4.6)$$

Proof. It suffices to show

$$W_0' = f_1N + f_2B, \quad (4.7)$$

because the vector W_0 is a constant if and only if $f_1 = f_2 = 0$ in (4.7). By using (2.1), we obtain

$$W_0' = (a_1' + \kappa a_2)T + (a_2' + \kappa a_1 - \lambda\kappa a_3)\,N + (a_3' + \lambda\kappa a_2)\,B. \quad (4.8)$$

On the other hand, from (4.4) and (4.5), we find

$$a_2 = \frac{1}{\kappa}a_1'.$$

It follows that the coefficient of T in Eq (4.8) is zero. By using (4.4)–(4.6), the coefficients of N and B are found as follows:

$$\begin{aligned} a_2' + \kappa a_1 - \lambda\kappa a_3 &= f_1, && (4.9)\\ a_3' + \lambda\kappa a_2 &= f_2, && (4.10)\end{aligned}$$

respectively. (4.9) and (4.10) show that W_0 is a constant if and only if $f_1 = f_2 = 0$. □

Theorem 3. *For a timelike hyperelastic strip F_α, the torque vector $W_1 = s_1T + s_2N + s_3B - \alpha \times W_0$ is constant, where*

$$\begin{aligned} s_1 &= -p\lambda\kappa^{p-1}\left(1+\lambda^2\right)^{p-1}, && (4.11)\\ s_2 &= \frac{1}{\kappa}\left(p\kappa^{p-1}\lambda\left(1+\lambda^2\right)^{p-1}\right)', && (4.12)\end{aligned}$$

and

$$s_3 = \frac{p}{2}\kappa^{p-1}\left(1+\lambda^2\right)^{p-1}\left(1-\lambda^2\right). \quad (4.13)$$

Moreover, if W_1 is a constant but α does not define a timelike hyperelastic strip, then $\|\gamma\|$ is conserved.

Proof. We calculate the first derivative of W_1 as follows:

$$W_1' = \underbrace{(s_1' + \kappa s_2)}_{0}T + \left(\underbrace{\kappa s_1 + s_2' - \lambda\kappa s_3}_{-a_3} - (-a_3)\right)N + \left(\underbrace{s_3' + \lambda\kappa s_2}_{a_2} - a_2\right)B - \alpha \times W_0'. \quad (4.14)$$

Substituting (4.11)–(4.13) and the derivatives s_1', s_2' and s_3' in (4.14), we see that coefficients of T, N and B are zero. Thus, Eq (4.14) can be written as follows:

$$W_1' = -\alpha \times W_0'. \tag{4.15}$$

Equation (4.15) implies that $W_1' = 0$ when W_0 is a constant, so we can see from Theorem 2, W_1 is a constant when α defines a timelike hyperelastic strip.

Now, we assume that W_1 is a constant vector field but α does not define a timelike hyperelastic strip. Using (4.7), we get

$$0 = W_1' = -\alpha \times (f_1 N + f_2 B).$$

This means that $\alpha \in Span\,\{N, B\}$. So we have

$$< \alpha, \alpha >' = 2 < \alpha, T >= 0.$$

□

5. Conclusions

The elastic behavior of developable rectifying strips is analyzed through the minimization of the Sadowsky functional. Wunderlich demonstrated that this functional is proportional to the Willmore functional, a key element in minimal surface theory. More recently, Tükel extended this work, proving that the Sadowsky-type functional is proportional to the p-Willmore functional. Additionally, Tükel and Turhan examined developable rectifying strips in Minkowski 3-space with non-null base curves and showed a similar relationship. This study focuses on examining and advancing the geometry of timelike rectifying strips, which were first introduced by Tükel and Turhan in their work [29]. It is proven, following Wunderlich's [32] approach, that for such strips, the Sadowsky-type functional is proportional to the p-Willmore functional. In this case, the Euler-Lagrange equations characterize timelike hyperelastic strips represented by the critical points of the Sadowsky-type functional. This process has been obtained through classical variational calculus.

Considering that the Sadowsky-type functional is proportional to the p-Willmore functional, it can be concluded that hyperelastic strips also serve as examples of p-Willmore surfaces. In this case, the existence of p-Willmore surfaces in Minkowski 3-space is thereby guaranteed.

In Minkowski 3-space, we derived the internal force and torque vectors for timelike hyperelastic strips using Lorentzian translations and rotations. Through these conserved quantities, we established geometric conservation laws and analyzed the structural behavior of timelike strips. This approach provides a fundamental framework that ensures the existence and stability of such surfaces. We hope that these conserved quantities will enable the definition of new types of elastic strips and establish connections between timelike hyperelastic strips and certain hyperelastic curves lying on de Sitter 2-space and pseudo-hyperbolic 2-space.

To the best of our knowledge, this is the first study that establishes the connection between timelike hyperelastic strips and p-Willmore surfaces in Minkowski 3-space through conserved geometric quantities. The use of Lorentzian transformations to derive internal force and torque equations for hyperelastic strips offers a novel variational framework, which can be extended to other pseudo-Riemannian geometries. This approach may lead to new methods for modeling elastic structures within the framework of general relativity and spacetime geometry.

Author contributions

G. Ö. Tükel and T. Turhan contributed equally to all stages of the study, including conceptualization, methodology, investigation, formal analysis, data curation, writing-original draft, writing-review and editing, visualization, supervision, and project administration.

Use of AI tools declaration

During the preparation of this work, the author used AI tools in order to improve language and readability in the abstract, introduction and conclusions sections. After using these tools/services, the authors reviewed and edited the content as needed and takes full responsibility for the content of the publication.

Conflict of interest

The authors declare no conflict of interest.

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