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Research article

On a generation of degenerate Daehee polynomials

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Abstract: Recently, probabilistic versions of certain special polynomials have been introduced, leading to the discovery of many interesting properties of these polynomials by many researchers. In this paper, we define the probabilistic degenerate Daehee polynomials, denoted by $D^{Y}_{n,\lambda}(x)$, and explore their properties along with several notable identities. We demonstrate that $D^{Y}_{n,\lambda}(x)$ and related special numbers can be expressed in terms of (degenerate) Stirling numbers of the first and second kinds, as well as falling factorial sequences.

Keywords: degenerate Daehee polynomials; probabilistic Daehee polynomials; stirling numbers of the first and the second kind; probabilistic Daehee polynomials; probabilistic Stirling numbers of the first and the second kind; probabilistic degenerate Stirling numbers of the second kind
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1. Introduction

The significance of special functions—such as Bernoulli polynomials, Euler polynomials, Hermite polynomials, the Gamma function, and others—cannot be overstated, as they play a central role in pure and applied mathematics, combinatorics, economics, engineering, mathematical physics, and related fields (see [1–3]). One notable example of such special polynomials is the *Daehee polynomials*, which are defined by the generating function

$$\frac{\log(1+t)}{t}(1+t)^{x} = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}, \text{ (see [4, 5]).}$$

When $x = 0$, the value $D_n = D_n(0)$ is known as the *Daehee numbers*.

Daehee polynomials are actively studied by many researchers, and their extensions are actively introduced. In [6], Kim et al. introduced the higher-order Daehee polynomials and numbers of the first kind and derived some interesting identities from umbral calculus. Park [7] defined the twisted Daehee polynomials with q-parameter by using p-adic invariant integral, and Cho et al. [8] extended these polynomials to the higher-order q-Daehee polynomials by using p-adic q-integral on $\mathbb{Z}_p$. Simsek [9] introduced lambda-Daehee polynomials, a generalization of Daehee polynomials, and found some interesting formulas and generating functions, including the Stirling number, falling factorial sequence, and more, see [10]. In [11], Do and Lim defined the (h, q)-Daehee polynomials, and Simsek and Ahmet [12] investigated applications on the Apostol—Daehee numbers by using classical techniques of p-adic theory. Lim [13] found a differential equation satisfied by the generating function of these polynomials and derived some interesting identities. Yun and Park defined the degenerate poly-Daehee polynomials arising from λ-umbral calculus [14], and Kim and Dolgy introduced the degenerate Catalan–Daehee polynomials of order r by using degenerate umbral calculus [15].

For $n, k \in \mathbb{N} \cup \{0\}$ with $n \geq k$, the *Stirling numbers of the first kind* and *second kind*, denoted $S_1(n, k)$, $S_2(n, k)$, respectively, are defined by the following generating functions:

$$\frac{1}{k!}\left(\log(1+t)\right)^k = \sum_{l=k}^{\infty} S_1(l,k)\frac{t^l}{l!}, \text{ and } \frac{1}{k!}\left(e^t-1\right)^k = \sum_{l=k}^{\infty} S_2(l,k)\frac{t^l}{l!}, \text{ (see [2])}. \tag{1.1}$$

For a given $\lambda \in \mathbb{R} - \{0\}$, the *degenerate exponential function* is defined by Carlitz [16] to be

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \text{ and } e_\lambda(t) = (1+\lambda t)^{\frac{1}{\lambda}}. \tag{1.2}$$

By using the degenerate exponential function, Kim et al. [17, 18] defined the *degenerate Stirling numbers of the first kind* and the *second kind* $S_{1,\lambda}(n, k)$ and $S_{2,\lambda}(n, k)$, respectively, defined as follows:

$$\frac{1}{k!}\left(\log_\lambda(1+t)\right)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!}, \text{ and } \frac{1}{k!}\left(e_\lambda(t)-1\right)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}, \tag{1.3}$$

where $\log_\lambda(t)$ is the compositional inverse of $e_\lambda(t)$ such that $e_\lambda\left(\log_\lambda(t)\right) = \log_\lambda\left(e_\lambda(t)\right) = t.$.

The *degenerate Daehee polynomials* were introduced by Kim et al. [5] to be

$$\frac{\log_\lambda(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_{n,\lambda}(x)\frac{t^n}{n!}. \tag{1.4}$$

In the special case $x = 0$, $D_{n,\lambda} = D_{n,\lambda}(0)$ are called the *degenerate Daehee numbers*.

By replacing t by $e_\lambda(t) - 1$ in (1.4), we obtain the *degenerate Bernoulli polynomials* as follows:

$$\frac{t}{e_\lambda(t)-1}e_\lambda^x(t) = \sum_{n=0}^{\infty} B_{n,\lambda}(x)\frac{t^n}{n!}, \text{(see [5, 16, 19])}.$$

In the special case of $x = 0$, $B_{n,\lambda} = B_{n,\lambda}(0)$ are called the *degenerate Bernoulli numbers*.

The *degenerate Bernoulli polynomials of the second kind of order* $r \in \mathbb{R}$ are defined as follows:

$$\left(\frac{t}{\log_\lambda(1+t)}\right)^r (1+t)^x = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}, \text{ (see [16])}. \tag{1.5}$$

When $x = 0$, $\beta_{n,\lambda}^{(r)} = \beta_{n,\lambda}^{(r)}(0)$ are called the *degenerate Bernoulli numbers of the second kind of order* r.

Let Y be a random variable with the moment generating function of Y,

$$E\left[e^{Yt}\right]=\sum_{n=0}^{\infty}E\left[Y^{n}\right]\frac{t^{n}}{n!},\ (|t|<r)\text{ exists for some } r>0,\ \text{(see [20,21])}. \tag{1.6}$$

Note that if $Y=1$, then $E\left[e^{Yt}\right]=e^{t}$.

As the generalization of the Stirling numbers of the first and the second kind, the *probabilistic Stirling numbers of the first* and *the second kind* associated with random variable Y, which were defined by Adell in [20, 21], are defined as follows:

$$\frac{1}{k!}\left(\log E\left[e^{Yt}\right]\right)^{k}=\sum_{n=k}^{\infty}(-1)^{n-k}S_{1}^{Y}(n,k)\frac{t^{n}}{n!}\text{ and }\frac{1}{k!}\left(E\left[e^{Yt}\right]-1\right)^{k}=\sum_{n=k}^{\infty}S_{2}^{Y}(n,k)\frac{t^{n}}{n!}. \tag{1.7}$$

In the special case $Y=1$, $S_{1}^{Y}(l,k)=S_{1}(n,k)$ and $S_{2}^{Y}(l,k)=S_{2}(n,k)$.

In the viewpoint of (1.6), T. Kim and D. S. Kim [22] defined the probabilistic degenerate Bell polynomials associated with random variable Y as follows,

$$\sum_{n=0}^{\infty}Bel_{n,\lambda}^{Y}(x)\frac{t^{n}}{n!}=e^{x\left(E\left[e_{\lambda}^{Y}(t)\right]-1\right)},$$

and found some interesting identities for those polynomials and their relationship to various special polynomials, such as probabilistic Stirling numbers, partial Bell polynomials, and others (see [22]). Moreover, T. Kim and D. S. Kim [23] defined the probabilistic Bernoulli and Euler polynomials $B_{n}^{Y}(x)$ and $E_{n}^{Y}(x)$, respectively, to be

$$\frac{t}{E\left[e^{Yt}\right]-1}\left(e\left[e^{Yt}\right]\right)^{x}=\sum_{n=0}^{\infty}B_{n}^{Y}(x)\frac{t^{n}}{n!}\text{ and }\frac{2}{E\left[e^{Yt}\right]+1}\left(e\left[e^{Yt}\right]\right)^{x}=\sum_{n=0}^{\infty}E_{n}^{Y}(t)\frac{t^{n}}{n!}.$$

In [24], Luo et al. defined the probabilistic degenerate Bernoulli numbers associated with random variable Y as follows:

$$\frac{t}{E\left[e_{\lambda}^{Y}(t)\right]-1}=\sum_{n=0}^{\infty}B_{n,\lambda}^{Y}\frac{t^{n}}{n!}. \tag{1.8}$$

In the special case of $Y=1$, $B_{n,\lambda}=B_{n,\lambda}^{Y}$ are the degenerate Bernoulli numbers.

Recently, probabilistic versions of various special functions have been defined and their properties and applications are being actively studied by many researchers (see [22–26]). In particular, Kim et al. [26] defined the probabilistic degenerate Stirling numbers of the second kind as follows:

$$\frac{1}{k!}\left(E\left[e_{\lambda}^{Y}(t)\right]-1\right)^{k}=\sum_{n=k}^{\infty}S_{2,\lambda}^{Y}(n,k)\frac{t^{n}}{n!}.$$

In this paper, we define probabilistic degenerate Daehee polynomials, a probabilistic approach to degenerate Daehee polynomials, and investigate properties of those polynomials and derive some interesting identities. Moreover, we show that the probabilistic degenerate Daehee polynomials are closely related to the (degenerate) Stirling numbers of the first and the second kind, rising factorial sequences and binomial coefficients, higher order degenerate Bernoulli polynomials of the second kind, and degenerate Bernoulli polynomials when Y is a specific random variable.

2. Probabilistic degenerate Daehee polynomials

Let $\left(Y_j\right)_{j\in\mathbb{N}}$ be the sequence of mutually independent copies of the random variable Y, and let

$$S_0 = 0,\ S_k = Y_1 + Y_2 + \cdots + Y_k,\ k \in \mathbb{N}.$$

In the viewpoint of (1.8), we define the *probabilistic degenerate Daehee polynomials* by the generating function to be

$$\frac{\log_\lambda(1+t)}{t}\left(E\left[e^{Y\log_\lambda(1+t)}\right]\right)^x = \sum_{n=0}^{\infty} D_{n,\lambda}^{Y}(x)\frac{t^n}{n!}. \tag{2.1}$$

Note that if we put $Y = 1$ and $\lambda \to 0$, then $D_{n,\lambda}^{Y}(x) = D_n(x)$.

Since

$$\begin{aligned}\left(E\left[e^{Y\log_\lambda(1+t)}\right]\right)^x &= \left(E\left[e^{Y\log_\lambda(1+t)}\right] - 1 + 1\right)^x \\ &= \sum_{n=0}^{\infty}(x)_n\frac{1}{n!}\left(E\left[e^{Y\log_\lambda(1+t)}\right] - 1\right)^n \\ &= \sum_{n=0}^{\infty}(x)_n\sum_{l=n}^{\infty} S_2^{Y}(l,n)\frac{1}{l!}\left(\log_\lambda(1+t)\right)^l \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{r=0}^{l} S_2^{Y}(n-r,n-l)S_{1,\lambda}(n,n-r)(x)_{n-l}\right)\frac{t^n}{n!},\end{aligned} \tag{2.2}$$

by (2.1), we obtain

$$\begin{aligned}&\frac{\log_\lambda(1+t)}{t}\left(E\left[e^{Y\log_\lambda(1+t)}\right]\right)^x \\ &= \left(\sum_{n=0}^{\infty} D_{n,\lambda}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{r=0}^{l} S_2^{Y}(n-r,n-l)S_{1,\lambda}(n,n-r)(x)_{n-l}\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{r=0}^{l}\binom{n}{m}S_2^{Y}(m-r,m-l)S_{1,\lambda}(m,m-r)D_{n-m,\lambda}(x)_{m-l}\right)\frac{t^n}{n!},\end{aligned} \tag{2.3}$$

where $(x)_0 = 1$ and $(x)_n = x(x-1)\cdots(x-(n-1))$, $n \geq 1$ are the falling factorial sequences.

Moreover,

$$\begin{aligned}\frac{t}{\log_\lambda(1+t)}\sum_{n=0}^{\infty} D_{n,\lambda}^{Y}(x)\frac{t^n}{n!} &= \left(\sum_{n=0}^{\infty}\beta_{n,\lambda}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} D_{n,\lambda}^{Y}(x)\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\beta_{m,\lambda}D_{n-m,\lambda}^{Y}(x)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.4}$$

By (2.2), (2.3), and (2.7), we obtain the following theorem.

Theorem 2.1. *For each $n \in \mathbb{N} \cup \{0\}$, we have*

$$D_{n,\lambda}^{Y}(x) = \sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{r=0}^{l}\binom{n}{m}S_{2}^{Y}(m-r,m-l)S_{1,\lambda}(m,m-r)D_{n-m,\lambda}(x)_{m-l}$$

and

$$\sum_{m=0}^{n}\binom{n}{m}\beta_{m,\lambda}D_{n-m,\lambda}^{Y}(x) = \sum_{l=0}^{n}\sum_{r=0}^{l}S_{2}^{Y}(n-r,n-l)S_{1,\lambda}(n,n-r)(x)_{n-l}.$$

Since

$$t\sum_{n=0}^{\infty}D_{n,\lambda}^{Y}(x)\frac{t^n}{n!} = \sum_{n=1}^{\infty}nD_{n-1,\lambda}^{Y}(x)\frac{t^n}{n!}, \tag{2.5}$$

by (2.2), we obtain

$$\begin{aligned}
&\log_{\lambda}(1+t)\left(E\left[e^{Y\log_{\lambda}(1+t)}\right]\right)^{x}\\
=&\left(\sum_{n=1}^{\infty}(\lambda-1)_{n-1}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{r=0}^{l}S_{2}^{Y}(n-r,n-l)S_{1,\lambda}(n,n-r)(x)_{n-l}\frac{t^n}{n!}\right)\\
=&\sum_{n=1}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{r=0}^{l}\binom{n}{m}S_{2}^{Y}(m-r,m-l)S_{1,\lambda}(m,m-r)\ (\lambda-1)_{n-m-1}(x)_{m-l}\right)\frac{t^n}{n!}\ .
\end{aligned} \tag{2.6}$$

By (2.5) and (2.6), we obtain the following corollary.

Corollary 2.2. *For each $n \in \mathbb{N}$, we have*

$$nD_{n-1,\lambda}^{Y}(x) = \sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{r=0}^{l}\binom{n}{m}S_{2}^{Y}(m-r,m-l)S_{1,\lambda}(m,m-r)(\lambda-1)_{n-m-1}(x)_{m-l}.$$

Replacing t by $e_{\lambda}(t)-1$ in (2.1), we obtain

$$\begin{aligned}
\frac{t}{e_{\lambda}(t)-1}\left(E\left[e^{Yt}\right]\right)^{x} &= \frac{t}{e_{\lambda}(t)-1}\left(E\left[e^{Yt}\right]-1+1\right)^{x}\\
&=\left(\sum_{n=0}^{\infty}B_{n,\lambda}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}(x)_{m}S_{2}^{Y}(n,m)\frac{t^n}{n!}\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\sum_{m=0}^{r}\binom{n}{r}S_{2}^{Y}(r,m)B_{n-r,\lambda}(x)_{m}\right)\frac{t^n}{n!},
\end{aligned} \tag{2.7}$$

and

$$\sum_{n=0}^{\infty}D_{n,\lambda}^{Y}(x)\frac{1}{n!}\left(e_{\lambda}(t)-1\right)^{n} = \sum_{n=0}^{\infty}D_{n,\lambda}^{Y}(x)\sum_{l=n}^{\infty}S_{2,\lambda}(l,n)\frac{t^l}{l!} = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}S_{2,\lambda}(n,m)D_{m,\lambda}^{Y}(x)\right)\frac{t^n}{n!}. \tag{2.8}$$

By (2.7) and (2.8), we obtain the following theorem.

Theorem 2.3. *For each $n \in \mathbb{N} \cup \{0\}$, we have*

$$\sum_{m=0}^{n} S_{2,\lambda}(n,m) D^{Y}_{m,\lambda}(x) = \sum_{r=0}^{n}\sum_{m=0}^{r}\binom{n}{r} S^{Y}_{2}(r,m) B_{n-r,\lambda}(x)_{m}.$$

If we take $\lambda \to 0$ in (1.8), we obtain

$$\frac{t}{E\left[e^{Yt}\right]-1} = \sum_{n=0}^{\infty} B^{Y}_{n}\frac{t^{n}}{n!}, \tag{2.9}$$

and so, by replacing t by $\log_{\lambda}(1+t)$ in (2.9), we have

$$\frac{\log_{\lambda}(1+t)}{E\left[e^{Y\log_{\lambda}(1+t)}\right]-1} = \sum_{n=0}^{\infty} B^{Y}_{n}\frac{1}{n!}\left(\log_{\lambda}(1+t)\right)^{n} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} B^{Y}_{m} S_{1,\lambda}(n,m)\frac{t^{n}}{n!}. \tag{2.10}$$

Note that

$$\begin{aligned}
\sum_{n=0}^{m}\left(E\left[e^{Y\log_{\lambda}(1+t)}\right]\right)^{n} =& \frac{\left(E\left[e^{Y\log_{\lambda}(1+t)}\right]\right)^{m+1}-1}{E\left[e^{Y\log_{\lambda}(1+t)}\right]-1}\\
=&\frac{1}{t}\frac{\log_{\lambda}(1+t)}{E\left[e^{Y\log_{\lambda}(1+t)}\right]-1}\left(\frac{t}{\log_{\lambda}(1+t)}\right)^{2}\frac{\log_{\lambda}(1+t)}{t}\left(\left(E\left[e^{Y\log_{\lambda}(1+t)}\right]\right)^{m+1}-1\right)\\
=&\frac{1}{t}\left(\sum_{n=0}^{\infty}\sum_{m=0}^{n} B^{Y}_{m} S_{1,\lambda}(n,m)\frac{t^{n}}{n!}\right)\left(\sum_{n=0}^{\infty}\beta^{(2)}_{n,\lambda}\frac{t^{n}}{n!}\right)\\
&\times\left(\sum_{n=0}^{\infty}\left(D^{Y}_{n,\lambda}(m+1)-D_{n,\lambda}\right)\frac{t^{n}}{n!}\right)\\
=&\sum_{n=1}^{\infty}\left(\sum_{a=0}^{n}\sum_{b=0}^{a}\sum_{m=0}^{b}\binom{n}{a}\binom{a}{b} S_{1,\lambda}(b,m) B^{Y}_{m}\beta^{(2)}_{a-b,\lambda}\left(D^{Y}_{n-a,\lambda}(m+1)-D_{n-a,\lambda}\right)\right)\frac{t^{n-1}}{n!}\\
=&\sum_{n=0}^{\infty}\left(\sum_{a=0}^{n+1}\sum_{b=0}^{a}\sum_{m=0}^{b}\binom{n+1}{a}\binom{a}{b}\frac{S_{1,\lambda}(b,m) B^{Y}_{m}\beta^{(2)}_{a-b,\lambda}}{n+1}\right.\\
&\left.\times\left(D^{Y}_{n-a+1,\lambda}(m+1)-D_{n-a+1,\lambda}\right)\right)\frac{t^{n}}{n!},
\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
\sum_{n=0}^{m}\left(E\left[e^{Y\log_{\lambda}(1+t)}\right]\right)^{n} &= \sum_{n=0}^{m} E\left[e^{S_{n}\log_{\lambda}(1+t)}\right]\\
&=\sum_{n=0}^{\infty}\sum_{a=0}^{m} E\left[(S_{a})^{n}\right]\frac{1}{n!}\left(\log_{\lambda}(1+t)\right)^{n}\\
&=\sum_{n=0}^{\infty}\left(\sum_{b=0}^{n}\sum_{a=0}^{m} S_{1,\lambda}(n,b) E\left[(S_{a})^{b}\right]\right)\frac{t^{n}}{n!}.
\end{aligned} \tag{2.12}$$

By (2.11) and (2.12), we obtain the following theorem.

Theorem 2.4. *For each $n \in \mathbb{N} \cup \{0\}$, we have*

$$\sum_{b=0}^{n} \sum_{a=0}^{m} S_{1,\lambda}(n,b) E\left[(S_a)^b\right] = \sum_{a=0}^{n+1} \sum_{b=0}^{a} \sum_{m=0}^{b} \binom{n+1}{a} \binom{a}{b} \frac{S_{1,\lambda}(b,m) B_m^Y \beta_{a-b,\lambda}^{(2)}}{n+1} \left(D_{n-a+1,\lambda}^Y(m+1) - D_{n-a+1,\lambda}\right).$$

Let Y be a uniform distribution on $[0,\ 1]$, which is a continuous random variable on $[0,\ 1]$, denoted by $Y \sim U(0,1)$ with probability density function

$$f(y) = 1,\ 0 \le y \le 1, f \text{ (see [25,27])}.$$

Then

$$E\left[e^{Y \log_\lambda(1+t)}\right] = \int_0^1 e^{y \log_\lambda(1+t)} dy = \frac{e^{\log_\lambda(1+t)} - 1}{\log_\lambda(1+t)},\ \text{(see [25,27])}. \tag{2.13}$$

Let $m \in \mathbb{N}$. By (2.13), we obtain

$$\begin{aligned}
&\frac{\log_\lambda(1+t)}{t} \left(E\left[e^{Y \log_\lambda(1+t)}\right]\right)^m \\
=&\frac{\log_\lambda(1+t)}{t} \left(\frac{e^{\log_\lambda(1+t)} - 1}{\log_\lambda(1+t)}\right)^m \\
=&\frac{1}{t} \left(\log_\lambda(1+t)\right)^{1-m} m! \frac{1}{m!} \left(e^{\log_\lambda(1+t)} - 1\right)^m \\
=&\frac{1}{t} \sum_{l=m}^{\infty} S_2(l,m) \frac{m!}{l!} \left(\log_\lambda(1+t)\right)^{l-m+1} \\
=&\frac{1}{t} \sum_{l=m}^{\infty} S_2(l,m) \frac{l+1}{\binom{l+1}{m}} \sum_{r=l+1-m}^{\infty} S_{1,\lambda}(r, l+1-m) \frac{t^r}{r!} \\
=&\sum_{l=1}^{\infty} \sum_{r=1}^{l} S_2(l-r+m, m) \frac{l-r+m+1}{\binom{l-r+m+1}{m}} S_{1,\lambda}(l, l-r+1) \frac{t^{l-1}}{l!} \\
=&\sum_{l=0}^{\infty} \left(\sum_{r=1}^{l+1} \frac{(l-r+m+2) S_2(l-r+m+1, m) S_{1,\lambda}(l+1, l-r+2)}{(l+1)\binom{l-r+m+2}{m}}\right) \frac{t^l}{l!}.
\end{aligned} \tag{2.14}$$

By (2.14), we obtain the following theorem.

Theorem 2.5. *Let $Y \sim U(0,1)$ be the uniform distribution, and let $m \in \mathbb{N}$. For each $n \in \mathbb{N} \cup \{0\}$, we have*

$$D_{n,\lambda}^Y(m) = \sum_{r=1}^{l+1} \frac{(l-r+m+2) S_2(l-r+m+1, m) S_{1,\lambda}(l+1, l-r+2)}{(l+1)\binom{l-r+m+2}{m}}.$$

Let Y be the binomial distribution with parameter m and p. The probability density function of Y is

$$f(y) = \binom{m}{y} p^y (1-p)^{m-y},\ \text{(see [25,27])},$$

and thus

$$E\left[e^{Y\log_\lambda(1+t)}\right] = \sum_{y=0}^{m} e^{y\log_\lambda(1+t)}\binom{m}{y}p^y(1-p)^{m-y} = \left(p\left(e^{\log_\lambda(1+t)}-1\right)+1\right)^m. \tag{2.15}$$

By (2.15), we obtain

$$\begin{aligned}
&\frac{\log_\lambda(1+t)}{t}\left(E\left[e^{Y\log_\lambda(1+t)}\right]\right)^x\\
=&\frac{\log_\lambda(1+t)}{t}\left(p\left(e^{\log_\lambda(1+t)}-1\right)+1\right)^{mx}\\
=&\frac{\log_\lambda(1+t)}{t}\sum_{n=0}^{\infty}(mx)_n p^n\frac{1}{n!}\left(e^{\log_\lambda(1+t)}-1\right)^n\\
=&\frac{1}{t}\sum_{n=0}^{\infty}(mx)_n p^n\sum_{l=n}^{\infty}S_2(l,n)\frac{1}{l!}\left(\log_\lambda(1+t)\right)^{l+1}\\
=&\frac{1}{t}\sum_{n=0}^{\infty}(mx)_n p^n\sum_{l=n}^{\infty}(l+1)S_2(l,n)\sum_{r=l+1}^{\infty}S_{1,\lambda}(r,l+1)\frac{t^r}{r!}\\
=&\sum_{n=1}^{\infty}\sum_{l=1}^{n}\sum_{r=1}^{l}(mx)_{n-l}p^{n-l}(n-r+1)S_2(n-r,n-l)S_{1,\lambda}(n,n-r+1)\frac{t^{n-1}}{n!}\\
=&\sum_{n=0}^{\infty}\sum_{l=1}^{n+1}\sum_{r=1}^{l}\frac{(n-r+2)(mx)_{n-l+1}p^{n-l+1}}{n+1}\\
&\times S_2(n-r+1,n-l+1)S_{1,\lambda}(n+1,n-r+2)\frac{t^n}{n!}.
\end{aligned} \tag{2.16}$$

By (2.16), we obtain the following theorem.

Theorem 2.6. *Let Y be the binomial distribution with parameters m and p. For each $n \in \mathbb{N}\cup\{0\}$, we have*

$$\begin{aligned}
D_{n,\lambda}^{Y}(x) = &\sum_{l=1}^{n+1}\sum_{r=1}^{l}\frac{(n-r+2)p^{n-l+1}}{n+1}\\
&\times S_2(n-r+1,n-l+1)S_{1,\lambda}(n+1,n-r+2)(mx)_{n-l+1}.
\end{aligned}$$

In particular,

$$\begin{aligned}
(n+1)D_{n,\lambda}^{Y}(1) = &\sum_{l=1}^{n+1}\sum_{r=1}^{l}(n-r+2)p^{n-l+1}(m)_{n-l+1}\\
&\times S_2(n-r+1,n-l+1)S_{1,\lambda}(n+1,n-r+2).
\end{aligned}$$

Let Y be the Bernoulli random variable with the success possibility p. The probability density function of Y is

$$f(y) = p^y(1-p)^{1-y},\ \text{(see [25,27])},$$

and so

$$E\left[e^{Y\log_\lambda(1+t)}\right] = \sum_{y\in\{0,1\}} e^{y\log_\lambda(1+t)} p^y(1-p)^{1-y} = p\left(e^{\log_\lambda(1+t)}-1\right)+1. \tag{2.17}$$

By (2.17), we obtain

$$\begin{aligned}
&\frac{\log_\lambda(1+t)}{t}\left(E\left[e^{Y\log_\lambda(1+t)}\right]\right)^x \\
=&\frac{\log_\lambda(1+t)}{t}\left(p\left(e^{\log_\lambda(1+t)}-1\right)+1\right)^x \\
=&\frac{\log_\lambda(1+t)}{t}\sum_{n=0}^{\infty}(x)_n p^n \frac{1}{n!}\left(e^{\log_\lambda(1+t)}-1\right)^n \\
=&\frac{1}{t}\sum_{n=0}^{\infty}(x)_n p^n \sum_{l=n}^{\infty}(l+1)S_2(l,n)\frac{1}{(l+1)!}\left(\log_\lambda(1+t)\right)^{l+1} \\
=&\sum_{n=0}^{\infty}(x)_n p^n \sum_{l=n}^{\infty}(l+1)S_2(l,n)\sum_{r=l+1}^{\infty}S_{1,\lambda}(r,l+1)\frac{t^{r-1}}{r!} \\
=&\sum_{n=1}^{\infty}\sum_{l=1}^{n}\sum_{r=1}^{l}(n-r+1)p^{n-l}S_2(n-r,n-l)S_{1,\lambda}(n,n-r+1)(x)_{n-l}\frac{t^{n-1}}{n!} \\
=&\sum_{n=0}^{\infty}\left(\sum_{l=1}^{n+1}\sum_{r=1}^{l}\frac{(n-r+2)p^{n-l+1}}{n+1}\right. \\
&\left.\times S_2(n-r+1,n-l+1)S_{1,\lambda}(n+1,n-r+2)(x)_{n-l+1}\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.18}$$

By (2.18), we obtain the following theorem.

Theorem 2.7. *Let Y be the binomial distribution with parameter m and p. For each $n \in \mathbb{N}\cup\{0\}$, we have*

$$\begin{aligned}
D_{n,\lambda}^{Y}(x) = &\sum_{l=1}^{n+1}\sum_{r=1}^{l}\frac{(n-r+2)p^{n-l+1}}{n+1} \\
&\times S_2(n-r+1,n-l+1)S_{1,\lambda}(n+1,n-r+2)(x)_{n-l+1}.
\end{aligned}$$

Let Y be the Poisson random variable with parameter $\alpha > 0$. The probability density function of Y is

$$f(y) = \frac{e^{-\alpha}\alpha^y}{y!}, \text{ (see [27])},$$

and so

$$E\left[e^{Y\log_\lambda(1+t)}\right] = \sum_{y=0}^{\infty} e^{y\log_\lambda(1+t)}\frac{e^{-\alpha}\alpha^y}{y!} = e^{-\alpha}\sum_{y=0}^{\infty}\frac{\left(\alpha e^{\log_\lambda(1+t)}\right)^y}{y!} = e^{\alpha\left(e^{\log_\lambda(1+t)}-1\right)}. \tag{2.19}$$

By (2.19), we obtain

$$
\begin{aligned}
&\frac{\log_{\lambda}(1+t)}{t}\left(E\left[e^{Y\log_{\lambda}(1+t)}\right]\right)^{x}\\
=&\frac{\log_{\lambda}(1+t)}{t}e^{\alpha x\left(e^{\log_{\lambda}(1+t)}-1\right)}\\
=&\frac{\log_{\lambda}(1+t)}{t}\sum_{n=0}^{\infty}\alpha^{n}x^{n}\frac{1}{n!}\left(e^{\log_{\lambda}(1+t)}-1\right)^{n}\\
=&\frac{1}{t}\sum_{n=0}^{\infty}\alpha^{n}x^{n}\sum_{l=n}^{\infty}(l+1)S_{2}(l,n)\frac{1}{(l+1)!}\left(\log_{\lambda}(1+t)\right)^{l+1}\\
=&\sum_{n=0}^{\infty}\alpha^{n}x^{n}\sum_{l=n}^{\infty}(l+1)S_{2}(l,n)\sum_{r=l+1}^{\infty}S_{1,\lambda}(r,l+1)\frac{t^{r-1}}{r!}\\
=&\sum_{n=1}^{\infty}\sum_{l=1}^{n}\sum_{r=1}^{l}\alpha^{n-l}(n-r+1)S_{2}(n-r,n-l)S_{1,\lambda}(n,n-r+1)x^{n-l}\frac{t^{n-1}}{n!}\\
=&\sum_{n=0}^{\infty}\left(\sum_{l=1}^{n+1}\sum_{r=1}^{l}\frac{\alpha^{n-l+1}(n-r+2)}{n+1}\right.\\
&\left.\times S_{2}(n-r+1,n-l+1)S_{1,\lambda}(n+1,n-r+2)x^{n-l+1}\right)\frac{t^{n}}{n!}.
\end{aligned}
\tag{2.20}
$$

By (2.20), we obtain the following theorem.

Theorem 2.8. *Let Y be the Poisson random variable with parameter $\alpha > 0$. For each $n \in \mathbb{N} \cup \{0\}$, we have*

$$
\begin{aligned}
D_{n,\lambda}^{Y}(x)=&\sum_{l=1}^{n+1}\sum_{r=1}^{l}\frac{\alpha^{n-l+1}(n-r+2)}{n+1}\\
&\times S_{2}(n-r+1,n-l+1)S_{1,\lambda}(n+1,n-r+2)x^{n-l+1}.
\end{aligned}
$$

In particular

$$
(n+1)D_{n,\lambda}^{Y}(1)=\sum_{l=1}^{n+1}\sum_{r=1}^{l}\alpha^{n-l+1}(n-r+2)S_{2}(n-r+1,n-l+1)S_{1,\lambda}(n+1,n-r+2).
$$

Let Y be the gamma random variable with parameters α, β with probability density function

$$
f(y)=\begin{cases}\beta e^{-\beta y}\frac{(\beta y)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } y\geq 0,\\ 0, & \text{if } y<0,\end{cases}
$$

and denoted by $Y \sim \Gamma(\alpha,\beta)$ (see [23, 25, 27]).

Let $Y \sim \Gamma(1, 1)$. Then

$$
E\left[e^{Y\log_{\lambda}(1+t)}\right]=\int_{0}^{\infty}e^{y\log_{\lambda}(1+t)}e^{-y}dy=\frac{1}{1-\log_{\lambda}(1+t)},
\tag{2.21}
$$

and thus, by (2.21), we obtain

$$\begin{aligned}
&\frac{\log_\lambda(1+t)}{t}\left(E\left[e^{Y\log_\lambda(1+t)}\right]\right)^x\\
=&\frac{\log_\lambda(1+t)}{t}\left(1-\log_\lambda(1+t)\right)^{-x}\\
=&\frac{1}{t}\sum_{n=0}^{\infty}(n+1)<x>_n\frac{1}{(n+1)!}\left(\log_\lambda(1+t)\right)^{n+1}\\
=&\frac{1}{t}\sum_{n=0}^{\infty}(n+1)<x>_n\sum_{l=n+1}^{\infty}S_{1,\lambda}(l,n+1)\frac{t^l}{l!}\\
=&\sum_{n=1}^{\infty}\sum_{l=1}^{n}(n-l+1)<x>_{n-l}S_{1,\lambda}(n,n-l+1)\frac{t^{n-1}}{n!}\\
=&\sum_{n=0}^{\infty}\left(\sum_{l=1}^{n+1}\frac{(n-l+2)<x>_{n-l+1}S_{1,\lambda}(n+1,n-l+2)}{n+1}\right)\frac{t^n}{n!},
\end{aligned}\tag{2.22}$$

where $<x>_0=1$ and $<x>_n=x(x+1)\cdots(x-(n-1))$, $n\geq 1$ are the rising factorial sequences.

By (2.22), we obtain the following theorem.

Theorem 2.9. *Let $Y\sim\Gamma(1,1)$ be the gamma random variable with parameters 1 and 1. For each $n\in\mathbb{N}\cup\{0\}$, we have*

$$D_{n,\lambda}^Y(x)=\sum_{l=1}^{n+1}\frac{(n-l+2)S_{1,\lambda}(n+1,n-l+2)}{n+1}<x>_{n-l+1}.$$

In particular,

$$(n+1)D_{n,\lambda}^Y(1)=\sum_{l=1}^{n+1}S_{1,\lambda}(n+1,n-l+2)(n-l+2)!.$$

3. Conclusions

Special polynomials play a significant role in various fields, including pure and applied mathematics, engineering, economics, and mathematical physics. In particular, the Daehee polynomials and numbers, introduced by T. Kim, have been shown to be closely related to many special polynomials and numbers, and their applications have been extensively studied.

In this paper, we introduce probabilistic degenerate Daehee polynomials, a probabilistic analogue of the classical Daehee polynomials, and investigate their properties by deriving various new and interesting identities. We show that these polynomials are closely related to the (degenerate) Stirling numbers of the first and second kinds, rising factorial sequences, higher-order degenerate Bernoulli polynomials of the second kind, and degenerate Bernoulli polynomials.

As further research, we believe that if the useful tools used in this paper are applied to the expansion of various special functions, such as poly-Daehee polynomials using polyexponential functions, new and interesting identities can be found.

Author contributions

Both authors of this article have been contributed equally. Both authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare no conflicts of interest.

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