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Research article

The Chevalley-Weil formula on nodal curves

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Abstract: In this paper, we studied the eigenspace of the regular differentials on a connected nodal curve X under the action of a finite automorphism group G. We proved that the dimension of the space of G-invariant regular differentials is the arithmetic genus of the quotient nodal curve X/G. In addition, we generalized the Chevalley-Weil formula to nodal curves in the case where X/G is smooth and gave some examples.

Keywords: Chevalley-Weil formula; nodal curves; regular differentials; G-invariant; characters
Mathematics Subject Classification: 14H05, 14H20, 14H37

1. Introduction

Let X be a connected projective smooth curve over an algebraically closed field k and $G \subseteq \mathrm{Aut}(X)$ be a finite subgroup. Then G acts in a natural way on the space of the holomorphic differential forms on X, and thus we obtain a k-linear representation $G \to \mathrm{GL}\left(H^0(X, \Omega_X)\right)$. In other words, $H^0(X, \Omega_X)$ is a $k[G]$-module.

If we want to study the $k[G]$-module structure on $H^0(X, \Omega_X)$, a basic problem is to determine the multiplicity of each indecomposable representation on it. The group algebra $k[G]$ is reductive (or we say semisimple) if and only if the characteristic $\mathrm{char}(k)$ is either zero or coprime with the order of G; in this case the indecomposable representations of G are exactly its irreducible representations.

The Chevalley-Weil formula tells us that the multiplicity of each irreducible representation can be characterized by the genus of the quotient curve X/G and the ramification data in the quotient map $\pi : X \to X/G$, which is a branched (or ramified) cover.

1.1. Origins and development of the Chevalley-Weil formula

The study of the Chevalley-Weil formula can be traced back to the late 19th century when Hurwitz [8] studied finite cyclic automorphism groups on compact Riemann surfaces ($k = \mathbb{C}$). Later, in the 1930s, Chevalley and Weil [2] calculated the irreducible multiplicities for a general finite group

G in the unramified case of π. Shortly after, Weil independently resolved the ramified case [18]. This result is therefore named the Chevalley-Weil formula, and the formula also holds for the characteristic p case with $k[G]$ semisimple [9].

When $\mathrm{char}(k) = p \mid \#G$, the structure of $H^0(X, \Omega_X)$ becomes more complicated, so people usually impose some restrictions on the cover π and the group G. There has been some research on the case where the ramified cover is tamely ramified [9, 16] or weakly ramified [10]. In addition, there has been research on some special automorphism groups, such as cyclic groups [17], finite p-groups [5], or groups with cyclic Sylow subgroups [1]. There are also some studies over perfect fields [12].

For general theory, in 1980, Ellingsrud and Lønsted [4] used the language of equivariant K-theory to study the Lefschetz trace of coherent G-sheaves on projective varieties for $k[G]$ semisimple. They obtained precise results of the $k[G]$-module structure on $H^0(X, \mathscr{L})$ where $\mathscr{L}$ is an invertible G-sheaf and X is a smooth projective curve [4, Theorem 3.8], which is a generalization of the Chevalley-Weil formula.

For higher dimensions, Nakajima gave some basic results in the 1980s. Consider a finite étale Galois cover $f : X \to Y$ between two projective algebraic varieties over a field k with $G = \mathrm{Gal}(X/Y)$. Then for any coherent G-sheaf $\mathcal{F}$, there exists a finitely generated free $k[G]$-module complex

$$L^\bullet : \ 0 \to L^0 \to L^1 \to \cdots \to L^m \to 0,$$

such that $H^i(X, \mathcal{F}) \simeq H^i(L^\bullet)$ as $k[G]$-modules [15]. The approach basically follows Mumford's method in [14, II.5 Lemma 1].

When the étale condition on the cover f is replaced by the requirement that f be tamely ramified, the result will be weaker: the above finitely generated free $k[G]$-module complex is just a finitely generated projective $k[G]$-module complex [16]. It leads to a corollary that $H^n(X, \mathcal{F})$ is $k[G]$-projective when $H^i(X, \mathcal{F}) = 0$ for all other $i \neq n$. Now for a non-empty finite G-invariant set S in the curve X, we have $H^1(X, \Omega_X(S)) = 0$ (and therefore $H^i(X, \Omega_X(S)) = 0$ for all $i > 0$). As a direct consequence of the above result, when π is tamely ramified, the logarithmic differential space $H^0(X, \Omega_X(S))$ is a projective $k[G]$-module. It should be noted that the field above need not be algebraically closed.

Almost at the same time, Kani described $H^0(X, \Omega_X)$ through the study of logarithmic differential space $H^0(X, \Omega_X(S))$, also obtaining that $H^0(X, \Omega_X(S))$ is projective [9]. However, at the end of his paper, Kani also pointed out that most of his results were covered by Nakajima's work. Nevertheless, Kani's proof process is more precise and also provides valuable tools for the main results of this paper.

For smooth curves, the Chevalley-Weil formula was well understood by now, but no attempt has been made for singular curves. The case of a nodal curve is a good frame in which to generalize the Chevalley-Weil formula.

1.2. Key findings and goals

Now let X be a connected projective nodal curve over an algebraically closed field k, $G \subseteq \mathrm{Aut}(X)$ is a finite subgroup of order n with $k[G]$ semisimple, and $\pi : X \to Y = X/G$ is the quotient map. Then Y is also a nodal curve.

In Section 2.1, we demonstrate that the regular differentials of nodal curves are suitable generalizations of the holomorphic differentials. A regular differential $\varphi \in H^0(X, \omega_X)$ is essentially a log differential in $H^0(\hat{X}, \Omega_{\hat{X}}(\hat{S}_X))$ that satisfies certain residue relations, where $\hat{S}_X$ is the preimage of the singular points under the normalization $\hat{X} \to X$. Therefore G acts (right) naturally on the regular differential space $H^0(X, \omega_X)$. Here ω_X is the dualizing sheaf of X.

In this paper, we set the goal to calculate the multiplicity of every character $\chi : G \to k^\times$, that is, the dimension of the eigenspace:

$$H^0(X, \omega_X)_\chi := \{\varphi \in H^0(X, \omega_X) \mid \sigma\varphi = \chi(\sigma)\varphi,\ \forall \sigma \in G\}.$$

Remark 1.1. *Every* 1*-dimensional representation is its own character. Although characters are the simplest irreducible representations, it still contains a lot of information that we are concerned about. There are mainly two reasons why we study it:*

- *When G is cyclic (or abelian), all irreducible representations of G are* 1*-dimensional;*
- *The space of G-invariant regular differentials is just the eigenspace of trivial character $1_G : G \to k^\times$, namely $H^0(X, \omega_X)^G = H^0(X, \omega_X)_{1_G}$.*

Recall that when X is smooth, every G-invariant holomorphic differential on X is the pullback of a holomorphic differential on the quotient curve $Y = X/G$ (see Proposition 3.2). Consequently,

$$\dim_k H^0(X, \Omega_X)^G = g_Y,$$

where g_Y denotes the (geometric) genus of Y.

When X is a nodal curve, we establish that

$$H^0(X, \omega_X)^G = \pi^* H^0(Y, \omega_Y),$$

proven for the irreducible case in Proposition 3.5 and extended to the general case (where X has multiple irreducible components) in Proposition 4.3. This leads to

$$\dim_k H^0(X, \omega_X)^G = p_a(Y),$$

where $p_a(Y)$ is the arithmetic genus of Y. This result generalizes the smooth case mentioned earlier.

For an arbitrary character $\chi : G \to k^\times$, when X is smooth, we have:

$$\dim_k H^0(X, \Omega_X)_\chi = g_Y - 1 + m_\chi + \langle \chi, 1_G \rangle.$$

In this expression:

- g_Y is the genus of Y.
- The term m_χ is defined as

$$m_\chi = \sum_{Q \in Y} \left\lfloor \frac{e_Q - 1}{e_Q} + \frac{1}{n} \langle \chi, R_{G,Q} \rangle_G \right\rfloor - \frac{1}{n} \langle \chi, R_G \rangle_G .$$

  Here:

  - The sum runs over all points $Q \in Y$.
  - $e_Q := e_P$ is the ramification index at any $P \in \pi^{-1}(Q)$. Note that $e_Q > 1$ if and only if Q is a branch point, with $R_{G,Q} = 0$ at all other points; therefore the sum is finite.

- R_G denotes the ramification module of π and $R_{G,Q}$ is the ramification module at the point Q (see Section 2.4).
- The notation $\langle \chi, V \rangle_G := \dim_k \mathrm{Hom}_{k[G]}(W, V)$ represents the multiplicity of χ in a G-representation V, where W is the irreducible $k[G]$-module corresponding to χ.
- $\lfloor a \rfloor$ denotes the greatest integer less than or equal to a.

When X is a nodal curve, we require the quotient curve $Y = X/G$ to be smooth* (see Remark 3.9). For the irreducible case, we have

$$H^0(X, \omega_X)_\chi \xrightarrow{\sim} H^0(\hat{X}, \Omega_{\hat{X}}(\hat{S}_X^\chi))_\chi\ ,$$

where $\hat{X} \to X$ is the normalization. We present the Chevalley-Weil formula for irreducible nodal curves in Theorem 3.13:

$$\dim_k H^0(X, \omega_X)_\chi = g_Y - 1 + m_\chi(\hat{S}_X^\chi) + \langle \chi, 1_G \rangle.$$

In this expression:

- $\hat{S}_X^\chi \subset \hat{S}_X$ is the singular χ-set of π (see Proposition 3.7).
- The term $m_\chi(\hat{S}_X^\chi)$ is an integer defined as

$$m_\chi(\hat{S}_X^\chi) = \#\hat{\pi}(\hat{S}_X^\chi) + \sum_{Q \notin \hat{\pi}(\hat{S}_X^\chi)} \left\lfloor \frac{e_Q - 1}{e_Q} + \frac{1}{n} \langle \chi, R_{G,Q} \rangle_G \right\rfloor - \frac{1}{n} \langle \chi, R_G \rangle_G\ .$$

  Here:
  - $\hat{\pi} : \hat{X} \to Y$ is the projection map after normalization.
  - $\#\hat{\pi}(\hat{S}_X^\chi)$ denotes the number of points in the image of $\hat{S}_X^\chi$ under $\hat{\pi}$.
  - e_Q and $R_G, R_{G,Q}$ are defined as before, but for $\hat{\pi}$ this time.

Remark 1.2. *Compared to the smooth case, the difference in* $\dim_k H^0(X, \omega_X)_\chi$ *arises between* m_χ *and* $m_\chi(\hat{S}_X^\chi)$*. The latter incorporates additional ramification data at singularities (the singular* χ*-set). Note that when* $\hat{S}_X^\chi = \emptyset$*, we have*

$$H^0(X, \omega_X)_\chi \xrightarrow{\sim} H^0(\hat{X}, \Omega_{\hat{X}})_\chi,$$

and $m_\chi(\hat{S}_X^\chi) = m_\chi$*, just as in the smooth case.*

In particular, when X *is smooth, we have* $\hat{X} = X$*, so this formula is a direct generalization of the smooth one.*

In the general case, let $X = \cup_{i=1}^d X_i$ be the decomposition into irreducible components. We can reduce the computation to the irreducible case.

Let $\alpha_1 : \hat{X}_1 \to X_1$ be the normalization of X_1, and then $\hat{X}_1 \xrightarrow{\hat{\pi}_1} Y \cong X_1/G_1$ is the induced covering map of smooth curves and we have

$$H^0(X, \omega_X)_\chi \xrightarrow{\sim} H^0(X_1, \omega_{X_1}(I_1^\chi))_{\chi_1} \xrightarrow{\sim} H^0(\hat{X}_1, \Omega_{\hat{X}_1}[\hat{S}_{X_1}^{\chi_1} \sqcup I_1^\chi])_{\chi_1}.$$

Here:

*Note that the smoothness assumption is imposed for a general character χ, while in the case of the trivial character $\chi = 1_G$, the quotient $Y = X/G$ is permitted to be nodal.

- $G_1 := \{\sigma \in G \mid \sigma(X_1) = X_1\}$ is the stabilizer subgroup of G on the component X_1.
- $\chi_1 : G_1 \to k^\times$ is the restriction of χ to G_1.
- $\hat{S}^{\chi_1}_{X_1}$ is the singular χ_1-set of $\hat{\pi}_1$.
- I_1 denotes the intersection locus in X (comprising all intersection points of the irreducible components) restricted to X_1 and

$$I^\chi_1 = \{P \in I_1 \mid \exists\, \tau \in G_P - G_1 \text{ such that } \chi(\tau) = -1\}.$$

  Here G_P is the stabilizer subgroup.
- $\hat{S}^{\chi_1}_{X_1} \cap I^\chi_1 = \varnothing$, since the points in I_1 are all nodes in X, but smooth points in X_1.

By applying the same argument as in the irreducible case, we obtain the Chevalley-Weil formula on connected nodal curves in Theorem 4.5:

$$\dim_k H^0(X, \omega_X)_\chi = g_Y - 1 + m_{\chi_1}(\hat{S}^{\chi_1}_{X_1} \sqcup I^\chi_1) + \delta_\chi.$$

- See the term $m_{\chi_1}(\hat{S}^{\chi_1}_{X_1} \sqcup I^\chi_1)$ in (4.14).
- The final term $\delta_\chi = 0$ or 1 where $\delta_\chi = 1$ if and only if $I^\chi_1 = \varnothing$ and $\chi_1 = 1_{G_1}$.

Remark 1.3. *Compared to the $H^0(X_1, \omega_{X_1})_{\chi_1}$, the difference lies between $m_{\chi_1}(\hat{S}^{\chi_1}_{X_1})$ and $m_{\chi_1}(\hat{S}^{\chi_1}_{X_1} \sqcup I^\chi_1)$. The latter includes additional ramification data at intersection points. Note that when $d = 1$, we have $X = X_1$ and $I^\chi_1 = \varnothing$. Therefore,*

$$H^0(X, \omega_X)_\chi \xrightarrow{\sim} H^0(X_1, \omega_{X_1})_{\chi_1}.$$

Hence, this formula directly generalizes the irreducible case mentioned above.

2. Preliminaries

In this paper, we consider a finite group G acting faithfully on a connected nodal curve X over an algebraically closed field k. Let $\#G = n$ and suppose that either $\mathrm{char}(k) = p \nmid n$ or $\mathrm{char}(k) = 0$ so that $k[G]$ is semisimple. A curve means an equidimensional reduced projective scheme of finite type of dimension 1 over k.

2.1. Regular differentials on nodal curves

Let X be a connected nodal curve. Since X is a projective variety, the dualizing sheaf ω_X always exists and can be explicitly described. Generally speaking, if $X \subseteq \mathbb{P}^N_k$ with codimension r, then the dualizing sheaf of X is given by $\omega_X = \mathcal{E}xt^r_{\mathbb{P}^N_k}(\mathcal{O}_X, \omega_{\mathbb{P}^N_k})$ [7, III.7.5]. Once the dualizing sheaf exists, it is unique together with its trace map [7, III.7.2].

It can be seen that the construction above involves a choice of ambient space $\mathbb{P}^N_k$. However, for the nodal curve, there is a simpler description based on its normalization.

Let $\alpha : \hat{X} \to X$ be the normalization of reduced curves [11, 7.5.1], namely we have $\hat{X} = \coprod_{1 \le i \le d} \hat{X}_i$, where each $\hat{X}_i$ is the normalization of the irreducible component X_i in X. Each $\hat{X}_i$ is a smooth curve, so $\omega_{\hat{X}} = \Omega_{\hat{X}}$ is the canonical sheaf of X. Let $\hat{S}_X \subset \hat{X}$ be the preimage of the nodes in X, and then we define the sheaf of regular differentials on an open subset $V \subseteq X$ by

$$\omega^{\mathrm{reg}}_X(V) := \{\eta \in \Gamma(\alpha^{-1}(V), \Omega_{\hat{X}}(\hat{S}_X)) \mid \sum_{\hat{P} \in \alpha^{-1}(P)} \mathrm{Res}_{\hat{P}}(\eta) = 0,\ \forall P \in V\}. \tag{2.1}$$

We can see that ω_X^{reg} is an O_X-module, and we call its global sections the regular differentials on X. We say a meromorphic differential φ_0 in $\hat{X}$ with at worst simple poles is regular at a point $P \in X$ if $\sum_{\hat{P}\in\alpha^{-1}(P)} \text{Res}_{\hat{P}}(\varphi_0) = 0$, namely the residue relation holds at P.

Remark 2.1. *The definition and properties of residues for differentials on curves can be found in [7, III.7.14]. Note that when X is a smooth curve, the regular differentials on X are exactly the holomorphic differentials. Therefore, regular differentials are the natural generalization of holomorphic differentials.*

In fact, the following fact holds for dualizing sheaves on nodal curves:

Theorem 2.2. *Let X be a nodal curve, and then we have the canonical isomorphism $\omega_X^{reg} \simeq \omega_X$.*

This is a highly nontrivial result. In the exercises of [6, 3.A], one can verify the universal property of the dualizing sheaf by endowing ω_X^{reg} with a trace map. For further verification on the compatibility, the readers are referred to the detailed proof in [3, 5.2].

In summary, the dualizing sheaf on a nodal curve has a precise characterization that depends only on the structure of meromorphic differentials with at worst simple poles in $\Omega_{\hat{X}}(\hat{S}_X)$.

It is known that $H^0(X, \omega_X)$ is a k-vector space of dimension $p_a(X)$, the arithmetic genus of X. Since the finite group G acts on X, both the rational function field $K(X)$ and $H^0(X, \omega_X)$ are naturally (right) $k[G]$-modules. Every 1-dimensional representation is its own character. Our goal is to compute the multiplicity of every character $\chi : G \to k^\times$, that is, the dimension of the eigenspace $H^0(X, \omega_X)_\chi := \{\varphi \in H^0(X, \omega_X) \mid \sigma\varphi = \chi(\sigma)\varphi,\ \forall\sigma \in G\}$ over k.

2.2. Hilbert's Theorem 90

For the sake of discussion, let X be smooth for the rest of this section. Let $G \subseteq \text{Aut}(X)$ be a finite group of automorphisms of order n. Then the quotient map $\pi : X \to Y := X/G$ is a Galois covering, i.e., $K(X)/K(Y)$ is a Galois field extension, where $K(X)$ and $K(Y)$ are the rational function fields of the corresponding curves.

Proposition 2.3. *Let $\chi : G \to k^\times$ be a character, and then there exists a rational function $f_\chi \in K(X)^\times$ such that $\sigma f_\chi = \chi(\sigma) f_\chi,\ \forall\sigma \in G$.*

This result is a special case of the following theorem (in [13, 5.23]):

Theorem 2.4. *Let E/F be a Galois field extension, and let $G = Gal(E/F)$. Then $H^1(G, E^\times) = 0$.*

Remark 2.5 (NOTES below Corollary 5.25 in [13])**.** *This theorem is a generalization of the famous Hilbert's Theorem 90, which was first discovered by Kummer in the case of $\mathbb{Q}[\xi_p]/\mathbb{Q}$, and later generalized by Emmy Noether. This theorem and its various generalizations are all referred to as Hilbert's Theorem 90.*

Here we only prove Proposition 2.3. Before that, we first prove a lemma:

Lemma 2.6 (Dedekind's independence theorem)**.** *Let F be a field, and G be a group. Then any finite number of different group homomorphisms $\chi_1, \ldots, \chi_m : G \to F^\times$ are linearly independent over F, i.e.,*

$$\sum a_i\chi_i = 0 \Longrightarrow a_1 = 0, \ldots, a_m = 0\ .$$

Proof. We use induction on m. The statement is obvious when $m = 1$. Assume it holds for $m - 1$. Suppose there exist $a_i \in F$ such that

$$a_1\chi_1(x) + a_2\chi_2(x) + \cdots + a_m\chi_m(x) = 0, \ \forall x \in G.$$

Next, we prove $a_i = 0$. Without loss of generality, suppose for some $g \in G, \chi_1(g) \neq \chi_2(g)$, and then we have

$$a_1\chi_1(g)\chi_1(x) + a_2\chi_2(g)\chi_2(x) + \cdots + a_m\chi_m(g)\chi_m(x) = 0, \ \forall x \in G.$$

Subtracting the first equation multiplied by $\chi_1(g)$ from this equation, we get

$$a_2'\chi_2 + \cdots + a_m'\chi_m = 0, \ a_i' = a_i\left(\chi_i(g) - \chi_1(g)\right).$$

By induction, $a_i' = 0, \ i = 2, \cdots, m$. Since $\chi_2(g) - \chi_1(g) \neq 0$, $a_2 = 0$, thus we have

$$a_1\chi_1 + a_3\chi_3 + \cdots + a_m\chi_m = 0.$$

By the induction hypothesis, the rest of the $a_i = 0$. □

Proof of Proposition 2.3. Consider the mapping

$$\sum_{\tau\in G}\chi(\tau)\tau : K(X) \to K(X).$$

By the above lemma, this mapping is not zero, so there exists a rational function $g \in K(X)$ such that

$$f := \sum_{\tau\in G}\chi(\tau)\tau g \neq 0.$$

Note that, for $\forall\sigma \in G$, we have

$$\sigma f = \sum_{\tau\in G}\chi(\tau)\cdot\sigma\tau(g) = \sum_{\tau\in G}\chi(\sigma)^{-1}\chi(\sigma\tau)\cdot\sigma\tau(g) = \chi(\sigma)^{-1}f.$$

Thus $f_\chi := f^{-1}$ is the desired function. □

2.3. Ramification data on curve quotients

The quotient map $\pi : X \to Y$ is also a ramified cover. Let e_P be the ramification index at $P \in X$, and then we have the ramification divisor

$$R_\pi = \sum_{P\in X}(e_P - 1)P.$$

For a divisor $D = \sum a_iP_i \in \mathrm{Div}(X)$, define $\pi_*D \in \mathrm{Div}(Y)$ by

$$\pi_*D = \sum a_i\pi(P_i).$$

If $D = \sum a_iQ_i \in \mathrm{Div}(Y)$ is a divisor and $r \in \mathbb{R}$, then define $\lfloor rD\rfloor \in \mathrm{Div}(Y)$ by

$$\lfloor rD\rfloor = \sum\lfloor ra_i\rfloor Q_i,$$

where $\lfloor ra_i\rfloor$ denotes the greatest integer $\leq ra_i$. Define $\pi^*D \in \mathrm{Div}(Y)$ by

$$\pi^*D := \sum_i\sum_{P\in\pi^{-1}(Q_i)}(a_ie_P)P \in \mathrm{Div}(Y).$$

Proposition 2.7 (Kani [9])**.** *Let G be a finite group (of order n) acting on a smooth curve X with R_π the ramification divisor of $\pi : X \to Y = X/G$. Suppose $D \in \mathrm{Div}(X)$ is a G-invariant divisor, and then for the trivial character $\chi = 1_G$, we have*

$$H^0(X, \mathcal{O}_X(D))^G = \pi^* H^0(Y, \mathcal{O}_Y \left\lfloor n^{-1}\pi_* D\right\rfloor), \tag{2.2}$$

$$H^0(X, \Omega_X(D))^G = \pi^* H^0(Y, \Omega_Y \left\lfloor n^{-1}\pi_*(D + R_\pi)\right\rfloor). \tag{2.3}$$

For any character χ, let $f_\chi \in K(X)^*$ be such that $\sigma f_\chi = \chi(\sigma) f_\chi$ for all $\sigma \in G$ (whose existence is guaranteed by Hilbert's Theorem 90). Then

$$H^0(X, \mathcal{O}_X(D))_\chi = f_\chi \cdot \pi^* H^0(Y, \mathcal{O}_Y \left\lfloor n^{-1}\pi_*\left(D + \left(f_\chi\right)\right)\right\rfloor), \tag{2.4}$$

$$H^0(X, \Omega_X(D))_\chi = f_\chi \cdot \pi^* H^0(Y, \Omega_Y \left\lfloor n^{-1}\pi_*(D + \left(f_\chi\right) + R_\pi)\right\rfloor). \tag{2.5}$$

Proof. (More details are given here than in [9].) First, it is easily deduced from the definition that $\pi^*\pi_* D = nD$ since D is G-invariant.

For (2.2), we have $D \geq \pi^* \left\lfloor n^{-1}\pi_* D\right\rfloor$ and hence $H^0(X, \mathcal{O}_X(D))^G \supseteq \pi^* H^0(Y, \mathcal{O}_Y \left\lfloor n^{-1}\pi_* D\right\rfloor)$. Conversely, if $f \in H^0(X, \mathcal{O}_X(D))^G$, then $f = \pi^* e$ with some $e \in K(Y)$. Hence

$$\pi_*((f) + D) = n(e) + \pi_* D \geq 0,$$

which implies $(e) + \left\lfloor n^{-1}\pi_* D\right\rfloor \geq 0$.

To prove (2.3), fix a meromorphic differential $0 \neq \varphi \in \Omega(Y)$, which exists by Riemann-Roch provided that the pole multiplicities are sufficiently high. By

$$H^0(X, \Omega_X(D)) = H^0(X, \mathcal{O}_X(D + (\pi^*\varphi))) \cdot \pi^*\varphi = H^0(X, \mathcal{O}_X(D + \pi^*(\varphi) + R_\pi)) \cdot \pi^*\varphi,$$

we have

$$\begin{aligned} H^0(X, \Omega_X(D))^G &= H^0(X, \mathcal{O}_X(D + \pi^*(\varphi) + R_\pi))^G \cdot \pi^*\varphi \\ &= \pi^* H^0(Y, \mathcal{O}_Y \left\lfloor n^{-1}\pi_*((D + \pi^*(\varphi) + R_\pi))\right\rfloor) \cdot \pi^*\varphi \\ &= \pi^*\left(H^0(Y, \mathcal{O}_Y(\left\lfloor n^{-1}\pi_*(D + R_\pi)\right\rfloor) + (\varphi)) \cdot \varphi\right) \\ &= \pi^* H^0(Y, \Omega_Y \left\lfloor n^{-1}\pi_*(D + R_\pi)\right\rfloor). \end{aligned}$$

Finally, (2.4) and (2.5) for general χ follow from

$$H^0(X, \mathcal{O}_X(D))_\chi = f_\chi \cdot H^0(X, \mathcal{O}_X(D + \left(f_\chi\right)))^G,$$

$$H^0(X, \Omega_X(D))_\chi = f_\chi \cdot H^0(X, \Omega_X(D + \left(f_\chi\right)))^G.$$

Note that the divisor $\left(f_\chi\right)$ is G-invariant, so it suffices to repeat the discussion above with D replaced by $D + \left(f_\chi\right)$. □

2.4. Ramification modules

The constructions in this section are based on Kani's work [9]. Fix a point $P \in X$, and let $G_P := \{\sigma \in G \mid \sigma(P) = P\}$ be the stabilizer subgroup of G at P, which is a cyclic group of order e_P. Then there is a unique character $\theta_P : G_P \to k^\times$ such that for any $f \in K(X)^\times$,

$$\frac{\sigma f}{f} \equiv \theta_P(\sigma)^{v_P(f)} (\mathrm{mod}\ \mathfrak{m}_P), \quad \forall \sigma \in G_P,$$

where v_P denotes the valuation at P and $\mathfrak{m}_P$ the maximal ideal of the local ring O_P.

Set

$$R_{G,P} := \mathrm{Ind}_{G_P}^{G} \left(\bigoplus_{d=0}^{e_P-1} d \cdot \theta_P^d \right).^*$$

Definition 2.8. Let $Bl(Y)$ be the branch locus of $\pi : X \to Y$, namely the subset of all branch points in Y. For a point $Q \in Y$, we define the ramification module of Q by

$$R_{G,Q} := \bigoplus_{P \in \pi^{-1}(Q)} R_{G,P},$$

and the ramification module of π by

$$R_G := \bigoplus_{Q \in Y} R_{G,Q_i}.$$

Note that this is a finite sum because $R_{G,Q} = 0$ for $Q \notin Bl(Y)$.

Let $\chi : G \to k^\times$ be a character and $f_\chi \in K(X)_\chi$ as in Proposition 2.3. Since $\chi^n = 1_G$, we have $f_\chi^n \in k(X)^G = \pi^* k(Y)$. Write $\left(f_\chi^n\right) = \pi^*(nA + B)$ where $A, B \in \mathrm{Div}(Y)$ and $\lfloor n^{-1} B \rfloor = 0$. Note that $\mathrm{Supp}(B) \subseteq Bl(Y)$, so we write $B = \sum_{Q \in Bl(Y)} b_Q Q$. By definition, we have

$$b_Q = n \left\langle \frac{v_Q(f_\chi^n)}{n} \right\rangle,^\dagger$$

where $\langle r \rangle = r - \lfloor r \rfloor$ denotes the fractional part of r.

The following lemma shows that this B is independent of the choices of f_χ:

Lemma 2.9. *Let $\chi : G \to k^\times$ be a character. Then for any $Q \in Bl(Y)$, we have*

$$n \left\langle \frac{v_Q(f_\chi^n)}{n} \right\rangle = \langle \chi, R_{G,Q} \rangle_G . \tag{2.6}$$

Proof. Let $P \in \pi^{-1}(Q)$. Then by Frobenius reciprocity, we have

$$\langle \chi, R_{G,P} \rangle_G = \left\langle \chi|_{G_P}, \bigoplus_{d=0}^{e_P-1} d \cdot \theta_P^d \right\rangle_{G_P} . \tag{2.7}$$

---

*Here $d \cdot \theta_P^d$ means $\oplus^d \theta_P^d$.

†In the expression $v_Q(f_\chi^n)$, the function f_χ^n is regarded as a rational function on Y. Consequently, we have $v_P(f_\chi^n) = e_Q v_Q(f_\chi^n)$.

Note that θ_P^d runs through are all the irreducible representations of G_P, and hence we have

$$\langle \chi, R_{G,P} \rangle_G = a \Leftrightarrow \chi|_{G_P} = \theta_P^a, \tag{2.8}$$

for $0 \le a < e_P$. Choose a generator σ of G_P, and then by the definition of f_χ, we have

$$\sigma f_\chi = \chi(\sigma) f_\chi = \theta_P(\sigma)^a f_\chi.$$

Furthermore, by the definition of θ_P, we have

$$\theta_P(\sigma)^a = \frac{\sigma f_\chi}{f_\chi} \equiv \theta_P(\sigma)^{v_P(f_\chi)} (\mathrm{mod}\ \mathfrak{m}_P),$$

which implies $a \equiv v_P(f_\chi) (\mathrm{mod}\ e_P)$ since $\theta_P(\sigma)$ has order e_P in $k^\times$. Finally,

$$\frac{\langle \chi, R_{G,P} \rangle_G}{e_P} = \left\langle \frac{v_P(f_\chi)}{e_P} \right\rangle = \left\langle \frac{v_Q(f_\chi^n)}{n} \right\rangle = \frac{b_Q}{n},$$

and hence we have

$$\langle \chi, R_{G,Q} \rangle_G = \left\langle \chi, \bigoplus_{P \in \pi^{-1}(Q)} R_{G,P} \right\rangle_G = \frac{n}{e_P} \langle \chi, R_{G,P} \rangle_G = b_Q.$$

□

3. Irreducible nodal curves

Here is a basic result on the nodal curve quotient [11, 10.3.48]:

Proposition 3.1. *Let X be a reduced nodal curve, and $G \subseteq \mathrm{Aut}(X)$ be a finite automorphism group on X. Then the quotient curve $Y = X/G$ is a nodal curve. More precisely, let $P \in X$ be a closed point, and Q be its image in Y. Then we have the following results:*

(a) If X is smooth at P, then Y is smooth at Q;
(b) If P is an ordinary double point on X, then Q is either a smooth point or an ordinary double point.

Let X be an irreducible nodal curve in this section.

3.1. The G-invariant differentials

Let G be a finite group acting on an irreducible nodal curve X and $Y = X/G$ the quotient curve, which is also an irreducible nodal curve. We want to study the G-invariant space of regular differentials $H^0(X, \omega_X)^G$. It is a classical fact that:

Proposition 3.2. *If X is smooth, then*

$$\dim_k H^0(X, \Omega_X)^G = g_Y.$$

Proof. With the notations of Section 2.3, let $e_Q := e_P$ for any $P \in \pi^{-1}(Q)$. Note that

$$\lfloor n^{-1}\pi_* R_\pi \rfloor = \sum_{Q \in Y} \left\lfloor \frac{e_Q - 1}{e_Q} \right\rfloor Q = 0.$$

By Proposition 2.7 (2.3), we have

$$H^0(X, \Omega_X)^G = \pi^* H^0(Y, \Omega_Y \lfloor n^{-1}\pi_* R_\pi \rfloor) = \pi^* H^0(Y, \Omega_Y),$$

which is of dimension g_Y, the geometric genus of Y. $\square$

Here comes a natural question: For the covering $\pi : X \to Y$ of irreducible nodal curves, do we still have the equality

$$\dim_k H^0(X, \omega_X)^G = p_a(Y)? \tag{3.1}$$

The answer is yes.

Consider the normalizations $\hat{X} \to X$ and $\hat{Y} \to Y$, respectively. We have a canonical isomorphism $\hat{X}/G = \hat{Y}$, so there is a commutative diagram:

$$\begin{tikzcd} \hat{X} \arrow[r, "\hat{\pi}"] \arrow[d] & \hat{Y} \arrow[d] \\ X \arrow[r, "\pi"] & Y. \end{tikzcd}$$

This induces the corresponding morphisms of differentials

$$\begin{tikzcd} H^0(X, \omega_X) \arrow[d, hook] & \pi^* H^0(Y, \omega_Y) \arrow[l, dashed, "?"'] \arrow[d, hook] \\ H^0(\hat{X}, \Omega_{\hat{X}}(\hat{S}_X)) & \hat{\pi}^* H^0(\hat{Y}, \Omega_{\hat{Y}}(\hat{S}_Y)). \arrow[l, hook'] \end{tikzcd} \tag{3.2}$$

The lower row is obtained by $\hat{\pi}^{-1}(\hat{S}_Y) \subseteq \hat{S}_X$, since $X \to Y$ maps smooth points to smooth points.

Lemma 3.3. *There is a canonical inclusion*

$$\pi^* H^0(Y, \omega_Y) \subseteq H^0(X, \omega_X),$$

that makes (3.2) commute.

Proof. We say $\{P_1, P_2\} \subseteq \hat{S}_X$ is a *pair* if the two points are precisely the preimages of the same node in X. In this context, a differential $\varphi \in H^0(\hat{X}, \Omega_{\hat{X}}(\hat{S}_X))$ is regular if and only if

$$\mathrm{Res}_{P_1} \varphi = -\mathrm{Res}_{P_2} \varphi,$$

for every pair $\{P_1, P_2\}$.

In this situation, we say that $\{P_1, P_2\}$ is the preimage of the corresponding node, denoted by P. Note that for any pair $\{P_1, P_2\}$, the ramification indices are equal, i.e.,

$$e_{P_1} = e_{P_2},$$

since the orbits of both points have the same cardinality, namely, $\#\{\sigma(P) \in X \mid \sigma \in G\}$.

Now, consider a regular differential $\varphi_Y \in H^0(Y, \omega_Y)$. For any pair $\{P_1, P_2\} = \alpha^{-1}(P) \subseteq \hat{S}_X$, we have

$$\mathrm{Res}_{\hat{\pi}(P_1)}\,\varphi_Y = -\,\mathrm{Res}_{\hat{\pi}(P_2)}\,\varphi_Y.$$

Moreover, for any point $P_0 \in \hat{X}$, the pullback satisfies

$$\mathrm{Res}_{P_0}(\hat{\pi}^*\varphi_Y) = e_{P_0} \cdot \mathrm{Res}_{\hat{\pi}(P_0)}\,\varphi_Y.$$

Thus, applying these equalities for the pair $\{P_1, P_2\}$, we obtain

$$\mathrm{Res}_{P_1}(\hat{\pi}^*\varphi_Y) = e_{P_1} \cdot \mathrm{Res}_{\hat{\pi}(P_1)}\,\varphi_Y = -e_{P_2} \cdot \mathrm{Res}_{\hat{\pi}(P_2)}\,\varphi_Y = -\,\mathrm{Res}_{P_2}(\hat{\pi}^*\varphi_Y). \tag{3.3}$$

This shows that $\hat{\pi}^*\varphi_Y$ is regular at P, and hence is a regular differential in X. □

Furthermore, we have

Lemma 3.4. *For the left column of (3.2), we have*

$$H^0(X, \omega_X)^G \hookrightarrow H^0(\hat{X}, \Omega_{\hat{X}}(\pi^{-1}(\hat{S}_Y)))^G.$$

Proof. All we need is to show a G-invariant regular differential has no poles at $\hat{S}_X - \hat{\pi}^{-1}(\hat{S}_Y)$. Suppose $\hat{S}_X - \hat{\pi}^{-1}(\hat{S}_Y) \neq \varnothing$, otherwise there is nothing to prove.

Given a pair $\{P_1, P_2\} \subseteq \hat{S}_X - \hat{\pi}^{-1}(\hat{S}_Y)$, we know that the points of $\hat{S}_X - \hat{\pi}^{-1}(\hat{S}_Y)$ are mapped to the smooth locus of Y, and hence there exists a $\sigma \in G$ such that $\sigma(P_1) = P_2$. So for any regular differential $\varphi \in H^0(X, \omega_X)^G$, we have $\mathrm{Res}_{P_1}\,\varphi = \mathrm{Res}_{P_1}\,\sigma\varphi = \mathrm{Res}_{\sigma(P_1)}\,\varphi = \mathrm{Res}_{P_2}\,\varphi$. As $\mathrm{Res}_{P_1}\,\varphi = -\,\mathrm{Res}_{P_2}\,\varphi$ by definition, it forces that $\mathrm{Res}_{P_1}\,\varphi = \mathrm{Res}_{P_2}\,\varphi = 0$, which implies φ has no poles at $\{P_1, P_2\}$. □

Now we can give a positive answer to (3.1).

Theorem 3.5. *With the notations above, the lower row of the canonical commutative diagram*

$$\begin{tikzcd} H^0(X, \omega_X)^G \arrow[d, hook] & H^0(Y, \omega_Y) \arrow[l, "\pi^*"'] \arrow[d, hook] \\ H^0(\hat{X}, \Omega_{\hat{X}}(\hat{\pi}^{-1}(\hat{S}_Y)))^G & H^0(\hat{Y}, \Omega_{\hat{Y}}(\hat{S}_Y)) \arrow[l, "\hat{\pi}^*"'] \end{tikzcd}$$

is an isomorphism. Moreover, both $H^0(X, \omega_X)^G$ and $H^0(Y, \omega_Y)$ are the subspaces of log differentials satisfying the residue relations, so we have the isomorphism for the upper row. In particular, we have $\dim_k H^0(X, \omega_X)^G = p_a(Y)$.

Back to the smooth case, we have the following:

Proposition 3.6. *Suppose $\pi : X \to Y$ is the quotient morphism of smooth curves and $S \subset Y$ is a finite set. Then*

$$H^0(Y, \Omega_Y(S)) \to H^0(X, \Omega_X(\pi^{-1}(S))^G \tag{3.4}$$

is an isomorphism.

Proof. By Proposition 2.7 (2.3), we have

$$H^0(X, \Omega_X(\pi^{-1}(S))^G = \pi^* H^0(Y, \Omega_Y(\left\lfloor n^{-1}\pi_*(\pi^{-1}(S) + R_\pi)\right\rfloor)).^*$$

First, note that for the divisor $n^{-1}\pi_*\pi^{-1}(S)$, the coefficient corresponding to a point $Q \in S$ is exactly $1/e_Q$. Now, consider

$$\left\lfloor n^{-1}\pi_*\pi^{-1}(S) + n^{-1}\pi_* R_\pi \right\rfloor = \sum a_Q Q,$$

and analyze the coefficient a_Q for each prime divisor Q by considering three cases:

- If $Q \in Bl(Y) - S$ (i.e., Q belongs to the branch locus but is not in S), then

$$a_Q = \left\lfloor \frac{e_Q - 1}{e_Q} \right\rfloor = 0.$$

- If $Q \in S - Bl(Y)$ (i.e., Q is in S but not in the branch locus), then

$$a_Q = 1.$$

- If $Q \in S \cap Bl(Y)$ (i.e., Q is both in S and in the branch locus), then

$$a_Q = \left\lfloor \frac{1}{e_Q} + \frac{e_Q - 1}{e_Q} \right\rfloor = \lfloor 1 \rfloor = 1.$$

Since these are the only possibilities, it follows that $\left\lfloor n^{-1}\pi_*(\pi^{-1}(S) + R_\pi) \right\rfloor = S$. □

Proof of Theorem 3.5. Apply Proposition 3.6 to $\hat{\pi} : \hat{X} \to \hat{Y}$ and $\hat{S}_Y \subset \hat{Y}$. □

3.2. Chevalley-Weil formula for irreducible nodal curves

Let χ be a character of G, and $\alpha : \hat{X} \to X$ the normalization. With the notations in (3.2), consider the embedding

$$H^0(X, \omega_X)_\chi \hookrightarrow H^0(\hat{X}, \Omega_{\hat{X}}(\hat{S}_X))_\chi.$$

We want to determine the image of $H^0(X, \omega_X)_\chi$ in $H^0(\hat{X}, \Omega_{\hat{X}}(\hat{S}_X))_\chi$.

Proposition 3.7. *Suppose that Y is smooth. Define*

$$\hat{S}_X^\chi := \left\{\hat{P} \in \hat{S}_X \mid \exists \tau \in G_{\alpha(\hat{P})} \text{ s.t. } \tau(\hat{P}) \neq \hat{P} \text{ and } \chi(\tau) = -1\right\}. \tag{3.5}$$

Then the image of $H^0(X, \omega_X)_\chi$ in $H^0(\hat{X}, \Omega_{\hat{X}}(\hat{S}_X))_\chi$ is equal to $H^0(\hat{X}, \Omega_{\hat{X}}(\hat{S}_X^\chi))_\chi$. So we have an isomorphism

$$H^0(X, \omega_X)_\chi \xrightarrow{\sim} H^0(\hat{X}, \Omega_{\hat{X}}(\hat{S}_X^\chi))_\chi\,. \tag{3.6}$$

We call $\hat{S}_X^\chi$ the singular χ-set *of π.*

[*] Here and there, we adopt the convention that every finite subset is interpreted as an effective divisor when needed.

Proof. Let $\varphi_0 \in H^0(X, \omega_X)_\chi$ a regular differential. If it has poles on a pair $\{P_1, P_2\} \subseteq \hat{S}_X$, then there is some $T \in G_P$ such that $T(P_1) = P_2$[†], which is guaranteed by the smoothness of Y. So we have

$$\chi(T)\,\mathrm{Res}_{P_1}(\varphi_0) = \mathrm{Res}_{P_1}(T\varphi_0) = \mathrm{Res}_{P_2}(\varphi_0) = -\,\mathrm{Res}_{P_1}(\varphi_0),$$

which implies $\chi(T) = -1$ and $P_1, P_2 \in \hat{S}_X^\chi$. In particular, when $\hat{S}_X^\chi = \varnothing$, we see that any regular differential in $H^0(X, \omega_X)_\chi$ has no pole on $\hat{X}$.

Conversely, assume $\varphi \in H^0(\hat{X}, \Omega_{\hat{X}}(\hat{S}_X^\chi))_\chi$ and $\hat{S}_X^\chi \neq \varnothing$. Let $\{P_1, P_2\} \subseteq \hat{S}_X^\chi$ be a pair. By definition, there is some automorphism $T \in G_P$ such that $T(P_1) = P_2$ and $\chi(T) = -1$. Hence

$$-\,\mathrm{Res}_{P_1}\varphi = \mathrm{Res}_{P_1} T\varphi = \mathrm{Res}_{T^{-1}(P_1)}\varphi = \mathrm{Res}_{P_2}\varphi.$$

This means φ is regular. □

Remark 3.8. *If $\chi = 1_G$ is the trivial representation, then $\chi(\sigma) \neq -1$ for any $\sigma \in G$, and hence $\hat{S}_X^{1_G} = \varnothing$. By Propositions 3.7 and 3.2, we obtain*

$$H^0(X, \omega_X)^G \xrightarrow{\sim} H^0(\hat{X}, \Omega_{\hat{X}})^G = \pi^* H^0(Y, \Omega_Y).$$

This conclusion is consistent with Proposition 3.5 in the case that Y is smooth, in which case $\hat{S}_Y = \varnothing$.

Remark 3.9. *Here we explain why the condition of Y being smooth is needed. Suppose the meromorphic differential $\varphi_0 \in H^0(\hat{X}, \Omega_{\hat{X}}(S))$ has a simple pole at some point $P_1 \in S$, i.e.,* $\mathrm{Res}_{P_1}\varphi_0 \neq 0$. *If $\varphi_0 \in H^0(X, \omega_X)_\chi$, then φ_0 is regular at $\alpha(P_1)$ (α is the normalization), so that there exists a pair $\{P_1, P_2\}$ on $\hat{X}$ satisfying the following two conditions:*

(a) $v_{P_2}(\varphi_0) = v_{P_1}(\varphi_0) = -1$;
(b) $\mathrm{Res}_{P_2}\varphi_0 = -\,\mathrm{Res}_{P_1}\varphi_0$.

For condition (a), generally, we cannot determine the value $v_{P_2}(\varphi_0)$ from $v_{P_1}(\varphi_0)$. However, when Y is smooth, we have the following commutative diagram:

$$\begin{tikzcd} \hat{X} \arrow[d, "\alpha"'] \arrow[dr, "\hat{\pi}"] & \\ X \arrow[r, "\pi"] & Y \simeq \hat{X}/G. \end{tikzcd}$$

Here $\hat{\pi}$ is a ramified cover of smooth curves. Since P_1, P_2 are mapped to the same point in Y (through X), there exists an automorphism τ that permutes the pair $\{P_1, P_2\}$ and satisfies $\tau\varphi_0 = \chi(\tau)\varphi_0$. Consequently, we must have

$$v_{P_2}(\varphi_0) = v_{P_1}(\varphi_0)$$

and

$$\mathrm{Res}_{P_2}\varphi_0 = \chi(\tau)\,\mathrm{Res}_{P_1}\varphi_0.$$

For condition (b), under the hypothesis Y being smooth, it is equivalent to say $\chi(\tau) = -1$ for some (and hence any) automorphism τ permuting $\{P_1, P_2\}$. Furthermore, this criterion works for the intersection points in a nodal curve with several irreducible components (see Section 4.2).

[†]Hence $T(P_2) = P_1$ by symmetry.

Assume $Y = X/G$ is smooth for the rest of this section, and then $\hat{\pi} : \hat{X} \to Y$ is the induced ramified cover of $\pi : X \to Y$. Now we compute the dimension of $H^0(\hat{X}, \Omega_{\hat{X}}(\hat{S}_X^\chi))_\chi$ through Proposition 2.7.

Let f_χ be a rational function on $\hat{X}$ such that $\sigma f_\chi = \chi(\sigma) f_\chi$, $\forall \sigma \in G$. Set

$$D_\chi := \left\lfloor n^{-1}\hat{\pi}_*(\hat{S}_X^\chi + \left(f_\chi\right) + R_{\hat{\pi}})\right\rfloor.$$

By Proposition 2.7 (2.5), we have

$$H^0(\hat{X}, \Omega_{\hat{X}}(\hat{S}_X^\chi))_\chi = f_\chi \cdot \hat{\pi}^* H^0(Y, \Omega_Y(D_\chi)).$$

At this point, we reduce the problem to calculating the dimension of $H^0(Y, \Omega_Y(D_\chi))$.

Using the Riemann-Roch theorem on Y, we have

$$\dim_k H^0(Y, \Omega_Y(D_\chi)) = \dim_k H^0(Y, \mathcal{O}_Y(-D_\chi)) + \deg D_\chi + g_Y - 1. \tag{3.7}$$

For $\dim_k H^0(Y, \mathcal{O}_Y(-D_\chi))$, we have:

Lemma 3.10. *The space $H^0(Y, \mathcal{O}_Y(-D_\chi))$ vanishes except when $\chi = 1_G$, and in this case, we have* $\dim_k H^0(Y, -D_{1_G}) = 1$.

Proof. We will prove the following:

1) If $\hat{S}_X^\chi \neq \varnothing$, then $\deg D_\chi > 0$.
2) The divisor D_χ is principal if and only if $\chi = 1_G$; in this case, $\hat{S}_X^\chi = \varnothing$.

Recall that we have

$$\hat{\pi}_*\left(f_\chi\right) = \sum_{Q\in Y} \frac{n}{e_Q} a_Q \cdot Q,$$

where $a_Q = v_P(f_\chi)$, $\forall P \in \hat{\pi}^{-1}(Q)$. Note that

$$\left\lfloor n^{-1}\hat{\pi}_*(\left(f_\chi\right) + R_{\hat{\pi}})\right\rfloor = \sum_Q \left\lfloor \frac{a_Q + e_Q - 1}{e_Q} \right\rfloor Q \geq \sum_Q \frac{a_Q}{e_Q} Q = n^{-1}\hat{\pi}_*\left(f_\chi\right).$$

Therefore, we have the inequalities

$$\deg\left\lfloor n^{-1}\hat{\pi}_*\left(\hat{S}_X^\chi + \left(f_\chi\right) + R_{\hat{\pi}}\right)\right\rfloor \geq \deg\left\lfloor n^{-1}\hat{\pi}_*(\left(f_\chi\right) + R_\pi)\right\rfloor \geq \deg n^{-1}\hat{\pi}_*\left(f_\chi\right) = 0. \tag{3.8}$$

Now, write

$$\hat{\pi}_*(\hat{S}_X^\chi) = \sum_{Q\in Y} \frac{n}{e_Q} c_Q \cdot Q.$$

If $\hat{S}_X^\chi \neq \varnothing$, then there exists some point Q' such that $c_{Q'} \geq 1$. Hence, we can write

$$\begin{aligned} \deg D_\chi &= \sum_{Q\neq Q'} \left\lfloor \frac{c_Q + a_Q + e_Q - 1}{e_Q} \right\rfloor + \left\lfloor \frac{c_{Q'} + a_{Q'} + e_{Q'} - 1}{e_{Q'}} \right\rfloor \\ &\geq \sum_{Q\neq Q'} \left\lfloor \frac{c_Q + a_Q + e_Q - 1}{e_Q} \right\rfloor + \left\lfloor \frac{a_{Q'} + e_{Q'}}{e_{Q'}} \right\rfloor > \sum_Q \frac{a_Q}{e_Q} = 0. \end{aligned} \tag{3.9}$$

Thus we have $\deg D_\chi > 0$ provided $\hat{S}_X^\chi \neq \emptyset$, and consequently

$$\dim_k H^0(Y, \mathcal{O}_Y(-D_\chi)) = 0.$$

Next, assume that $\chi = 1_G$. It follows that $\hat{S}_X^\chi = \emptyset$ and $f_\chi = \hat{\pi}^* h$ for some rational function $h \in K(Y)$. Then we have

$$\begin{aligned} D_\chi &= \left\lfloor n^{-1}\hat{\pi}_*\left(\hat{\pi}^* h\right) + n^{-1}\hat{\pi}_* R_{\hat{\pi}} \right\rfloor \\ &= n^{-1}\hat{\pi}_*\left(\hat{\pi}^* h\right) + \left\lfloor n^{-1}\hat{\pi}_* R_{\hat{\pi}} \right\rfloor \\ &= n^{-1}\hat{\pi}_*\left(\hat{\pi}^* h\right) + 0 = (h)\,, \end{aligned} \tag{3.10}$$

which shows that D_χ is principal. Therefore, in this case, $\dim_k H^0(Y, \mathcal{O}_Y(-D_\chi)) = 1$.

Conversely, suppose that D_χ is a principal divisor; that is, there exists some rational function $h \in K(Y)$ such that $D_\chi = (h)$. Then $\deg D_\chi = 0$, which forces $\hat{S}_X^\chi = \emptyset$. Then, by the previous calculations,

$$0 = \deg \left\lfloor n^{-1}\hat{\pi}_*(\left(f_\chi\right) + R_{\hat{\pi}}) \right\rfloor = \sum_Q \left\lfloor \frac{a_Q + e_Q - 1}{e_Q} \right\rfloor \geq \sum_Q \frac{a_Q}{e_Q} = \deg n^{-1}\hat{\pi}_*\left(f_\chi\right) = 0.$$

This chain of equalities implies that

$$\left\lfloor \frac{a_Q + e_Q - 1}{e_Q} \right\rfloor = \frac{a_Q}{e_Q},$$

which means that $e_Q \mid a_Q$. Therefore,

$$D_\chi = n^{-1}\hat{\pi}_*\left(f_\chi\right) = (h)\,.$$

Applying $\hat{\pi}^*$ to both sides, we obtain

$$n^{-1}\hat{\pi}_*\hat{\pi}^*\left(f_\chi\right) = \hat{\pi}^*(h)\,,$$

which implies that $\left(f_\chi\right) = (\hat{\pi}^* h)$. In other words, $f_\chi \in K(X)^G$, i.e., $\chi = 1_G$. □

Now we want to calculate $\deg D_\chi$.

Definition 3.11. Let $S \subseteq \hat{X}$ be a finite subset stable by G, and define

$$m_\chi(S) := \#\hat{\pi}(S) + \sum_{Q \notin \hat{\pi}(S)} \left\lfloor \frac{e_Q - 1}{e_Q} + \frac{1}{n}\langle \chi, R_{G,Q}\rangle_G \right\rfloor - \frac{1}{n}\langle \chi, R_G\rangle_G\,, \tag{3.11}$$

where $R_{G,Q}$ and R_G are ramification modules of $\hat{\pi} : \hat{X} \to Y$.

Lemma 3.12. *We have* $\deg D_\chi = m_\chi(\hat{S}_X^\chi)$, *which is independent of the choice of* f_χ.

Proof. Let $\left(f_\chi^n\right) = \hat{\pi}^*(nA + B)$ as in the discussion before Lemma 2.9 satisfying $\lfloor n^{-1}B \rfloor = 0$. Then we have

$$D_\chi = \left\lfloor n^{-1}\hat{\pi}_*\hat{S}_X^\chi + n^{-1}\hat{\pi}_*\left(f_\chi\right) + n^{-1}\hat{\pi}_* R_{\hat{\pi}} \right\rfloor$$

$$= \left\lfloor n^{-1}\hat{\pi}_*\hat{S}_X^\chi + n^{-2}\hat{\pi}_*\hat{\pi}^*(nA+B) + n^{-1}\hat{\pi}_*R_{\hat{\pi}} \right\rfloor.$$

Note that

$$n^{-1}\hat{\pi}_*\hat{S}_X^\chi = \sum_{Q\in\hat{\pi}(\hat{S}_X^\chi)} \frac{1}{e_Q}Q\ ,\ n^{-1}\hat{\pi}_*R_{\hat{\pi}} = \sum_{Q\in Bl(Y)} \frac{e_Q-1}{e_Q}Q.$$

So we decompose $n^{-1}\hat{\pi}_*(\hat{S}_X^\chi) = U + V$ into two parts according to whether the point is branched, so that V is a $\mathbb{Q}$-divisor which is supported on the branch locus of $\hat{\pi}$, i.e., $\mathrm{Supp}(V) = Bl(Y) \cap \hat{\pi}(\hat{S}_X^\chi)$. So U is an integer divisor with all coefficients equal to 1. Now we have

$$D_\chi = U + A + \left\lfloor V + n^{-1}B + n^{-1}\hat{\pi}_*R_{\hat{\pi}} \right\rfloor. \tag{3.12}$$

Let $B = \sum_Q b_Q Q$ (supported on $Bl(Y)$) and according to $\deg B = -\deg nA$, we get

$$\begin{aligned}\deg D_\chi &= \deg U + \sum_{Q\in\mathrm{Supp}(V)} \left\lfloor 1 + \frac{b_Q}{n} \right\rfloor + \sum_{Q\notin\hat{\pi}(\hat{S}_X^\chi)} \left\lfloor \frac{e_Q-1}{e_Q} + \frac{b_Q}{n} \right\rfloor - \sum_Q \frac{b_Q}{n} \\ &= \#\hat{\pi}(\hat{S}_X^\chi) + \sum_{Q\notin\hat{\pi}(\hat{S}_X^\chi)} \left\lfloor \frac{e_Q-1}{e_Q} + \frac{b_Q}{n} \right\rfloor - \sum_Q \frac{b_Q}{n}.\end{aligned} \tag{3.13}$$

By (2.6) in Lemma 2.9, we have $b_Q = \langle\chi, R_{G,Q}\rangle_G$, so

$$\deg D_\chi = \#\hat{\pi}(\hat{S}_X^\chi) + \sum_{Q\notin\hat{\pi}(\hat{S}_X^\chi)} \left\lfloor \frac{e_Q-1}{e_Q} + \frac{1}{n}\langle\chi, R_{G,Q}\rangle_G \right\rfloor - \frac{1}{n}\langle\chi, R_G\rangle_G = m_\chi(\hat{S}_X^\chi). \tag{3.14}$$

□

We summarize the discussion in this section.

Let X be an irreducible nodal curve over k, G be a finite automorphism group of order n on X, and $Y = X/G$ be a smooth curve. Let $\hat{\pi} : \hat{X} \to Y$ be the ramified cover induced by the normalization $\pi : X \to Y$, and $\hat{S}_X \subseteq \hat{X}$ be the preimage of the singular points of X on $\hat{X}$. Denote R_G the ramification module of $\hat{\pi}$, and $R_{G,Q}$ the ramification module of $Q \in Y$.

Theorem 3.13 (The Chevalley-Weil formula on irreducible nodal curves)**.** *Let $\chi : G \to k^\times$ be a character. If the quotient curve $Y = X/G$ is smooth, then*

$$\dim_k H^0(X, \omega_X)_\chi = g_Y - 1 + m_\chi(\hat{S}_X^\chi) + \langle\chi, 1_G\rangle. \tag{3.15}$$

Here $\hat{S}_X^\chi$ is the singular χ-set of π as in (3.5)*, and in Definition 3.11,*

$$m_\chi(\hat{S}_X^\chi) = \#\hat{\pi}(\hat{S}_X^\chi) + \sum_{Q\notin\hat{\pi}(\hat{S}_X^\chi)} \left\lfloor \frac{e_Q-1}{e_Q} + \frac{1}{n}\langle\chi, R_{G,Q}\rangle_G \right\rfloor - \frac{1}{n}\langle\chi, R_G\rangle_G\,.$$

In particular, when $\chi = 1_G$, we have $\hat{S}_X^\chi = \varnothing$ and $\dim_k H^0(X, \omega_X)^G = g_Y$*. This is the special case that $\hat{S}_Y = \varnothing$ in Proposition 3.5.*

Example 3.14 (Hyperelliptic stable curves)**.** We call a stable curve C a hyperelliptic stable curve if there exists a 2-order automorphism $J : C \to C$ such that $C/\langle J\rangle = \mathbb{P}^1$. Such an automorphism of C is called an involution.

Let C be an irreducible hyperelliptic stable curve with N (≥ 1) nodes, and $\hat{C}$ be the normalization of C with genus g. Then $\hat{\pi} : \hat{C} \to \mathbb{P}^1$ has $2g+2$ fixed points, which are also all the ramification points of $\hat{\pi}$, with ramification index all equal to 2. So the number of branch points on $\mathbb{P}^1$ is $\#Bl(\mathbb{P}^1) = 2g+2$ and no branch point lies in $\hat{\pi}(\hat{S}_C)$.

Note that the Galois group $G = \langle J\rangle \cong \mathbb{Z}_2$ has two characters 1_G and χ^-, where $\chi^-(J) = -1$.

For $\chi = 1_G$, we have $\dim_k H^0(C, \omega_C)^G = g(\mathbb{P}^1) = 0$.

For χ^-, we have $\hat{S}_C^{\chi^-} = \hat{S}_C$, so $s_{\chi^-} = N$. The induced character at any branch point P is

$$\theta_P : G_P = G \to k^*, J \mapsto -1.$$

Therefore $R_{G,Q} = \theta_P$, so for all $Q \in Bl(\mathbb{P}^1)$, we have $\langle \chi^-, R_{G,Q}\rangle_G = 1$, and then $\langle \chi^-, R_G\rangle_G = 2g+2$. We calculate

$$\begin{aligned} m_{\chi^-}(\hat{S}_C^{\chi^-}) &= s_{\chi^-} + \sum_{Q\in Bl(\mathbb{P}^1)} \left\lfloor \frac{e_Q - 1}{e_Q} + \frac{1}{2}\langle \chi_{-1}, R_{G,Q}\rangle_G \right\rfloor - \frac{1}{2}\langle \chi^-, R_G\rangle_G \\ &= N + 2g + 2 - (g+1) = p_a(C) + 1. \end{aligned} \tag{3.16}$$

Finally, we get $\dim_k H^0(C, \omega_C)_{\chi^-} = g(\mathbb{P}^1) - 1 + m_{\chi^-}(\hat{S}_C^{\chi^-}) = p_a(C)$.

4. Nodal curves with several irreducible components

In this section, let X be a connected nodal curve with several irreducible components. Suppose that X admits a finite automorphism group G, and let $\pi : X \to Y = X/G$ be the quotient morphism. Consequently, $Y = X/G$ is also a connected nodal curve.

We will prove that $H^0(X, \omega_X)^G = \pi^* H^0(Y, \omega_Y)$ and then generalize Theorem 3.13 to the Chevalley-Weil formula for connected nodal curves, assuming the smoothness of Y for the same reasons stated in Remark 3.9.

Due to the abundance of symbols, in the following, we will replace some round brackets with square brackets, such as using $H^0(Y, \omega_Y[I_Y])$ instead of $H^0(Y, \omega_Y(I_Y))$.

4.1. Reduction

Let $Y = \cup_j Y_j$ be the decomposition into irreducible components. Then

$$\coprod Y_j \to Y$$

is the normalization at the intersection points. Let I_{Y_j} be the set of intersection points of Y_j with other components, and we call it the intersection locus of Y_j. Note that these points are nodes in the whole Y, but smooth points in each Y_j.

Then we can decompose the regular differential of Y by the irreducible components:

$$H^0(Y, \omega_Y) \hookrightarrow \oplus_j H^0(Y_j, \omega_{Y_j}[I_{Y_j}]),\ \varphi \mapsto (\varphi|_{Y_j})_j. \tag{4.1}$$

Furthermore let $\hat{Y}_j \to Y_j$ be the normalization. Then we have

$$H^0(Y_j, \omega_{Y_j}[I_{Y_j}]) \hookrightarrow H^0(\hat{Y}_j, \Omega_{\hat{Y}_j}[\hat{S}_{Y_j} \sqcup I_{Y_j}]).$$

Now let $X'_j := \pi^{-1}(Y_j)$ be the preimage of an irreducible component. This is a nodal curve but could be disconnected and we have*

$$H^0(X, \omega_X) \hookrightarrow \oplus_j H^0(X'_j, \omega_{X'_j}[I_{X'_j}]), \tag{4.2}$$

where $I_{X'_j} = \pi^{-1}(I_{Y_j})$. In fact, the information of the eigenspace $H^0(X'_j, \omega_{X'_j})_\chi$ of the regular differentials is contained in any single irreducible component.

Let X be a (possibly disconnected) nodal curve decomposed into its irreducible components

$$X = \bigcup_{i=1}^{d} X_i,$$

and suppose that $Y = X/G$ is irreducible. Then the action of G on the set of irreducible components $\{X_1, \ldots, X_d\}$ is transitive, and in particular, all these irreducible components are isomorphic. Let $S \subset X$ be a G-invariant divisor, and set $S_i := S|_{X_i}$. Consequently, G acts on

$$\oplus_{i=1}^{d} H^0(X_i, \omega_{X_i}[S_i]).$$

More precisely, for any $\sigma \in G$ we define the permutation of the indices by setting $\sigma(i)$ via the relation $\sigma(X_i) = X_{\sigma(i)}$. Then for any $\varphi \in H^0(X_i, \omega_{X_i}[S_i])$ we have:

$$\sigma\varphi \in H^0(X_{\sigma^{-1}(i)}, \omega_{X_{\sigma^{-1}(i)}}[S_{\sigma^{-1}(i)}]).^\dagger$$

Let $G_i := \{\sigma \in G \mid \sigma(X_i) = X_i\}$ denote the stabilizer subgroup of X_i. Then we have:

Proposition 4.1. *Given a character $\chi : G \to k^\times$ with restriction $\chi_i := \chi|_{G_i} : G_i \to k^\times$ to G_i, we have a canonical projection*

$$p_i : \left(\oplus_{i=1}^{d} H^0(X_i, \omega_{X_i}[S_i])\right)_\chi \to H^0(X_i, \omega_{X_i}[S_i])_{\chi_i}, \tag{4.3}$$

which is an isomorphism.

Proof. We show this holds for p_1.

Suppose

$$(\varphi_1, \cdots, \varphi_d) \in \left(\oplus_{i=1}^{d} H^0(X_i, \omega_{X_i}[S_i])\right)_\chi .$$

Then for any $\sigma \in G$ we have

$$\sigma(\varphi_1, \cdots, \varphi_d) = (\sigma\varphi_i)_{\sigma^{-1}(i)} = (\chi(\sigma)\varphi_i)_i .$$

Since G acts transitively on the set $\{X_1, \ldots, X_d\}$, for each i there exists some $\sigma_i : X_i \to X_1$. Hence, we can express the tuple as

$$(\varphi_1, \ldots, \varphi_d) = \left(\varphi_1, \chi(\sigma_2)^{-1}\sigma_2\varphi_1, \ \ldots, \chi(\sigma_d)^{-1}\sigma_d\varphi_1\right),$$

*If a curve X decomposes as $X = X_1 \sqcup X_2$, then the space of differentials decomposes as $H^0(X, \omega_X) = H^0(X_1, \omega_{X_1}) \oplus H^0(X_2, \omega_{X_2})$.

†σ acts by pullback on the differentials (that is, if $\sigma : X_j \to X_i$ and $\varphi \in H^0(X_i, \omega_{X_j})$, then $\sigma\varphi := \sigma^*\varphi \in H^0(X_j, \omega_{X_j})$).

which shows that the tuple $(\varphi_i)_i$ is completely determined by its first component φ_1. This proves that the projection

$$p_1 : \left(\oplus_{i=1}^d H^0(X_i, \omega_{X_i}[S_i])\right)_\chi \longrightarrow H^0(X_1, \omega_{X_1}[S_1])$$

is injective. It remains to show that φ_1 lies in the χ_1-eigenspace. For any element $\tau \in G_1$ (the stabilizer of X_1), we have $\tau(1) = 1$ and $\chi(\tau) = \chi_1(\tau)$. Therefore,

$$\tau\varphi_1 = \chi_1(\tau)\varphi_1,$$

which confirms that φ_1 indeed belongs to $H^0(X_1, \omega_{X_1}[S_1])_{\chi_1}$.

Conversely, let

$$\varphi_1 \in H^0(X_1, \omega_{X_1}[S_1])_{\chi_1},$$

and for each i choose an isomorphism $\sigma_i : X_i \to X_1$ as before. We need to show that

$$\left(\chi(\sigma_i)^{-1}\sigma_i\varphi_1\right)_i$$

defines an element of

$$\left(\oplus_{i=1}^d H^0(X_i, \omega_{X_i}[S_i])\right)_\chi .$$

First, note that if we choose two different isomorphisms $\sigma_i, \tau_i : X_i \to X_1$, then $\tau_i^{-1}\sigma_i$ is an element of the stabilizer G_1 of X_1. Since φ_1 lies in the χ_1-eigenspace, we have

$$\tau_i^{-1}\sigma_i\varphi_1 = \chi_1(\tau_i^{-1}\sigma_i)\varphi_1.$$

Because the character χ restricts on G_1 to χ_1, this can be rewritten as

$$\tau_i^{-1}\sigma_i\varphi_1 = \chi(\tau_i)^{-1}\chi(\sigma_i)\varphi_1.$$

Multiplying both sides on the left by $\chi(\sigma_i)^{-1}\chi(\tau_i)\tau_i$ yields

$$\chi(\sigma_i)^{-1}\sigma_i\varphi_1 = \chi(\tau_i)^{-1}\tau_i\varphi_1.$$

Thus, the section

$$\left(\varphi_1, \chi(\sigma_2)^{-1}\sigma_2\varphi_1, \ldots, \chi(\sigma_d)^{-1}\sigma_d\varphi_1\right)$$

is independent of the particular choice of the isomorphisms σ_i.

Next, let $\tau \in G$ be arbitrary. We may assume that τ sends the index j to i, that is, $\tau(j) = i$. Then, by the definition of the action, we have

$$\tau\left(\chi(\sigma_i)^{-1}\sigma_i\varphi_1\right) = \chi(\sigma_i)^{-1}(\tau\sigma_i)\varphi_1.$$

Since $\tau\sigma_i$ and σ_j are both isomorphisms from X_j to X_1, we have

$$\chi(\tau\sigma_i)^{-1}(\tau\sigma_i)\varphi_1 = \chi(\sigma_j)^{-1}\sigma_j\varphi_1.$$

Thus,

$$\tau\left(\chi(\sigma_i)^{-1}\sigma_i\varphi_1\right) = \chi(\sigma_i)^{-1}\chi(\tau\sigma_i)\chi(\sigma_j)^{-1}\sigma_j\varphi_1 = \chi(\tau)\chi(\sigma_j)^{-1}\sigma_j\varphi_1.$$

It follows that

$$\tau\left(\varphi_1, \chi(\sigma_2)^{-1}\sigma_2\varphi_1, \ldots, \chi(\sigma_d)^{-1}\sigma_d\varphi_1\right) = \chi(\tau)\left(\varphi_1, \chi(\sigma_2)^{-1}\sigma_2\varphi_1, \ldots, \chi(\sigma_d)^{-1}\sigma_d\varphi_1\right).$$

In conclusion, we obtain that p_1 is an isomorphism. $\square$

Let X_j be a chosen irreducible component of X'_j. By Proposition 4.1, there are immersions

$$H^0(X'_j, \omega_{X'_j})_\chi \hookrightarrow H^0(X_j, \omega_{X_j}[I_{X_j}])_{\chi_j},$$

and

$$H^0(X'_j, \omega_{X'_j}[I_{X'_j}])_\chi \hookrightarrow H^0(X_j, \omega_{X_j}[I'_j \sqcup I_{X_j}])_{\chi_j},$$

where $\chi_j = \chi|_{G_j}$. Consequently, we obtain the chain of inclusions

$$H^0(X, \omega_X)_\chi \hookrightarrow \oplus_j H^0(X_j, \omega_{X_j}[I'_j \sqcup I_{X_j}])_{\chi_j} \hookrightarrow \oplus_j H^0(\hat{X}_j, \Omega_{\hat{X}_j}[\hat{S}_{X_j} \sqcup I_{X_j} \sqcup I'_j])_{\chi_j}. \tag{4.4}$$

Here:

- $G_j = \{\sigma \in G \mid \sigma(X_j) = X_j\}$ and $|G_j| = |G|/d_j$ if X'_j has d_j irreducible components.
- $I'_j = I_{X'_j} \cap X_j$.
- I_{X_j} is the set of intersection points of X_j with other components in X'_j.
- $\hat{S}_{X_j}$ is the preimage of nodes in the normalization $\hat{\alpha}_j : \hat{X}_j \to X_j$.

Now we consider the G-invariant regular differentials. Suppose $\chi = 1_G$ and consider the following diagram as (3.2):

$$\begin{tikzcd}
H^0(X, \omega_X)^G \arrow[d, hook] & & H^0(Y, \omega_Y) \arrow[ll, dashed] \arrow[d, hook] \\
\oplus_j H^0(\hat{X}_j, \Omega_{\hat{X}_j}[\hat{S}_{X_j} \sqcup I_{X_j} \sqcup I'_j])^{G_j} & & \oplus_j H^0(\hat{Y}_j, \Omega_{\hat{Y}_j}[\hat{S}_{Y_j} \sqcup I_{Y_j}]). \arrow[ll, hook', "\oplus_j \hat{\pi}_j^*"']
\end{tikzcd} \tag{4.5}$$

Lemma 4.2. *There is a canonical inclusion*

$$\pi^* H^0(Y, \omega_Y) \subseteq H^0(X, \omega_X)$$

that makes the diagram above commute.

Proof. Let

$$\varphi_Y \in H^0(Y, \omega_Y)$$

be a regular differential on Y and consider its image in the lower left-hand corner:

$$\hat{\pi}^* \varphi_Y \in \oplus_j H^0(\hat{X}_j, \Omega_{\hat{X}_j}[\hat{S}_{X_j} \sqcup I_{X_j} \sqcup I'_j])^{G_j}.$$

We need to verify that the residues of $\hat{\pi}^* \varphi_Y$ satisfy the relation

$$\mathrm{Res}_{P_1}(\hat{\pi}^* \varphi_Y) = -\mathrm{Res}_{P_2}(\hat{\pi}^* \varphi_Y),$$

where $\{P_1, P_2\}$ is a pair in $\hat{X}$, i.e., P_1 and P_2 are points lying over the same node P in X.

Without loss of generality, assume that $P_1 \in \hat{X}_1$, the normalization of X_1, which is a chosen irreducible component of X'_1. Then

$$\mathrm{Res}_{P_1}(\hat{\pi}^* \varphi_Y) = e_{P_1} \cdot \mathrm{Res}_{\hat{\pi}(P_1)} \varphi_Y,$$

where e_{P_1} is the ramification index of P_1 in the cover of smooth curves

$$\hat{\pi}_1 : \hat{X}_1 \to \hat{Y}_1.$$

According to Lemma 3.3, the remaining task is to establish the equality

$$e_{P_1} = e_{P_2}.$$

To see this, let l_P denote the total number of elements in the orbit of the point P (which is common to both P_1 and P_2) under the action of the group G. Then the orbit of P_1 has exactly l_P/d_1 elements in $\hat{X}_1$, where d_1 is the number of irreducible components in X_1'. Hence, by the very definition of the ramification index we obtain

$$e_{P_1} = \frac{|G_1|}{l_P/d_1} = \frac{|G|}{l_P}.$$

By the same argument to P_2, we also have

$$e_{P_2} = \frac{|G|}{l_P}.$$

This completes the proof. □

The same argument as Lemma 3.4 tells us

$$H^0(X, \omega_X)^G \hookrightarrow \oplus_j H^0(\hat{X}_j, \Omega_{\hat{X}_j}[\hat{\pi}_j^{-1}(\hat{S}_{Y_j} \sqcup I_{Y_j})])^{G_j}.$$

With the notations above, we have:

Theorem 4.3. *The upper and lower row of the canonical commutative diagram*

$$\begin{tikzcd} H^0(X, \omega_X)^G \arrow[d, hook] & H^0(Y, \omega_Y) \arrow[l, "\sim"'] \arrow[d, hook] \\ \oplus_j H^0(\hat{X}_j, \Omega_{\hat{X}_j}[\hat{\pi}_j^{-1}(\hat{S}_{Y_j} \sqcup I_{Y_j})])^{G_j} & \oplus_j H^0(\hat{Y}_j, \Omega_{\hat{Y}_j}[\hat{S}_{Y_j} \sqcup I_{Y_j}]) \arrow[l, "\sim"'] \end{tikzcd}$$

are both isomorphisms. In particular, we have $\dim_k H^0(X, \omega_X)^G = p_a(Y)$.

Proof. By Proposition 3.6, we have

$$\hat{\pi}_j^* H^0(\hat{Y}_j, \Omega_{\hat{Y}_j}[\hat{S}_{Y_j} \sqcup I_{Y_j}]) \xrightarrow{\sim} H^0(\hat{X}_j, \Omega_{\hat{X}_j}[\hat{\pi}_j^{-1}(\hat{S}_{Y_j} \sqcup I_{Y_j})])^{G_j}.$$

So we have the isomorphism for the lower row. Moreover, both $H^0(X, \omega_X)^G$ and $H^0(Y, \omega_Y)$ are the subspaces of log differentials satisfying the residue relations, so we have the isomorphism for the upper row. □

4.2. Chevalley-Weil formula for connected nodal curves

Let Y be smooth for the remaining part of this section. We are going to calculate $\dim_k H^0(X,\omega_X)_\chi$ for general χ.

Let $X = \cup_{i=1}^d X_i$ be the decomposition of irreducible components, and then we can decompose the regular differential space on X into each irreducible component:

$$H^0(X,\omega_X) \hookrightarrow \oplus_{i=1}^d H^0(X_i,\omega_{X_i}[I_i]),\ \varphi \mapsto (\varphi|_{X_i}), \tag{4.6}$$

where I_i is the intersection locus of X_i.

Since Y is smooth, then by the criterion in Remark 3.9, set

$$I_i^\chi := \{P \in I_i \mid \exists\tau \in G_P - G_i \text{ s.t. } \chi(\tau) = -1\} \tag{4.7}$$

as those intersection points that could be the poles of $\varphi|_{X_i}$ for $\varphi \in H^0(X,\omega_X)_\chi$, and then we have the isomorphism

$$H^0(X,\omega_X)_\chi \xrightarrow{\sim} \left(\oplus_{i=1}^d H^0(X_i,\omega_{X_i}[I_i^\chi])\right)_\chi. \tag{4.8}$$

Applying Proposition 4.1 shows that

$$\left(\oplus_{i=1}^d H^0(X_i,\omega_{X_i}[I_i^\chi])\right)_\chi \xrightarrow{\sim} H^0(X_1,\omega_{X_1}[I_1^\chi])_{\chi_1}. \tag{4.9}$$

So far, our research object has been reduced to the action of the stabilizer subgroup G_1 on the irreducible nodal curve X_1.

Remark 4.4. *With the notations above, note that $I_i^\chi = \varnothing$ when $\chi = 1_G$. For the quotient morphism $\pi_1 : X_1 \to X_1/G_1 \cong Y$, we have*

$$H^0(X,\omega_X)^G \xrightarrow{\sim} H^0(X_1,\omega_{X_1})^{G_1} \xrightarrow{\sim} H^0(Y,\omega_Y). \tag{4.10}$$

This first isomorphism is by (4.9) and the second is by Theorem 3.5. This is a special case of Theorem 4.3, and we have

$$\dim_k H^0(X,\omega_X)^G = g(Y). \tag{4.11}$$

Now it remains to calculate $\dim_k H^0(X_1,\omega_{X_1}[I_1^\chi])_{\chi_1}$.

Let $\hat{\pi}_1 : \hat{X}_1 \to Y$ be the normalization of π_1, $Bl(Y)$ the branch locus, R_{G_1} the ramification module of $\hat{\pi}_1$, and $R_{G_1,Q}$ the ramification module of $Q \in Y$. If we let $\#G = n$, then $n_1 := \#G_1 = n/d$.

Let $\hat{S}_{X_1}^{\chi_1}$ be the singular χ_1-set of X_1 as (3.5), and then we have the isomorphism

$$H^0(X_1,\omega_{X_1}[I_1^\chi])_{\chi_1} \xrightarrow{\sim} H^0(\hat{X}_1,\Omega_{\hat{X}_1}[\hat{S}_{X_1}^{\chi_1} \cup I_1^\chi])_{\chi_1} \tag{4.12}$$

by the same argument as in Proposition 3.7.

By Proposition 2.7 (2.5) again, we have

$$H^0(\hat{X}_1,\Omega_{\hat{X}_1}[\hat{S}_{X_1}^{\chi_1} \cup I_1^\chi])_{\chi_1} = f_{\chi_1} \cdot \hat{\pi}_1^* H^0(Y,\Omega_Y\left\lfloor n_1^{-1}\pi_*\left(\hat{S}_{X_1}^{\chi_1} \cup I_1^\chi + (f_{\chi_1}) + R_{\hat{\pi}_1}\right)\right\rfloor),$$

where $f_{\chi_1} \in K(X_1)^\times$ satisfies $\sigma f_{\chi_1} = \chi_1(\sigma) f_{\chi_1},\ \forall\sigma \in G_1$.

Set $D_{\chi_1} := \left\lfloor n_1^{-1}\pi_*\left(\hat{S}_{X_1}^{\chi_1} \cup I_1^{\chi} + (f_{\chi_1}) + R_{\hat{\pi}_1}\right)\right\rfloor$. By the Riemann-Roch theorem, we have

$$\dim_k H^0(X, \omega_X)_\chi = \dim_k H^0(Y, \Omega_Y(D_{\chi_1})) = \dim_k H^0(Y, O_Y(-D_{\chi_1})) + \deg D_{\chi_1} + g_Y - 1. \tag{4.13}$$

Applying the calculation of Lemma 3.12, we have

$$\deg D_{\chi_1} = m_{\chi_1}(\hat{S}_{X_1}^{\chi_1} \cup I_1^{\chi}),$$

and see that

$$m_{\chi_1}(\hat{S}_{X_1}^{\chi_1} \cup I_1^{\chi}) = \#\hat{\pi}_1(\hat{S}_{X_1}^{\chi_1} \cup I_1^{\chi}) + \sum_{Q \notin \hat{\pi}_1(\hat{S}_{X_1}^{\chi_1} \cup I_1^{\chi})} \left\lfloor \frac{e_Q - 1}{e_Q} + \frac{d}{n}\langle \chi_1, R_{G_1,Q}\rangle_{G_1} \right\rfloor - \frac{d}{n}\langle \chi_1 R_{G_1}\rangle_{G_1} \tag{4.14}$$

in Definition 3.11.

Finally, through the discussion of Lemma 3.10, we get

$$\dim_k H^0(Y, O_Y(-D_{\chi_1})) = \delta_\chi, \tag{4.15}$$

where $\delta_\chi = 0$ or 1. $\delta_\chi = 1$ if and only if D_{χ_1} is principal; in this case, $I_1^{\chi} = \varnothing$ and $\chi_1 = 1_{G_1}$. ‡

In summary, we have obtained the following:

Let X be a connected nodal curve with d irreducible components, and G an automorphism group on X of order n. Assume that the quotient curve $Y = X/G$ is smooth. Then there exists a canonical ramified cover $\hat{\pi}_1 : \hat{X}_1 \to X_1/G_1 \simeq Y$ for an irreducible component X_1 with $G_1 = \{\sigma \in G \mid \sigma(X_1) = X_1\}$.

Theorem 4.5 (The Chevalley-Weil formula on connected nodal curves). *Let $\chi : G \to k^\times$ be a character. If the quotient curve $Y = X/G$ is smooth, then*

$$\dim_k H^0(X, \omega_X)_\chi = g_Y - 1 + m_{\chi_1}(\hat{S}_{X_1}^{\chi_1} \cup I_1^{\chi}) + \delta_\chi, \tag{4.16}$$

where χ_1 is the restriction of χ on G_1, and see $m_{\chi_1}(\hat{S}_{X_1}^{\chi_1} \cup I_1^{\chi})$ in (4.14) and δ_χ in (4.15).

In particular, when $\chi = 1_G$, we have $\dim_k H^0(X, \omega_X)^G = g_Y$.

Example 4.6. Let the curves $C_1 \simeq C_2 \simeq \mathbb{P}^1$ intersect transversely at $m > 2$ points, and consider the hyperelliptic stable curve $C = C_1 \cup C_2$, with the involution J which permutes C_1 and C_2. Then we have the quotient mapping $\pi : C \to C/\langle J\rangle = \mathbb{P}^1$. The arithmetic genus of C is $p_a(C) = m - 1$ (see [11, Proposition 7.5.4 and Lemma 10.3.18]) and the Galois group $\langle J\rangle$ has two characters 1_G and χ^-.

For χ^-, note that the cover $\pi_1 : C_1 \to \mathbb{P}^1$ induced by π is an isomorphism, so we have $\chi_1^- = id$. Therefore $\hat{S}_{C_1}^{id} = \varnothing$ and $m_{\chi_1^-}(\hat{S}_{C_1}^{id} \cup I_1^{\chi^-}) = \#\pi(I_1^{\chi^-}) = m$. By Theorem 4.5, we have

$$\dim_k H^0(C, \omega_C)_{\chi^-} = g_{\mathbb{P}^1} - 1 + m = m - 1 = p_a(C). \tag{4.17}$$

For $\chi = 1_G$, by (4.11), we have $\dim_k H^0(C, \omega_C)^G = p_a(\mathbb{P}^1) = 0$.

‡In this context, we require that $\hat{S}_{X_1}^{\chi_1} \cup I_1^{\chi}$ is empty. In particular, when $\chi_1 = 1_{G_1}$, note that $\hat{S}_{X_1}^{\chi_1} = \varnothing$; however, I_1^{χ} is determined by χ rather than χ_1. See Example 4.6 where even though $\chi_1^- = $ id, $I_1^{\chi^-}$ is nonempty.

5. Conclusions

In this work, we have systematically investigated the eigenspaces of regular differentials on a nodal curve X under the action of a finite automorphism group G. Our primary achievement is the demonstration that the dimension of the space of G-invariant regular differentials is the arithmetic genus of the quotient curve X/G, providing a non-trivial generalization of the smooth case. Furthermore, under the condition that X/G is smooth, we have successfully extended the Chevalley-Weil formula to the nodal setting, obtaining an exact expression for the dimension of the eigenspace for any one-dimensional character χ. This was accomplished by first addressing irreducible nodal curves and incorporating a singular χ-set, then reducing the analysis on a conncected nodal curve to that on one of its irreducible components. This reduction involves adding correction terms that account for certain intersection points on that component. These explicit formulas illuminate how nodal singularities (including intersection points) contribute to the dimensions of these eigenspaces, advancing our understanding of group actions on differentials of nodal curves.

Use of Generative-AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no competing interests.

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