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Research article

Stability analysis of fractional-order quaternion-valued neural networks with multiple delays and parameter uncertainties

Guoqing Jiang[1], **Xiaolan Liu**[1,2,3,*], **Lei Wang**[1] **and Chongwei Zheng**[4]

[1] College of Mathematics and Statistics, Sichuan University of Science and Engineering, Zigong, Sichuan, 643000, China

[2] Key Laboratory of Higher Education of Sichuan Province for Enterprise Informationalization and Internet of Things, Zigong, Sichuan, 643000, China

[3] South Sichuan Center for Applied Mathematics, Zigong, Sichuan, 643000, China

[4] Department of Navigation, Dalian Naval Academy, Chinese People's Liberation Army, Dalian, 116000, China

* **Correspondence:** Email: xiaolanliu@suse.edu.cn.

Abstract: This paper proposes a fractional-order quaternion-valued neural network (FOQVNN) model with dual proportional delays, neutral delays, and parameter uncertainties. By leveraging classical lemmas and a relaxed linear matrix inequality (LMI) condition, we prove not only the uniqueness of the equilibrium point in the proposed model, but also the global robust stability of this equilibrium. Finally, numerical simulations were provided to validate the theoretical results.

Keywords: double proportional delay; neutral delay; parameter uncertainty; globally robust stable; LMI
Mathematics Subject Classification: 34A34, 34D23, 37N25

1. Introduction

With Hopfield's seminal work on neural networks (NNs) in 1984 [1], NNs have been extensively applied to diverse domains such as image processing [2,3], pattern recognition [4,5], and combinatorial optimization [6–8]. In many practical uses, we always hope that NNs have a unique stable equilibrium point, because uniqueness of the equilibrium point can ensure the stability of the system. Therefore, it is valuable to study the stability of the equilibrium point of NNs.

In fact, the tendency of many systems to change is not only related to the current state, but also hinges on the past state, which is referred to as "delay". Time delay is ubiquitous in a wide range of practical systems. There are various reasons for the delay, such as the delay in the sensor measurement process and the delay in the signal transmission process [9,10]. Time delays tend to degrade system performance and even destabilize the system, leading to undesirable phenomena such as instability, oscillations, and chaos [11–14]. Because of its universality, it is necessary to consider time delay in neural network models. But time delay can be divided into various types, such as leakage delay, proportional delay, neutral delay, and ordinary delay, etc. Leakage delay is a phenomenon observed in a specific circuit environment. The existence of a leakage delay can affect the security of an encrypted device because it can lead to additional information leaked, and in practice, leakage delay is widespread [15,16]. Proportional latency is a concept related to server overload management and request prioritization. Its main purpose is to ensure that the quality of service for high-priority requests is no lower than that for low-priority requests when the server is busy. This is usually achieved by setting the distinguishing parameters of two types of data streams and controlling their delay time ratio. The application of proportional latency can help optimize server performance, especially when handling large and diverse requests [17–19]. A neural delay exists not only in the state itself, but also in the derivatives of past states [20–22]. A neutral neural network can better reflect the whole process of network transmission. Neural network models with time delays have been extensively studied in [23–29]. Zhang et al. [27] investigated a neutral-type delayed generalized Cohen-Grossberg Binomial associative memory (BAM) neural network. By applying the homeomorphism principle and inequality techniques, they constructed appropriate Lyapunov functions to establish criteria for the model's global asymptotic stability. In a related study, Yu et. al. [23] addressed a class of uncertain large-scale systems with time delays, where exponential stability was derived through convex combination techniques and time-varying Lyapunov functionals.

In real life, since we mainly deal with practical problems in a three-dimensional space, quaternions can more concisely represent any rotation in three-dimensional space than real and complex numbers, thereby reducing the dimensionality of the system and improving computational efficiency [30]. Therefore, quaternions have been widely used. Quaternion neural networks have some advantages over traditional real or complex neural networks [30–32]. The stability and synchronization phenomena in fractional-order quaternion-valued neural networks were systematically examined in [24–26], which extended the theoretical framework to non-integer-order dynamics and quaternion-valued systems.

In recent years, fractional calculus, due to its genetic and wireless storage characteristics, can better describe the memory and inheritance of various materials and processes accurately [33], so fractional calculus can achieve results and a wider range of applications that integer derivatives cannot achieve. Therefore, many scholars have innovated fractional derivative operators into NNs. In recent decades, the fractional-order neural network's stability has been studied by many scholars, and a large quantity of results have been obtained. For example, fractional-order real-value neural networks with time delays were discussed in [34,35], sufficient conditions for consistent stability were established, and the existence of a unique globally stable equilibrium point was proved. Rakkiyappan et al. [36] studied the asymptotic stability of fractional-order real-value neural networks with a fixed time delay. The sufficient conditions were obtained to ensure the asymptotic stability of the neural network. Fractional-order complex-value neural networks with delay have been investigated in [35–38], and sufficient conditions were provided to ensure the stability of these neural networks. Fractional-order quaternion-valued neural networks (FOQVNNs) with time delay were studied in [39–42], where Humphries et al. [39] used the real-valued decomposition method to provide a new sufficient condition. Ali et al. [40] investigated the dissipativity analysis of the FOQVNN with time delays. By employing a decomposition method, they proved the dissipativity of the model. Furthermore, the authors established a new criterion for verifying both global dissipativity and exponential stability of the FOQVNNs model using fractional calculus theory and inequality techniques. The direct method was used to derive sufficient conditions or criteria for stability in [33–36]. In reality, for the link weight matrix, perturbations often occur over time, so Song et al. [31] studied FOQVNNs with time-varying uncertain parameters . Under the proposed two assumptions, based on inequality techniques, the homeomorphism principle, and Lyapunov stability theory, they derived a stability criterion that guarantees the existence of a unique global stable equilibrium point for the model and gave it in the form of a linear matrix inequality (LMI).

Inspired by previous work [22], we propose a multi-delay FOQVNN model incorporating dual proportional delays, neutral delays, and parameter uncertainties. Compared with most existing literature on fractional-order quaternion-valued neural networks (FOQVNNs), the main innovations and contributions of this paper are as follows:

(i) By using direct methods, there is no need to decompose the model into four real-valued systems or two complex-valued systems;
(ii) Stability criteria are established for both delay-independent and delay-dependent cases;
(iii) Non-positive-definite linear matrix inequality criteria have been derived and can be efficiently verified via the MATLAB LMI Toolbox.

The remainder of this paper is organized as follows: In Preliminary, some symbol explanation, definitions and necessary lemmas of fractional derivatives are given. In our main results, by some lemmas and a weak condition of LMI, we derive the uniqueness of the equilibrium point and the stability of the model. Then some related results and remarks are also provided. Finally, a numerical example was provided to validate our results.

2. Preliminaries

This section includes some notations, definitions, and important lemmas needed for this paper. $\mathbb{Q}$ represents a quaternion skewed field. The symbols $\mathbb{Q}^n$ and $\mathbb{Q}^{n\times n}$ refer to n-dimensional and $n \times n$-dimensional quaternion systems, respectively. The conjugate transpose of matrix B is represented by F^*. The minimum eigenvalue of quaternion matrix B is denoted by $\lambda_{min}(B)$. For $B \in \mathbb{Q}^n$, if $B > 0$ and B is a Hermitian matrix, then $F^* = B$. $B > 0$ is expressed as $x^*Bx > 0$ for any non-zero vector x. $B \geq 0$ is expressed as $x^*Bx \geq 0$ for any non-zero vector x. $\|\cdot\|$ represents vector norms. I represents the identity matrix.

A quaternion is a type of hypercomplex number that is shaped like $a_1 + b_1 i + c_1 j + d_1 k$, where a_1, b_1, c_1, d_1 are real numbers, i, j, k represent imaginary units that separately comply with the following rules:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

Then, the addition of quaternions is defined as

$$a_1 + b_1 i + c_1 j + d_1 k + a_2 + b_2 i + c_2 j + d_2 k = (a_1 + a_2) + i(b_1 + b_2) + j(c_1 + c_2) + k(d_1 + d_2),$$

and the multiplication of quaternions is defined as

$$(a_1 + b_1 i + c_1 j + d_1 k) \times (a_2 + b_2 i + c_2 j + d_2 k)$$

$$= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + i(a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)$$
$$+ j(c_1a_2 + a_1c_2 + d_1b_2 - b_1d_2) + k(a_1d_2 + d_1a_2 + b_1c_2 - c_1b_2).$$

It is easy to observe that multiplication of quaternions does not follow commutative law. The conjugate definition of quaternions is

$$\overline{a + bi + cj + dk} = a - bi - cj - dk,$$

and the modulus of quaternions is defined as

$$|a + bi + cj + dk| = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

Next, we provide the definitions of the fractional derivatives of quaternions and the Caputo fractional derivatives.

Definition 2.1. [40] *Let $f(t) \in \mathbb{Q}^n$ be a continuously differentiable function in $(a, +\infty)$. The fractional integral order α of function f(t) can be expressed as*

$$_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} f(s)ds,$$

where $\Gamma(\cdot)$ is the gamma function, $\alpha \in (0, 1)$, and $\Gamma(s) = \int_0^{+\infty} t^{s-1}e^{-t}dt$.

Definition 2.2. [40] *The Caputo fractional derivative of order $\alpha \in (0, 1)$ for a differentiable function $f(t) \in \mathbb{Q}^n$ in the interval $(a, +\infty)$ can be expressed as:*

$${}_a^CD_t^{\alpha} f(t) = \frac{1}{\Gamma(1 - \alpha)} \int_a^t (t - s)^{-\alpha} f'(s)ds.$$

Obviously, ${}_a^CD_t^{\alpha} k = 0$ for any constant vector $k \in \mathbb{Q}^n$.

Definition 2.3. [43] *For the equilibrium solution q^*, let $q(t)$ denote the solution to the differential system with any initial condition $\varphi(t)$. If for every $\varepsilon > 0$, there exists a $\delta(\varepsilon, t_0) > 0$, when $|\varphi(t) - q^*| < \delta$, such that*

$$||q(t) - q^*|| < \varepsilon \;\; for \;\; t \geq t_0,$$

then this equilibrium point is called stable.

In order to obtain our results, we make some assumptions about the activation function and time-varying parameters.

(**H**1) There exist two positive diagonal matrices $L = \text{diag}\{l_1, l_2, \ldots, l_n\}$ and $K = \text{diag}\{k_1, k_2, \cdots, k_n\}$ such that

$$|f_i(u_1) - f_i(u_2)| \leqslant l_i|u_1 - u_2|, \quad |g_i(u_1) - g_i(u_2)| \leqslant k_i|u_1 - u_2|,$$

for all $u_1,\ u_2 \in \mathbb{Q}$.

(**H**2) The forms of time-varying parameters $\Delta D(t)$, $\Delta E(t)$, $\Delta F(t)$, and $\Delta C(t)$ are given by the following equations:

$$\Delta D(t) = G_DU_D(t)H_D, \quad \Delta E(t) = G_EU_E(t)H_E,$$
$$\Delta F(t) = G_FU_F(t)H_F, \quad \Delta C(t) = G_CU_C(t)H_C,$$

where $G_D, H_D, G_E, H_E, G_F, H_F, G_C$, and H_C are known constant matrices, $U_D(t), U_E(t), U_F(t)$, and $U_C(t)$ are time-varying matrices subjected by

$$U_D^*(t)U_D(t) \leqslant I, \quad U_E^*(t)U_E(t) \leqslant I, \quad U_F^*(t)U_F(t) \leqslant I, \quad U_C^*(t)U_C(t) \leqslant I,$$

$\Delta D(t)$ is a diagonal matrix, and $D + \Delta D(t)$ is a positive diagonal matrix.

Let $G = [G_D \quad G_E \quad G_F \quad G_C]$, $U(t) = \text{diag}\{U_D(t), U_E(t), U_F(t), U_C(t)\}$, $H = \text{diag}\{H_D, H_E, H_F, H_C\}$, and $GU(t)H = [-\Delta D(t)\ \Delta E(t)\ \Delta F(t)\ \Delta C(t)]$ with

$$U^*(t)U(t) \leqslant I.$$

Next we provide some necessary lemmas to facilitate the establishment of our main results.

Lemma 2.1. [22] *If $\mu, \nu \in \mathbb{Q}$ and $\varepsilon \in \mathbb{R}$, then*

$$\mu^*\nu + \nu^*\mu \leqslant \varepsilon\mu^*\mu + \varepsilon^{-1}\nu^*\nu.$$

Lemma 2.2. [22] *Let $A \in \mathbb{Q}^{n\times n}$, $C \in \mathbb{R}^{m\times m}$, $B \in \mathbb{Q}^{n\times m}$, A and C are symmetric, and then*

$$\begin{bmatrix} A & B \\ F^* & -C \end{bmatrix} \leq 0$$

is the same as in the following conditions: $C \geq 0$ and $A + BC^{-1}F^* \leq 0$.

Lemma 2.3. [22] *Let $f(t) : \mathbb{Q}^n \to \mathbb{Q}^n$ be a continuous map satisfying*
(i) $f(t)$ is injective on $\mathbb{Q}^n$,
(ii) $\lim\limits_{\|t\|\to+\infty} \|f(t)\| \to +\infty$.
Thus, $f(t)$ serves as a homeomorphism from $\mathbb{Q}^n$ to itself.

Lemma 2.4. [22] *Let $f(t) \in \mathbb{Q}^n$ be a differentiable function. Consider a Hermitian matrix Q belonging to $\mathbb{Q}^{n\times n}$, and assume that it is positive definite. Then*

$${}_0^C D_t^\alpha (f^*(t)Qf(t)) \leqslant f^*(t)Q({}_0^C D_t^\alpha f(t)) + \left({}_0^C D_t^\alpha f(t)\right)^* Qf(t).$$

In this paper, we propose the following FOQVNN:

$${}_0^C D_t^\alpha u(t) = -D(t)u(t) + E(t)f(u(\delta t)) + F(t)g(u(\sigma t)) + C(t) \times ({}_0^C D_t^\alpha u(t-\varrho)) + N(t), \tag{2.1}$$

for $t \geqslant 0$, where $u(t) \in \mathbb{Q}^n$ is a state vector that is associated with the neuron; $f(u(\delta t)) = (f_1(u_1(\delta t)), f_2(u_2(\delta t)), \cdots, f_n(u_n(\delta t)))^T \in \mathbb{Q}^n$ and $g(u(\sigma t)) = (g_1(u_1(\sigma t)), g_2(u_2(\sigma t)), \cdots, g_n(u_n(\sigma t)))^T \in \mathbb{Q}^n$ are vectors composed of activation functions; $0 < \delta < 1$ is the proportional delay, $0 < \sigma < 1$ is the proportional delay, ϱ is the neutral delay; and the $n \times n$ link matrices $D(t), E(t), F(t)$, and $C(t)$ illustrate the connectivity of neurons across different layers in a neural network. Here, matrix $D(t)$ is defined as a constant matrix, while matrices $E(t), F(t)$, and $C(t)$ are represented as quaternion matrices, allowing for a more complex representation of the interconnections. The link matrices $D(t), E(t), F(t)$, and $C(t)$ are obtained by adding the known matrices A, B, C, and D to the uncertain parameter matrices $\Delta D(t), \Delta E(t), \Delta F(t)$, and $\Delta C(t)$; D and $\Delta D(t)$ are positive diagonal matrices; $N(t)$ is the input vector, N is a known constant vector, and ΔN is an uncertain parameter.

3. Materials and methods

A weaker sufficient condition related to proportional delay is built below. Under this weak condition, we derive that system (2.1) has a unique globally robust stable equilibrium point.

Theorem 3.1. *Suppose that* (**H**1) *and* (**H**2) *hold. Let* $P_1, P_2 \in \mathbb{Q}^{n\times n}$ *be two positive definite matrices,* R_1 *and* $R_2 \in \mathbb{Q}^{n\times n}$ *be positive definite diagonal matrices, and* ρ , $\varepsilon_1 > 0, \varepsilon_2 > 0, \varepsilon_3 > 0, \varepsilon_4 > 0$ *be positive real numbers that satisfy the following LMI. Under these assumptions, then system (2.1) has one and only one equilibrium point, and it is globally robustly stable.*

$$\Pi(t) = \begin{bmatrix} \pi_{11} & \pi_{12} & 0 & \rho H^* & P_1 G \\ * & -P_2 & P_2 G & 0 & 0 \\ * & * & -\rho I & 0 & 0 \\ * & * & * & -\rho I & 0 \\ * & * & * & * & \pi_{55} \end{bmatrix} \leq 0,$$

where

$$\pi_{11} = \begin{bmatrix} \pi_{11}^{(11)} & P_1E & P_1F & P_1C \\ * & \pi_{11}^{(22)} & 0 & 0 \\ * & * & \pi_{11}^{(33)} & 0 \\ * & * & * & \pi_{11}^{(44)} \end{bmatrix}, \pi_{12} = \begin{bmatrix} -D^* \\ E^* \\ F^* \\ C^* \end{bmatrix} P_2, \pi_{55} = \mathrm{diag}\{-\varepsilon_1, -\varepsilon_2, -\varepsilon_3, -\varepsilon_4\},$$

with

$$\pi_{11}^{(11)} = -P_1D - DP_1 + \varepsilon_1 H_D^* H_D + LR_1L + KR_2K,\ \pi_{11}^{(22)} = \varepsilon_2 H_E^* H_E - \delta R_1,$$
$$\pi_{11}^{(33)} = \varepsilon_3 H_F^* H_F - \sigma R_2, \pi_{11}^{(44)} = \varepsilon_4 H_F^* H_F - P_2,$$

and the remaining part of π_{ij} *is zero.*

Proof. We aim to prove that system (2.1) has one and only one equilibrium point.

Assume that $\tilde{u} \in \mathbb{Q}^n$ represents an equilibrium point for system (2.1). Therefore, $\tilde{u}$ satisfies

$$-D(t)\tilde{u} + E(t)f(\tilde{u}) + F(t)g(\tilde{u}) + N(t) = 0, \tag{3.1}$$

for $t \geqslant 0$. The definition of the mapping is as follows:

$$\phi(v) = -D(t)v + E(t)f(v) + F(t)g(v) + C(t)\phi(v) + N(t). \tag{3.2}$$

First, it will be shown that $(\phi(u))$ is an injective map on $\mathbb{Q}^n$. For $\vartheta, \xi \in \mathbb{Q}^n$ and $\vartheta \neq \xi$, we only need to get $\phi(\vartheta) \neq \phi(\xi)$. From (3.2), we can obtain

$$\phi(\vartheta) - \phi(\xi) = -D(t)(\vartheta - \xi) + E(t)[f(\vartheta) - f(\xi)] + F(t)[g(\vartheta) - g(\xi)] + C(t)[\phi(\vartheta) - \phi(\xi)]. \tag{3.3}$$

Multiplying $(\vartheta - \xi)^* P_1$ on the left side in (3.3), then

$$\begin{aligned}(\vartheta - \xi)^* P_1[\phi(\vartheta) - \phi(\xi)] = &- (\vartheta - \xi)^* P_1 D(t)(\vartheta - \xi) + (\vartheta - \xi)^* P_1 E(t) \\ &\times [f(\vartheta) - f(\xi)] + (\vartheta - \xi)^* P_1 F(t)[g(\vartheta) \\ &- g(\xi)] + (\vartheta - \xi)^* P_1 C(t)[\phi(\vartheta) - \phi(\xi)].\end{aligned} \tag{3.4}$$

Performing the conjugate transpose on (3.4), it yields that

$$
\begin{aligned}
{}^{*}P_1(\vartheta-\xi) = &- (\vartheta-\xi)^{*}D^{*}(t)P_1(\vartheta-\xi) + [f(\vartheta)-f(\xi)]^{*} \\
&\times E^{*}(t)P_1(\vartheta-\xi) + [g(\vartheta)-g(\xi)]^{*}F^{*}(t)P_1(\vartheta-\xi) \\
&+ [\phi(\vartheta)-\phi(\xi)]^{*}C^{*}(t)P_1(\vartheta-\xi).
\end{aligned} \tag{3.5}
$$

Similarly, it deduces from (3.5) that

$$
\begin{aligned}
{}^{*}P_2[\phi(\vartheta)-\phi(\xi)] = &(-D(t)(\vartheta-\xi) + E(t)[f(\vartheta)-f(\xi)] \\
&+ F(t)[g(\vartheta)-g(\xi)] + C(t)[\phi(\vartheta)-\phi(\xi)])^{*} \\
&\times P_2(-D(t)(\vartheta-\xi) + E(t)[f(\vartheta)-f(\xi)]) \\
&+ F(t)[g(\vartheta)-g(\xi)] + C(t)[\phi(\vartheta)-\phi(\xi)].
\end{aligned} \tag{3.6}
$$

Combing (3.4), (3.5), and (3.6), we have that

$$
\begin{aligned}
&(\vartheta-\xi)^{*}P_1[\phi(\vartheta)-\phi(\xi)] + [\phi(\vartheta)-\phi(\xi)]^{*}P_1(\vartheta-\xi) \\
= &(\vartheta-\xi)^{*}(-P_1D(t) - D(t)P_1)(\vartheta-\xi) \\
&+ (\vartheta-\xi)^{*}P_1E(t)[f(\vartheta)-f(\xi)] + [f(\vartheta)-f(\xi)]^{*}E^{*}(t)P_1(\vartheta-\xi) \\
&+ (\vartheta-\xi)^{*}P_1F(t)[g(\vartheta)-g(\xi)] + [g(\vartheta)-g(\xi)]^{*}F^{*}(t)P_1(\vartheta-\xi) \\
&+ (\vartheta-\xi)^{*}P_1C(t)[\phi(\vartheta)-\phi(\xi)] + [\phi(\vartheta)-\phi(\xi)]^{*}C^{*}(t)P_1(\vartheta-\xi) \\
&- [\phi(\vartheta)-\phi(\xi)]^{*}P_2[\phi(\vartheta)-\phi(\xi)] \\
&+ (-D(t)(\vartheta-\xi) + E(t)[f(\vartheta)-f(\xi)] \\
&+ F(t)[g(\vartheta)-g(\xi)] + C(t)[\phi(\vartheta)-\phi(\xi)])^{*} \\
&\times P_2(-D(t)(\vartheta-\xi) + E(t)[f(\vartheta)-f(\xi)] \\
&+ F(t)[g(\vartheta)-g(\xi)] + C(t)[\phi(\vartheta)-\phi(\xi)]).
\end{aligned} \tag{3.7}
$$

By Lemma 2.1, we can derive the following four inequalities:

$$
\begin{aligned}
&(\vartheta-\xi)^{*}(-P_1D(t) - D^{*}(t)P_1)(\vartheta-\xi) \\
\leqslant &(\vartheta-\xi)^{*}(-P_1D - D^{*}P_1)(\vartheta-\xi) \\
&+ \varepsilon_1^{-1}(\vartheta-\xi)^{*}P_1G_DG_D^{*}P_1(\vartheta-\xi) + \varepsilon_1(\vartheta-\xi)^{*}N_D^{*}N_D(\vartheta-\xi),
\end{aligned} \tag{3.8}
$$

$$
\begin{aligned}
&(\vartheta-\xi)^{*}P_1E(t)[f(\vartheta)-f(\xi)] + [f(\vartheta)-f(\xi)]^{*}E^{*}(t)P_1(\vartheta-\xi) \\
\leqslant &(\vartheta-\xi)^{*}\,P_1A[f(\vartheta)-f(\xi)] + \left[f(\vartheta)-f(\xi)\right]^{*}\,E^{*}P_1(\vartheta-\xi) \\
&+ \varepsilon_2^{-1}(\vartheta-\xi)^{*}P_1G_EG_E^{*}P_1(\vartheta-\xi) + \varepsilon_2[f(\vartheta)-f(\xi)]^{*}N_E^{*}N_E[f(\vartheta)-f(\xi)],
\end{aligned} \tag{3.9}
$$

$$
\begin{aligned}
&(\vartheta-\xi)^{*}P_1F(t)[g(\vartheta)-g(\xi)] + [g(\vartheta)-g(\xi)]^{*}F^{*}(t)P_1(\vartheta-\xi) \\
\leqslant &(\vartheta-\xi)^{*}\,P_1B[g(\vartheta)-g(\xi)] + \left[g(\vartheta)-g(\xi)\right]^{*}\,F^{*}P_1(\vartheta-\xi) \\
&+ \varepsilon_3^{-1}(\vartheta-\xi)^{*}P_1G_FG_F^{*}P_1(\vartheta-\xi) + \varepsilon_3[g(\vartheta)-g(\xi)]^{*}H_F^{*}H_F[g(\vartheta)-g(\xi)],
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
&(\vartheta-\xi)^{*}P_1C(t)[\phi(\vartheta)-\phi(\xi)] + [\phi(\vartheta)-\phi(\xi)]^{*}C^{*}(t)P_1(\vartheta-\xi) \\
\leqslant &(\vartheta-\xi)^{*}P_1C[\phi(\vartheta)-\phi(\xi)] + [\phi(\vartheta)-\phi(\xi)]^{*}C^{*}P_1(\vartheta-\xi) \\
&+ \varepsilon_4^{-1}(\vartheta-\xi)^{*}P_1G_CG_C^{*}P_1(\vartheta-\xi) + \varepsilon_4[\phi(\vartheta)-\phi(\xi)]^{*}H_C^{*}H_C[\phi(\vartheta)-\phi(\xi)].
\end{aligned} \tag{3.11}
$$

Since R_1 and R_2 are positive diagonal matrices, by (**H**1), the following inequalities hold:

$$0 \leqslant (\vartheta - \xi)^* L R_1 L(\vartheta - \xi) - [f(\vartheta) - f(\xi)]^* R_1 [f(\vartheta) - f(\xi)], \tag{3.12}$$

and

$$0 \leqslant (\vartheta - \xi)^* K R_2 K(\vartheta - \xi) - [g(\vartheta) - g(\xi)]^* R_2 [g(\vartheta) - g(\xi)]. \tag{3.13}$$

From (3.7)–(3.13), it can be concluded that

$$(\vartheta - \xi)^* P_1 [\phi(\vartheta) - \phi(\xi)] + [\phi(\vartheta) - \phi(\xi)]^* P_1 (\vartheta - \xi) \leqslant \Lambda_1^* \psi(t) \Lambda_1, \tag{3.14}$$

where

$$\Lambda_1 = [(\vartheta - \xi)^*, (f(\vartheta) - f(\xi))^*, (g(\vartheta) - g(\xi))^*, (\phi(\vartheta) - \phi(\xi))^*]^*,$$

$$\psi(t) = \begin{bmatrix} \psi_{11}(t) & P_1 E - D^*(t) P_2 E(t) & P_1 F - D^*(t) P_2 F(t) & P_1 C - D^*(t) P_2 C(t) \\ * & \psi_{22}(t) & E^*(t) P_2 F(t) & E^*(t) P_2 C(t) \\ * & * & \psi_{33}(t) & F^*(t) P_2 C(t) \\ * & * & * & \psi_{44}(t) \end{bmatrix},$$

with

$$\begin{aligned} \psi_{11}(t) = &- P_1 D - D P_1 + \varepsilon_1^{-1} P_1 G_D G_D^* P_1 + \varepsilon_1 H_D^* H_D + \varepsilon_2^{-1} P_1 G_E G_E^* P_1 \\ &+ \varepsilon_3^{-1} P_1 G_F G_F^* P_1 + \varepsilon_4^{-1} P_1 G_C G_C^* P_1 + D(t) P_2 D(t) + L R_1 L + K R_2 K, \\ \psi_{22}(t) = &\ \varepsilon_2 H_E^* H_E + E^*(t) P_2 E(t) - R_1, \\ \psi_{33}(t) = &\ \varepsilon_3 H_F^* H_F + F^*(t) P_2 F(t) - R_2, \\ {}_{44}(t) = &\ \varepsilon_4 H_C^* H_C + C^*(t) P_2 C(t) - P_2. \end{aligned}$$

In view of Lemma 2.2 and $\Pi(t) \leq 0$, we can acquire that

$$\begin{bmatrix} \widehat{\pi}_{11} & \pi_{12} & 0 & \rho H^* \\ * & -P_2 & P_2 M & 0 \\ * & * & -\rho I & 0 \\ * & * & * & -\rho I \end{bmatrix} \leqslant 0, \tag{3.15}$$

where

$$\begin{aligned} \widehat{\pi}_{11}(t) = &- P_1 D - D P_1 + \varepsilon_1^{-1} P_1 G_D G_D^* P_1 + \varepsilon_1 H_D^* H_D \\ &+ \varepsilon_2^{-1} P_1 G_E G_E^* P_1 + \varepsilon_3^{-1} P_1 G_F G_F^* P_1 \\ &+ \varepsilon_4^{-1} P_1 G_C G_C^* P_1 + D(t) P_2 D(t) + L R_1 L + K R_2 K. \end{aligned}$$

By virtue of Lemma 2.2 in (3.15), it follows that

$$\begin{bmatrix} \widehat{\pi}_{11} & \pi_{12} \\ * & -P_2 \end{bmatrix} + \rho^{-1} \begin{bmatrix} 0 \\ P_2 G \end{bmatrix} \begin{bmatrix} 0 \\ P_2 G \end{bmatrix}^* + \rho \begin{bmatrix} H^* \\ 0 \end{bmatrix} \begin{bmatrix} H^* \\ 0 \end{bmatrix}^* \leqslant 0.$$

In view of (**H**2) and LMI, it is easy to conclude that

$$\begin{bmatrix} \widehat{\pi}_{11} & \pi_{12} \\ * & -P_2 \end{bmatrix} + \begin{bmatrix} 0 \\ P_2 G \end{bmatrix} U(t) \begin{bmatrix} H^* \\ 0 \end{bmatrix}^* + \begin{bmatrix} H^* \\ 0 \end{bmatrix} U^*(t) \begin{bmatrix} 0 \\ P_2 G \end{bmatrix}^* \leqslant 0,$$

that is,

$$\begin{bmatrix} \widehat{\pi}_{11} & \pi_{12} + H^* U^*(t) G^* P_2 \\ * & -P_2 \end{bmatrix} \leqslant 0.$$

Applying Lemma 2.2, we have

$$\widehat{\pi}_{11} + (\pi_{12} + H^* U^*(t) G^* P_2) P_2^{-1} (\pi_{12} + H^* U^*(t) G^* P_2)^* \leqslant 0.$$

According to the operation properties of matrices, it can be calculated that

$$\widehat{\pi}_{11} + (\pi_{12} + H^* U^*(t) G^* P_2) P_2^{-1} (\pi_{12} + H^* U^*(t) G^* P_2)^* = \psi_1(t), \tag{3.16}$$

where

$$\psi_1(t) = \begin{bmatrix} \psi_{11}(t) & P_1 E - D^*(t) P_2 E(t) & P_1 F - D^*(t) P_2 F(t) & P_1 C - D^*(t) P_2 C(t) \\ * & \widetilde{\psi}_{22}(t) & E^*(t) P_2 F(t) & E^*(t) P_2 C(t) \\ * & * & \widetilde{\psi}_{33}(t) & F^*(t) P_2 C(t) \\ * & * & * & \psi_{44}(t) \end{bmatrix},$$

$$\begin{aligned} \widetilde{\psi}_{22}(t) &= \varepsilon_2 H_E^* H_E + E^*(t) P_2 E(t) - \delta R_1, \\ \widetilde{\psi}_{33}(t) &= \varepsilon_3 H_F^* H_F + F^*(t) P_2 F(t) - \sigma R_2. \end{aligned}$$

Thus, we get that

$$\psi(t) = \psi_1(t) + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -(1-\delta) R_1 & 0 & 0 \\ 0 & 0 & -(1-\rho) R_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} < 0. \tag{3.17}$$

Obviously, $\Lambda \neq 0$ since $\vartheta \neq \xi$. From (3.14) and (3.17), we conclude that

$$(\vartheta - \xi)^* P_1 [\phi(\vartheta) - \phi(\xi)] + [\phi(\vartheta) - \phi(\xi)]^* P_1 (\vartheta - \xi) < 0. \tag{3.18}$$

This completes the proof that ϕ is an injective map on $\mathbb{Q}^n$.

Second, we show that $\phi(u)$ is radically unbounded, i.e., $\lim\limits_{\|u\| \to +\infty} \|\phi(u)\| \to +\infty$. From the definition of $\phi(u)$,

$$\begin{aligned} \phi(u) - \phi(0) = &- D(t) u + E(t)(f(u) - f(0)) \\ &+ F(t)(g(u) - g(0)) + C(t)(\phi(u) - \phi(0)). \end{aligned} \tag{3.19}$$

In the subsequent process, a method analogous to (3.14) is employed, and we can obtain

$$u^* P_1 [\phi(u) - \phi(0)] + [\phi(u) - \phi(0)]^* P_1 u \leqslant \eta^* \psi(t) \eta, \tag{3.20}$$

where

$$\eta = [u, f(u) - f(0), g(u) - g(0), \phi(u) - \phi(0)]^*.$$

Therefore

$$u^* P_1 [\phi(u) - \phi(0)] + [\phi(u) - \phi(0)]^* P_1 u \leqslant -\lambda_{\min}(-\psi(t)) \|u\|^2.$$

Thus, we have

$$\begin{aligned}\lambda_{\min}(-\psi(t))\|u\|^2 &\leqslant -u^*P_1[\phi(u)-\phi(0)]-[\phi(u)-\phi(0)]^*P_1u\\&= -2\mathrm{Re}(u^*P_1(\phi(u)-\phi(0)))\\&\leqslant 2|u^*P_1(\phi(u)-\phi(0))|\\&\leqslant 2\|u\|\cdot\|P_1\|\cdot\|\phi(u)-\phi(0)\|\\&\leqslant 2\|u\|\cdot\|P_1\|\cdot(\|\phi(u)\|+\|\phi(0)\|).\end{aligned}\tag{3.21}$$

That is, $\lim_{\|u\|\to+\infty}\|\phi(u)\|\to+\infty$. Now this mapping $\phi(u)$ is a homeomorphism to itself by Lemma 2.3. Thus system (2.1) has one and only one equilibrium point.

Third, we investigate and demonstrate the robust stability of the unique equilibrium point $\tilde{u}$ associated with system (2.1). Let $\upsilon(t)=u(t)-\tilde{u}$, and then system (2.1) can be rewritten as

$$ {}_0^CD_t^\alpha\upsilon(t)=-D(t)\upsilon(t)+E(t)\tilde{f}(\upsilon(\delta t))+F(t)\tilde{g}(\upsilon(\rho t))+C(t)({}_0^CD_t^\alpha\upsilon(t-\varrho)),\tag{3.22}$$

where

$$\tilde{f}(\upsilon(\delta t))=f(\upsilon(\delta t)+\tilde{u})-f(\tilde{u}),\tilde{g}(\upsilon(\sigma t))=g(\upsilon(\sigma t)+\tilde{u})-g(\tilde{u}).$$

Clearly, we can get from (3.1)–(3.13) that

$$0\leqslant\upsilon^*(\delta t)LR_1L\upsilon(\delta t)-\tilde{f}^*(\upsilon(\delta t))R_1\tilde{f}(\upsilon(\delta t)),\tag{3.23}$$

and

$$0\leqslant\upsilon^*(\sigma t)KR_2K\upsilon(\sigma t)-\tilde{g}^*(\upsilon(\sigma t))R_2\tilde{g}(\upsilon(\sigma t)).\tag{3.24}$$

Consider the following Lyapunov-Krasooski functional:

$$V(t)=V_1+V_2+V_3+V_4,$$

where

$$\begin{aligned}V_1&={}_0^CD_t^{-(1-\alpha)}[\upsilon(t)^*P_1\upsilon(t)],\\V_2&=\int_{t-\varrho}^t\left({}^C{}_0D_t^\alpha\upsilon(s)\right)^*P_2({}^C{}_0D_t^\alpha\upsilon(s))ds,\\V_3&=\int_{\rho t}^t\upsilon^*(s)KR_2K\upsilon(s)ds,\\V_4&=\int_{\delta t}^t\upsilon^*(s)LR_1L\upsilon(s)ds.\end{aligned}$$

Differentiate $V_i(t)$ $(i=1,2,3,4)$, respectively, and then

$$\begin{aligned}\dot{V}_1&={}_0^CD_t^\alpha[\upsilon^*(t)P_1\upsilon(t)]\\&\leqslant\upsilon^*(t)P_1[-D(t)\upsilon(t)+E(t)\tilde{f}(\upsilon(\delta t))+F(t)\tilde{g}(\upsilon(\sigma t))+C(t)({}_0^CD_t^\alpha\upsilon(t-\varrho))]\\&\quad+[-D(t)\upsilon(t)+E(t)\tilde{f}(\upsilon(\delta t))+F(t)\tilde{g}(\upsilon(\sigma t))+C(t)({}^C{}_0D_t^\alpha\upsilon(t-\varrho))]^*P_1\upsilon(t),\\\dot{V}_2&=[-D(t)\upsilon(t)+E(t)\tilde{f}(\upsilon(\delta t))+F(t)\tilde{g}(\upsilon(\sigma t))+C(t)({}_0^CD_t^\alpha\upsilon(t-\varrho))]^*P_2\\&\quad\times[-D(t)\upsilon(t)+E(t)\tilde{f}(\upsilon(\delta t))+F(t)\tilde{g}(\upsilon(\sigma t))+C(t)({}^C{}_0D_t^\alpha\upsilon(t-\varrho))]\\&\quad-({}_0^CD_t^\alpha\upsilon(t-\varrho))^*P_2({}_0^CD_t^\alpha\upsilon(t-\varrho)),\\\dot{V}_3&=\upsilon^*(t)KR_2Ke(t)-\rho(\upsilon^*(\rho t)KR_2Ke(\rho t)),\\\dot{V}_4&=\upsilon^*(t)LR_1Le(t)-\delta(\upsilon^*(\delta t)LR_1Le(\delta t)).\end{aligned}$$

Based on the above equations, we have

$$\dot{V}(t) \leq \Lambda(t)^* \psi_1(t)\Lambda(t),$$

where

$$\Lambda(t) = (\upsilon^*(t), \tilde{f}^*(\upsilon(\delta t)), \tilde{g}^*(\upsilon(\rho t)), C_0 D_t^\alpha \upsilon(t-\varrho))^*.$$

By (3.17), we acquire that

$$\dot{V}(t) \leq \Lambda(t)^* \psi_1(t)\Lambda(t) \leqslant 0. \tag{3.25}$$

The global robust stability of equilibrium point $\tilde{u}$ can be directly obtained from Lyapunov stability theory. The proof is complete. □

Next, a weaker sufficient condition related to mixed delays and the order of fractional derivatives is built. Under this weak condition, we derive that system (2.1) has a unique globally robust stable equilibrium point.

Theorem 3.2. *Suppose that* (**H**1) *and* (**H**2) *hold. Let $P_1, P_2, P_3, P_4 \in \mathbb{Q}^{n\times n}$ be four positive definite matrices, R_1 and $R_2 \in \mathbb{Q}^{n\times n}$ be positive definite diagonal matrices, and ρ , $\varepsilon_1 > 0, \varepsilon_2 > 0, \varepsilon_3 > 0, \varepsilon_4 > 0$ be positive real numbers that satisfy the following LMI. Under these assumptions, then system (2.1) has one and only one equilibrium point, and it is globally robustly stable.*

$$\Theta(t) = \begin{bmatrix} \Theta_{11} & \Theta_{12} & 0 & \rho H^* & P_1 G \\ * & \Theta_{22} & \Theta_{23} & 0 & 0 \\ * & * & -\rho I & 0 & 0 \\ * & * & * & -\rho I & 0 \\ * & * & * & * & \Theta_{55} \end{bmatrix} \leq 0,$$

where

$$\Theta_{11} = \begin{bmatrix} \Theta_{11}^{(11)} & P_1 E & P_1 F & P_1 C \\ * & \Theta_{11}^{(22)} & 0 & 0 \\ * & * & \Theta_{11}^{(33)} & 0 \\ * & * & * & \Theta_{11}^{(44)} \end{bmatrix},$$

$$\Theta_{12} = \begin{bmatrix} -D^* \\ E^* \\ F^* \\ C^* \end{bmatrix} (P_2 + \varrho^2 P_4 + \alpha P_3),\ \Theta_{22} = -P_2 - \varrho^2 P_4 - \alpha P_3,$$

$$\Theta_{23} = (P_2 + \varrho^2 P_4 + \alpha P_3)G, \quad \Theta_{55} = \mathrm{diag}\{-\varepsilon_1, -\varepsilon_2, -\varepsilon_3, -\varepsilon_4\},$$

with

$$\Theta_{11}^{(11)} = -P_1 D - D P_1 + \varepsilon_1 H_D^* H_D + L R_1 L + K R_2 K,\ \Theta_{11}^{(22)} = \varepsilon_2 H_E^* H_E - \delta R_1,$$
$$\Theta_{11}^{(33)} = \varepsilon_3 H_F^* H_F - \sigma R_2, \Theta_{11}^{(44)} = \varepsilon_4 H_F^* H_F - P_2,$$

and the remaining part of Θ_{ij} is zero.

Proof. Due to

$$\Theta(t) - \Pi(t) \geq 0,$$

from Theorem 3.1, it can be conclued that system (2.1) has a globally unique equilibrium point. Let $\upsilon(t) = u(t) - \tilde{u}$, and then system (2.1) can be rewritten as

$${}_{0}^{C}D_{t}^{\alpha}\upsilon(t) = -D(t)\upsilon(t) + E(t)\tilde{f}(\upsilon(\delta t)) + F(t)\tilde{g}(\upsilon(\rho t)) + C(t)({}_{0}^{C}D_{t}^{\alpha}\upsilon(t-\varrho)),$$

where

$$\tilde{f}(\upsilon(\delta t)) = f(\upsilon(\delta t) + \tilde{u}) - f(\tilde{u}), \tilde{g}(\upsilon(\sigma t)) = g(\upsilon(\sigma t) + \tilde{u}) - g(\tilde{u}).$$

Consider the following Lyapunov-Krasooski functional:

$$\begin{aligned}
V_1(t) = \quad & {}_{0}^{C}D_{t}^{-(1-\alpha)}(\upsilon(t)^{*}P_1\upsilon(t)) + \int_{t-\varrho}^{t}\left({}_{0}^{C}D_{t}^{\alpha}\upsilon(s)\right)^{*}P_2({}_{0}^{C}D_{t}^{\alpha}\upsilon(s))ds \\
& + \alpha\int_{0}^{t}({}_{0}^{C}D_{t}^{\alpha}\upsilon(s))^{*}P_3({}_{0}^{C}D_{t}^{\alpha}\upsilon(s))ds \\
& + \varrho\int_{-\varrho}^{0}\int_{t+\varsigma}^{t}({}_{0}^{C}D_{t}^{\alpha}\upsilon(s))^{*}P_4({}_{0}^{C}D_{t}^{\alpha}\upsilon(s))dsd\varsigma, \\
V_2(t) = & \int_{\rho t}^{t}\upsilon^{*}(s)KR_2K\upsilon(s)ds + \int_{\delta t}^{t}\upsilon^{*}(s)LR_1L\upsilon(s)ds.
\end{aligned}$$

Differentiate $V_i(t)$ $(i = 1, 2)$, respectively, and then

$$\begin{aligned}
\dot{V}_1 + \dot{V}_2 = & {}_{0}^{C}D_{t}^{\alpha}[\upsilon^{*}(t)P_1\upsilon(t)] \\
\leqslant & \upsilon^{*}(t)P_1[-D(t)\upsilon(t) + E(t)\tilde{f}(\upsilon(\delta t)) + F(t)\tilde{g}(\upsilon(\sigma t)) + C(t) \\
& \times({}_{0}^{C}D_{t}^{\alpha}\upsilon(t-\varrho))] + [-D(t)\upsilon(t) + E(t)\tilde{f}(\upsilon(\delta t)) + F(t)\tilde{g}(\upsilon(\sigma t)) \\
& + C(t)({}^{C}{}_{0}D_{t}^{\alpha}\upsilon(t-\varrho))]^{*}P_1\upsilon(t), +[-D(t)\upsilon(t) + E(t)\tilde{f}(\upsilon(\delta t)) \\
& + F(t)\tilde{g}(\upsilon(\sigma t)) + C(t)({}_{0}^{C}D_{t}^{\alpha}\upsilon(t-\varrho))]^{*}(P_2 + \varrho^2P_4 + \alpha P_3) \\
& \times[-D(t)\upsilon(t) + E(t)\tilde{f}(\upsilon(\delta t)) + F(t)\tilde{g}(\upsilon(\sigma t)) \\
& + C(t)({}^{C}{}_{0}D_{t}^{\alpha}\upsilon(t-\varrho))] - ({}_{0}^{C}D_{t}^{\alpha}\upsilon(t-\varrho))^{*}P_2({}_{0}^{C}D_{t}^{\alpha}\upsilon(t-\varrho)) \\
& - \int_{t-\varrho}^{t}({}_{0}^{C}D_{t}^{\alpha}\upsilon(s))^{*}P_4\int_{t-\varrho}^{t}({}_{0}^{C}D_{t}^{\alpha}\upsilon(s))ds \\
& + \upsilon^{*}(t)KR_2Ke(t) - \rho(\upsilon^{*}(\rho t)KR_2Ke(\rho t)) \\
& + \upsilon^{*}(t)LR_1Le(t) - \delta(\upsilon^{*}(\delta t)LR_1Le(\delta t)).
\end{aligned}$$

It follows that

$$\dot{V}_1 + \dot{V}_2 \leq \nu(t)\eta(t)\nu(t),$$

where

$$\nu(t) = (\upsilon^{*}(t), \tilde{f}^{*}(\upsilon(\delta t)), \tilde{g}^{*}(\upsilon(\rho t)), C_0D_{t}^{\alpha}\upsilon(t-\varrho), \int_{t-\varrho}^{t}({}_{0}^{C}D_{t}^{\alpha}\upsilon(s))^{*}),$$

$$\eta(t) = \begin{bmatrix} \eta_{11}(t) & \eta_{12}(t) & \eta_{13}(t) & \eta_{14}(t) & 0 \\ * & \eta_{22}(t) & \eta_{23}(t) & \eta_{24}(t) & 0 \\ * & * & \eta_{33}(t) & \eta_{34}(t) & 0 \\ * & * & * & \eta_{44}(t) & 0 \\ * & * & * & * & \eta_{55}(t) \end{bmatrix},$$

with

$$\begin{aligned}
\eta_{11}(t) =& - P_1D - DP_1 + \varepsilon_1^{-1}P_1G_DG_D^*P_1 + \varepsilon_1H_D^*H_D + \varepsilon_2^{-1}P_1G_EG_E^*P_1 \\
&+ \varepsilon_3^{-1}P_1G_FG_F^*P_1 + \varepsilon_4^{-1}P_1G_CG_C^*P_1 + D(t)(P_2 + \alpha P_3 + \varrho^2P_4)D(t), \\
\eta_{12}(t) =& P_1E - D^*(t)(P_2 + \alpha P_3 + \varrho^2P_4)E(t), \\
\eta_{13}(t) =& P_1F - D^*(t)(P_2 + \alpha P_3 + \varrho^2P_4)F(t), \\
\eta_{14}(t) =& P_1C - D^*(t)(P_2 + \alpha P_3 + \varrho^2P_4)C(t), \\
\eta_{22}(t) =& \varepsilon_2H_E^*H_E + E^*(t)(P_2 + \alpha P_3 + \varrho^2P_4)E(t), \\
\eta_{23}(t) =& E^*(t)(P_2 + \varrho^2P_4)F(t),\ \eta_{24}(t) = E^*(t)(P_2 + \varrho^2P_4)C(t), \\
\eta_{33}(t) =& \varepsilon_3H_F^*H_F + F^*(t)(P_2 + \alpha P_3 + \varrho^2P_4)F(t), \\
\eta_{34}(t) =& F^*(t)(P_2 + \alpha P_3 + \varrho^2P_4)C(t), \\
\eta_{44}(t) =& \varepsilon_4H_C^*H_C + C^*(t)(P_2 + \alpha P_3 + \varrho^2P_4)C(t) - P_2,\ \eta_{55}(t) = -P_4.
\end{aligned}$$

Similarly as in (3.4)–(3.15), we can obtain

$$\dot{V}_1 + \dot{V}_2 \le \nu(t)\eta(t)\nu(t) \le 0.$$

The global robust stability of equilibrium point $\tilde{u}$ can be directly obtained from Lyapunov stability theory. The proof is complete. □

Remark 3.1. *(i) If $C(t) = 0$, then system (2.1) reduces to the following model:*

$$ {}_0^C D_t^{\alpha} u(t) = -D(t)u(t) + E(t)f(u(\delta t)) + F(t)g(u(\sigma t)) + N(t). \tag{3.26}$$

(ii) When $\delta = \sigma = 1$, model (3.26) turns into the following model:

$$ {}_0^C D_t^{\alpha} u(t) = -D(t)u(t) + E(t)f(u(t)) + F(t)g(u(t)) + N(t). \tag{3.27}$$

(iii) If time-varying parameters vanish model (3.27), model (3.27) becomes the following model:

$$ {}_0^C D_t^{\alpha} u(t) = -Du(t) + Ef(u(t)) + Bg(u(t)) + N. \tag{3.28}$$

Corollary 3.1. *Suppose that* (**H**1) *and* (**H**2) *hold. Let $P_1, P_2 \in \mathbb{Q}^{n\times n}$ be two positive definite matrices, R_1 and $R_2 \in \mathbb{Q}^{n\times n}$ be positive definite diagonal matrices, and ρ , $\varepsilon_1 > 0, \varepsilon_2 > 0, \varepsilon_3 > 0$ be positive real numbers that satisfy the following LMI. Under these assumptions, model (27) has one and only one equilibrium point that is globally robustly stable.*

$$\Pi(t) = \begin{bmatrix} \pi_{11} & \pi_{12} & 0 & \rho H^* & P_1G \\ * & -P_2 & P_2G & 0 & 0 \\ * & * & -\rho I & 0 & 0 \\ * & * & * & -\rho I & 0 \\ * & * & * & * & \pi_{55} \end{bmatrix} \le 0,$$

$$\pi_{11}=\begin{bmatrix}\pi_{11}^{(11)} & P_1E & P_1F\\ * & \pi_{11}^{(22)} & 0\\ * & * & \pi_{11}^{(33)}\end{bmatrix},\pi_{12}=\begin{bmatrix}-D^*\\ E^*\\ F^*\end{bmatrix}P_2,\pi_{55}=\mathrm{diag}[-\varepsilon_1-\varepsilon_2-\varepsilon_3],M=\begin{bmatrix}G_D & G_E & G_F\end{bmatrix},$$

$$\pi_{11}^{(11)}(t)=-P_1D-DP_1+\varepsilon_1H_D^*H_D+LR_1L+KR_2K,\ \pi_{11}^{(22)}=\varepsilon_2H_E^*H_E-\delta R_1,\quad \pi_{11}^{(33)}=\varepsilon_3H_F^*H_F-\rho R_2.$$

Corollary 3.2. *Suppose that* (**H**1) *and* (**H**2) *hold. Let* $P_1\in\mathbb{Q}^{n\times n}$ *be a positive definite matrices,* R_1 *and* $R_2\in\mathbb{Q}^{n\times n}$ *be positive definite diagonal matrices, and satisfy the following LMI. Under these assumptions, model (3.28) has one and only one equilibrium point that is globally robustly stable.*

$$\pi(t)=\begin{bmatrix}\pi_{11}(t) & P_1E & P_1F\\ * & -R_1 & 0\\ * & * & -R_2\end{bmatrix}<0,$$

where

$$\pi_{11}(t)=-P_1D-DP_1+LR_1L+KR_2K.$$

4. Numerical simulation

Below, we present numerical examples that are unrelated to delay, as well as examples related to delay.

Example 4.1. *Assume that* (**H**1) *and* (**H**2) *hold. We provide the following parameters:*

$$\varrho=0.1,\delta=0.6,\sigma=0.5,f_1(u(t))=f_2(u(t))=g_1(u(t))=g_2(u(t))=0.5\tanh(u(t)),$$

$$D=\begin{bmatrix}11 & 0\\ 0 & 11\end{bmatrix},G_D=\begin{bmatrix}0.8 & 0\\ 0 & 0.8\end{bmatrix},N_D=\begin{bmatrix}0.91 & 0\\ 0 & 0.91\end{bmatrix},\ L=K=\begin{bmatrix}0.5 & 0\\ 0 & 0.5\end{bmatrix},$$

$$E=\begin{bmatrix}0.4+0.35i+0.0457j+0.0579k & 0.3+0.021i-0.074j+0.05k\\ 0.0286-0.0207i+0.0354j+0.0171k & 0.051+0.0789i-0.0237j+0.0765k\end{bmatrix},$$

$$G_E=\begin{bmatrix}0.5761+0.3025i+0.3321j+0.3095k & -0.1654-0.7121i-0.2583j+0.9057k\\ -0.1856-0.1128i+0.2752j+1.165k & 0.0012+0.4570i-0.4577j+0.5180k\end{bmatrix},$$

$$H_E=\begin{bmatrix}-0.3854+0.53737i-0.1223j+0.2024k & 0.9208+0.4679i-0.3231j+0.1415k\\ 0.3526-0.2024i+0.0799j+0.0546k & 0.4657+0.4679i-0.1517j+0.1415k\end{bmatrix},$$

$$F=\begin{bmatrix}0.1215+0.2065i+0.1726j+0.3579k & 0.2328+0.1779i-0.2345j-0.2597k\\ 0.2031-0.0365i-0.2746j+0.0827k & -0.27261+0.3571i-0.2001j+0.2979k\end{bmatrix},$$

$$G_F=\begin{bmatrix}0.1422+0.1617i+0.1309j+0.0836k & 0.0501+0.0501i+0.5024j+0.2514k\\ 0302+0.1018i+0.0124j+0.0821k & -0.1208-0.0605i-0.0502j-0.0214k\end{bmatrix},$$

$$H_F=\begin{bmatrix}-0.3017+0.2654i-0.041j+0.041k & 0.8012+0.08021i-0.215j+0.021k\\ 0.9752-0.9544i+0.072j+0.217k & -0.1651+0.0821i-0.271j-0.0138k\end{bmatrix},$$

$$C=\begin{bmatrix}0.0767+0.1223i+0.0729j+0.2016k & 0.238+0.0238i-0.0699j+0.2153k\\ 0.0352-0.0471i-0.2216j+0.1427k & -0.0641-0.0281i-0.3541j+0.3025k\end{bmatrix},$$

$$G_C=\begin{bmatrix}-0.0027+0.0039i-0.0361j-0.081k & 0.0327+0.0027i+0.5142j-0.2016k\\ 0.0028+0.0716i-0.3216j+0.1427k & -0.0061-0.0071i+0.0421j+0.0281k\end{bmatrix},$$

$$H_C = \begin{bmatrix} 0.081 - 0.035i - 0.0138j + 0.0218k & 0.01171 + 0.1164i + 0.0911j + 0.0691k \\ 0.0118 + 0.0581i + 0.1451j + 0.0131k & 0.021 - 0.01171i - 0.021j - 0.11k \end{bmatrix},$$
$$N = \begin{bmatrix} 0.2189 - 0.3299i + 0.2721j + 0.1213k \\ 0.6579 + 0.6571i + 0.1311j + 0.5241k \end{bmatrix}, U_D(t) = \begin{bmatrix} 0.11\sin t & 0 \\ 0 & 0.021\cos t \end{bmatrix},$$
$$U_E(t) = \begin{bmatrix} U_{E_{11}}(t) & U_{E_{12}}(t) \\ U_{A_{21}}(t) & U_{E_{22}}(t) \end{bmatrix}, U_F(t) = \begin{bmatrix} U_{B_{11}}(t) & U_{B_{12}}(t) \\ U_{B_{21}}(t) & U_{B_{22}}(t) \end{bmatrix}, \quad U_C(t) = \begin{bmatrix} U_{C_{11}}(t) & U_{C_{12}}(t) \\ U_{C_{21}}(t) & U_{C_{22}}(t) \end{bmatrix},$$

where

$$\begin{aligned}
U_{E_{11}}(t) &= 0.11\sin^2 t + 0.11i\sin^2 t + 0.11j\sin^2 t + 0.11k\sin^2 t,\\
U_{E_{12}}(t) &= 0.11\sin^2 2t + 0.21i\cos^2 ti + 0.11j\sin^2 2t + 0.21k\cos^2 t,\\
U_{E_{21}}(t) &= 0.11\sin^2 t + 0.11i\cos^2 t + 0.11j\sin^2 t + 0.11k\cos^2 t,\\
U_{E_{22}}(t) &= 0.21\cos^2 t + 0.31i\sin^2 t + 0.21j\cos^2 t + 0.31k\sin^2 t,\\
U_{F_{11}}(t) &= 0.11\sin^2 t + 0.11i\sin^2 ti + 0.11j\sin^2 t + 0.11k\sin^2 t,\\
U_{F_{12}}(t) &= 0.11\sin^2 2t + 0.21i\cos^2 t + 0.11j\sin^2 2t + 0.21k\cos^2 t,\\
U_{F_{21}}(t) &= 0.11\sin^2 t + 0.11i\cos^2 t + 0.11j\sin^2 t + 0.11k\cos^2 t,\\
U_{F_{22}}(t) &= 0.21\cos^2 t + 0.31i\sin^2 t + 0.21j\cos^2 t + 0.31k\sin^2 t,\\
U_{C_{11}}(t) &= 0.11\sin^2 t + 0.11i\sin^2 t + 0.11j\sin^2 t + 0.11k\sin^2 t,\\
U_{C_{12}}(t) &= 0.11\sin^2 2t + 0.21i\cos^2 t + 0.11j\sin^2 2t + 0.21k\cos^2 t,\\
U_{C_{21}}(t) &= 0.11\sin^2 t + 0.11i\cos^2 t + 0.11j\sin^2 t + 0.11k\cos^2 t,\\
U_{C_{22}}(t) &= 0.21\cos^2 t + 0.31i\sin^2 t + 0.21j\cos^2 t + 0.31k\sin^2 t.
\end{aligned}$$

We can easily obtain P_1, P_2, R_1, and R_2 through the yalmap toolbox in MATLAB R2023a, where

$$P_1 = \begin{bmatrix} 124.8833 & -10.6604 + 41.7934i - 28.0679j - 7.8028k \\ -10.6604 - 41.7934i + 28.0679j + 7.8028k & 141.0081 \end{bmatrix},$$
$$P_2 = \begin{bmatrix} 6.6348 & -1.5516 + 2.4717i - 1.1168j - 0.6125k \\ -1.5516 - 2.4717i + 1.1168j + 0.6125k & 8.5973 \end{bmatrix},$$
$$R_1 = \begin{bmatrix} 910.7343 & 0 \\ 0 & 1776.6482 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 1305.3855 & 0 \\ 0 & 1027.9514 \end{bmatrix},$$
$$\rho = 96.4255, \quad \varepsilon_1 = 189.6456, \quad \varepsilon_2 = 315.2880, \quad \varepsilon_3 = 119.4362, \quad \varepsilon_4 = 22.6492.$$

From Theorem 3.1, it can be concluded that system (2.1) has a unique globally robust stable equilibrium point. From Figure 1, we can also see that system (2.1) is globally robust stable, where

$$P_0 = [1.5 + 1.2i - 0.5j + 1.6k,\ -1.8 - 1.2i + 1.5j + 0.2k].$$

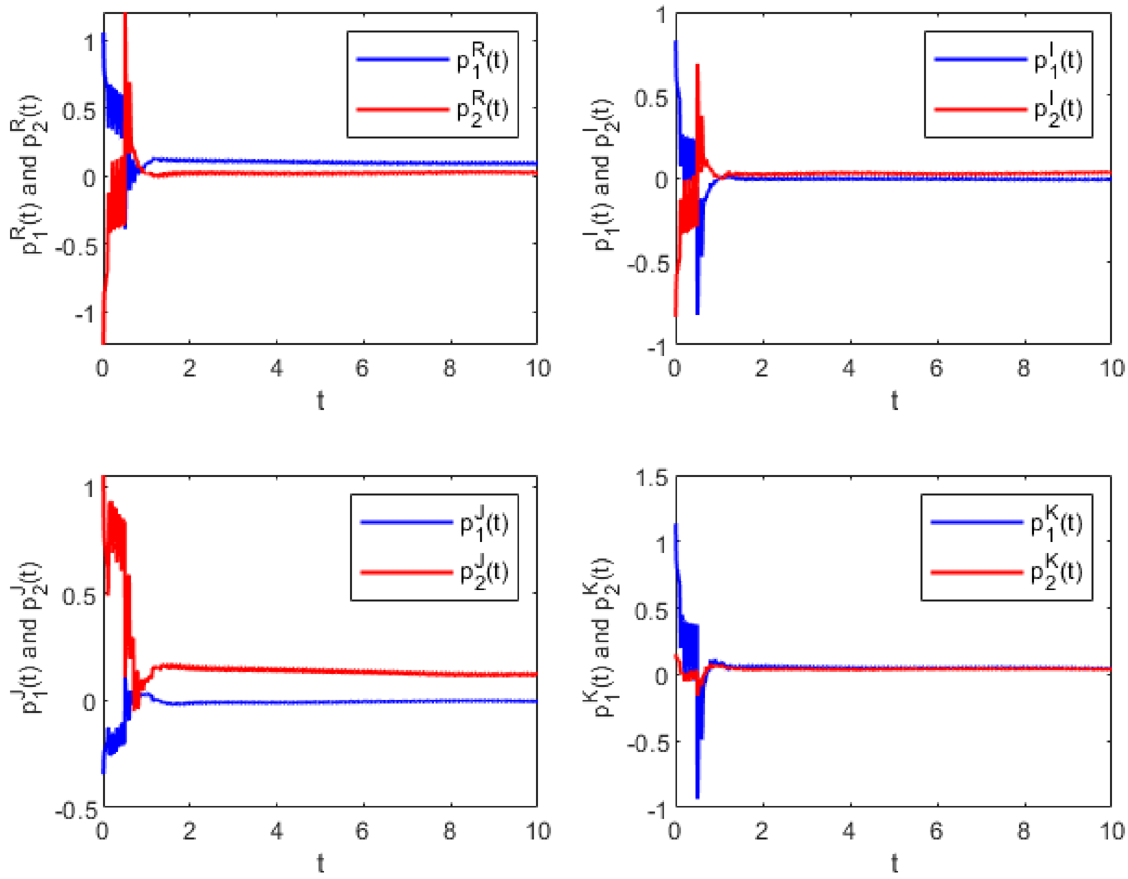

Figure 1. Stability of four different components of the FOQVNN.

Remark 4.1. *Compared with [22], we propose FOQVNNs with double proportional delays, neural delay, and uncertain parameters, and obtain a robust stable result. Song [22] studied the stability of a FOQVNN model with neutral delay, classical delay, and parameter uncertainty. In Example 4.1, we set* $\varrho = 0.1, \delta = 0.6, \sigma = 0.5,$ *and* $f_1(u(t)) = f_2(u(t)) = g_1(u(t)) = g_2(u(t)) = 0.5\tanh(u(t))$. *Figure 1 shows that system (2.1) tends to stability, while Song showed that the system (2.1) he proposed in [22] tends to stability when the neural delay* $\tau = 0.1$ *and* $\sigma = 0.04$ *in Example 1. At the same time, the activation function* $f_1(u(t)) = f_2(u(t)) = g_1(u(t)) = g_2(u(t)) = 0.5\tanh(u(t))$ *in Example 1 of [22]. Although these images are similar in [22] and in this paper, Song [22] discussed fractional-order quaternion-value neural networks with neutral delay, classical delay, and parameter uncertainty, while we studied dual proportional delays, neutral delay, and uncertain parameters. The differential system studied in this article is different than that in [22].*

Example 4.2. *Assume that* (**H**1) *and* (**H**2) *hold, and we provide the following parameters:*

$$\varrho = 0.1, \delta = 0.96, \sigma = 0.95, f_1(u(t)) = f_2(u(t)) = g_1(u(t)) = g_2(u(t)) = 0.5\tanh(u(t)),$$

$$D = \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix}, G_D = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix}, N_D = \begin{bmatrix} 0.91 & 0 \\ 0 & 0.91 \end{bmatrix}, \ L = K = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix},$$

$$E = \begin{bmatrix} 0.4 + 0.3312i + 0.0567j + 0.0379k & 0.3 + 0.0214i - 0.3783j + 0.4521k \\ 0.2431 - 0.3342i + 0.1041j + 0.5124k & 0.3563 + 0.5054i - 0.5174j + 0.0271k \end{bmatrix},$$

$$G_E = \begin{bmatrix} -0.6356 - 0.3654i - 0.3453j + 0.4462k & -0.2567 - 0.8572i - 0.3236j + 0.8522k \\ -0.2456 - 0.3645i - 0.3342j + 0.1463k & 0.1786 + 0.5235i - 0.5246j + 0.6335k \end{bmatrix},$$

$$H_E = \begin{bmatrix} -0.4325 + 0.6573i - 0.1253j + 0.24k & 0.9543 + 0.5235i - 0.32j + 0.16k \\ 0.4574 + 0.2235i + 0.08j + 0.05k & -0.5246 + 0.4742i - 0.15j + 0.14k \end{bmatrix},$$

$$F = \begin{bmatrix} 0.1526 + 0.2153i + 0.5433j + 0.3673k & 0.2475 + 0.3354i - 0.5423j - 0.3735k \\ 0.2553 - 0.0351i - 0.3837j + 0.4347k & -0.2854 + 0.3523i - 0.5534j + 0.2224k \end{bmatrix},$$

$$G_F = \begin{bmatrix} -0.1406 + 0.1667i + 0.1309j + 0.0836k & 0.0556 + 0.0553i + 0.5024j + 0.2514k \\ 0.0352 + 0.1032i + 0.0124j + 0.0821k & -0.1243 - 0.0653i - 0.5j - 0.2423k \end{bmatrix},$$

$$H_F = \begin{bmatrix} -0.3274 + 0.6545i - 0.4345 + 0.4234k & 0.8122 + 0.8021i + 0.3345j + 0.6532k \\ 0.9523 - 0.9542i + 0.7422j + 0.6532k & -0.0163 + 0.0843i - 0.4425j - 0.2342k \end{bmatrix},$$

$$C = \begin{bmatrix} 0.1837 + 0.2347i + 0.5786j + 0.3783k & 0.1383 + 0.1383i + 0.1786j + 0.3456k \\ 0.0315 - 0.0417i - 0.3687j + 0.2563k & -0.0612 - 0.0309i - 0.6456j + 0.4374k \end{bmatrix},$$

$$G_C = \begin{bmatrix} -0.0032 + 0.0054i - 0.0361j - 0.0817k & 0.0395 + 0.0032i + 0.0178j - 0.0561k \\ 0.0031 + 0.0715i - 0.0586j + 0.0768k & -0.0061 - 0.0081i + 0.0479j + 0.0268k \end{bmatrix},$$

$$H_C = \begin{bmatrix} 0.0832 - 0.0354i - 0.0314j + 0.0414k & 0.0117 + 0.0116i + 0.0784j + 0.0855k \\ 0.1182 + 0.0665i + 0.2452j + 0.02632k & 0.3243 - 0.0255i - 0.2245j - 0.0175k \end{bmatrix},$$

$$N = \begin{bmatrix} 0.1534 - 0.3678i + 0.2721j + 0.1213k \\ 0.7876 + 0.8435i + 0.021j + 0.6211k \end{bmatrix}, \quad U_D(t) = \begin{bmatrix} 0.2145\sin^2 t & 0 \\ 0 & 0.3254\cos^2 t \end{bmatrix},$$

$$U_E(t) = \begin{bmatrix} U_{E_{11}}(t) & U_{E_{12}}(t) \\ U_{E_{21}}(t) & U_{E_{22}}(t) \end{bmatrix}, \quad U_F(t) = \begin{bmatrix} U_{F_{11}}(t) & U_{F_{12}}(t) \\ U_{F_{21}}(t) & U_{F_{22}}(t) \end{bmatrix}, \quad U_C(t) = \begin{bmatrix} U_{C_{11}}(t) & U_{C_{12}}(t) \\ U_{C_{21}}(t) & U_{C_{22}}(t) \end{bmatrix},$$

where

$$\begin{aligned}
U_{E_{11}}(t) &= 0.1234\sin^2 t + 0.2245i\sin^2 t + 0.2424j\sin^2 t + 0.1243k\sin^2 t,\\
U_{E_{12}}(t) &= 0.1546\sin^2 2t + 0.3321i\cos^2 t + 0.2543j\sin^2 2t + 0.3234k\cos^2 t,\\
U_{E_{21}}(t) &= 0.2432\sin^2 t + 0.2441i\cos^2 t + 0.2564j\sin^2 t + 0.2234k\cos^2 t,\\
U_{E_{22}}(t) &= 0.3156\cos^2 t + 0.2246i\sin^2 t + 0.2245j\cos^2 t + 0.3643k\sin^2 t,\\
U_{F_{11}}(t) &= 0.1154\sin^2 t + 0.1356i\sin^2 t + 0.1623j\sin^2 t + 0.2362k\sin^2 t,\\
U_{F_{12}}(t) &= 0.1235\sin^2 2t + 0.3324i\cos^2 t + 0.2543j\sin^2 2t + 0.3234k\cos^2 t,\\
U_{F_{21}}(t) &= 0.2653\sin^2 t + 0.0152i\cos^2 t + 0.2342j\sin^2 t + 0.5512k\cos^2 t,\\
U_{F_{22}}(t) &= 0.2632\cos^2 t + 0.0325i\sin^2 t + 0.0223j\cos^2 t + 0.3597k\sin^2 t,\\
U_{C_{11}}(t) &= 0.1687\sin^2 t + 0.2234i\sin^2 t + 0.1357j\sin^2 t + 0.2865k\sin^2 t,\\
U_{C_{12}}(t) &= 0.2856\sin^2 2t + 0.3684i\cos^2 t + 0.1985j\sin^2 2t + 0.2547k\cos^2 t,\\
U_{C_{21}}(t) &= 0.1457\sin^2 t + 0.3456i\cos^2 t + 0.1786j\sin^2 t + 0.1456k\cos^2 t,\\
U_{C_{22}}(t) &= 0.2542\cos^2 t + 0.3479i\sin^2 t + 0.2267j\cos^2 t + 0.3474k\sin^2 t.
\end{aligned}$$

We can easily obtain P_1, P_2, P_3, P_4, R_1, and R_2 through the yalmap toolbox in MATLAB R2023a, where

$$P_1 = \begin{bmatrix} 52.8711 & -1.5496 - 4.5683i - 1.4139j - 0.8693k \\ -1.5496 + 4.5683i + 1.4139j + 0.8693k & 50.2395 \end{bmatrix},$$
$$P_2 = \begin{bmatrix} 33.9865 & -8.4712 + 8.5020i + 15.7041j + 4.4958k \\ -8.4712 - 8.5020i - 15.7041j - 4.4958k & 52.1771 \end{bmatrix},$$
$$P_3 = \begin{bmatrix} -140.19 & 34.94 - 35.07i - 64.7796j - 18.5453k \\ 34.94 + 35.07i + 64.7796j + 18.5453k & -215.23 \end{bmatrix},$$
$$P_4 = \begin{bmatrix} 106.21 & -26.47 + 26.57i + 49.0754j + 14.0495k \\ -26.47 - 26.57i - 49.0754j - 14.0495k & 163.05 \end{bmatrix},$$
$$R_1 = \begin{bmatrix} 764.7 & 0 \\ 0 & 1164.4 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 1709.9 & 0 \\ 0 & 1260.8 \end{bmatrix},$$
$$\rho = 107.1262, \quad \varepsilon_1 = 114.0418, \quad \varepsilon_2 = 124.4127, \quad \varepsilon_3 = 55.9324, \quad \varepsilon_4 = 25.1004.$$

Remark 4.2. *Figure 2 illustrates the trajectory of the neural vector, demonstrating that the system remains stable even after parameter perturbations, which confirms the conclusion of Theorem 3.2.*

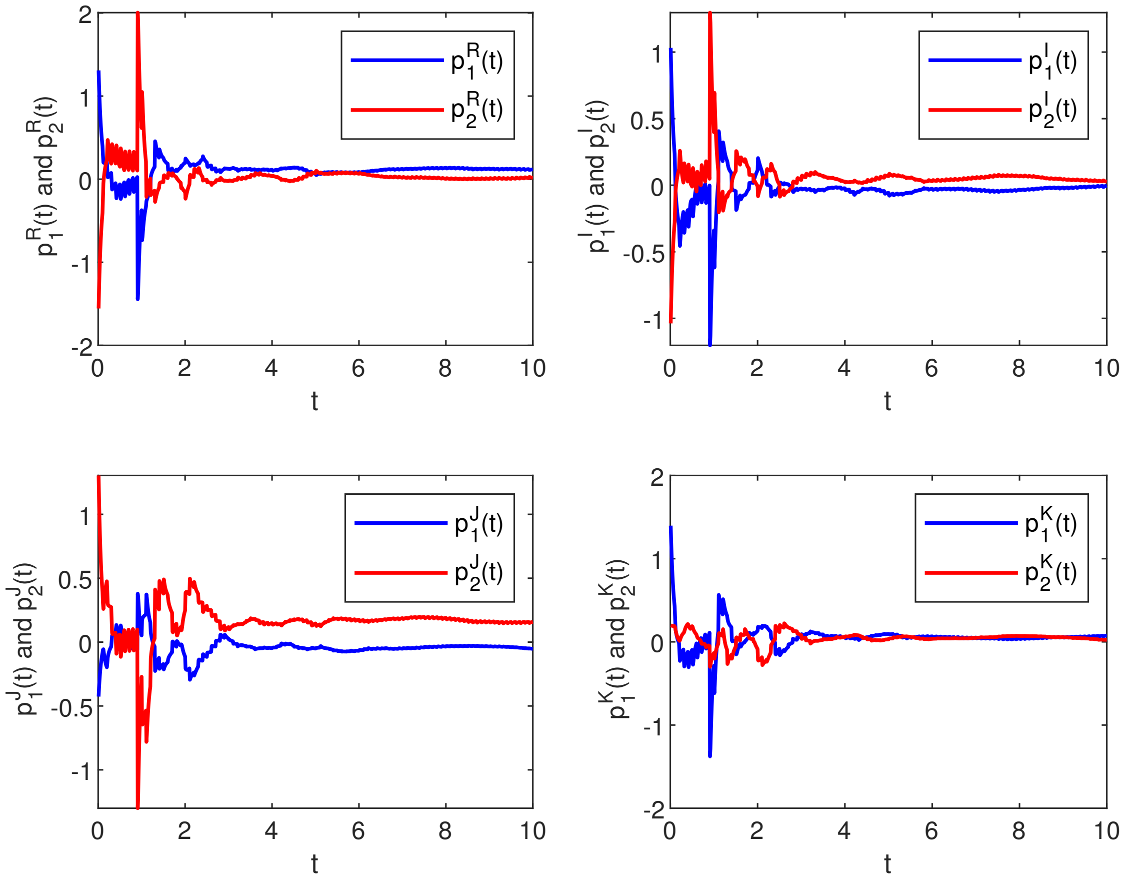

Figure 2. Stability of four different components of the FOQVNN.

5. Conclusions

We introduced two proportional delays in the FOQVNN model with neutral delay and parameter uncertainty. The robust stability of the FOQVNN model was studied without directly decomposing it into RVNNs or CVNNs. Through some existing lemmas and Lyapunov stability theory, a sufficient condition for the non-positive definite LMI related to proportional delay has been obtained, and this linear matrix inequality can ensure that the FOQVNN model has one and only one equilibrium point. The non-positive definite LMI we obtained can be validated using the quaternion toolbox, sdpt3 toolbox, and existing toolboxes in MATLAB. Finally, a numerical example was used to validate the correctness of our main results. This article obtained the stability and parameter uncertainty of the FOQVNN model with dual proportional and neutral delays. This is already very close to the operation of real-world networks, but in reality, latency can also change over time. Therefore, we hope to study FOQVNN models with time-varying delays in the future.

Author contributions

Guoqing Jiang: Question raised, research investigation, initial draft writing, MATLAB, validation, review and editing; Xiaolan Liu: Analysis, review, revise, initial draft; Lei Wang: Review, examination; Chongwei Zheng: Review, revise. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Availability of data and materials

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare no conflicts of interest regarding the publication of this paper.

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