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Research article

Using multi-attribute decision-making technique for the selection of agribots via newly defined fuzzy sets

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Abstract: Reference parameter mapping (passing arguments by reference) is a technique where the reference (like to find physical meaning, memory address) of a parameter is passed to a function or procedure, rather than a copy of the parameter's value. This approach enables changes made to the parameter within the function to affect the original data. In decision-making systems, reference parameter mapping (passing arguments by reference) offers several key advantages that enhance flexibility, consistency, and efficiency. This is especially useful in scenarios where decisions are based on shared data, complex interactions, and iterative updates. In this paper, a new class of fuzzy set was introduced that is known as the (q_1, q_2)-rung Diophantine fuzzy set, where q_1 and q_2 are reference parameter mappings. Most of the classical and new generalized fuzzy sets are exceptional classes of (q_1, q_2)-rung Diophantine fuzzy set ((q_1, q_2)- $RDFS$) like intuitionistic fuzzy set (IFS), Pythagorean fuzzy Sets ($PyFS$s) and q-rung Orthopair fuzzy sets (q-$ROFS$s), linear Diophantine fuzzy sets ($LDFS$), and so on. It is commonly seen in multi-criteria decision-making ($MCDM$) scenarios that the presence of imprecise information and ambiguity in the decision maker's judgment affects the resolution technique. Fuzzy models that are now in use are unable to effectively manage these uncertainties to provide an appropriate balance during the decision-making process. Using control (reference) parameter mappings, (q_1, q_2)- $RDFS$s are potent fuzzy model that can handle these challenging problems. Two more novel ideas are presented in this work: (q_1, q_2)-rung Diophantine fuzzy averaging and geometric aggregation operators with newly defined score and accuracy functions. An agricultural field robot $MCDM$ framework was proposed, incorporating (q_1, q_2)-rung Diophantine fuzzy averaging and geometric aggregation operators. This strategy's efficacy and adaptability in addressing real-world issues were demonstrated by its application to get more benefits. This study has a lot of potential to handle difficult socioeconomic issues and offer vital information to academic, government, and analysts searching for fresh approaches in a variety of fields.

Keywords: fuzzy set; q-rung orthopair fuzzy set; reference parameter mappings; (q_1, q_2)-rung Diophantine fuzzy set; aggregation operators; $MCDM$ problem
Mathematics Subject Classification: 90B50

Abbreviations: (q_1, q_2)–$RDFS$: (q_1, q_2)-rung Diophantine fuzzy set; IFS: Intuitionistic fuzzy set; $PyFS$s: Pythagorean fuzzy sets; q-$ROFS$s: q-rung Orthopair fuzzy sets; $LDFS$s: linear Diophantine fuzzy sets; $MCDM$: multi-criteria decision-making; MD: membership degree; NMD: non-membership degree; BFS: Bipolar fuzzy sets; m-PFS: m-polar fuzzy sets; $(3,4)$-$QOFS$: $(3,4)$-quaisrung orthopair fuzzy set; q-$ROFS$s: q-rung orthopair fuzzy sets; (p, q)-$QOFS$: (p, q)-quaisrung orthopair fuzzy set; $CIFS$: Complex Intuitionistic fuzzy sets; $CLDFS$s: Complex linear diophantine fuzzy sets; Cq-$ROFS$s: Complex q-rung Orthopair fuzzy sets; $CPyFS$: Complex pythagorean fuzzy sets; $MADM$: Multi-attribute decision making; $IFWA$: Intuitionistic fuzzy weighted average; $IFOWA$: Intuitionistic fuzzy ordered weighted averaging; $IVFFS$s: Interval-valued Fermatean fuzzy sets

1. Introduction

Real-world problems are often challenging to analyze mathematically due to the presence of ambiguity, uncertainty, and imprecision. To address these challenges, Zadeh [1] introduced fuzzy sets (FSs) and the concept of the membership function. Fuzzy set theory, originally proposed by Zadeh, has since become a key framework for handling uncertainty and ambiguity in decision-making and knowledge representation. Unlike traditional crisp sets, which only permit binary membership values of 0 or 1, fuzzy sets enable partial membership, with values ranging between 0 and 1.

Atanassov [2] extended fuzzy set theory into intuitionistic fuzzy set (IFS) theory. Similar to fuzzy sets, IFS employs a membership degree (MD), but it also introduces a non-membership degree (NMD). The uncertainty or hesitation between MD and NMD is represented by the degree of indeterminacy in IFS. This added layer of indeterminacy enhances IFS's effectiveness in handling complex decision-making scenarios with conflicting or incomplete information. Zhang [3] later introduced Bipolar Fuzzy Sets (BFS), a concept designed to address bipolarity in information systems. Chen et al. [4] advanced this concept by introducing m-polar fuzzy sets (m-PFS), which are defined as [0, 1]m within a non-empty universal set U, allowing for the representation of elements with multiple membership values and facilitating the handling of multipolarity. Smarandache [5] made a notable contribution by expanding fuzzy set theory to include Neutrosophic Sets (NSs). Yager [6,7] further developed generalized models of IFS, such as Pythagorean Fuzzy Sets ($PyFS$s), (3,4)-quaisrung orthopair fuzzy set ((3,4)-$QOFS$) [8] and q-rung orthopair Fuzzy Sets (q-$ROFS$s), and (p, q)-quaisrung orthopair fuzzy set ((p, q)-$QOFS$) [9]. For more applications, see [10,11] and the references therein

Spherical fuzzy sets have been developed and introduced in the literature, offering novel approaches to multi-attribute decision-making [12], with notable applications in medical diagnosis [13]. The incorporation of Power Muirhead mean operators has further enhanced decision-making methods [14]. Additionally, the introduction of bipolar soft sets [15] has broadened the scope of applications in this area.

Figure 1 and Table 1 present the visual comparison of IFS alongside $PyFS$, (3,4)-$QOFS$, q-$ROFS$, and (p,q)-$QOFS$ as well as theoretical, respectively.

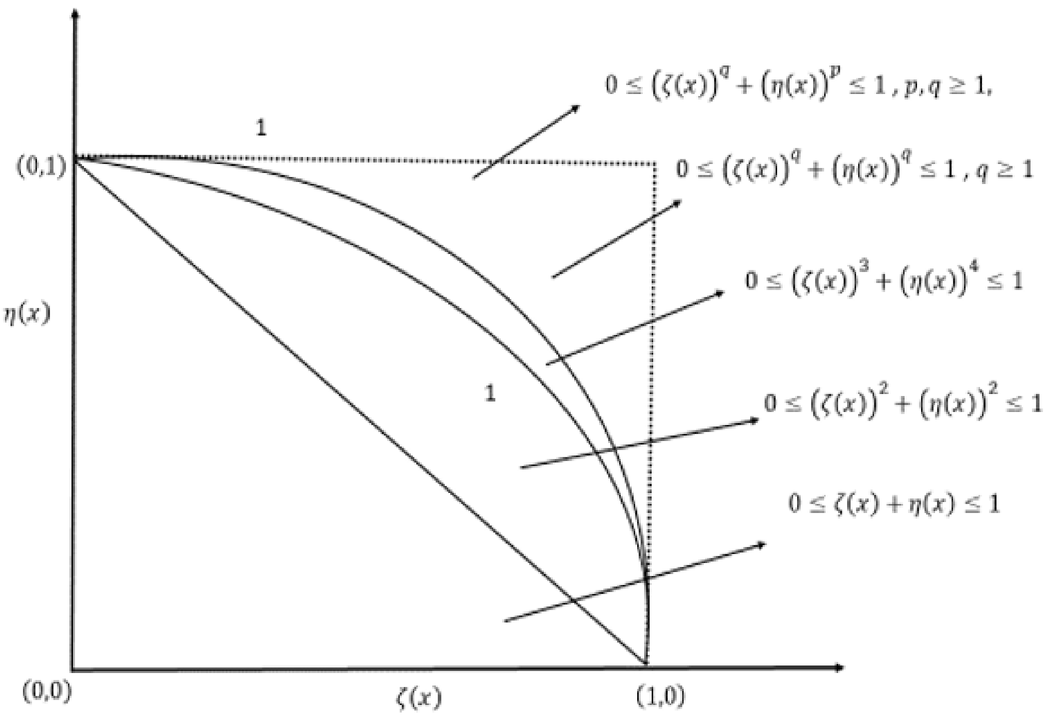

Figure 1. A comparison between IFS, $PyFS$, (3,4)-$QOFS$, q-$ROFS$, and (p,q)-$QOFS$.

Table 1. Some FSs' benchmarks and their limitations.

Collections	Remarks	Parameterization
FS (Zadeh 1965)	Unable to handle non-membership $\eta(x)$	No
IFS (Atanassov 1986)	cannot deal with the condition, $\zeta(x)+\eta(x)>1$	No
$PyFS$ (Yager 2013)	cannot deal with the condition, $(\zeta(x))^2+(\eta(x))^2>1$	No
(3,4)-$QROFS$ (Seikh and Mandal 2022)	cannot deal with the condition, $(\zeta(x))^3+(\eta(x))^4>1$	No
q-$ROFS$ (Yager 2016)	Unable to deal with smaller "q" values with the condition, $(\zeta(x))^{\mathrm{q}}+(\eta(x))^{\mathrm{q}}>1$ and for $\zeta(x)=1, \eta(x)=1$	No

Continued on next page

Collections	Remarks	Parameterization
p,q-$QROFS$ (Seikh and Mandal 2022)	Unable to deal with smaller “q” values with the condition, $\sqrt[k]{\left(\zeta(x)\right)^{p}+\left(\eta(x)\right)^{\mathrm{q}}}>1$ and for $\zeta(x)=1,\eta(x)=1$	No
$LDFS$ (Riaz 2019)	This collection covers upon this situation, $0\leq(\alpha)\zeta(x)+(\beta)\eta(x)\leq 1$, and don’t work under the influence of reference parameters (α,β). $\alpha+\beta>1$	yes
q-$ROLDFS$ (Almagrabi 2021)	This collection covers upon this situation, $0\leq\alpha^{\mathrm{q}}\zeta(x)+\beta^{\mathrm{q}}\eta(x)\leq 1$, and don’t work under the influence of reference parameters $\langle\alpha,\beta)$. $\alpha^{\mathrm{q}}+\beta^{\mathrm{q}}>1$ and for $\alpha=1,\beta=1$	yes

Recent studies have made notable progress in both the theoretical underpinnings and practical applications of fuzzy set theory within dynamic domains, particularly in multi-criteria decision-making. The concept of linear Diophantine fuzzy sets was introduced to highlight its importance in addressing multi-attribute decision-making challenges [16]. This concept was further refined, leading to a novel decision-making approach demonstrated through the use of linear Diophantine fuzzy graphs [17]. The researchers in [18] suggest a q-linear Diophantine fuzzy decision-making model to address the unique challenges posed by the COVID-19 pandemic and support crisis management. This model enhances the $LDFS$ framework by integrating q-$LDFS$s and introducing specialized aggregation operators for emergency decision-making systems. Complex linear Diophantine fuzzy sets ($CLDFS$s), introduced by [19], offer an intriguing extension of traditional membership grades. These fuzzy sets have applications in various fields, including complex intuitionistic fuzzy sets (CIFS), complex q-rung Orthopair fuzzy sets (Cq-$ROFS$s), and complex Pythagorean fuzzy sets ($CPyFS$). The researchers in [20] explored interval-valued linear Diophantine fuzzy dets (IV-$LDFS$s), establishing a solid foundation for decision-making using algebraic principles and Muirhead mean operators. This approach is well-suited for multi-attribute decision making ($MADM$) as it offers an objective, quantifiable framework for decision processes. Additionally, [21] introduces linear Diophantine fuzzy relations (LDF-relations), which link binary relations with $LDFS$s, creating a more flexible framework for analyzing object symmetry and supporting the development of algebraic structures like semi-rings and semi-groups. The use of a scoring system in LDF-relations enhances ranking and decision optimization. Moreover, [22] emphasizes complex scoring functions and optimality criteria, illustrating how $LDFS$ theory can be applied to solve the shortest path problem in linear Diophantine fuzzy graphs.

Intuitionistic fuzzy aggregation operators play a crucial role in data fusion and decision-making when dealing with uncertainty and ambiguity. Unlike traditional aggregation methods, these operators improve functionality by incorporating intuitionistic fuzzy numbers, which represent both membership and non-membership degrees, offering a more nuanced portrayal of uncertainty. Operators like the intuitionistic fuzzy weighted average (IFWA) and intuitionistic fuzzy ordered weighted averaging (IFOWA) are valuable tools for synthesizing data from various sources. They are particularly useful in areas such as risk assessment, expert systems, and multi-criteria decision analysis, where uncertain or imprecise information is common. Riaz and Farid [23] introduced soft-max aggregation operators within the framework of a linear Diophantine fuzzy setting. Peng et al. [24], Yan et al. [25], and Yin et al. [26,27] discussed the application of picture fuzzy sets in decision makings using various type of aggregation operators. A key advancement is the linear Diophantine fuzzy reasonable averaging operator, which proves highly useful in selecting biomedical materials [28]. Additionally, the generalized linear Diophantine fuzzy Choquet integral has been applied effectively in risk analysis and project management [29]. Interval-valued linear Diophantine fuzzy Frank aggregation operators have also demonstrated advantages in various decision-making scenarios [30]. In the textile industry, multi-criteria decision-making ($MCDM$) has been extensively explored. For scenarios involving multiple criteria, advanced methods such as ELECTRE or PROMETHEE are utilized, while the Pareto optimality concept is applied to bicriteria scenarios [31]. Practical applications of the AHP approach and the MMASSI technique are illustrated in the painting department of an automobile manufacturing plant [32,33], showcasing the effectiveness of various MCDA methods in enhancing confidence and robustness in outcomes. Jabeen et al. [34] used new approach to multi-attribute decision-making ($MADM$) based on Aczel-Alsina power Bonferroni aggregation operators for q-rung orthopair fuzzy sets (q-$ROFS$) that is a novel method designed to handle uncertainty and imprecision in decision-making scenarios. Using Einstein prioritized linear Diophantine fuzzy aggregation operators, Farid et al. [35] revolved around a decision-making methodology tailored for selecting suppliers of thermal power equipment with a focus on sustainability. This approach integrates advanced mathematical tools and fuzzy logic to address the complexities and uncertainties inherent in supplier evaluation processes. With the help of Einstein linear Diophantine fuzzy aggregation operators in multicriteria decision-making, Lampan et al. [36] introduced a sophisticated method for addressing complex problems involving uncertainty, multiple conflicting criteria, and interdependence among criteria. Ibrahim and Alshammari [37] initiated to introduce the concept of (n,m)-rung orthopair fuzzy sets and studied their applications to multi-criteria decision making. After that, Panpho and Yiarayong [38] presented the concept of (p, q)-Rung linear Diophantine fuzzy sets with application in decision making. Moreover, Riaz et al. [39] also given some application in decision making via linear Diophantine fuzzy approach. Seikh and Chatterjee [40,41] discussed that the growing challenge of managing electronic waste (e-waste) has necessitated the identification of sustainable strategies that consider diverse perspectives and uncertainties in decision-making. Interval-valued Fermatean fuzzy sets ($IVFFS$s) provide a robust mathematical framework to model uncertainties and imprecise preferences often encountered in group decision-making scenarios. By combining confidence-based group decision-making methods with $IVFFS$s, this approach aims to support decision-makers in evaluating and selecting optimal e-waste management strategies as well as given the new approach that help in determination of best renewable energy sources in India using SWARA-ARAS in confidence level-based interval-valued Fermatean fuzzy environment. Recently, Khan et al. [42] proposed the new fuzzy sets, known as diamond intuitionistic fuzzy sets. Moreover, some applications are also provided in medical diagnosis via diamond intuitionistic fuzzy information. It is well known fact that geographic Information Systems involve integrating spatial and non-spatial data for decision-making across diverse applications like urban planning, environmental management, and resource allocation. Coupling analysis in GIS focuses on understanding and improving the interaction between multiple subsystems or datasets, see [43–46]. This analysis can benefit from advanced fuzzy decision-making tools to handle uncertainty and subjectivity inherent in geographic data, see [47–49] and the references therein. Using the weighted aggregated Sum Product Assessment method, enhanced with bipolar complex fuzzy Aczel-Alsina [50–52] and linguistic Aczel-Alsina power aggregation operators, see [53–56]. Ali et al. [57,58] provided a robust framework for evaluating coupling in GIS. These operators integrate linguistic expressions, complex numbers, and bipolar fuzzy logic, enabling a more nuanced decision-making process. The concepts of q-rung orthopair fuzzy soft sets [59] and q-rung orthopair fuzzy N-soft sets [60] provided an advanced mathematical framework for addressing uncertainty and vagueness in multi-criteria group decision-making problems. By combining the properties of q-Rung Orthopair Fuzzy Sets with soft set theory [61–64], this approach enhances flexibility in modeling membership and non-membership degrees, especially in scenarios involving multiple decision-makers [65–67]. The development of new aggregation operators tailored for q-Rung Orthopair Fuzzy Soft Sets and q-rung orthopair fuzzy N-soft sets further strengthens their application in real-world decision-making problems.

On the other hand, “reference parameter mappings” extend the concept of “reference parameters” by establishing structured relationships between parameters across different contexts, systems, or decision criteria. Mappings accommodate varying conditions and criteria that affect reference parameters, maintaining the model’s flexibility and accuracy even when inputs or constraints evolve. In short, reference parameter mappings expand the utility of reference parameters by capturing complex relationships and dependencies, ensuring consistency, adaptability, and precision across contexts. Reference parameter mappings in decision-making involve the systematic alignment of key decision variables with specific parameters to structure the decision-making process. This technique enhances clarity by linking inputs such as costs, benefits, risks, or performance indicators to the corresponding decision criteria or objectives. By mapping these relationships, it becomes possible to analyze the impact of each parameter on outcomes, assign appropriate weights to factors, and optimize decisions. Advantages of reference parameter mappings include improved decision accuracy, consistency, and transparency. They enable better prioritization of resources, support the development of predictive models, and enhance communication among stakeholders by illustrating the rationale behind choices. Applications of this concept are found across various domains, including business strategy (aligning costs and revenues to financial goals), policy formulation (linking economic indicators to societal objectives), project management (mapping milestones to resource allocation), and healthcare (connecting treatment plans to patient outcomes).

Thus, inspired by ongoing research work and especially by the concept of reference parameter mappings, the following are the key innovations and technical methods employed in this study, expanding on this driving force. Within the realm of $MCDM$ in (q_1, q_2)-$RDFS$ contexts, we focus on the primary objectives outlined below.

(i) The (q_1, q_2)- $RDFS$ is a robust fuzzy model that employs control (reference) parameters mappings to mitigate the impact of ambiguous information.

(ii) This study introduces new concepts related to the score and accuracy functions.

(iii) Innovative (q_1, q_2)-rung Diophantine fuzzy aggregation operators are proposed for information aggregation.

(iv) A comprehensive $MADM$ framework utilizing (q_1, q_2)-rung Diophantine fuzzy averaging and geometric aggregation operators is outlined.

(v) The proposed agricultural field robots $MCDM$ framework is demonstrated in the context of advanced technology, with its application to addressing its effectiveness and adaptability in real-world scenarios. They are increasingly being used in modern farming to improve efficiency, reduce labor costs, and address the growing challenges of food production, such as labor shortages, sustainability, and climate change. These robots can handle tasks such as planting, watering, weeding, monitoring crop health, and even harvesting.

The remainder of this paper is structured as follows: In Section 2, we provide an overview of fundamental concepts related to fuzzy sets, including FS, IFS, $Py FS$s, ${\rm q}$-$ROFS$s, and $LDFS$. In Section 3, we introduce the novel concept of generalized class of linear Diophantine fuzzy set, which is a known (q_1, q_2)-rung Diophantine fuzzy set. Additionally, some new basic operation and relation are defined as well as new score and accuracy functions are presented. Some nontrivial examples are also presented to compare (q_1, q_2)-rung Diophantine fuzzy number. In Section 4, we cover the concept of (q_1, q_2)-rung Diophantine fuzzy averaging aggregation operators, including the (q_1, q_2)-rung Diophantine fuzzy averaging hybrid aggregation operator. In Section 5, the (q_1, q_2)-rung Diophantine fuzzy geometric aggregation operators are also introduced using the (q_1, q_2)-rung Diophantine fuzzy numbers. Section 6 features a case study on an agricultural field robot $MCDM$ framework using (q_1, q_2)-rung Diophantine fuzzy information. Additionally, some advantages and drawback are also discussed in this section. In Sections 7 and 8, we conclude with a summary of the proposed methods and outline future directions.

2. Preliminaries

In this section, we begin by discussing the core concept and its relevant interpretive features before introducing a new approach. We start with the fundamental definition of a fuzzy set, as follows:

Definition 1. (Yager 2016) Suppose E be a fixed set. A ${\rm q}$-rung orthopair fuzzy set (${\rm q}$-$ROFS$) B on E have the following mathematical symbol;

$$B = \{(x, \zeta(x), \eta(x)) : x \in E\},$$

where $\zeta(x)$ and $\eta(x) \in [0,1]$ are MG and NMG functions with subject to $0 \le (\zeta(x))^{\rm q} + (\eta(x))^{\rm q} \le 1$; ${\rm q} \ge 1$, see Figure 1. The hesitancy part is denoted as

$$\pi(x) = \sqrt[{\rm q}]{1 - (\zeta(x))^{\rm q} - (\eta(x))^{\rm q}},$$

Note that, it is not possible to introduce ${\rm q} \to \infty$ because it is not easy to give this value for decision makers.

Definition 2. (Riaz and Hashmi 2019) Suppose E be a fixed non-empty reference set and the linear Diophantine fuzzy set ($LDFS$) is denoted by G_D and mathematical defined as:

$$G_D = \{(x, \langle\zeta(x), \eta(x)\rangle, \langle\alpha, \beta\rangle) : x \in E\},$$

where $\zeta(x), \eta(x), \alpha, \beta \in [0,1]$ are MG, NMG and reference parameters (RPs), respectively, and hold the condition $0 \le \alpha\zeta(x) + \beta\eta(x) \le 1,\ \forall\, x \in E$ with $0 \le \alpha + \beta \le 1$. Such reference parameters may help to describe or identify a specific model. Indeterminacy degree can be defined as:

$$\gamma\pi(x) = 1 - \alpha\zeta(x) - \beta\eta(x),$$

where $\curlyvee$ is the reference parameter of the indeterminacy degree.

Definition 3. (Almagrabi 2022) Suppose E be a fixed non-empty reference set and the linear Diophantine fuzzy set ($LDFS$) is denoted by G_D and mathematical defined as:

$$G_D = \{(x, \langle \zeta(x), \eta(x) \rangle, \langle \alpha, \beta \rangle) : x \in E\},$$

where $\zeta(x), \eta(x), \alpha, \beta \in [0,1]$ are MG, NMG and references parameters (RPs) respectively, and hold the condition $\leq \alpha^{\mathrm{q}}\zeta(x) + \beta^{\mathrm{q}}\eta(x) \leq 1$, with $0 \leq \alpha^{\mathrm{q}} + \beta^{\mathrm{q}} \leq 1$. Such reference parameters may help to describe or identify a specific model. Indeterminacy degree can be defined as:

$$\curlyvee\pi(x) = 1 - \alpha\zeta(x) - \beta\eta(x),$$

where $\curlyvee$ is the reference parameter of the indeterminacy degree.

3. The (q_1, q_2)-rung Diophantine fuzzy set

In this section, we firstly start with the main definition of (q_1, q_2)-rung Diophantine fuzzy set such that:

Definition 4. Let us have a fixed universe E and its sub-set T. The set

$$T = \{(x, \langle \zeta(x), \eta(x) \rangle, \langle q_1(\alpha), q_2(\beta) \rangle) : \text{for all } x \in E\},$$

where $0 \leq \left(q_1(\alpha)\right)^{\mathrm{q}}\zeta(x) + \left(q_2(\beta)\right)^{\mathrm{q}}\eta(x) \leq 1$, with $0 \leq \left(q_1(\alpha)\right)^{\mathrm{q}} + \left(q_2(\beta)\right)^{\mathrm{q}} \leq 1$, is called (q_1, q_2)-rung Diophantine fuzzy set ((q_1, q_2)-$RDFS$) and functions $\zeta, \eta : E \to [0, 1]$ indicate the degree of membership (validity, etc.) and non-membership (non-validity, etc.) of element $x \in E$ to a fixed set $T \subseteq E$ as well as $q_1 : J_1 \to [0, 1]$, and $q_2 : J_2 \to [0, 1]$ are parameter mappings which are known as reference parameters mappings, where $\alpha \in J_1$, and $\beta \in J_2$. Now, we can define also function $\pi : E \to [0, 1]$ by means of

$$q_3(\curlyvee)\pi(x) = \sqrt[\mathrm{q}]{1 - \left(q_1(\alpha)\right)^{\mathrm{q}}\zeta(x) - \left(q_2(\beta)\right)^{\mathrm{q}}\eta(x)},$$

and it corresponds to degree of indeterminacy (uncertainty, etc.), where $q_3 : J_3 \to [0, 1]$ and $\curlyvee \in J_3$. Note that if J_1, J_2, and J_3 are sets of reference parameters. Then, the q_1, q_3, and q_3 are representing the special criteria to choose suitable values for reference parameters. Additionally, one can define the criteria through the following way such that $q_1, q_2, q_3 : J \to [0, 1]$, where $J = J_1 \cup J_2 \cup J_3$.

Here, some novel and classical exceptional cases are acquired such that:

If $q_1 = q_2 = q$, then (q_1, q_2)-$RDFS$ reduces to q-$RDFS$.

Let $J_1, J_2 =$ set of references parameters and $\alpha \in J_1$, $\beta \in J_2$. If $q_1(\alpha) = \alpha^{\frac{1}{\mathrm{q}}}$ and $q_2(\beta) = \beta^{\frac{1}{\mathrm{q}}}$, then one can acquired the definition of $LDFS$.

Let $J =$ set of references parameters and $\alpha, \beta \in J$. If $q_1(\alpha) = \alpha$ and $q_2(\beta) = \beta$, then one can acquired the definition of q-$RLDFS$.

Using Remarks (2) and (4) approaches, if someone takes $q_1(\alpha) = \left(\frac{\alpha}{\mathrm{q}}\right)^{\frac{1}{\mathrm{q}}}$ and $q_2(\beta) = \left(\frac{\beta}{\mathrm{q}}\right)^{\frac{1}{\mathrm{q}}}$, with $\mathrm{q} \geq 2$, then we have another version of (q_1, q_2)-$RDFS$ which is known as q-fractional linear Diophantine fuzzy set (q-$FLDFS$).

Using Remark (3) approaches, then one can take $q_1(\alpha) = \frac{\alpha^{\mathrm{q}}}{1+\mathrm{q}}$ and $q_2(\beta) = \frac{\beta^{\mathrm{q}}}{1+\mathrm{q}}$, with $\mathrm{q} \geq 1$, then, we have new version of (q_1, q_2)-$RDFS$.

If someone takes $q_1(\alpha)=\frac{\alpha}{\mathfrak{q}}$ and $q_2(\beta)=\frac{\beta}{\mathfrak{q}}$, with $\mathfrak{q}\geq 2$, then, we have another version of (q_1,q_2)-$RDFS$.

Let $J_1, J_2 =$ set of references parameters and $\alpha\in J_1,\ \beta\in J_2$. If $q_1(\alpha)=\alpha^{\mathfrak{q}^{\frac{1}{\mathfrak{q}}}}$ and $q_2(\beta)=\beta^{\mathfrak{p}^{\frac{1}{\mathfrak{q}}}}$ and $\mathfrak{q},\mathfrak{p}\geq 1$, then one can acquired the definition of $(\mathfrak{q},\mathfrak{p})$-$RLDFS$.

Similarly, the new exceptional cases have been obtained like Remarks (1)–(7). It's mean that (q_1,q_2)-$RDFS$ is best outcome in filed of fuzzy theory as compare to other classical fuzzy sets like IFS, $LDFS$, -$RLDFS$, q-rung orthopair fuzzy sets and q-fractional fuzzy sets, etc.

Definition 5. A (q_1,q_2)-rung Diophantine fuzzy number is a collection of

$$\Upsilon=\{\langle\zeta,\eta\rangle,\langle q_1,q_2\rangle\},$$

where Υ represent the (q_1,q_2)-rung Diophantine fuzzy number with conditions;
(i) $0\leq\left(q_1(\alpha)\right)^{\mathfrak{q}}+\left(q_2(\beta)\right)^{\mathfrak{q}}\leq 1$;
(ii) $0\leq\left(q_1(\alpha)\right)^{\mathfrak{q}}\zeta(x)+\left(q_2(\beta)\right)^{\mathfrak{q}}\eta(x)\leq 1$;
(iii) $0\leq q_1(\alpha),\zeta(x),q_2(\beta),\eta(x)\leq 1$.

For the sake of simplicity, the set of (q_1,q_2)-rung Diophantine fuzzy numbers ((q_1,q_2)-$RDFNs$).

Next definition is about absolute (q_1,q_2)-rung Diophantine fuzzy set, and null or empty (q_1,q_2)-rung Diophantine fuzzy set.

Definition 6. A (q_1,q_2)-rung Diophantine fuzzy set on E of the form

$$^{1}\Upsilon=\{(x,(1,0),(1,0)):x\in E\}$$

is called absolute (q_1,q_2)-rung Diophantine fuzzy set, and

$$^{0}\Upsilon=\{(x,(0,1),(0,1)):x\in E\}$$

is called empty or null (q_1,q_2)-rung Diophantine fuzzy set.

3.1. Basic operations on (q_1,q_2)-rung Diophantine fuzzy sets

For the sake of easy understanding, we will take the following three (q_1,q_2)-$RDFS$s over fixed universe E:

$$\Upsilon=\left\{\left(x,\langle\zeta_\Upsilon(x),\eta_\Upsilon(x)\rangle_{(p,q)},\langle q_{1_\Upsilon}(\alpha),q_{2_\Upsilon}(\beta)\rangle\right):\text{for all } x\in E\right\},$$

$$Y=\left\{\left(x,\langle\zeta_Y(x),\eta_Y(x)\rangle_{(p,q)},\langle q_{1_Y}(\alpha),q_{2_Y}(\beta)\rangle\right):\text{for all } x\in E\right\},$$

$$Z=\left\{\left(x,\langle\zeta_Z(x),\eta_Z(x)\rangle_{(p,q)},\langle q_{1_Z}(\alpha),q_{2_Z}(\beta)\rangle\right):\text{for all } x\in E\right\}.$$

Definition 7. Let Υ and Y be two (q_1,q_2)-rung Diophantine fuzzy sets. Then,

- $\Upsilon\subseteq Y$ iff $\zeta_\Upsilon(x)\leq\zeta_Y(x)$, $\eta_\Upsilon(x)\geq\eta_Y(x)$, $q_{1_\Upsilon}(\alpha)\leq q_{1_Y}(\alpha)$ and $q_{2_\Upsilon}(\beta)\geq q_{2_Y}(\beta)$,
- $\Upsilon=Y$ iff $\Upsilon\subseteq Y$ and $\Upsilon\supseteq Y$,

- $\Upsilon \cup Y = \Big\{\Big(x,\ \langle \curlyvee\left(\zeta_\Upsilon(x), \zeta_Y(x)\right), \curlywedge\left(\eta_\Upsilon(x), \eta_Y(x)\right)\rangle, \langle \curlyvee\left(q_{1_\Upsilon}(\alpha), q_{1_Y}(\alpha)\right), \curlywedge$
  $\left(q_{2_\Upsilon}(\beta), q_{2_Y}(\beta)\right)\rangle\Big)$: for all $x \in E\Big\}$,
- $\Upsilon \cap Y = \Big\{\Big(x,\ \langle \curlywedge\left(\zeta_\Upsilon(x), \zeta_Y(x)\right), \curlyvee\left(\eta_\Upsilon(x), \eta_Y(x)\right)\rangle, \langle \curlywedge\left(q_{1_\Upsilon}(\alpha), q_{1_Y}(\alpha)\right), \curlyvee$
  $\left(q_{2_\Upsilon}(\beta), q_{2_Y}(\beta)\right)\rangle\Big)$: for all $x \in E\Big\}$,
- $\Upsilon^c = \{(x,\ \langle \eta_\Upsilon(x), \zeta_\Upsilon(x)\rangle, \langle\ q_{2_\Upsilon}(\beta), q_{1_\Upsilon}(\alpha)\rangle)$: for all $x \in E\}$.

Proposition 1. Let Υ, Y, and Z be three (q_1, q_2)-rung Diophantine fuzzy sets. Then, following properties holds such that

1) $\Upsilon \subseteq Y$ and $Y \subseteq Z$ implies $\Upsilon \subseteq Z$; (Inclusion property),
2) $\Upsilon \cup Y = Y \cup \Upsilon$ and $\Upsilon \cap Y = Y \cap \Upsilon$; (Commutative law),
3) $\Upsilon \cup (Y \cup Z) = (\Upsilon \cup Y) \cup Z$ and $\Upsilon \cap (Y \cap Z) = (\Upsilon \cap Y) \cap Z$; (Associative law)
4) $\Upsilon \cup (Y \cap Z) = (\Upsilon \cup Y) \cap (\Upsilon \cup Z)$ and $\Upsilon \cap (Y \cup Z) = (\Upsilon \cap Y) \cup (\Upsilon \cap Z)$; (Distributive laws)
5) De-Morgan's Laws holds for Υ and Y.

Proof: (1) Consider $\Upsilon \subseteq Y$ and $Y \subseteq Z$, then by Definition 7, we have

$$\begin{gathered}\zeta_\Upsilon(x) \le \zeta_Y(x),\ \eta_\Upsilon(x) \ge \eta_Y(x),\\ q_{1_\Upsilon}(\alpha) \le q_{1_Y}(\alpha),\ q_{2_\Upsilon}(\beta) \ge q_{2_Y}(\beta),\end{gathered} \tag{1}$$

and

$$\begin{gathered}\zeta_Y(x) \le \zeta_Z(x),\ \eta_Y(x) \ge \eta_Z(x),\\ q_{1_Y}(\alpha) \le q_{1_Z}(\alpha),\ q_{2_Y}(\beta) \ge q_{2_Z}(\beta).\end{gathered} \tag{2}$$

Combining (1) and (2), we have

$$\begin{gathered}\zeta_\Upsilon(x) \le \zeta_Y(x) \le\ \zeta_Z(x),\ \eta_\Upsilon(x) \ge \eta_Y(x) \ge \eta_Z(x),\\ q_{1_\Upsilon}(\alpha) \le q_{1_Y}(\alpha) \le\ q_{1_Z}(\alpha),\ q_{2_\Upsilon}(\beta) \ge q_{2_Y}(\beta) \ge q_{2_Z}(\beta),\end{gathered} \tag{3}$$

From (3), we conclude that

$$\begin{gathered}\zeta_\Upsilon(x) \le\ \zeta_Z(x), \eta_\Upsilon(x) \ge \eta_Z(x),\\ q_{1_\Upsilon}(\alpha) \le\ q_{1_Z}(\alpha), q_{2_\Upsilon}(\beta) \ge q_{2_Z}(\beta),\end{gathered}$$

Hence, $\Upsilon \subseteq Z$.

Similarly, the remailing results 2–5 can be proved easily.

3.2. Comparison between two (q_1, q_2)-rung Diophantine fuzzy sets

It is well known fact that comparison laws in fuzzy theory play a critical role, especially in the field of decision making and some other optimization problems. These laws enable us to differentiate the two (q_1, q_2)-rung Diophantine fuzzy sets as well as sometime these rules tell us the worth of the relation between these two (q_1, q_2)-rung Diophantine fuzzy sets that this relation is very strong.

Definition 8. Let $\Upsilon = \{\langle \zeta_\Upsilon, \eta_\Upsilon\rangle, \langle q_{1_\Upsilon}, q_{2_\Upsilon}\rangle\}$ be a (q_1, q_2)-$RDFN$. Then, score function

$\left(\mathcal{S}_{(q_1,q_2)\text{-}RDFN}(\Upsilon)\right)$ and accuracy functions $\left(H_{(q_1,q_2)\text{-}RDFN}(\Upsilon)\right)$ of Υ are denoted and defined as:

$$\mathcal{S}_{(q_1,q_2)\text{-}RDFN}(\Upsilon) = \frac{1}{2}\left[\zeta_\Upsilon - \eta_\Upsilon + \left(q_{1\Upsilon}\right)^{\mathrm{q}} - \left(q_{2\Upsilon}\right)^{\mathrm{q}}\right], \tag{4}$$

where, $-1 \leq \mathcal{S}_{(q_1,q_2)\text{-}RDFN}(\Upsilon) \leq 1$.

$$H_{(q_1,q_2)\text{-}RDFN}(\Upsilon) = \frac{1}{2}\left[\frac{\zeta_\Upsilon + \eta_\Upsilon}{2} + \left(q_{1\Upsilon}\right)^{\mathrm{q}} + \left(q_{2\Upsilon}\right)^{\mathrm{q}}\right],$$

where, $0 \leq H_{(q_1,q_2)\text{-}RDFN}(\Upsilon) \leq 1$, respectively. These rules define the comparison between two $(q_1, q_2)\text{-}RDFNs$ Υ_1 and Υ_2 such that

a) Υ_1 is higher ranked than Υ_2 if $\mathcal{S}_{(q_1,q_2)\text{-}RDFN}(\Upsilon_1) > \mathcal{S}_{(q_1,q_2)\text{-}RDFN}(\Upsilon_2)$.

b) Υ_1 is lower ranked than Υ_2 if $\mathcal{S}_{(q_1,q_2)\text{-}RDFN}(\Upsilon_1) < \mathcal{S}_{(q_1,q_2)\text{-}RDFN}(\Upsilon_2)$.

When $\mathcal{S}_{(q_1,q_2)\text{-}RDFN}(\Upsilon_1) = \mathcal{S}_{(q_1,q_2)\text{-}RDFN}(\Upsilon_2)$ for two $(q_1, q_2)\text{-}RDFNs$, then

c) Υ_1 is higher ranked than Υ_2 if $H_{(q_1,q_2)\text{-}RDFN}(\Upsilon_1) > H_{(q_1,q_2)\text{-}RDFN}(\Upsilon_2)$.

d) Υ_1 is lower ranked than Υ_2 if $H_{(q_1,q_2)\text{-}RDFN}(\Upsilon_1) < H_{(q_1,q_2)\text{-}RDFN}(\Upsilon_2)$.

e) Υ_1 is similar Υ_2 if $H_{(q_1,q_2)\text{-}RDFN}(\Upsilon_1) = H_{(q_1,q_2)\text{-}RDFN}(\Upsilon_2)$.

Example 1. Let $\Upsilon_1 = (\langle 1, .9\rangle, \langle 0.3, 0.3\rangle)$ and $\Upsilon_2 = (\langle 0.4, 0.9\rangle, \langle 0.5, 0.5\rangle)$ be two alternatives with $(q_1, q_2)\text{-}RDFNs$. Then, score function is utilized to determine the preferred option such that

$$\mathcal{S}_{(q_1,q_2)\text{-}RDFN}(\Upsilon_1) = \frac{1}{2}[1 - .9 + 0] = .05,$$

$$\mathcal{S}_{(q_1,q_2)\text{-}RDFN}(\Upsilon_2) = \frac{1}{2}[4 - .9 + 0] = -0.2,$$

hence, option Υ_2 is preferable to option Υ_1.

Example 2. If $(q_1, q_2)\text{-}RDFNs$ for two alternatives are $\Upsilon_2 = (\langle 1, .5\rangle, \langle 0.4, 0.4\rangle)$ and $\Upsilon_2 = (\langle .9, .4\rangle, \langle .5, .5\rangle)$, then score function is utilized to determine the preferred option such that

$$\mathcal{S}_{(q_1,q_2)\text{-}RDFN}(\Upsilon_1) = \frac{1 - .5}{2} = .25,$$

$$\mathcal{S}_{(q_1,q_2)\text{-}RDFN}(\Upsilon_2) = \frac{.9 - .4}{2} = .25.$$

So, we are unsure of which option is preferable in this situation. However, by using Eq (5), we can get

$$H_{(q_1,q_2)\text{-}RDFN}(\Upsilon_1) = \frac{1 + .5}{4} + \frac{.4 + .4}{2} = .78,$$

$$H_{(q_1,q_2)\text{-}RDFN}(\Upsilon_2) = \frac{.9 + .4}{4} + \frac{.5 + .5}{2} = .83,$$

as a result, alternative Υ_1 is superior to alternative Υ_2.

In decision-making, multi-criteria analysis, and fuzzy set applications, *quadratic score functions* are sometimes preferred over linear score functions due to their enhanced sensitivity to differences in values, better differentiation, and ability to emphasize larger deviations. In short, the quadratic score function is useful when differentiation, sensitivity to extremes, and robustness in the scoring process are prioritized, making it advantageous for applications that demand higher accuracy in ranking or decision-making. The quadratic score function for (q_1, q_2)-$RDFN$ is defined in the next subsection.

Definition 9. Let $\Upsilon = \{\langle \zeta_\Upsilon, \eta_\Upsilon \rangle, \langle q_{1_\Upsilon}, q_{2_\Upsilon} \rangle\}$ be a (q_1, q_2)-$RDFN$. Then, quadratic score function $\left(\mathcal{Q}_{(q_1,q_2)\text{-}RDFN}(\Upsilon)\right)$ and quadratic accuracy functions $\left(И_{(q_1,q_2)\text{-}RDFN}(\Upsilon)\right)$ of Υ are denoted and defined as:

$$\mathcal{Q}_{(q_1,q_2)\text{-}RDFN}(\Upsilon) = \frac{1}{2}\left[{\zeta_\Upsilon}^2 - {\eta_\Upsilon}^2 + \left(\left(q_{1_\Upsilon}\right)^{\mathrm{q}}\right)^2 - \left(\left(q_{2_\Upsilon}\right)^{\mathrm{q}}\right)^2\right], \tag{5}$$

where, $-1 \leq \mathcal{Q}_{(q_1,q_2)\text{-}RDFN}(\Upsilon) \leq 1$.

$$И_{(q_1,q_2)\text{-}RDFN}(\Upsilon) = \frac{1}{2}\left[\frac{{\zeta_\Upsilon}^2 + {\eta_\Upsilon}^2}{2} + \left(\left(q_{1_\Upsilon}\right)^{\mathrm{q}}\right)^2 + \left(\left(q_{2_\Upsilon}\right)^{\mathrm{q}}\right)^2\right],$$

where, $0 \leq И_{(q_1,q_2)\text{-}RDFN}(\Upsilon) \leq 1$, respectively. These rules define the comparison between two (q_1, q_2)-$RDFNs$ Υ_1 and Υ_2 such that

1) Υ_1 is higher ranked than Υ_2 if $\mathcal{Q}_{(q_1,q_2)\text{-}RDFN}(\Upsilon_1) > \mathcal{Q}_{(q_1,q_2)\text{-}RDFN}(\Upsilon_2)$.
2) Υ_1 is lower ranked than Υ_2 if $\mathcal{Q}_{(q_1,q_2)\text{-}RDFN}(\Upsilon_1) < \mathcal{Q}_{(q_1,q_2)\text{-}RDFN}(\Upsilon_2)$.
3) When $\mathcal{Q}_{(q_1,q_2)\text{-}RDFN}(\Upsilon_1) = \mathcal{Q}_{(q_1,q_2)\text{-}RDFN}(\Upsilon_2)$ for two (q_1, q_2)-$RDFNs$, then
4) Υ_1 is higher ranked than Υ_2 if $И_{(q_1,q_2)\text{-}RDFN}(\Upsilon_1) > И_{(q_1,q_2)\text{-}RDFN}(\Upsilon_2)$.
5) Υ_1 is lower ranked than Υ_2 if $И_{(q_1,q_2)\text{-}RDFN}(\Upsilon_1) < И_{(q_1,q_2)\text{-}RDFN}(\Upsilon_2)$.

Υ_1 is similar Υ_2 if $И_{(q_1,q_2)\text{-}RDFN}(\Upsilon_1) = И_{(q_1,q_2)\text{-}RDFN}(\Upsilon_2)$.

In a fuzzy decision-making scenario, where the satisfaction level of each decision outcome varies, the expectation score function calculates the average satisfaction score, factoring in the likelihood of each satisfaction level. This helps decision-makers evaluate the overall desirability of each option while accounting for the fuzziness of outcomes. In essence, the expectation score function is valuable when we need to assess options with probabilistic or fuzzy outcomes, giving a fair and comprehensive measure for informed decision-making. We now introduce the expectation score function (ESF), which is an additional generalized score function.

Definition 10. Let $\Upsilon = \{\langle \zeta_\Upsilon, \eta_\Upsilon \rangle, \langle q_{1_\Upsilon}, q_{2_\Upsilon} \rangle\}$ be a (q_1, q_2)-$RDFN$. Then, ESF $\left(\check{\mathrm{E}}_{(q_1,q_2)\text{-}RDFN}(\Upsilon)\right)$ of Υ are denoted and defined as:

$$\check{\mathrm{E}}_{(q_1,q_2)\text{-}RDFN}(\Upsilon) = \frac{1}{2}\left[\frac{\zeta_\Upsilon - \eta_\Upsilon + 1}{2} + \frac{\left(q_{1_\Upsilon}\right)^{\mathrm{q}} - \left(q_{2_\Upsilon}\right)^{\mathrm{q}} + 1}{2}\right], \tag{6}$$

where, $0 \leq \check{\mathrm{E}}_{(q_1,q_2)\text{-}RDFN}(\Upsilon) \leq 1$.

4. The (q_1, q_2)-rung Diophantine fuzzy weighted averaging aggregation operators

In this section, we propose some different types of (q_1, q_2)-rung Diophantine fuzzy weighted averaging aggregation operators. First, we define the (q_1, q_2)-rung Diophantine fuzzy weighted averaging aggregation operator.

Definition 11. Let $\Upsilon_1 = \{\langle \zeta_1, \eta_1 \rangle, \langle q_{1_1}, q_{2_1} \rangle\}$ and $\Upsilon_2 = \{\langle \zeta_2, \eta_2 \rangle, \langle q_{1_2}, q_{2_2} \rangle\}$ be two (q_1, q_2)-rung

Diophantine fuzzy numbers and $\lambda > 0$,

$$\Upsilon_1 \oplus \Upsilon_2 = \left(\langle \sqrt[q]{(\zeta_1)^q + (\zeta_2)^q - (\zeta_1)^q (\zeta_2)^q}, \eta_1 \eta_2 \rangle, \langle \sqrt[q]{\left(q_{1_1}\right)^q + \left(q_{1_2}\right)^q - \left(q_{1_1}\right)^q \left(q_{1_2}\right)^q}, q_{2_1} q_{2_2} \rangle \right),$$

$$\lambda \Upsilon_1 = \left(\langle \sqrt[q]{1 - (1 - (\zeta_1)^q)^\lambda}, {\eta_1}^\lambda \rangle, \langle \sqrt[q]{1 - \left(1 - \left(q_{1_1}\right)^q\right)^\lambda}, {q_{2_1}}^\lambda \rangle \right).$$

For the sake of simplicity, the set of (q_1, q_2)-rung Diophantine fuzzy numbers ((q_1, q_2)-$RDFNs$) on E is denoted by (q_1, q_2)-$RDFN(E)$.

Example 3. Let $\Upsilon_1 = (\langle 1, 0.8 \rangle, \langle 0.3, 0.5 \rangle)$ and $\Upsilon_2 = (\langle 0.5, 0.9 \rangle, \langle 0.5, 0.3 \rangle)$ be two alternatives with (q_1, q_2)-$RDFNs$.

$$\Upsilon_1 \oplus \Upsilon_2 = (\langle 0.5, 0.72 \rangle, \langle 0.5, 0.15 \rangle).$$

If $\lambda = 0.6$, then,

$$\lambda \Upsilon_1 = (\langle 1, 0.875 \rangle, \langle 0.193, 0.66 \rangle).$$

Definition 12. The (q_1, q_2)-rung Diophantine fuzzy weighted averaging aggregation ((q_1, q_2)-$RDFWAA$) operator on "n" numbers of (q_1, q_2)-$RDFNs$ on the set E is defined with the help of this transformation $\Omega: (q_1, q_2)\text{-}RDFN(E) \rightarrow (q_1, q_2)\text{-}RDFN(E)$ associated with weight vector $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)^T$ with $\sum_{j=1}^{n} \omega_j = 1$ and it can be computed as follows:

When $\{\Upsilon_1 = \left(\langle \zeta_1, \eta_1 \rangle, \langle q_{1_1}, q_{2_1} \rangle\right), \Upsilon_2 = \left(\langle \zeta_2, \eta_2 \rangle, \langle q_{1_2}, q_{2_2} \rangle\right), \ldots, \Upsilon_n = \left(\langle \zeta_n, \eta_n \rangle, \langle q_{1_n}, q_{2_n} \rangle\right)\}$ are (q_1, q_2)-$RDFNs$,

$$(q_1, q_2)\text{-}RDFWAA_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots, \Upsilon_n) = \prod_{j=1}^{n} \omega_j \Upsilon_j.$$

Note that the proofs of all the next results in this section are straightforward, so we have omitted them.

Theorem 1. The (q_1, q_2)-rung Diophantine fuzzy weighted averaging aggregation ((q_1, q_2)-$RDFWAA$) operator on "n" numbers of (q_1, q_2)-$RDFNs$ on the set E is defined with the help of this transformation $\Omega: (q_1, q_2)\text{-}RDFN(E) \rightarrow (q_1, q_2)\text{-}RDFN(E)$ associated with weight vector $\omega = (\omega_1, \omega_2, \omega_3, \ldots \ldots \ldots, \omega_n)^T$ with $\sum_{j=1}^{n} \omega_j = 1$ and it can be computed as follows:

When $\{\Upsilon_1 = \left(\langle \zeta_1, \eta_1 \rangle, \langle q_{1_1}, q_{2_1} \rangle\right), \Upsilon_2 = \left(\langle \zeta_2, \eta_2 \rangle, \langle q_{1_2}, q_{2_2} \rangle\right), \ldots, \Upsilon_n = \left(\langle \zeta_n, \eta_n \rangle, \langle q_{1_n}, q_{2_n} \rangle\right)\}$ are (q_1, q_2)-$RDFNs$,

$$(q_1, q_2)\text{-}RDFWAA_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots, \Upsilon_n)$$

$$= \prod_{j=1}^{n} \omega_j \Upsilon_j$$

$$= \left(\langle \sqrt[q]{1 - \prod_{j=1}^{n} \left(1 - \left(\zeta_j\right)^q\right)^{\omega_j}}, \prod_{j=1}^{n} {\eta_j}^{\omega_j} \rangle, \langle \sqrt[q]{1 - \prod_{j=1}^{n} \left(1 - \left(q_{1_j}\right)^q\right)^{\omega_j}}, \prod_{j=1}^{n} {q_{2_j}}^{\omega_j} \rangle \right).$$

This operator can easily be proved with support of (q_1, q_2)-$RDFNs$ operations and mathematical-induction. Here, μ and ν are representing the membership and non-membership function. ω is called weight function, Υ_j are (q_1, q_2)-$RDFNs$, where $j \in N$.

It is simple to demonstrate that the weighted averaging operator possesses the following characteristics:

Theorem 2. (Idempotency) If all Υ_j are equal that is $\Upsilon_j = \Upsilon$ for all j, then,

$$(q_1, q_2)\text{-}RDFWAA_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots, \Upsilon_n) = \Upsilon.$$

Theorem 3. (Boundedness) Let Υ_j $(j = 1,2,3, \ldots, n)$ be a collection of (q_1, q_2)-$RDFNs$, and let

$\Upsilon^- = \min_j \Upsilon_j$, $\Upsilon^+ = \max_j \Upsilon_j$.

Then,

$$\Upsilon^- \le (q_1, q_2)\text{-}RDFWAA_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots, \Upsilon_n) \le \Upsilon^+.$$

Theorem 4. (Monotonicity) Let Υ_j $(j = 1,2,3, \ldots, n)$ and Υ_j' $(j = 1,2,3, \ldots, n)$ be two collections of (q_1, q_2)-$RDFNs$. If $\Upsilon_j \le \Upsilon_j'$ for all j, then,

$$(q_1, q_2)\text{-}RDFWAA_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots, \Upsilon_n) \le (q_1, q_2)\text{-}RDFWAA_\omega(\Upsilon_1', \Upsilon_2', \Upsilon_3', \ldots, \Upsilon_n').$$

We would now like to introduce the (q_1, q_2)-rung Diophantine fuzzy ordered weighted averaging aggregation operator.

Definition 13. The (q_1, q_2)-rung Diophantine fuzzy ordered weighted averaging aggregation ((q_1, q_2)-$RDFOWAA$) operator on "n" numbers of (q_1, q_2)-$RDFNs$ is defined with the help of this transformation Ω: (q_1, q_2)-$RDFN(E) \to (q_1, q_2)$-$RDFN(E)$ associated with $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)^T$ with $\sum_{j=1}^n \omega_j = 1$, and it can be computed as follows: When $\{\Upsilon_1 = (\langle \zeta_1, \eta_1 \rangle, \langle q_{1_1}, q_{2_1} \rangle), \Upsilon_2 = (\langle \zeta_2, \eta_2 \rangle, \langle q_{1_2}, q_{2_2} \rangle), \ldots, \Upsilon_n = (\langle \zeta_n, \eta_n \rangle, \langle q_{1_n}, q_{2_n} \rangle)\}$ are (q_1, q_2)-$RDFN$,

$$(q_1, q_2)\text{-}RDFOWAA_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots, \Upsilon_n) = \prod_{j=1}^n \omega_j \Upsilon_{\sigma(j)},$$

where $(\sigma(1), \sigma(2), \sigma(3), \ldots \ldots \ldots \ldots, \sigma(n))$ is the arrangement of $j \in N$, for which $\Upsilon_{\sigma(j-1)} \ge \Upsilon_{\sigma(j)}$, for all $j \in N$.

Theorem 5. The (q_1, q_2)-rung Diophantine fuzzy ordered weighted averaging aggregation ((q_1, q_2)-$RDFOWAA$) operator on "n" numbers of (q_1, q_2)-$RDFNs$ is defined with the help of this transformation Ω: (q_1, q_2)-$RDFN(E) \to (q_1, q_2)$-$RDFN(E)$ associated with $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)^T$ with $\sum_{j=1}^n \omega_j = 1$, and it can be computed as follows:

When $\{\Upsilon_1 = (\langle \zeta_1, \eta_1 \rangle, \langle q_{1_1}, q_{2_1} \rangle), \Upsilon_2 = (\langle \zeta_2, \eta_2 \rangle, \langle q_{1_2}, q_{2_2} \rangle), \ldots, \Upsilon_n = (\langle \zeta_n, \eta_n \rangle, \langle q_{1_n}, q_{2_n} \rangle)\}$ are (q_1, q_2)-$RDFNs$,

$$(q_1, q_2)\text{-}RDFOWAA_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots, \Upsilon_n)$$

$$= \prod_{j=1}^{n} \omega_j \Upsilon_{\sigma(j)}$$

$$= \left(\langle \sqrt[q]{1 - \prod_{j=1}^n \left(1 - \left(\zeta_{\sigma(j)}\right)^q\right)^{\omega_j}}, \prod_{j=1}^n \eta_{\sigma(j)}{}^{\omega_j} \rangle, \langle \sqrt[q]{1 - \prod_{j=1}^n \left(1 - \left(q_{1_{\sigma(j)}}\right)^q\right)^{\omega_j}}, \prod_{j=1}^n q_{2_{\sigma(j)}}{}^{\omega_j} \rangle \right),$$

where $(\sigma(1),\sigma(2),\sigma(3),\ldots\ldots\ldots\ldots,\sigma(n))$ is the arrangement of $j\in N$, for which $\Upsilon_{\sigma(j-1)}\geq\Upsilon_{\sigma(j)}$, for all $j\in N$.

It is simple to demonstrate that the weighted averaging operator possesses the following characteristics:

Theorem 6. (Idempotency) If all Υ_j are equal that is $\Upsilon_j=\Upsilon$ for all j, then,

$$(q_1,q_2)\text{-}RDFOWAA_{\omega}(\Upsilon_1,\Upsilon_2,\Upsilon_3,\ldots,\Upsilon_n)=\Upsilon.$$

Theorem 7. (Boundedness) Let Υ_j $(j=1,2,3,\ldots,n)$ be a collection of (q_1,q_2)-$RDFNs$, and let

$$\Upsilon^-=\min_j\Upsilon_j,\ \Upsilon^+=\max_j\Upsilon_j.$$

Then,

$$\Upsilon^-\leq(q_1,q_2)\text{-}RDFOWAA_{\omega}(\Upsilon_1,\Upsilon_2,\Upsilon_3,\ldots,\Upsilon_n)\leq\Upsilon^+.$$

Theorem 8. (Monotonicity) Let Υ_j $(j=1,2,3,\ldots,n)$ and Υ_j' $(j=1,2,3,\ldots,n)$ be two collections of (q_1,q_2)-$RDFNs$. If $\Upsilon_j\leq\Upsilon_j'$ for all j, then,

$$(q_1,q_2)\text{-}RDFOWAA_{\omega}(\Upsilon_1,\Upsilon_2,\Upsilon_3,\ldots,\Upsilon_n)\leq(q_1,q_2)\text{-}RDFOWAA_{\omega}(\Upsilon_1',\Upsilon_2',\Upsilon_3',\ldots,\Upsilon_n').$$

Theorem 9. (Commutativity) Let Υ_j $(j=1,2,3,\ldots,n)$ and Υ_j' $(j=1,2,3,\ldots,n)$ be two collections of (q_1,q_2)-$RDFNs$. If $\Upsilon_j\leq\Upsilon_j'$ for all j, then,

$$(q_1,q_2)\text{-}RDFOWAA_{\omega}(\Upsilon_1,\Upsilon_2,\Upsilon_3,\ldots,\Upsilon_n)=(q_1,q_2)\text{-}RDFOWAA_{\omega}(\Upsilon_1',\Upsilon_2',\Upsilon_3',\ldots,\Upsilon_n'),$$

where Υ_j' $(j=1,2,3,\ldots,n)$ is any permutation of Υ_j $(j=1,2,3,\ldots,n)$.

The (q_1,q_2)-rung Diophantine fuzzy hybrid weighted averaging aggregation operator is now ready for introduction.

Definition 14. The (q_1,q_2)-rung Diophantine fuzzy hybrid weighted averaging aggregation ((q_1,q_2)-$RDFHWAA$) operator on "n" numbers of (q_1,q_2)-$RDFNs$ is defined with the help of this transformation $\Omega:(q_1,q_2)\text{-}RDFN(E)\to(q_1,q_2)\text{-}RDFN(E)$ associated with $\omega=(\omega_1,\omega_2,\omega_3,\ldots,\omega_n)^T$ with $\sum_{j=1}^{n}\omega_j=1$ and it can be computed as follows:

When $\{\Upsilon_1=(\langle\zeta_1,\eta_1\rangle,\langle q_{1_1},q_{2_1}\rangle),\Upsilon_2=(\langle\zeta_2,\eta_2\rangle,\langle q_{1_2},q_{2_2}\rangle),\ldots,\Upsilon_n=(\langle\zeta_n,\eta_n\rangle,\langle q_{1_n},q_{2_n}\rangle)\}$ are (q_1,q_2)-$RDFNs$,

$$(q_1,q_2)\text{-}RDFHWAA_{\omega}(\Upsilon_1,\Upsilon_2,\Upsilon_3,\ldots,\Upsilon_n)$$

$$=\prod_{j=1}^{n}\omega_j\,\Upsilon_{\sigma(j)}^{\star},$$

where $\Upsilon_{\sigma(j)}^{\star}$ is biggest jth weighted (q_1,q_2)-rung Diophantine fuzzy values $\Upsilon_j^{\star}$ $\left(\Upsilon_j^{\star}=(\Upsilon_j)^{n\omega_j},j\in N\right)$ and $\omega=(\omega_1,\omega_2,\omega_3,\ldots\ldots\ldots,\omega_n)^T$ be the weights of $\Upsilon_j^{\star}$ by means of $\omega>0$ with $\sum_{j=1}^{n}\omega_j=1$.

Theorem 10. The (q_1,q_2)-rung Diophantine fuzzy hybrid weighted averaging aggregation ((q_1,q_2)-$RDFHWAA$) operator on "n" numbers of (q_1,q_2)-$RDFNs$ is defined with the help of this transformation $\Omega:(q_1,q_2)\text{-}RDFN(E)\to(q_1,q_2)\text{-}RDFN(E)$ associated with $\omega=(\omega_1,\omega_2,\omega_3,\ldots\ldots\ldots,\omega_n)^T$ with $\sum_{j=1}^{n}\omega_j=1$, and it can be computed as follows:

When $\{\Upsilon_1=(\langle\zeta_1,\eta_1\rangle,\langle q_{1_1},q_{2_1}\rangle),\Upsilon_2=(\langle\zeta_2,\eta_2\rangle,\langle q_{1_2},q_{2_2}\rangle),\ldots,\Upsilon_n=(\langle\zeta_n,\eta_n\rangle,\langle q_{1_n},q_{2_n}\rangle)\}$ are

(q_1, q_2)-$RDFNs$,

$$(q_1, q_2)\text{-}RDFHWAA_{\omega}(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots \ldots \ldots, \Upsilon_n)$$

$$= \prod_{j=1}^{n} \omega_j \, \Upsilon^{\star}_{\sigma(j)}$$

$$= \left(\langle \sqrt[q]{1 - \prod_{j=1}^{n} \left(1 - \left(\zeta^{\star}{}_{\sigma(j)}\right)^{q}\right)^{\omega_j}}, \prod_{j=1}^{n} \eta^{\star}_{\sigma(j)}{}^{\omega_j} \rangle, \langle \sqrt[q]{1 - \prod_{j=1}^{n} \left(1 - \left(q_1{}^{\star}{}_{\sigma(j)}\right)^{q}\right)^{\omega_j}}, \prod_{j=1}^{n} q_2{}^{\star}{}_{\sigma(j)}{}^{\omega_j} \rangle \right),$$

where $\Upsilon^{\star}_{\sigma(j)}$ is biggest jth weighted (q_1, q_2)-rung Diophantine fuzzy values $\Upsilon_j^{\star}$ $\left(\Upsilon_j^{\star} = n\omega_j \Upsilon_j, j \in N\right)$ and $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)^T$ be the weights of $\Upsilon_j^{\star}$ by means of $\omega > 0$ with $\sum_{j=1}^{n} \omega_j = 1$.

It is interesting to note that if $\omega = \left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$, then (q_1, q_2)-$RDFWAA$, and (q_1, q_2)-$RDFOWAA$ operators are considered to be exceptional cases of (q_1, q_2)-$RDFHWAA$ operator. Thus, it concludes that (q_1, q_2)-$RDFHWAA$ operators are the extension of (q_1, q_2)-$RDFWAA$ and (q_1, q_2)-$RDFOWAA$ operators.

5. The (q_1, q_2)-rung Diophantine fuzzy weighted geometric aggregation operators

In this section, we propose some different types of (q_1, q_2)-rung Diophantine fuzzy weighted geometric aggregation operators. First, we define the (q_1, q_2)-rung Diophantine fuzzy weighted geometric aggregation operator.

Definition 15. Let $\Upsilon_1 = \{\langle \zeta_1, \eta_1 \rangle, \langle q_{1_1}, q_{2_1} \rangle\}$ and $\Upsilon_2 = \{\langle \zeta_2, \eta_2 \rangle, \langle q_{1_2}, q_{2_2} \rangle\}$ be two (q_1, q_2)-rung Diophantine fuzzy numbers and $\lambda > 0$.

$$\Upsilon_1 \otimes \Upsilon_2 = \left(\langle \zeta_1 \zeta_2, \eta_1 + \eta_2 - \eta_1 \eta_2 \rangle, \langle q_{1_1} q_{1_2}, q_{2_1} + q_{2_2} - q_{2_1} q_{2_2} \rangle\right),$$

$$\Upsilon_1{}^{\lambda} = \left(\langle \zeta_1{}^{\lambda}, 1 - (1 - \eta_1)^{\lambda} \rangle, \langle q_{1_1}{}^{\lambda}, 1 - \left(1 - q_{2_1}\right)^{\lambda} \rangle\right).$$

Example 4. Let $\Upsilon_1 = (\langle 0.8,1 \rangle, \langle 0.5,0.3 \rangle)$ and $\Upsilon_2 = (\langle 0.9,0.5 \rangle, \langle 0.3,0.5 \rangle)$ be two alternatives with (q_1, q_2)-$RDFNs$.

$$\Upsilon_1 \otimes \Upsilon_2 = (\langle 0.72,0.5 \rangle, \langle 0.15,0.5 \rangle).$$

If $\lambda = 0.6$, then,

$$\Upsilon_1{}^{\lambda} = (\langle\ 0.875, 1 \rangle, \langle 0.66\ ,0.193 \rangle).$$

Definition 16. The (q_1, q_2) -rung Diophantine fuzzy weighted geometric aggregation ((q_1, q_2)-$RDFWGA$) operator on "n" numbers of (q_1, q_2)-$RDFNs$ on the set E is defined with the help of this transformation $\Omega: (q_1, q_2)\text{-}RDFN(E) \rightarrow (q_1, q_2)\text{-}RDFN(E)$ associated with weight vector $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)^T$ with $\sum_{j=1}^{n} \omega_j = 1$, and it can be computed as follows:

When $\{\Upsilon_1 = \left(\langle \zeta_1, \eta_1 \rangle, \langle q_{1_1}, q_{2_1} \rangle\right), \Upsilon_2 = \left(\langle \zeta_2, \eta_2 \rangle, \langle q_{1_2}, q_{2_2} \rangle\right), \ldots, \Upsilon_n = \left(\langle \zeta_n, \eta_n \rangle, \langle q_{1_n}, q_{2_n} \rangle\right)\}$ are (q_1, q_2)-$RDFNs$,

$$(q_1,q_2)\text{-}RDFWGA_{\omega}(\Upsilon_1,\Upsilon_2,\Upsilon_3,\dots,\Upsilon_n)=\prod_{j=1}^{n}\Upsilon_j^{\omega_j}.$$

It is worth mentioning that the proofs for all the results in this section are straightforward, and therefore, have been omitted.

Theorem 11. The (q_1,q_2)-rung Diophantine fuzzy weighted geometric aggregation ((q_1,q_2)-$RDFWGA$) operator on "n" numbers of (q_1,q_2)-$RDFNs$ on the set E is defined with the help of this transformation $\Omega: (q_1,q_2)\text{-}RDFN(E)\to(q_1,q_2)\text{-}RDFN(E)$ associated with weight vector $\omega=(\omega_1,\omega_2,\omega_3,\dots\dots\dots,\omega_n)^T$ with $\sum_{j=1}^{n}\omega_j=1$, and it can be computed as follows:

When $\{\Upsilon_1=(\langle\zeta_1,\eta_1\rangle,\langle q_{1_1},q_{2_1}\rangle),\Upsilon_2=(\langle\zeta_2,\eta_2\rangle,\langle q_{1_2},q_{2_2}\rangle),\dots,\Upsilon_n=(\langle\zeta_n,\eta_n\rangle,\langle q_{1_n},q_{2_n}\rangle)\}$ are (q_1,q_2)-$RDFNs$,

$$(q_1,q_2)\text{-}RDFWGA_{\omega}(\Upsilon_1,\Upsilon_2,\Upsilon_3,\dots\dots\dots,\Upsilon_n)$$

$$=\prod_{j=1}^{n}\Upsilon_j^{\omega_j}$$

$$=\left(\langle\textstyle\prod_{j=1}^{n}\zeta_j^{\,\omega_j},\sqrt[q]{1-\prod_{j=1}^{n}\left(1-(\eta_j)^{q}\right)^{\omega_j}}\rangle,\langle\prod_{j=1}^{n}q_{1_j}^{\,\omega_j},\sqrt[q]{1-\prod_{j=1}^{n}\left(1-\left(q_{2_j}\right)^{q}\right)^{\omega_j}}\rangle\right).$$

This operator can be proved with support of (q_1,q_2)-$RDFNs$ operations and mathematical induction. Here, μ and ν are representing the membership and non-membership function. ω is called the weight function, Υ_j are (q_1,q_2)-$RDFNs$, where $j\in N$.

It is simple to demonstrate that the weighted averaging operator possesses the following characteristics:

Theorem 12. (Idempotency) If all Υ_j are equal that is $\Upsilon_j=\Upsilon$ for all j, then,

$$(q_1,q_2)\text{-}RDFWGA_{\omega}(\Upsilon_1,\Upsilon_2,\Upsilon_3,\dots,\Upsilon_n)=\Upsilon.$$

Theorem 13. (Boundedness) Let Υ_j $(j=1,2,3,\dots\dots\dots,n)$ be a collection of (q_1,q_2)-$RDFNs$, and let

$$\Upsilon^-=\min_j\Upsilon_j,\ \Upsilon^+=\max_j\Upsilon_j.$$

Then,

$$\Upsilon^-\le(q_1,q_2)\text{-}RDFWGA_{\omega}(\Upsilon_1,\Upsilon_2,\Upsilon_3,\dots,\Upsilon_n)\le\Upsilon^+.$$

Theorem 14. (Monotonicity) Let Υ_j $(j=1,2,3,\dots\dots\dots,n)$ and Υ_j' $(j=1,2,3,\dots\dots\dots,n)$ be two collections of (q_1,q_2)-$RDFNs$. If $\Upsilon_j\le\Upsilon_j'$ for all j, then,

$$(q_1,q_2)\text{-}RDFOWAA_{\omega}(\Upsilon_1,\Upsilon_2,\Upsilon_3,\dots,\Upsilon_n)\le(q_1,q_2)\text{-}RDFWGA_{\omega}(\Upsilon_1',\Upsilon_2',\Upsilon_3',\dots,\Upsilon_n').$$

Definition 17. The (q_1,q_2)-rung Diophantine fuzzy ordered weighted geometric aggregation ((q_1,q_2)-$RDFOWGA$) operator on "n" numbers of (q_1,q_2)-$RDFNs$ is defined with the help of this transformation $\Omega: (q_1,q_2)\text{-}RDFN(E)\to(q_1,q_2)\text{-}RDFN(E)$ associated with $\omega=$

$(\omega_1, \omega_2, \omega_3, \dots, \omega_n)^T$ with $\sum_{j=1}^{n} \omega_j = 1$, and it can be computed as follows:
When $\{\Upsilon_1 = (\langle \zeta_1, \eta_1 \rangle, \langle q_{1_1}, q_{2_1} \rangle), \Upsilon_2 = (\langle \zeta_2, \eta_2 \rangle, \langle q_{1_2}, q_{2_2} \rangle), \dots \dots \dots, \Upsilon_n = (\langle \zeta_n, \eta_n \rangle, \langle q_{1_n}, q_{2_n} \rangle)\}$ are (q_1, q_2)-$RDFN$,

$$(q_1, q_2)\text{-}RDFOWGA_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \dots, \Upsilon_n)$$

$$= \prod_{j=1}^{n} \Upsilon_{\sigma(j)}{}^{\omega_j},$$

where $(\sigma(1), \sigma(2), \sigma(3), \dots \dots \dots, \sigma(n))$ is the arrangement of $j \in N$, for which $\Upsilon_{\sigma(j-1)} \geq \Upsilon_{\sigma(j)}$, for all $j \in N$.

Theorem 15. The (q_1, q_2)-rung Diophantine fuzzy ordered weighted geometric aggregation ((q_1, q_2)-$RDFOWGA$) operator on "n" numbers of (q_1, q_2)-$RDFNs$ is defined with the help of this transformation $\Omega: (q_1, q_2)\text{-}RDFN(E) \to (q_1, q_2)\text{-}RDFN(E)$ associated with $\omega = (\omega_1, \omega_2, \omega_3, \dots, \omega_n)^T$ with $\sum_{j=1}^{n} \omega_j = 1$ and it can be computed as follows:
When $\{\Upsilon_1 = (\langle \zeta_1, \eta_1 \rangle, \langle q_{1_1}, q_{2_1} \rangle), \Upsilon_2 = (\langle \zeta_2, \eta_2 \rangle, \langle q_{1_2}, q_{2_2} \rangle), \dots \dots \dots, \Upsilon_n = (\langle \zeta_n, \eta_n \rangle, \langle q_{1_n}, q_{2_n} \rangle)\}$ are (q_1, q_2)-$RDFN$,

$$(q_1, q_2)\text{-}RDFOWGA_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \dots, \Upsilon_n)$$

$$= \prod_{j=1}^{n} \Upsilon_{\sigma(j)}{}^{\omega_j}$$

$$= \left(\langle \prod_{j=1}^{n} \zeta_{\sigma(j)}{}^{\omega_j}, \sqrt[\mathrm{q}]{1 - \prod_{j=1}^{n} \left(1 - \eta_{\sigma(j)}{}^{\mathrm{q}}\right)^{\omega_j}} \rangle, \langle \prod_{j=1}^{n} q_{1_{\sigma(j)}}{}^{\omega_j}, \sqrt[\mathrm{q}]{1 - \prod_{j=1}^{n} \left(1 - \left(q_{2_{\sigma(j)}}\right)^{\mathrm{q}}\right)^{\omega_j}} \rangle \right),$$

where $(\sigma(1), \sigma(2), \sigma(3), \dots \dots \dots, \sigma(n))$ is the arrangement of $j \in N$, for which $\Upsilon_{\sigma(j-1)} \geq \Upsilon_{\sigma(j)}$, for all $j \in N$.

It is simple to demonstrate that the weighted averaging operator possesses the following characteristics:

Theorem 16. (Idempotency) If all Υ_j are equal that is $\Upsilon_j = \Upsilon$ for all j, then,

$$(q_1, q_2)\text{-}RDFOWGA_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \dots, \Upsilon_n) = \Upsilon.$$

Theorem 17. (Boundedness) Let Υ_j $(j = 1,2,3, \dots \dots \dots, n)$ be a collection of (q_1, q_2)-$RDFNs$, and let

$$\Upsilon^- = \min_j \Upsilon_j, \; \Upsilon^+ = \max_j \Upsilon_j.$$

Then,

$$\Upsilon^- \leq (q_1, q_2)\text{-}RDFOWGA_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \dots, \Upsilon_n) \leq \Upsilon^+.$$

Theorem 18. (Monotonicity) Let Υ_j $(j = 1,2,3, \dots, n)$ and Υ_j' $(j = 1,2,3, \dots, n)$ be two collections of (q_1, q_2)-$RDFNs$. If $\Upsilon_j \leq \Upsilon_j'$ for all j, then,

$$(q_1, q_2)\text{-}RDFOWGA_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \dots, \Upsilon_n) \leq (q_1, q_2)\text{-}RDFOWAA_\omega(\Upsilon_1', \Upsilon_2', \Upsilon_3', \dots, \Upsilon_n').$$

Theorem 19. (Commutativity) Let Υ_j $(j = 1,2,3, \dots \dots \dots, n)$ and Υ_j' $(j = 1,2,3, \dots \dots \dots, n)$ be two collections of (q_1, q_2)-$RDFNs$. If $\Upsilon_j \leq \Upsilon_j'$ for all j, then,

$$(q_1, q_2)\text{-}RDFOWGA_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \dots, \Upsilon_n) = (q_1, q_2)\text{-}RDFOWAA_\omega(\Upsilon_1', \Upsilon_2', \Upsilon_3', \dots, \Upsilon_n'),$$

where Υ_j' $(j = 1,2,3, \dots, n)$ is any permutation of Υ_j $(j = 1,2,3, \dots, n)$.

Definition 18. The (q_1, q_2)-rung Diophantine fuzzy hybrid weighted geometric averaging aggregation ((q_1, q_2)-$RDFHWGA$) operator on "n" numbers of (q_1, q_2)-$RDFNs$ is defined with the help of this transformation Ω: (q_1, q_2)-$RDFN(E) \to (q_1, q_2)$-$RDFN(E)$ associated with $\omega = (\omega_1, \omega_2, \omega_3, \dots, \omega_n)^T$ with $\sum_{j=1}^n \omega_j = 1$, and it can be computed as follows:

When $\{\Upsilon_1 = (\langle \zeta_1, \eta_1 \rangle, \langle q_{1_1}, q_{2_1} \rangle), \Upsilon_2 = (\langle \zeta_2, \eta_2 \rangle, \langle q_{1_2}, q_{2_2} \rangle), \dots, \Upsilon_n = (\langle \zeta_n, \eta_n \rangle, \langle q_{1_n}, q_{2_n} \rangle)\}$ are (q_1, q_2)-$RDFNs$,

$$(q_1, q_2)\text{-}RDFHWGA_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \dots, \Upsilon_n) = \prod_{j=1}^n {\Upsilon_{\sigma(j)}^{\star}}^{\omega_j},$$

where $\Upsilon_{\sigma(j)}^{\star}$ is biggest jth weighted (q_1, q_2)-rung Diophantine fuzzy values $\Upsilon_j^{\star}$ $\left(\Upsilon_j^{\star} = (\Upsilon_j)^{n\omega_j}, j \in N\right)$ and $\omega = (\omega_1, \omega_2, \omega_3, \dots, \omega_n)^T$ be the weights of $\Upsilon_j^{\star}$ by means of $\omega > 0$ with $\sum_{j=1}^n \omega_j = 1$.

Theorem 20. The (q_1, q_2)-$RDFHWGA$ operator on "n" numbers of (q_1, q_2)-$RDFNs$ is defined with the help of this transformation Ω: (q_1, q_2)-$RDFN(E) \to (q_1, q_2)$-$RDFN(E)$ associated with $\omega = (\omega_1, \omega_2, \omega_3, \dots, \omega_n)^T$ with $\sum_{j=1}^n \omega_j = 1$, and it can be computed as follows:

When $\{\Upsilon_1 = (\langle \zeta_1, \eta_1 \rangle, \langle q_{1_1}, q_{2_1} \rangle), \Upsilon_2 = (\langle \zeta_2, \eta_2 \rangle, \langle q_{1_2}, q_{2_2} \rangle), \dots, \Upsilon_n = (\langle \zeta_n, \eta_n \rangle, \langle q_{1_n}, q_{2_n} \rangle)\}$ are (q_1, q_2)-$RDFNs$,

$$(q_1, q_2)\text{-}RDFHWGA_\omega(\Upsilon_1, \Upsilon_2, \Upsilon_3, \dots, \Upsilon_n)$$

$$= \prod_{j=1}^n {\Upsilon_{\sigma(j)}^{\star}}^{\omega_j}$$

$$= \left(\langle \prod_{j=1}^n {\zeta_{\sigma(j)}^{\star}}^{\omega_j}, \sqrt[\mathrm{q}]{1 - \prod_{j=1}^n \left(1 - \left(\eta_{\sigma(j)}^{\star}\right)^{\mathrm{q}}\right)^{\omega_j}} \rangle, \langle \prod_{j=1}^n {q_1{}_{\sigma(j)}^{\star}}^{\omega_j}, \sqrt[\mathrm{q}]{1 - \prod_{j=1}^n \left(1 - \left(q_2{}_{\sigma(j)}^{\star}\right)^{\mathrm{q}}\right)^{\omega_j}} \rangle \right),$$

where $\Upsilon_{\sigma(j)}^{\star}$ is biggest jth weighted (q_1, q_2)-rung Diophantine fuzzy values $\Upsilon_j^{\star}$ $\left(\Upsilon_j^{\star} = (\Upsilon_j)^{n\omega_j}, j \in N\right)$ and $\omega = (\omega_1, \omega_2, \omega_3, \dots, \omega_n)^T$ be the weights of $\Upsilon_j^{\star}$ by means of $\omega > 0$ with $\sum_{j=1}^n \omega_j = 1$.

It is interesting to note that if $\omega = \left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)$, then (q_1, q_2)-$RDFWGA$ and (q_1, q_2)-$RDFOWGA$ operators are considered to be exceptional cases of (q_1, q_2)-$RDFHWGA$ operator. So, it concludes that (q_1, q_2)-$RDFHWGA$ operators are the extension of (q_1, q_2)-$RDFWGA$ and (q_1, q_2)-$RDFOWGA$ operators.

6. $MADM$ approach using suggested techniques

The $MADM$ technique is highly effective and well-suited for selecting the optimal choice from a limited set of possibilities due to its structure. To enhance the effectiveness and quality of previously proposed methods, we introduce a section on the $MADM$ technique procedure incorporating four appropriate operators: The (q_1, q_2)-$RDFWAA$ operator, (q_1, q_2)-$RDFOWAA$ operator, and (q_1, q_2)-$RDFHWAA$ operator. To assess some real-world issues, our objective is to calculate the decision-making process.

As a collection of finite values of alternatives, we take into consideration $\hat{A} = \{\hat{A}_1, \hat{A}_2, \dots, \hat{A}_m\}$. In addition, we choose a finite set of attributes, including, $\hat{U} = \{\hat{U}_1, \hat{U}_2, \dots, \hat{U}_n\}$, which are chosen with a weight vector $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ such that $\omega_j > 0$ with $\sum_{j=1}^{n} \omega_j = 1$, for every alternative. Additionally, to calculate the matrix that assesses the optimal choice after taking the decision-making process into account, we hope to assign the (q_1, q_2)-$RDFN$ values to each alternative, observed that ζ_j and η_j denote the positive and negative grades, where α_j and β_j are reference parameters corresponding to alternative $(\hat{A}_j)$ that satisfy the attribute $(\hat{U}_j)$ provided by the decision makers, where $0 \le \left(q_{1j}\right)^{q} \zeta_j(x) + \left(q_{2j}\right)^{q} \eta_j(x) \le 1$ and $0 \le \left(q_{1j}\right)^{q} + \left(q_{2j}\right)^{q} \le 1$. Additionally, we stated the refusal degree $q_{3j}\pi_j(x) = \sqrt[q]{1 - \left(q_{1j}\right)^{q} \zeta_j(x) - \left(q_{1j}\right)^{q} \eta_j(x)}$. As a result, in order to accomplish the aforementioned approach, we take into account a few real-world applications and attempt to assess them using theoretical frameworks.

6.1. *The suggested algorithm*

The primary impact of this subsection is to assess a process for illustrating the problem that is addressed in the following section. The primary steps of the decision-making approach are outlined below:

Step 1. Determine a team matrix by incorporating their values into the (q_1, q_2)-$RDFN$ form.

Step 2. During assigning the values, we have two opinions "profit and cost", such as if we have cost-type data, then our first priority is to normalize it otherwise not.

Step 3. Using the six various types of operators "(q_1, q_2)-$RDFWAA$ operator, (q_1, q_2)-$RDFOWAA$ operator, (q_1, q_2)-$RDFHWAA$ operator, (q_1, q_2)-$RDFWGA$ operator, (q_1, q_2)-$RDFOWGA$ operator, and (q_1, q_2)-$RDFHWGA$ operator" aggregate the collection of data into a singleton set such that

$$(q_1, q_2)\text{-}RDFWAA_{\omega}(\Upsilon_1, \Upsilon_2, \Upsilon_3, \dots, \Upsilon_n)$$

$$= \prod_{j=1}^{n} \omega_j \Upsilon_j$$

$$= \left(\langle \sqrt[q]{1 - \prod_{j=1}^{n} \left(1 - (\zeta_j)^{q}\right)^{\omega_j}}, \prod_{j=1}^{n} \eta_j^{\omega_j} \rangle, \langle \sqrt[q]{1 - \prod_{j=1}^{n} \left(1 - \left(q_{1j}\right)^{q}\right)^{\omega_j}}, \prod_{j=1}^{n} q_{2j}^{\omega_j} \rangle \right).$$

$$(q_1, q_2)\text{-}RDFOWAA_{\omega}(\Upsilon_1, \Upsilon_2, \Upsilon_3, \dots, \Upsilon_n)$$

$$= \prod_{j=1}^{n} \omega_j \Upsilon_{\sigma(j)}$$

$$= \left(\langle \sqrt[q]{1 - \prod_{j=1}^{n} \left(1 - \left(\zeta_{\sigma(j)}\right)^{q}\right)^{\omega_j}}, \prod_{j=1}^{n} \eta_{\sigma(j)}{}^{\omega_j} \rangle, \langle \sqrt[q]{1 - \prod_{j=1}^{n} \left(1 - \left({q_1}_{\sigma(j)}\right)^{q}\right)^{\omega_j}}, \prod_{j=1}^{n} {q_2}_{\sigma(j)}{}^{\omega_j} \rangle \right).$$

$$(q_1, q_2)\text{-}RDFHWAA_{\omega}(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots, \Upsilon_n)$$

$$= \prod_{j=1}^{n} \omega_j \Upsilon^{\star}_{\sigma(j)}$$

$$= \left(\langle \sqrt[q]{1 - \prod_{j=1}^{n} \left(1 - \left(\zeta^{\star}{}_{\sigma(j)}\right)^{q}\right)^{\omega_j}}, \prod_{j=1}^{n} \eta^{\star}_{\sigma(j)}{}^{\omega_j} \rangle, \langle \sqrt[q]{1 - \prod_{j=1}^{n} \left(1 - \left({q_1}^{\star}{}_{\sigma(j)}\right)^{q}\right)^{\omega_j}}, \prod_{j=1}^{n} {q_2}^{\star}{}_{\sigma(j)}{}^{\omega_j} \rangle \right).$$

$$(q_1, q_2)\text{-}RDFWGA_{\omega}(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots, \Upsilon_n)$$

$$= \prod_{j=1}^{n} \Upsilon_j^{\omega_j}$$

$$= \left(\langle \prod_{j=1}^{n} \zeta_j^{\omega_j}, \sqrt[q]{1 - \prod_{j=1}^{n} \left(1 - \left(\eta_j\right)^{q}\right)^{\omega_j}} \rangle, \langle \prod_{j=1}^{n} {q_1}_j^{\omega_j}, \sqrt[q]{1 - \prod_{j=1}^{n} \left(1 - \left({q_2}_j\right)^{q}\right)^{\omega_j}} \rangle \right)$$

$$(q_1, q_2)\text{-}RDFOWGA_{\omega}(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots, \Upsilon_n)$$

$$= \prod_{j=1}^{n} \Upsilon_{\sigma(j)}{}^{\omega_j}$$

$$= \left(\langle \prod_{j=1}^{n} \zeta_{\sigma(j)}{}^{\omega_j}, \sqrt[q]{1 - \prod_{j=1}^{n} \left(1 - \eta_{\sigma(j)}{}^{q}\right)^{\omega_j}} \rangle, \langle \prod_{j=1}^{n} {q_1}_{\sigma(j)}{}^{\omega_j}, \sqrt[q]{1 - \prod_{j=1}^{n} \left(1 - \left({q_2}_{\sigma(j)}\right)^{q}\right)^{\omega_j}} \rangle \right).$$

$$(q_1, q_2)\text{-}RDFHWGA_{\omega}(\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots, \Upsilon_n)$$

$$= \prod_{j=1}^{n} \Upsilon^{\star}_{\sigma(j)}{}^{\omega_j}$$

$$= \left(\langle \prod_{j=1}^{n} \zeta^{\star}_{\sigma(j)}{}^{\omega_j}, \sqrt[q]{1 - \prod_{j=1}^{n} \left(1 - \left(\eta^{\star}_{\sigma(j)}\right)^{q}\right)^{\omega_j}} \rangle, \langle \prod_{j=1}^{n} {q_1}^{\star}_{\sigma(j)}{}^{\omega_j}, \sqrt[q]{1 - \prod_{j=1}^{n} \left(1 - \left({q_2}^{\star}_{\sigma(j)}\right)^{q}\right)^{\omega_j}} \rangle \right).$$

Step 4. Determine the aggregated theories with respect to different score values, such as

$$\mathcal{S}_{(q_1,q_2)\text{-}RDFN}(\Upsilon) = \frac{1}{2}[\zeta_\Upsilon - \eta_\Upsilon + (q_1)^{\mathrm{q}} - (q_2)^{\mathrm{q}}],$$

$$\mathcal{Q}_{(q_1,q_2)\text{-}RDFN}(\Upsilon) = \frac{1}{2}[\zeta_\Upsilon{}^2 - \eta_\Upsilon{}^2 + ((q_1)^{\mathrm{q}})^2 - ((q_2)^{\mathrm{q}})^2],$$

$$\check{\mathrm{E}}_{(q_1,q_2)\text{-}RDFN}(\Upsilon) = \frac{1}{2}\left[\frac{\zeta_\Upsilon - \eta_\Upsilon + 1}{2} + \frac{(q_1)^{\mathrm{q}} - (q_2)^{\mathrm{q}} + 1}{2}\right],$$

where, $-1 \le \mathcal{S}_{(q_1,q_2)\text{-}RDFN}(\Upsilon), \mathcal{Q}_{(q_1,q_2)\text{-}RDFN}(\Upsilon) \le 1,\ 0 \le \check{\mathrm{E}}_{(q_1,q_2)\text{-}RDFN}(\Upsilon) \le 1.$

In the event that the score function is not successful, then the accuracy function will be used like

$$H_{(q_1,q_2)\text{-}RDFN}(\Upsilon) = \frac{1}{2}\left[\frac{\zeta_\Upsilon + \eta_\Upsilon}{2} + (q_1)^{\mathrm{q}} + (q_2)^{\mathrm{q}}\right],$$

$$И_{(q_1,q_2)\text{-}RDFN}(\Upsilon) = \frac{1}{2}\left[\frac{\zeta_\Upsilon{}^2 + \eta_\Upsilon{}^2}{2} + ((q_1)^{\mathrm{q}})^2 + ((q_2)^{\mathrm{q}})^2\right],$$

where, $0 \le H_{(q_1,q_2)\text{-}FN}(\Upsilon), И_{(q_1,\ \ _2)\text{-}FN}(\Upsilon) \le 1.$

Step 5. Try to identify the standout among the alternatives by analyzing the ranking values based on the score values.

To improve the value of the assessed techniques and enable the practical application of the aforementioned procedure, we consider a number of numerical examples that demonstrate the superiority and validity of the invented operators. The suggested algorithm's geometrical interpretation is presented in the form of Figure 2.

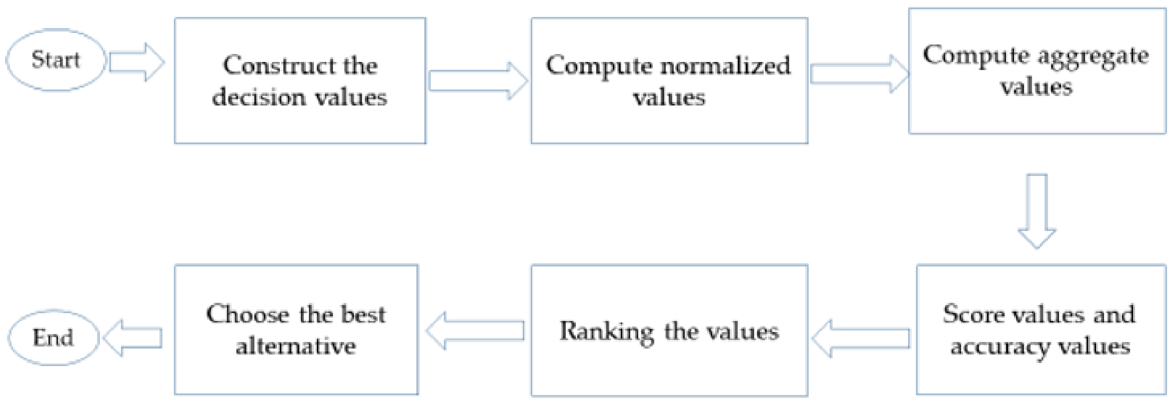

Figure 2. Geometrical interpretation of the proposed algorithm.

6.2. Types of agricultural field robots

Agricultural field robots, often referred to as **agribots**, are specialized autonomous machines designed to perform a variety of tasks in agricultural fields. They are increasingly being used in modern farming to improve efficiency, reduce labor costs, and address the growing challenges of food production, such as labor shortages, sustainability, and climate change. These robots can handle tasks such as planting, watering, weeding, monitoring crop health, and harvesting. Here are some key types

and applications of agricultural field robots:
$\hat{A}_1$: **Harvesting robots** are advanced machines designed to automate the picking and harvesting of crops in agriculture. These robots use technologies like artificial intelligence (AI), computer vision, and robotic arms to identify, select, and harvest ripe produce with precision. By reducing reliance on manual labor, they enhance efficiency, minimize crop damage, and enable around-the-clock operations, making them an essential innovation in modern farming. Harvesting robots are transforming modern agriculture by boosting productivity, minimizing waste, and addressing labor shortages.
$\hat{A}_2$: **Spraying and irrigation robots** are autonomous machines used in agriculture to optimize the application of water, fertilizers, and pesticides. These robots are equipped with sensors, GPS, and AI technologies to accurately deliver water and chemicals based on the specific needs of crops. They can move through fields autonomously, reducing waste, preventing overuse, and ensuring even distribution. By improving efficiency and precision, spraying and irrigation robots help conserve resources, protect the environment, and support sustainable farming practices. Spraying and irrigation robots are a crucial part of precision agriculture, ensuring optimal resource use while supporting crop health and sustainability.
$\hat{A}_3$: **Crop monitoring robots** are autonomous machines designed to track the health, growth, and conditions of crops in real time. These robots are equipped with sensors, cameras, and AI technologies to gather data on soil moisture, plant health, pest infestations, and nutrient levels. By providing precise, up-to-date information, crop monitoring robots help farmers make informed decisions, optimize resource use, and improve crop yields. They are an essential tool in precision agriculture, enabling early detection of issues and enhancing overall farm productivity. By offering real-time, accurate data, **crop monitoring robots** are transforming modern agriculture, helping farmers optimize their practices and improve sustainability while maximizing yields.
$\hat{A}_4$: **Soil analysis robots** are autonomous machines designed to assess soil conditions in agricultural fields. Equipped with sensors and sampling tools, these robots measure factors like soil moisture, pH levels, nutrient content, and temperature. By providing precise, real-time data, soil analysis robots help farmers understand the health of their soil and make informed decisions about irrigation, fertilization, and crop management. They play a crucial role in precision agriculture, enhancing resource efficiency, improving crop yields, and supporting sustainable farming practices. Soil analysis robots are revolutionizing how farmers manage their soil, providing detailed insights that lead to improved productivity, resource conservation, and sustainable agricultural practices.
$\hat{A}_5$: **Swarming robots** are groups of autonomous machines that work together collaboratively, often mimicking the behavior of natural swarms like ants or bees. In agriculture, these robots communicate and coordinate to perform tasks such as planting, weeding, harvesting, and monitoring crops. Swarming robots rely on decentralized control, AI, and sensor networks to efficiently cover large areas, making them ideal for precision farming. Their ability to work in groups enhances productivity, reduces labor costs, and increases the overall efficiency of farming operations.
Selecting the right agricultural robot involves considering a variety of factors to ensure it meets the specific needs of your farm and integrates well into existing operations. Thus, $\alpha = reliable \in J_1,$ and $\beta = not\ reliable \in J_2$. Here are some key factors to take into account:
$\hat{U}_1$: **Compatibility** for an agriculture robot refers to its ability to seamlessly integrate with existing farming systems, technologies, and operations. It ensures that the robot can work alongside current equipment, software, and infrastructure without issues. Key aspects include: The robot must connect with other farm machinery, like tractors or irrigation systems. It should work with existing farm management software, GPS systems, and data platforms for smooth operation and data sharing. The robot must be

suited to the farm's specific conditions, such as crop type, soil, and terrain. It should easily adapt to the farm's size and operations, whether for small-scale or large-scale farming. Ensuring compatibility enhances efficiency, reduces costs, and maximizes the robot's effectiveness in agricultural tasks.

$\hat{U}_2$: **Technology and Features** refer to the advanced tools and capabilities integrated into a robot to enhance its performance and efficiency. In the context of robotics, these include: Devices for detecting environmental conditions, monitoring crop health, and navigating autonomously. Enables the robot to analyze data, adapt to its environment, and improve decision-making over time. GPS, LiDAR, and other technologies that allow robots to move and operate independently in fields. Robotic arms, sprayers, or harvesting tools that enable accurate and efficient task execution. IoT, Wi-Fi, or Bluetooth capabilities that enable communication with other systems and data sharing. These technologies and features ensure that robots are capable, efficient, and suited for complex tasks, particularly in sectors like agriculture.

$\hat{U}_3$: **Maintenance and support** refer to the ongoing care and services required to keep a robot functioning optimally throughout its lifespan. For agricultural robots, this includes: Routine checks to ensure the robot's hardware, such as sensors and moving parts, are in good working condition. Keeping the robot's software and AI algorithms up to date for improved functionality and bug fixes. Timely repairs to address any mechanical or electrical issues that may arise. Access to experts or service teams to help troubleshoot problems or provide guidance on usage. Providing operators with manuals, tutorials, and support for efficient use and maintenance of the robot. Proper maintenance and support ensure the robot's reliability, extend its lifespan, and prevent costly breakdowns, making it essential for sustained operation.

$\hat{U}_4$: **Regulatory compliance** refers to the adherence of agricultural robots to laws, guidelines, and standards set by governing bodies. It ensures that the robot operates safely and ethically within legal frameworks. Key aspects include: Ensuring the robot meets safety requirements to protect users, workers, and the environment. Complying with laws regarding pesticide use, emissions, and sustainable farming practices. Adhering to rules on data collection and storage, especially when robots gather information from fields or connected devices. Obtaining necessary approvals and certifications from regulatory agencies before the robot is deployed. Meeting regulatory compliance ensures the robot is legally permitted for use and operates safely and responsibly in agricultural settings.

Step 1. Determine a team matrix by incorporating their values into the (q_1, q_2)-$RDFN$ form Table 2.

Table 2. Decision matrix of (q_1, q_2)-RDF information.

	$\hat{U}_1$	$\hat{U}_2$	$\hat{U}_3$	$\hat{U}_4$	$\hat{U}_5$
$\hat{A}_1$	$\begin{pmatrix}\langle 1,0.966\rangle,\\ \langle 0.978,0.633\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 1,0.934\rangle,\\ \langle 0.767,0.845\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 1,0.916\rangle,\\ \langle 0.9,0.758\rangle)\end{pmatrix}$	$\begin{pmatrix}\langle 0.928,0.873\rangle,\\ \langle 0.636,0.791\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 1,0.974\rangle,\\ \langle 0.87,0.737\rangle)\end{pmatrix}$
$\hat{A}_2$	$\begin{pmatrix}\langle 0.876,0.831\rangle,\\ \langle 0.8,0.916\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.895,0.789\rangle,\\ \langle 0.78,0.641\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.679,0.88\rangle,\\ \langle 0.88,0.59\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.348,0.812\rangle,\\ \langle 0.647,0.858\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.98,0.89\rangle,\\ \langle 0.92,0.8\rangle\end{pmatrix}$
$\hat{A}_3$	$\begin{pmatrix}\langle 0.754,0.789\rangle,\\ \langle 0.762,0.789\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.69,0.99\rangle,\\ \langle 0.345,0.863\rangle\end{pmatrix}$	$\begin{pmatrix}(\langle 0.328,0.987\rangle,\\ \langle 0.947,0.77\rangle)\end{pmatrix}$	$\begin{pmatrix}\langle 0.756,0.865\rangle,\\ \langle 0.645,0.563\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 1,0.96\rangle,\\ \langle 0.78,0.8\rangle\end{pmatrix}$
$\hat{A}_4$	$\begin{pmatrix}\langle 1,0.959\rangle,\\ \langle 0.903,0.644\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.895,0.925\rangle,\\ \langle 0.835,0.706\rangle\end{pmatrix}$	$\begin{pmatrix}(\langle 1,0.92\rangle,\\ \langle 0.645,0.9\rangle)\end{pmatrix}$	$\begin{pmatrix}\langle 0.882,0.949\rangle,\\ \langle 0.642,0.645\rangle\end{pmatrix}$	$\begin{pmatrix}\langle 0.862,0.92\rangle,\\ \langle 0.724,0.9\rangle\end{pmatrix}$

Step 2. During assign the values, we have two opinions "same type of data and different type of data", such as if we have different type of data, then our first priority is to normalize such that

$$\mathcal{L}_j = \begin{cases} \left(\langle \zeta_j, \eta_j \rangle, \langle q_{1j}, q_{2j} \rangle\right), & \text{same type input data;} \\ \left(\langle \eta_j, \zeta_j \rangle, \langle q_{2j}, q_{1j} \rangle\right), & \text{different type input data.} \end{cases}$$

In this case, since the input data for all attributes is identical, there is no need to normalize the data. All alternatives and criteria in our specific problem are of the same nature.

Step 3. Using the six various types of operators "(q_1, q_2)-$RDFWAA$ operator, (q_1, q_2)-$RDFOWAA$ operator, (q_1, q_2)-$RDFHWAA$ operator, (q_1, q_2)-$RDFWGA$ operator, (q_1, q_2)-$RDFOWGA$ operator, and (q_1, q_2)-$RDFHWGA$ operator" aggregate the collection of data into a singleton set, see Tables 3–16.

Aggregation operator with respect to $\mathcal{S}_{(q_1,q_2)\text{-FN}}$.

Table 3. (q_1, q_2)-$RDFWAA$.

$\hat{A}_1$	$(\langle 1,0.9294\rangle, \langle 0.9067,0.7551\rangle)$
$\hat{A}_2$	$(\langle 0.8821,0.8388\rangle, \langle 0.8434,0.7163\rangle)$
$\hat{A}_3$	$(\langle 1,0.9398\rangle, \langle 0.8652,0.7626\rangle)$
$\hat{A}_4$	$(\langle 1,0.9329\rangle, \langle 0.8128,0.7561\rangle)$

Table 4. (q_1, q_2)-$RDFOWAA$.

$\hat{A}_1$	$(\langle 1,0.9377\rangle, \langle 0.9103,0.7398\rangle)$
$\hat{A}_2$	$(\langle 0.9045,0.8318\rangle, \langle 0.8572,0.7861\rangle)$
$\hat{A}_3$	$(\langle 1,0.8433\rangle, \langle 0.8239,0.7738\rangle)$
$\hat{A}_4$	$(\langle 1,0.9372\rangle, \langle 0.8129,0.7192\rangle)$

Table 5. (q_1, q_2)-$RDFHWAA$.

$\hat{A}_1$	$(\langle 1,0.9439\rangle, \langle 0.8866,0.7896\rangle)$
$\hat{A}_2$	$(\langle 0.8291,0.8706\rangle, \langle 0.8108,0.7559\rangle)$
$\hat{A}_3$	$(\langle 1,0.9375\rangle, \langle 0.8554,0.8171\rangle)$
$\hat{A}_4$	$(\langle 1,0.9377\rangle, \langle 0.7779,0.7921\rangle)$

Table 6. (q_1, q_2)-$RDFWGA$.

$\hat{A}_1$	$(\langle 0.9889,0.9498\rangle, \langle 0.8232,0.7704\rangle)$
$\hat{A}_2$	$(\langle 0.7192,0.8482\rangle, \langle 0.8027,0.7755\rangle)$
$\hat{A}_3$	$(\langle 0.5929,0.9721\rangle, \langle 0.6524,0.7869\rangle)$
$\hat{A}_4$	$(\langle 0.9390,0.9359\rangle, \langle 0.7423,0.8032\rangle)$

Table 7. (q_1, q_2)-$RDFOWGA$.

$\hat{A}_1$	$(\langle 0.9889,0.9507\rangle, \langle 0.8232,0.7626\rangle)$
$\hat{A}_2$	$(\langle 0.7530,0.9974\rangle, \langle 0.8172,0.9901\rangle)$
$\hat{A}_3$	$(\langle 0.6959,0.9498\rangle, \langle 0.6893,0.7774\rangle)$
$\hat{A}_4$	$(\langle 0.9215,0.9400\rangle, \langle 0.7418,0.8022\rangle)$

Table 8. (q_1, q_2)-$RDFHWGA$.

$\hat{A}_1$	$(\langle 0.9866, 0.8858\rangle, \langle 0.8623, 0.7279\rangle)$
$\hat{A}_2$	$(\langle 0.8141, 0.9493\rangle, \langle 0.8386, 0.7614\rangle)$
$\hat{A}_3$	$(\langle 0.7079, 0.9334\rangle, \langle 0.7571, 0.7462\rangle)$
$\hat{A}_4$	$(\langle 0.9459, 0.8829\rangle, \langle 0.8066, 0.7712\rangle)$

Aggregation operator with respect to $\mathcal{Q}_{(q_1,q_2)\text{-FN}}$.

Table 9. (q_1, q_2)-$RDFOWAA$.

$\hat{A}_1$	$(\langle 1, 0.9403\rangle, \langle 0.9107, 0.7417\rangle)$
$\hat{A}_2$	$(\langle 0.9114, 0.8361\rangle, \langle 0.8464, 0.7589\rangle)$
$\hat{A}_3$	$(\langle 1, 0.8385\rangle, \langle 0.8239, 0.7194\rangle)$
$\hat{A}_4$	$(\langle 1, 0.9331\rangle, \langle 0.8168, 0.7469\rangle)$

Table 10. (q_1, q_2)-$RDFHWAA$.

$\hat{A}_1$	$(\langle 1, 0.9439\rangle, \langle 0.8868, 0.7896\rangle)$
$\hat{A}_2$	$(\langle 0.8192, 0.8749\rangle, \langle 0.8270, 0.7465\rangle)$
$\hat{A}_3$	$(\langle 1, 0.9301\rangle, \langle 0.8776, 0.8033\rangle)$
$\hat{A}_4$	$(\langle 1, 0.9388\rangle, \langle 0.7832, 0.7834\rangle)$

Table 11. (q_1, q_2)-$RDFOWGA$.

$\hat{A}_1$	$(\langle 0.9918, 0.9517\rangle, \langle 0.8505, 0.7667\rangle)$
$\hat{A}_2$	$(\langle 0.7823, 0.8462\rangle, \langle 0.8099, 0.8446\rangle)$
$\hat{A}_3$	$(\langle 0.6545, 0.9498\rangle, \langle 0.6894, 0.7775\rangle)$
$\hat{A}_4$	$(\langle 0.9338, 0.9360\rangle, \langle 0.7519, 0.8256\rangle)$

Table 12. (q_1, q_2)-$RDFHWGA$.

$\hat{A}_1$	$(\langle 0.9866, 0.8858\rangle, \langle 0.8623, 0.7279\rangle)$
$\hat{A}_2$	$(\langle 0.8141, 0.9493\rangle, \langle 0.8386, 0.7614\rangle)$
$\hat{A}_3$	$(\langle 0.7079, 0.9334\rangle, \langle 0.7571, 0.7462\rangle)$
$\hat{A}_4$	$(\langle 0.9459, 0.8829\rangle, \langle 0.8066, 0.7712\rangle)$

Aggregation operator with respect to $\breve{E}_{(q_1,q_2)\text{-FN}}$.

Table 13. (q_1, q_2)-$RDFOWAA$.

$\hat{A}_1$	$(\langle 1, 0.9377\rangle, \langle 0.9103, 0.7398\rangle)$
$\hat{A}_2$	$(\langle 0.9045, 0.8318\rangle, \langle 0.8572, 0.7861\rangle)$
$\hat{A}_3$	$(\langle 1, 0.8433\rangle, \langle 0.8239, 0.7738\rangle)$
$\hat{A}_4$	$(\langle 1, 0.9372\rangle, \langle 0.8129, 0.7192\rangle)$

Table 14. (q_1, q_2)-$RDFHWAA$.

$\hat{A}_1$	$(\langle 1,0.9439\rangle, \langle 0.8866,0.7896\rangle)$
$\hat{A}_2$	$(\langle 0.8291,0.8706\rangle, \langle 0.8108,0.7559\rangle)$
$\hat{A}_3$	$(\langle 1,0.9375\rangle, \langle 0.8554,0.8171\rangle)$
$\hat{A}_4$	$(\langle 1,0.9377\rangle, \langle 0.7779,0.7921\rangle)$

Table 15. (q_1, q_2)-$RDFOWGA$.

$\hat{A}_1$	$(\langle 0.9889,0.9507\rangle, \langle 0.8232,0.7626\rangle)$
$\hat{A}_2$	$(\langle 0.7530,0.9974\rangle, \langle 0.8172,0.9901\rangle)$
$\hat{A}_3$	$(\langle 0.6959,0.9498\rangle, \langle 0.6893,0.7774\rangle)$
$\hat{A}_4$	$(\langle 0.9215,0.9400\rangle, \langle 0.7418,0.8022\rangle)$

Table 16. (q_1, q_2)-$RDFHWGA$.

$\hat{A}_1$	$(\langle 0.9866,0.8858\rangle, \langle 0.8623,0.7279\rangle)$
$\hat{A}_2$	$(\langle 0.8141,0.9493\rangle, \langle 0.8386,0.7614\rangle)$
$\hat{A}_3$	$(\langle 0.7079,0.9334\rangle, \langle 0.7571,0.7462\rangle)$
$\hat{A}_4$	$(\langle 0.9459,0.8829\rangle, \langle 0.8066,0.7712\rangle)$

Step 4. Refer to Tables 17 and 18 to find the aggregated theory's score values using (4), (5), and (6) such that:

Table 17. (q_1, q_2)-$RDFWAA$ Score values.

	$\mathcal{S}_{(q_1,q_2)\text{-}FN}$			
	$\mathcal{S}_{(q_1,q_2)\text{-}RDFN}(\hat{A}_1)$	$\mathcal{S}_{(q_1,q_2)\text{-}RDFN}(\hat{A}_2)$	$\mathcal{S}_{(q_1,q_2)\text{-}RDFN}(\hat{A}_3)$	$\mathcal{S}_{(q_1,q_2)\text{-}RDFN}(\hat{A}_4)$
(q_1, q_2)-$RDFWAA$	0.2025	0.1051	0.1272	0.0699
(q_1, q_2)-$RDFOWAA$	0.2126	0.1048	0.1160	0.0823
(q_1, q_2)-$RDFHWAA$	0.1381	0.0165	0.0727	0.0214
	$\mathcal{Q}_{(q_1,q_2)\text{-}FN}(\Upsilon)$			
	$\mathcal{Q}_{(q_1,q_2)\text{-}RDFN}(\hat{A}_1)$	$\mathcal{Q}_{(q_1,q_2)\text{-}RDFN}(\hat{A}_2)$	$\mathcal{Q}_{(q_1,q_2)\text{-}RDFN}(\hat{A}_3)$	$\mathcal{Q}_{(q_1,q_2)\text{-}RDFN}(\hat{A}_4)$
(q_1, q_2)-$RDFWAA$	0.1508	0.0600	0.1007	0.0734
(q_1, q_2)-$RDFOWAA$	0.1624	0.0871	0.1486	0.0751
(q_1, q_2)-$RDFHWAA$	0.1055	−0.0334	0.1054	0.0593
	$\check{E}_{(q_1,q_2)\text{-}FN}$			
	$\check{E}_{(q_1,q_2)\text{-}RDFN}(\hat{A}_1)$	$\check{E}_{(q_1,q_2)\text{-}RDFN}(\hat{A}_2)$	$\check{E}_{(q_1,q_2)\text{-}RDFN}(\hat{A}_3)$	$\check{E}_{(q_1,q_2)\text{-}RDFN}(\hat{A}_4)$
(q_1, q_2)-$RDFWAA$	0.6013	0.5526	0.5636	0.5349
(q_1, q_2)-$RDFOWAA$	0.6063	0.5529	0.5580	0.5412
(q_1, q_2)-$RDFHWAA$	0.5690	0.5082	0.5363	0.5107

Table 18. (q_1, q_2)-$RDFWGA$ Score values.

	$\mathcal{S}_{(q_1,q_2)\text{-}FN}$			
	$\mathcal{S}_{(q_1,q_2)\text{-}RDFN}(\hat{A}_1)$	$\mathcal{S}_{(q_1,q_2)\text{-}RDFN}(\hat{A}_2)$	$\mathcal{S}_{(q_1,q_2)\text{-}RDFN}(\hat{A}_3)$	$\mathcal{S}_{(q_1,q_2)\text{-}RDFN}(\hat{A}_4)$
(q_1, q_2)-$RDFWGA$	0.0634	−0.0461	−0.2368	−0.0337
(q_1, q_2)-$RDFOWGA$	0.0623	−0.4980	−0.1612	−0.0440
(q_1, q_2)-$RDFHWGA$	0.1539	−0.0080	−0.1077	0.0555
	$\mathcal{Q}_{(q_1,q_2)\text{-}FN}$			
	$\mathcal{Q}_{(q_1,q_2)\text{-}RDFN}(\hat{A}_1)$	$\mathcal{Q}_{(q_1,q_2)\text{-}RDFN}(\hat{A}_2)$	$\mathcal{Q}_{(q_1,q_2)\text{-}RDFN}(\hat{A}_3)$	$\mathcal{Q}_{(q_1,q_2)\text{-}RDFN}(\hat{A}_4)$
(q_1, q_2)-$RDFWGA$	0.0603	−0.0962	−0.3032	−0.0043
(q_1, q_2)-$RDFOWGA$	0.0618	−0.0649	−0.2417	−0.0150
(q_1, q_2)-$RDFHWGA$	0.1275	−0.1094	−0.1641	0.0621
	$\mathcal{E}_{(q_1,q_2)\text{-}FN}$			
	$\mathcal{E}_{(q_1,q_2)\text{-}RDFN}(\hat{A}_1)$	$\mathcal{E}_{(q_1,q_2)\text{-}RDFN}(\hat{A}_2)$	$\mathcal{E}_{(q_1,q_2)\text{-}RDFN}(\hat{A}_3)$	$\mathcal{E}_{(q_1,q_2)\text{-}RDFN}(\hat{A}_4)$
(q_1, q_2)-$RDFWGA$	0.5317	0.4779	0.3816	0.4831
(q_1, q_2)-$RDFOWGA$	0.5311	0.2519	0.4194	0.4789
(q_1, q_2)-$RDFHWGA$	0.5769	0.4969	0.4461	0.5277

Step 5. Analyze the ranking values based on the score values and look for the standout alternative among the four; refer to Tables 19 and 20.

Table 19. Ranking of (q_1, q_2)-$RDFWAA$ operators w.r.t. $\mathcal{S}_{(q_1,q_2)\text{-}RDFN}$, $\mathcal{Q}_{(q_1,q_2)\text{-}RDFN}$, and $\mathcal{E}_{(q_1,q_2)\text{-}RDFN}$.

	$\mathcal{S}_{(q_1,q_2)\text{-}RDFN}$	$\mathcal{Q}_{(q_1,q_2)\text{-}RDFN}$	$\mathcal{E}_{(q_1,q_2)\text{-}RDFN}$
(q_1, q_2)-$RDFWAA$	$\hat{A}_1 > \hat{A}_3 > \hat{A}_2 > \hat{A}_4$	$\hat{A}_1 > \hat{A}_3 > \hat{A}_4 > \hat{A}_2$	$\hat{A}_1 > \hat{A}_3 > \hat{A}_2 > \hat{A}_4$
(q_1, q_2)-$RDFOWAA$	$\hat{A}_1 > \hat{A}_3 > \hat{A}_2 > \hat{A}_4$	$\hat{A}_1 > \hat{A}_3 > \hat{A}_2 > \hat{A}_4$	$\hat{A}_1 > \hat{A}_3 > \hat{A}_2 > \hat{A}_4$
(q_1, q_2)-$RDFHWAA$	$\hat{A}_1 > \hat{A}_3 > \hat{A}_4 > \hat{A}_2$	$\hat{A}_1 > \hat{A}_3 > \hat{A}_4 > \hat{A}_2$	$\hat{A}_1 > \hat{A}_3 > \hat{A}_4 > \hat{A}_2$

Table 20. Ranking of (q_1, q_2)-$RDFWGA$ operators w.r.t. $\mathcal{S}_{(q_1,q_2)\text{-}RDFN}$, $\mathcal{Q}_{(q_1,q_2)\text{-}RDFN}$, and $\mathcal{E}_{(q_1,q_2)\text{-}RDFN}$.

	$\mathcal{S}_{(q_1,q_2)\text{-}RDFN}$	$\mathcal{Q}_{(q_1,q_2)\text{-}RDFN}$	$\mathcal{E}_{(q_1,q_2)\text{-}RDFN}$
(q_1, q_2)-$RDFWGA$	$\hat{A}_1 > \hat{A}_4 > \hat{A}_2 > \hat{A}_3$	$\hat{A}_1 > \hat{A}_4 > \hat{A}_3 > \hat{A}_2$	$\hat{A}_1 > \hat{A}_4 > \hat{A}_2 > \hat{A}_3$
(q_1, q_2)-$RDFOWGA$	$\hat{A}_1 > \hat{A}_4 > \hat{A}_3 > \hat{A}_2$	$\hat{A}_1 > \hat{A}_4 > \hat{A}_2 > \hat{A}_3$	$\hat{A}_1 > \hat{A}_4 > \hat{A}_3 > \hat{A}_2$
(q_1, q_2)-$RDFHWGA$	$\hat{A}_1 > \hat{A}_4 > \hat{A}_2 > \hat{A}_3$	$\hat{A}_1 > \hat{A}_4 > \hat{A}_2 > \hat{A}_3$	$\hat{A}_1 > \hat{A}_4 > \hat{A}_2 > \hat{A}_3$

According to the geometric representation of Tables 19 and 20 with respect to Tables 17 and 18 (see Figures 3–5, respectively), we have,

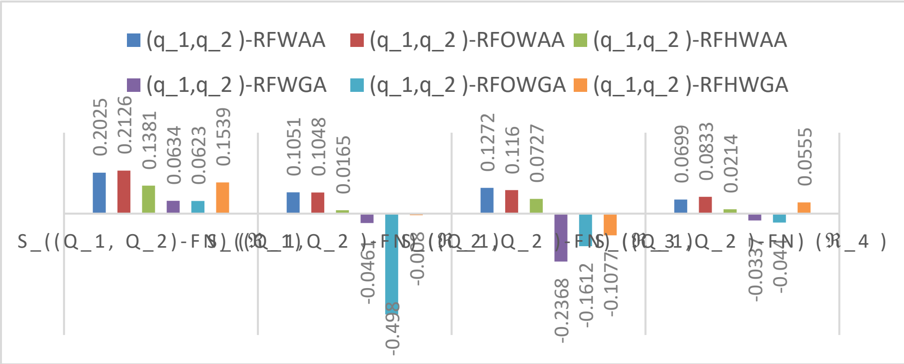

Figure 3. Scores of alternatives based on the six aggregation operators.

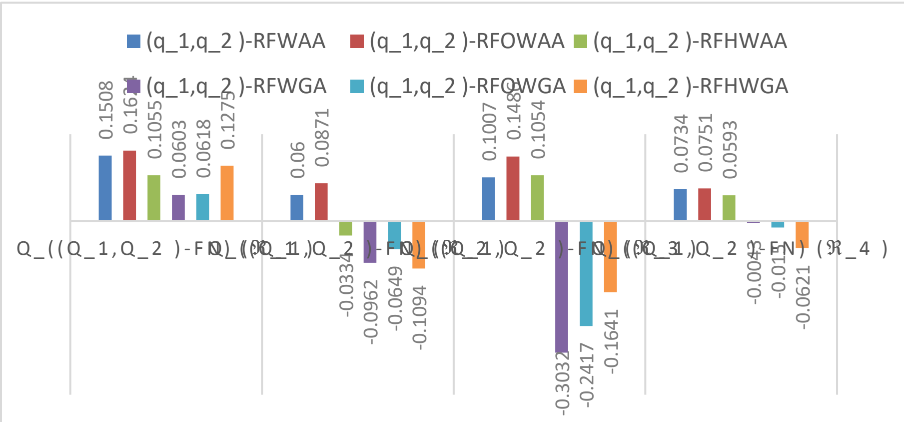

Figure 4. Quadratic scores of alternatives based on the six aggregation operators.

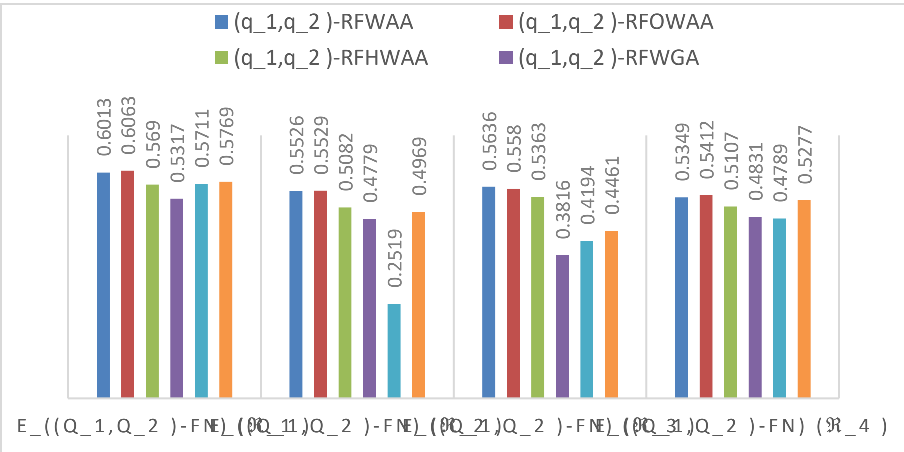

Figure 5. Expectation scores of alternatives based on the six aggregation operators.

We can find that the most desirable decision is $\hat{A}_1$. Note that, each operator receives the same rating results, and these operators are also steady.

6.3. Advantages of agricultural robots

- **Labor Efficiency**: Robots can work around the clock, reducing the need for human labor, which is becoming increasingly scarce in rural areas.
- **Precision Agriculture**: Robots can perform tasks with high precision, reducing waste of resources like seeds, water, and pesticides.
- **Sustainability**: By optimizing resource use and reducing chemical inputs, robots contribute to more environmentally friendly farming practices.
- **Yield Improvement**: Early detection of pests, diseases, or nutrient deficiencies enables timely intervention, improving crop yield and quality.

6.4 Challenges and considerations

- **High Initial Costs**: The cost of acquiring and maintaining agricultural robots can be prohibitive for small farms.
- **Field Variability**: Different crops and field conditions may require tailored robotic solutions, limiting the general applicability of a single robot model.
- **Technology Integration**: Successful integration of robots into farming requires compatible software systems and trained personnel to manage them.
- **Power Supply**: Energy efficiency and battery life are limitations for many agricultural robots, especially in large-scale operations.

In the future, AI-powered agribots are expected to play an even bigger role in precision agriculture, leveraging machine learning and data analysis to enhance food security and optimize farming in the face of climate change and population growth.

7. Discussions

The Table 21 lists the following benefits that the suggested operators have over the present operators:

- From Table 21, it can be easily seen that our proposed method is more flexible than other methods due to the reference parameter mappings.
- Given its greater versatility, expertise, and generality, the (q_1, q_2)-$RDFMADM$ model can handle a greater number of decision-making problems with varying values while meeting the requirements of $MADM$ problems.
- Information in (q_1, q_2)-$RDFWAA$ operators and (q_1, q_2)-$RDFWGA$ -operators is represented using (q_1, q_2)-$RDFN$s. Combining FNs with the pair of reference parameters mappings "(q_1, q_2)" yields the (q_1, q_2)-$RDFN$s, which provides all evaluation information. The suggested operator is more universal because the (q_1, q_2)-$RDFS$s manages both quantitative and qualitative data.
- It can be noticed from Table 21, when data are in the (q_1, q_2)-rung Diophantine fuzzy information form, the classical operators presented are unable to address the problems. The suggested approach yields more precise and accurate results.
- These developed operators are exceptional cases. However, due to the ongoing ambiguity of decision data, these operators have some restrictions. Our improved operators are therefore far more efficient.

Table 21. The distinctive analyses of various techniques.

Methods	Information correlation	Monotonicity	Flexibility	Deal with (q_1, q_2)-$RDFS$
Ibrahim and Alshammari [37]	No	No	No	No
Riaz and Hashmi [16]	No	No	No	No
Farid et al. [35]	No	No	No	No
Riaz et al. [16]	No	No	No	No
Iampan et al. [36]	No	No	No	No
Almagrabi et al. [18]	No	No	No	No
Panpho and Yiarayong [38]	No	No	No	No
(q_1, q_2)-$FWAA$-operator	Yes	Yes	Yes	Yes
(q_1, q_2)-$FOWAA$-operator	Yes	Yes	Yes	Yes
(q_1, q_2)-$FHWAA$-operator	Yes	Yes	Yes	Yes
(q_1, q_2)-$FWGA$-operator	Yes	Yes	Yes	Yes
(q_1, q_2)-$FOWGA$-operator	Yes	Yes	Yes	Yes
(q_1, q_2)-$FHWGA$-operator	Yes	Yes	Yes	Yes

8. Conclusions

We have explored fuzzy set extensions, including IFSs, $Py FS$s, and q-$ROFS$s. In this work, we introduce a new extension called the (q_1, q_2)-rung Diophantine fuzzy set ((q_1, q_2)-$RDFS$), which provides a more flexible and efficient framework for managing uncertainty. The inclusion of control parameters mappings q_1 and q_2 adds additional features to the (q_1, q_2)-$RDFS$ model. The geometric properties of (q_1, q_2)-$RDFS$ have been outlined to compare it with other fuzzy set extensions. (q_1, q_2)-$RDFS$ offers a broader framework than FS, $Py FS$, $(3,4)$-$QROFS$, q-$ROFS$, and p, q-$QROFS$. We introduced several scoring systems and accuracy functions for comparing (q_1, q_2)-rung Diophantine fuzzy numbers ((q_1, q_2)-$RDFN$s). Additionally, the concept of $LDFS$ has been extended to include the (q_1, q_2)-rung Diophantine fuzzy weighted averaging aggregation ((q_1, q_2)-$RDFSWAA$) and (q_1, q_2)-rung Diophantine fuzzy weighted geometric aggregation ((q_1, q_2)-$RDFSWGA$) operators. Through examples, we demonstrated key findings within the (q_1, q_2)-$RDFSWAA$ and (q_1, q_2)-$RDFSWGA$ frameworks. We proposed two unique approaches for multi-attribute decision-making ($MADM$) based on (q_1, q_2)-$RDFSWAA$ and (q_1, q_2)-$RDFSWGA$. Through a numerical example, we illustrated how these $MADM$ methods can be applied to selecting the appropriate agricultural field robots. Additionally, we analyzed the influence of scoring functions on the overall outcomes and offered a brief comparison between the proposed methods and existing ones. We anticipate that the insights presented in this publication will assist decision-makers and researchers in tackling various real-world challenges. Moving forward, we plan to explore the practical applications of concepts such as the (q_1, q_2)-rung Diophantine fuzzy rough set ((q_1, q_2)-$RDFRS$), hesitant (q_1, q_2)-rung Diophantine fuzzy set (H-(q_1, q_2)-$RDFS$), (q_1, q_2)-rung Diophantine fuzzy graphs ((q_1, q_2)-RDF graphs), and interval-valued (q_1, q_2)-rung Diophantine fuzzy set (IV-(q_1, q_2)-$RDFS$). Our proposed model can be further extended by incorporating techniques such as TOPSIS, VIKOR, AHP, and methods for information aggregation, correlation, and distance and similarity metrics.

Author contributions

Muhammad Bilal Khan: Conceptualization, formal analysis, investigation, resources, writing-original draft, writing-review and editing, visualization, project administration; Dragan Pamucar: Conceptualization, resources, supervision, project administration; Mohamed Abdelwahed: Validation, resources, supervision, project administration; Nurnadiah Zamri: Validation, formal analysis, investigation, visualization; Loredana Ciurdariu: Validation, formal analysis. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors claim to have no conflicts of interest.

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