Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(5): 12149–12167.
DOI:10.3934/math.2025551
Received: 07 February 2025
Revised: 21 April 2025
Accepted: 27 April 2025
Published: 27 May 2025

Research article

New subclass of generalized close-to-convex function related with quasi-subordination

Aoen[1,2]**, Shuhai Li**[2]**, Tula**[3]**, Shuwen Li**[1,*]**and Hang Gao**[1]

[1] College of Mathematics and Statistics, Northeast Normal University, Jilin 130024, China; cfxyaoen@sina.com, lisw739@nenu.edu.cn, hangg@nenu.edu.cn

[2] College of Mathematics and Computer Science, Chifeng University, Inner Mongolia 024000, China; cfxyaoen@sina.com, lishms66@163.com

[3] College of Continuing Education, Chifeng University, Inner Mongolia 024000, China; 248390750@qq.com

* **Correspondence:** Email: lisw739@nenu.edu.cn.

Abstract: In this paper, we introduce a new class of generalized close-to-convex functions, which are defined by quasi-subordination relationship. The integral expression, the estimation of the first two terms and the Fekete-Szegö problem of functions belonging to the class are obtained. Our results extend and unify previous work on starlike, convex, and close-to-convex functions under quasi-subordination. Examples are provided to demonstrate sharpness.

Keywords: close-to-convex function; quasi-convex function; quasi-subordination; integral expression; coefficient estimate; Fekete-Szegö problem
Mathematics Subject Classification: 30C45

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the normalized form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the unit open disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

and all coefficients are complex numbers. Let Φ denote the set of analytic function with positive real part on $\mathbb{D}$ with

$$\phi(0) = 1, \ \ \phi'(0) > 0$$

and $\phi(z)$ maps $\mathbb{D}$ onto a region starlike with respect to 1 and symmetric with respect to the x-axis. And, the function $\phi(z)$ has a series expansion of the form

$$\phi(z) = 1 + \sum_{k=1}^{\infty} A_k z^k,$$

where all coefficients $A_k (k \geq 1)$ are real number and $A_1 > 0$. Also let $\mathcal{U}$ denote the class of Schwartz functions, which is analytic in $\mathbb{D}$ satisfying

$$u(0) = 0 \ \ \text{and} \ \ |u(z)| < 1.$$

In 1970, Robertson [1] introduced the concept of quasi-subordination. For two analytic functions $f_1(z)$ and $f_2(z)$, the function $f_1(z)$ is quasi-subordinate to $f_2(z)$ in $\mathbb{D}$, denoted by

$$f_1(z) \prec_q f_2(z), z \in \mathbb{D},$$

if there exists a Schwarz function $u(z) \in \mathcal{U}$ and an analytic function $h(z)$ with

$$|h(z)| \leq 1$$

such that

$$f_1(z) = h(z) f_2(u(z)).$$

Observe that when

$$h(z) = 1,$$

then

$$f_1(z) = f_2(u(z))$$

and it is said that $f_1(z)$ is subordinate to $f_2(z)$ and written

$$f_1(z) \prec f_2(z)$$

in $\mathbb{D}$. Also notice that if

$$u(z) = z,$$

then

$$f_1(z) = h(z) f_2(z)$$

and it is said that $f_1(z)$ is majorized by $f_2(z)$ and written

$$f_1(z) \ll f_2(z)$$

in $\mathbb{D}$. Hence it is obvious that quasi-subordination is a generalization of subordination as well as majorization. For works related to early study of the quasi-subordination concept, see [2–4].

In order to further explore the concept of quasi-subordination, some researchers have extended the construction of function classes and obtained some geometric properties of function classes. In 2012, Mohd and Darus generalized Ma-Minda starlike and convex classes in [5] and defined the generalized starlike class $\mathcal{S}_q^*(\phi)$ and the generalized convex class $C_q(\phi)$ by using quasi-subordination, as below

$$\mathcal{S}_q^*(\phi) = \left\{ f(z) \in \mathcal{A} : \frac{zf'(z)}{f(z)} - 1 \prec_q \phi(z) - 1, \phi(z) \in \Phi, z \in \mathbb{D} \right\},$$
$$C_q(\phi) = \left\{ f(z) \in \mathcal{A} : \frac{zf''(z)}{f'(z)} \prec_q \phi(z) - 1, \phi(z) \in \Phi, z \in \mathbb{D} \right\}.$$

And, they defined the following function class (also see [6])

$$\mathcal{M}_q(\alpha;\phi) = \left\{ f(z) \in \mathcal{A} : (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1 \prec_q \phi(z) - 1, \phi(z) \in \Phi, \alpha \geq 0, z \in \mathbb{D} \right\}.$$

In 2015, El-Ashwah et al. [7] introduced the generalized starlike class $\mathcal{S}_q^*(\mu;\phi)$ of complex order and the generalized convex class $C_q(\mu;\phi)$ of complex order as follows,

$$\mathcal{S}_q^*(\mu;\phi) = \left\{ f(z) \in \mathcal{A} : 1 + \frac{1}{\mu}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec_q \phi(z), \phi(z) \in \Phi, \mu \in \mathbb{C}\backslash\{0\}, z \in \mathbb{D} \right\},$$
$$C_q(\mu;\phi) = \left\{ f(z) \in \mathcal{A} : 1 + \frac{1}{\mu}\frac{zf''(z)}{f'(z)} \prec_q \phi(z), \phi(z) \in \Phi, \mu \in \mathbb{C}\backslash\{0\}, z \in \mathbb{D} \right\}.$$

In 2020, Ramachandran et al. [8] defined the class $\mathcal{M}_q^*(\alpha,\beta,\lambda;\psi)$ by using quasi-subordination. The function $f(z) \in \mathcal{A}$ is in the class $\mathcal{M}_q^*(\alpha,\beta,\lambda;\phi)$ if

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f(z)}\right)\right]^{\beta} - 1 \prec_q \phi(z) - 1,$$

where $\phi(z) \in \Phi$ and $0 \leq \alpha,\beta,\lambda \leq 1$. Many authors have studied various function subclasses defined by quasi-subordination. For example, Vays et al. [9], Altinkaya et al. [10], Goyal et al. [11] and Choi et al. [12] studied bi-univalent functions using quasi-subordination. Shah et al. [13] and Aoen et al. [14] introduced meromorphic functions using quasi-subordination. Karthikeyan et al. [15] studied Bazilević function using quasi-subordination. Shah et al. [16] studied non-Bazilević function using quasi-subordination. And, there are some function subclasses of linear and nonlinear operators (such as, hohlov operator [17], difference operator [18] and derivative operator [19]) using quasi-subordination.

Recently, some researchers have begun to generalize close-to-convex function classes by using quasi-subordination relationship. In 2019, Gurmeet Singh et al. [20] introduced the subclass of bi-close-to-convex function defined by quasi-subordination. In 2023, Aoen et al. [21] introduced the class of generalized close-to-convex function with complex order written as $\mathcal{K}_q(\gamma;\phi,\psi)$. This class were defined as below

$$\mathcal{K}_q(\gamma;\phi,\psi) = \left\{ f(z) \in \mathcal{A} : \frac{1}{\gamma}\left(\frac{zf'(z)}{g(z)} - 1\right) \prec_q \psi(z) - 1, g(z) \in \mathcal{S}_q^*(\phi), \phi(z), \psi(z) \in \Phi, \gamma \in \mathbb{C}\backslash\{0\}, z \in \mathbb{D} \right\}.$$

In order to denote a new function class, we need to introduce the following function subclasses.

Definition 1.1. Let
$$\alpha \in [0,1], \quad \mu \in \mathbb{C}\backslash\{0\}.$$
Also let $\phi(z) \in \Phi$. A function $f(z) \in \mathcal{A}$ given by (1.1) is said to be in the class $\mathcal{M}_q(\alpha,\mu;\phi)$ if the following condition is satisfied
$$\frac{1}{\mu}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)} - 1\right] \prec_q \phi(z) - 1, \quad z \in \mathbb{D}.$$

Example 1.2. Let
$$\alpha \in [0,1], \quad \mu \in \mathbb{C}\backslash\{0\}, \quad \phi(z) \in \Phi.$$
The function
$$f(z) : \mathbb{D} \to \mathbb{C}$$
defined by the following
$$\frac{1}{\mu}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)} - 1\right] = z[\phi(z) - 1]$$
belongs to the class $\mathcal{M}_q(\alpha,\mu;\phi)$.

Remark 1.3. There are some suitable choices of α, μ which would provide some classical subclasses of analytic functions.

(1) By taking $\mu = 1$ in Definition 1.1, we have
$$\mathcal{M}_q(\alpha, 1; \phi) \equiv \mathcal{M}_q(\alpha; \phi)$$
which is introduced by Mohd et al. [5].

(2) By taking $\alpha = 0$ in Definition 1.1, we have
$$\mathcal{M}_q(0, \mu; \phi) \equiv \mathcal{S}_q^*(\mu; \phi)$$
which is introduced by El-Ashwah et al. [7]. Specially, for $\mu = 1$ we have
$$\mathcal{M}_q(0, 1; \phi) \equiv \mathcal{S}_q^*(\phi)$$
which is introduced and studied by Mohd et al. [5].

(3) By taking $\alpha = 1$ in Definition 1.1, we have
$$\mathcal{M}_q(1, \mu; \phi) \equiv C_q(\mu; \phi)$$
which is introduced by El-Ashwah et al. [7]. Specially, for $\mu = 1$ we have
$$\mathcal{M}_q(1, 1; \phi) \equiv C_q(\phi)$$
which is introduced and studied by Mohd et al. [5].

Now we define a generalization class of close-to-convex function by using quasi-subordination relationship.

Definition 1.4. Let

$$\alpha \in [0,1], \beta \in [0,1], \quad \mu \in \mathbb{C}\backslash\{0\}, \gamma \in \mathbb{C}\backslash\{0\}.$$

Also let

$$\psi(z) \in \Phi, \quad g(z) \in \mathcal{M}_q(\alpha, \mu; \phi).$$

A function $f(z) \in \mathcal{A}$ given by (1.1) is said to be in the class $C_q(\alpha, \beta, \mu, \gamma; \phi, \psi)$ if the following condition is satisfied

$$\frac{1}{\gamma}\left[(1-\beta)\frac{zf'(z)}{g(z)} + \beta\frac{(zf'(z))'}{g'(z)} - 1\right] \prec_q \psi(z) - 1, \quad z \in \mathbb{D}.$$

Example 1.5. Let

$$\alpha \in [0,1], \quad \beta \in [0,1], \quad \mu \in \mathbb{C}\backslash\{0\}, \quad \gamma \in \mathbb{C}\backslash\{0\}, \quad \phi(z) \in \Phi, \quad g(z) \in \mathcal{M}_q(\alpha, \mu; \phi).$$

The function

$$f(z) : \mathbb{D} \to \mathbb{C}$$

defined by the following

$$\frac{1}{\gamma}\left[(1-\beta)\frac{zf'(z)}{g(z)} + \beta\frac{(zf'(z))'}{g'(z)} - 1\right] = z[\psi(z) - 1]$$

belongs to the class $C_q(\alpha, \beta, \mu, \gamma; \phi, \psi)$.

Remark 1.6. There are some suitable choices of $\alpha, \beta, \mu, \gamma$ which would provide the following subclasses of the class $C_q(\alpha, \beta, \mu, \gamma; \phi, \psi)$.

(1) By taking $\beta = 0$ in Definition 1.4, the class $C_q(\alpha, \beta, \mu, \gamma; \phi, \psi)$ reduces to the new subclass $\mathcal{K}_q(\alpha, \mu, \gamma; \phi, \psi)$ which is the class of generalized close-to-convex function satisfied by

$$\frac{1}{\gamma}\left(\frac{zf'(z)}{g(z)} - 1\right) \prec_q \psi(z) - 1, \quad g(z) \in \mathcal{M}_q(\alpha, \mu; \phi), \phi(z), \psi(z) \in \Phi, z \in \mathbb{D}.$$

Specially, for $\alpha = 1, \mu = 1$ in the class $\mathcal{K}_q(\alpha, \mu, \gamma; \phi, \psi)$, we have

$$\mathcal{H}_q(\gamma; \phi, \psi) = \left\{f(z) \in \mathcal{A} : \frac{1}{\gamma}\left(\frac{zf'(z)}{g(z)} - 1\right) \prec_q \psi(z) - 1, g(z) \in C_q(\phi), \phi(z), \psi(z) \in \Phi, z \in \mathbb{D}\right\};$$

for $\alpha = 0, \mu = 1$ in the class $\mathcal{K}_q(\alpha, \mu, \gamma; \phi, \psi)$, we have

$$\mathcal{K}_q(0, 1, \gamma; \phi, \psi) \equiv \mathcal{K}_q(\gamma; \phi, \psi)$$

which is introduced and studied by Aoen et al. [21]. Also, for $\gamma = 1$ in the class $\mathcal{K}_q(\gamma; \phi, \psi)$, we have the class $\mathcal{K}_q(\phi, \psi)$ which is introduced and studied by Aoen et al. [21].

(2) By taking $\beta = 1$ in Definition 1.4, the class $C_q(\alpha, \beta, \mu, \gamma; \phi, \psi)$ reduces to the new subclass $C_q^*(\alpha, \mu, \gamma; \phi, \psi)$ which is the class of generalized quasi-convex function satisfied by

$$\frac{1}{\gamma}\left(\frac{(zf'(z))'}{g'(z)} - 1\right) \prec_q \phi(z) - 1, \quad g(z) \in \mathcal{M}_q(\alpha, \mu; \phi), \phi(z), \psi(z) \in \Phi, z \in \mathbb{D}.$$

Specially, for $\alpha = 1, \mu = 1$ in the class $C_q^*(\alpha, \mu, \gamma; \phi, \psi)$, we have

$$C_q^*(\gamma; \phi, \psi) = \left\{ f(z) \in \mathcal{A} : \frac{1}{\gamma}\left(\frac{(zf'(z))'}{g'(z)} - 1\right) \prec_q \phi(z) - 1, g(z) \in C_q(\phi), \phi(z), \psi(z) \in \Phi, z \in \mathbb{D} \right\};$$

for $\alpha = 0, \mu = 1$ in the class $K_q(\alpha, \mu, \gamma; \phi, \psi)$, we have

$$\mathcal{L}_q(\gamma; \phi, \psi) = \left\{ f(z) \in \mathcal{A} : \frac{1}{\gamma}\left(\frac{(zf'(z))'}{g'(z)} - 1\right) \prec_q \phi(z) - 1, g(z) \in \mathcal{S}_q^*(\phi), \phi(z), \psi(z) \in \Phi, z \in \mathbb{D} \right\}.$$

Studying the theory of analytic functions has been an area of concern for many researchers. The study of coefficients estimate is a more special and important field in complex analysis. For example, the bound for the second coefficient a_2 of normalized univalent functions readily yields the growth and distortion bounds for univalent functions. The coefficient functional $|a_3 - \mu a_2^2|$ (that is, Fekete-Szegö problem) also naturally arises in the investigation of univalency of analytic functions. There are now many results of this type in the literature, each of them dealing with coefficient estimate for various classes of functions. In particular, some authors start to study the coefficient estimates for various classes using quasi-subordination. For example, Arikan et al. [22] and Marut et al. [23] studied the Fekete-Szegö problem for some function subclasses using quasi-subordination. Aoen et al. [24] and Ahman et al. [25] obtained the results on coefficient estimates for various subclasses using quasi-subordination. The purpose of this paper is to study some properties of the class $C_q(\alpha, \beta, \mu, \gamma; \phi, \psi)$ and some of its subclasses, such as the integral expression, the first two coefficient estimate problems and Fekete-Szegö problem. Our results are new in this direction and they give birth to many corollaries.

In order to derive our main results, we have to recall here the following lemmas.

Lemma 1.7. Let $f(z) \in C_q(\phi)$, then

$$f(z) = \int_0^z \exp\left(\int_0^t \frac{h(\xi)[\phi(u(\xi)) - 1]}{\xi} d\xi\right) dt, \tag{1.2}$$

where

$$|h(z)| \le 1, \quad u(z) \in \mathcal{U}, \quad \phi(z) \in \Phi.$$

Proof. Since

$$f(z) \in C_q(\phi),$$

then there exist two analytic functions $h(z), u(z)$ with

$$|h(z)| \le 1, \quad |u(z)| < 1, \quad u(0) = 0$$

such that

$$\frac{zf''(z)}{f'(z)} = h(z)[\phi(u(z)) - 1]. \tag{1.3}$$

By substitution, the Eq (1.3) can be reduced to a first-order differential equation. According to the method of solving the first-order differential equations, we can obtain the general solution of the equation. That is,

$$f'(z) = \exp\left(\int_0^z \frac{h(t)[\phi(u(t)) - 1]}{t} dt\right). \tag{1.4}$$

Integrating both sides of Eq (1.4), we get (1.2). Thus, the proof of Lemma 1.7 is complete. □

Lemma 1.8. [21] Let $f(z) \in \mathcal{S}_q^*(\phi)$, then

$$f(z) = z\exp\left(\int_0^z \frac{h(\xi)[\phi(u(\xi)) - 1]}{\xi}d\xi\right),$$

where

$$|h(z)| \le 1, \quad u(z) \in \mathcal{U}, \quad \phi(z) \in \Phi.$$

Lemma 1.9. [26] Let

$$\varphi(z) = c_0 + \sum_{k=1}^{\infty} c_k z^k$$

be an analytic function in $\mathbb{D}$ with $|\varphi(z)| \le 1$, then

$$|c_0| \le 1, \quad |c_1| \le 1 - |c_0|^2.$$

Lemma 1.10. [27] Let

$$t(z) = \sum_{k=1}^{\infty} t_k z^k$$

be an analytic function in $\mathbb{D}$ with $|t(z)| < 1$, then

$$|t_1| \le 1, |t_2 - \mu t_1^2| \le \max\{1, |\mu|\},$$

where $\mu \in \mathbb{C}$. The result is sharp for the functions

$$t(z) = z \quad \text{or} \quad t(z) = z^2.$$

2. Integral expressions

In this section, we discuss the integral expressions for the class $C_q(\alpha, \beta, \mu, \gamma; \phi, \psi)$ and some of its subclasses by using methods for solving differential equations.

Theorem 2.1. Let

$$\alpha \in [0, 1], \quad \beta \in [0, 1], \quad \mu \in \mathbb{C}\backslash\{0\}, \quad \gamma \in \mathbb{C}\backslash\{0\},$$

the function $f(z) \in C_q(\alpha, \beta, \mu, \gamma; \phi, \psi)$ be given by (1.1). Then,

(i) If $\beta \ne 0$, then

$$f(z) = \frac{1}{\beta}\int_0^z \frac{[g(t)]^{1-\frac{1}{\beta}}}{t}\left(\int_0^t [g(\xi)]^{\frac{1}{\beta}-1} g'(\xi)[1 + \gamma h(\xi)(\psi(u(\xi)) - 1)]d\xi\right)dt. \tag{2.1}$$

(ii) If $\beta = 0$, then

$$f(z) = \int_0^z \frac{g(t)}{t}[1 + \gamma h(t)(\psi(u(t)) - 1)]dt,$$

where

$$|h(z)| \le 1, \quad u(z) \in \mathcal{U}, \quad \psi(z) \in \Phi, \quad g(z) \in \mathcal{M}_q(\alpha, \mu; \phi).$$

Proof. Since $f(z) \in C_q(\alpha,\beta,\mu,\gamma;\phi,\psi)$, then there exist two analytic functions $h(z), u(z)$ with

$$|h(z)| \leq 1, \quad |u(z)| < 1, \quad u(0) = 0$$

such that

$$\frac{1}{\gamma}\left[(1-\beta)\frac{zf'(z)}{g(z)} + \beta\frac{(zf'(z))'}{g'(z)} - 1\right] = h(z)[\psi(u(z)) - 1].$$

Then, we have

$$(zf'(z))' = \frac{(\beta-1)g'(z)}{\beta g(z)}zf'(z) + \frac{[1+\gamma h(z)(\psi(u(z))-1)]\,g'(z)}{\beta}.$$

Let

$$zf'(z) = F(z),$$

then we have

$$F'(z) = \frac{(\beta-1)g'(z)}{\beta g(z)}F(z) + \frac{[1+\gamma h(z)(\psi(u(z))-1)]\,g'(z)}{\beta}.$$

Then, the above equation is a first-order nonhomogeneous linear differential equation. According to the method of solving first-order linear differential equations, we can obtain the general solution of the equation. That is,

$$f'(z) = \frac{1}{\beta}\frac{[g(z)]^{1-\frac{1}{\beta}}}{z}\int_0^z [g(t)]^{\frac{1}{\beta}-1}\, g'(t)\,[1+\gamma h(t)\,(\psi(u(t))-1)]\,dt. \tag{2.2}$$

Integrating both sides of Eq (2.2), we get (2.1). Thus, the proof of Theorem 2.1 is complete. □

By taking $\beta = 1$ in Theorem 2.1, we obtain the following result.

Corollary 2.2. Let the function $f(z) \in C_q^*(\alpha,\mu,\gamma;\phi,\psi)$ be given by (1.1). Then

$$f(z) = \int_0^z \frac{1}{t}\left(\int_0^t g'(\xi)[1+\gamma h(\xi)\,(\psi(u(\xi))-1)]d\xi\right)dt,$$

where

$$|h(z)| \leq 1, \quad u(z) \in \mathcal{U}, \quad \psi(z) \in \Phi, \quad g(z) \in \mathcal{M}_q(\alpha,\mu;\phi).$$

According to Lemmas 1.7 and 1.8 and Corollary 2.2, we can obtain the following two results.

Corollary 2.3. Let the function $f(z) \in C_q^*(\gamma;\phi,\psi)$ be given by (1.1). Then

$$f(z) = \int_0^z \frac{1}{t}\left(\int_0^t [1+\gamma h(\xi)\,(\psi(u(\xi))-1)]\exp\left(\int_0^\xi \frac{h_1(\zeta)[\phi(u_1(\zeta))-1]}{\zeta}d\zeta\right)d\xi\right)dt,$$

where

$$|h(z)| \leq 1, \quad |h_1(z)| \leq 1, u(z), \quad u_1(z) \in \mathcal{U}, \quad \phi(z), \psi(z) \in \Phi.$$

Corollary 2.4. Let the function $f(z) \in \mathcal{L}_q(\gamma;\phi,\psi)$ be given by (1.1). Then

$$f(z) = \int_0^z \frac{1}{t}\left(\int_0^t [1+\gamma h(\xi)\,(\psi(u(\xi))-1)][1+h_1(\xi)(\phi(u_1(\xi))-1)]\exp\left(\int_0^\xi \frac{h_1(\zeta)[\phi(u_1(\zeta))-1]}{\zeta}d\zeta\right)d\xi\right)dt,$$

where

$$|h(z)| \leq 1, |h_1(z)| \leq 1, \quad u(z), u_1(z) \in \mathcal{U}, \quad \phi(z), \psi(z) \in \Phi.$$

3. Coefficient estimates problem

In this section, we obtain the first two coefficient estimate and Fekete-Szegö problem for the class $C_q(\alpha,\beta,\mu,\gamma;\phi,\psi)$ and some subclasses of this class by using algebraic operations, fundamental inequalities of analytic functions.

In addition to special statements, suppose the Taylor series expression for the following functions, as follows

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$
$$\phi(z) = 1 + \sum_{k=1}^{\infty} A_k z^k (A_1 \in \mathbb{R}, A_1 > 0), \quad \psi(z) = 1 + \sum_{k=1}^{\infty} B_k z^k (B_1 \in \mathbb{R}, B_1 > 0),$$
$$\varphi(z) = c_0 + \sum_{k=1}^{\infty} c_k z^k, \quad h(z) = h_0 + \sum_{k=1}^{\infty} h_k z^k,$$
$$u(z) = \sum_{k=1}^{\infty} u_k z^k, \quad v(z) = \sum_{k=1}^{\infty} v_k z^k.$$

In order to derive our main results, we have to discuss the first two coefficient estimates and Fekete-Szegö problem for the class $\mathcal{M}_q(\alpha,\mu;\phi)$.

Theorem 3.1. Let $\alpha \in [0,1], \mu \in \mathbb{C}\backslash\{0\}$, the function $f(z) \in \mathcal{M}_q(\alpha,\mu;\phi)$ be given by (1.1). Then

$$|a_2| \leq \frac{|\mu|A_1}{1+\alpha}, \tag{3.1}$$
$$|a_3| \leq \frac{|\mu|A_1}{2(1+2\alpha)} \max\left\{1, \left|\frac{\mu(1+3\alpha)}{(1+\alpha)^2}A_1 + \frac{A_2}{A_1}\right|\right\}, \tag{3.2}$$

and for any $\eta \in \mathbb{C}$,

$$|a_3 - \eta a_2^2| \leq \frac{|\mu|A_1}{2(1+2\alpha)} \max\left\{1, \left|MA_1 - \frac{A_2}{A_1}\right|\right\}, \tag{3.3}$$

where

$$M = \frac{\mu[2\eta(1+2\alpha) - (1+3\alpha)]}{(1+\alpha)^2}.$$

Proof. If $f(z) \in \mathcal{M}_q(\alpha,\mu;\phi)$, according to Definition 1.1, there exist analytic functions $\varphi(z)$ and $u(z)$, with

$$|\varphi(z)| \leq 1, \ \ u(0) = 0 \ \text{ and } \ |u(z)| < 1$$

such that

$$\frac{1}{\mu}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)} - 1\right] = \varphi(z)[\phi(u(z)) - 1]. \tag{3.4}$$

By substituting the Taylor series expression for the function $f(z)$ to the left of the above expression, we have

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + (2a_3 - a_2^2)z^2 + \cdots,$$

$$\frac{(zf'(z))'}{f'(z)} = 1 + 2a_2z + 2(3a_3 - 2a_2^2)z^2 + \cdots .$$

Thus we get the following expression

$$\frac{1}{\mu}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)} - 1\right] = \frac{1}{\mu}\left\{(1+\alpha)a_2z + [2(1+2\alpha)a_3 - (1+3\alpha)a_2^2]z^2 + \cdots\right\}. \quad (3.5)$$

And by substituting the power series expression of the functions $\varphi(z), \phi(z), u(z)$ to the right of (3.4), we can get the following expression

$$\begin{aligned}\varphi(z)[\phi(u(z)) - 1] &= (c_0 + c_1z + c_2z^2 + \cdots)\left[A_1(u_1z + u_2z_2^2 + \cdots) + A_2(u_1z + u_2z_2^2 + \cdots)^2 + \cdots\right] \\ &= A_1c_0u_1z + [A_1c_1u_1 + c_0\left(A_1u_2 + A_2u_1^2\right)]z^2 + \cdots . \quad (3.6)\end{aligned}$$

By substituting (3.5) and (3.6) into (3.4) and comparing the coefficients of the same power terms on both sides, we can get

$$a_2 = \frac{\mu A_1c_0u_1}{1+\alpha}, \quad (3.7)$$

$$a_3 = \frac{\mu A_1}{2(1+2\alpha)}\left[c_1u_1 + c_0\left(u_2 + \frac{A_2}{A_1}u_1^2\right) + \frac{(1+3\alpha)\mu}{(1+\alpha)^2}A_1c_0^2u_1^2\right].$$

Further,

$$a_3 - \eta a_2^2 = \frac{\mu A_1}{2(1+2\alpha)}\left[c_1u_1 + c_0\left(u_2 + \frac{A_2}{A_1}u_1^2\right) - \frac{2\eta(1+2\alpha) - (1+3\alpha)}{(1+\alpha)^2}\mu A_1c_0^2u_1^2\right]. \quad (3.8)$$

Applying Lemmas 1.9 and 1.10 to (3.7), we obtain

$$|a_2| \leq \frac{|\mu|A_1}{1+\alpha}.$$

Since $\varphi(z)$ is analytic and bounded in $\mathbb{D}$, using [28], for some

$$y, |y| < 1 : |c_0| \leq 1, c_1 = (1 - c_0^2)y.$$

Replacing the value of c_1 as defined above, we get

$$a_3 - \eta a_2^2 = \frac{\mu A_1}{2(1+2\alpha)}\left[yu_1 + c_0\left(u_2 + \frac{A_2}{A_1}u_1^2\right) - \left(\frac{2\eta(1+2\alpha) - (1+3\alpha)}{(1+\alpha)^2}\mu A_1u_1^2 + yu_1\right)c_0^2\right]. \quad (3.9)$$

If $c_0 = 0$, then applying Lemmas 1.9 and 1.10 to (3.9), we obtain

$$|a_3 - \eta a_2^2| \leq \frac{|\mu|A_1}{2(1+2\alpha)}.$$

If $c_0 \neq 0$, let

$$G(c_0) = yu_1 + c_0\left(u_2 + \frac{A_2}{A_1}u_1^2\right) - \left(\frac{2\eta(1+2\alpha) - (1+3\alpha)}{(1+\alpha)^2}\mu A_1u_1^2 + yu_1\right)c_0^2, \quad (3.10)$$

which is a polynomial in c_0 and have analytic in $|c_0| \leq 1$.

According to Maximum modulus principle, we get

$$\max |G(c_0)| = \max_{0\leq\theta\leq 2\pi} |G(e^{i\theta})| = |G(1)|.$$

Thus

$$|a_3 - \eta a_2^2| \leq \frac{|\mu| A_1}{2(1+2\alpha)} \left| u_2 - \left(\frac{2\eta(1+2\alpha) - (1+3\alpha)}{(1+\alpha)^2} \mu A_1 - \frac{A_2}{A_1} \right) u_1^2 \right|. \tag{3.11}$$

Applying Lemma 1.10 to (3.11), we can conclude (3.3). For $\eta = 0$ in (3.3), we have (3.2).

Let

$$\frac{1}{\mu}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)} - 1\right] = \phi(z) - 1$$

or

$$\frac{1}{\mu}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)} - 1\right] = z[\phi(z^2) - 1].$$

then the results of (3.1)–(3.3) are sharp. Thus, the proof of Theorem 3.1 is complete. □

Remark 3.2. (1) For $\mu = 1$ in Theorem 3.1, we can obtain the result which is Theorem 2.10 in [5].

(2) For $\mu = 1, \alpha = 0$ and $\mu = 1, \alpha = 1$ in Theorem 3.1, we can obtain the results which are Theorems 2.1 and 2.4 in [5], respectively.

(3) For $\alpha = 0$ and $\alpha = 1$ in Theorem 3.1, we improve the results which are Theorems 2.1 and 2.7 in [7], respectively.

Theorem 3.3. Let

$$\alpha \in [0,1], \quad \beta \in (0,1], \quad \mu \in \mathbb{C}\backslash\{0\}, \quad \gamma \in \mathbb{C}\backslash\{0\},$$

the function $f(z) \in C_q(\alpha,\beta,\mu,\gamma;\phi,\psi)$ be given by (1.1). Then

$$|a_2| \leq \frac{|\mu| A_1}{2(1+\alpha)} + \frac{|\gamma| B_1}{2(1+\beta)}, \tag{3.12}$$

$$|a_3| \leq \frac{|\mu| A_1}{6(1+2\alpha)} \max\left\{1, \left|\frac{\mu(1+3\alpha)}{(1+\alpha)^2} A_1 + \frac{A_2}{A_1}\right|\right\} + \frac{|\gamma| B_1}{3(1+2\beta)} \max\left\{1, \frac{|B_2|}{B_1}\right\} + \frac{|\mu\gamma|(1+3\beta)}{3(1+\alpha)(1+\beta)(1+2\beta)} A_1 B_1, \tag{3.13}$$

and for any $\tau \in \mathbb{C}$

$$|a_3 - \tau a_2^2| \leq \frac{|\mu| A_1}{6(1+2\alpha)} \max\left\{1, \left|PA_1 - \frac{A_2}{A_1}\right|\right\} + \frac{|\gamma| B_1}{3(1+2\beta)} \max\left\{1, \left|QB_1 - \frac{B_2}{B_1}\right|\right\} + \frac{|\mu\gamma[2(1+3\beta) - 3\tau(1+2\beta)]|}{6(1+\alpha)(1+\beta)(1+2\beta)} A_1 B_1, \tag{3.14}$$

where

$$P = \frac{\mu[3\tau(1+2\alpha) - 2(1+3\alpha)]}{2(1+\alpha)^2}, \quad Q = \frac{3\tau\gamma(1+2\beta)}{4(1+\beta)^2}.$$

Proof. If $f(z) \in C_q(\alpha, \beta, \mu, \gamma; \phi, \psi)$, then there exist analytic functions $h(z)$ and $v(z)$, with

$$|\varphi(z)| \leq 1, \quad v(0) = 0 \ \text{ and } \ |v(z)| < 1$$

such that

$$\frac{1}{\gamma}\left[(1-\beta)\frac{zf'(z)}{g(z)} + \beta\frac{(zf'(z))'}{g'(z)} - 1\right] = h(z)[\psi(v(z)) - 1]. \tag{3.15}$$

By substituting the Taylor series expression for the functions $f(z), g(z)$ to the left of the above expression, we have

$$\begin{aligned}\frac{zf'(z)}{g(z)} &= 1 + (2a_2 - b_2)z + (3a_3 - b_3 + b_2^2 - 2a_2b_2)z^2 + \cdots, \\ \frac{(zf'(z))'}{g'(z)} &= 1 + 2(2a_2 - b_2)z + 2(9a_3 - 3b_3 + 4b_2^2 - 8a_2b_2)z^2 + \cdots.\end{aligned}$$

Thus we get the following expression

$$\begin{aligned}&\frac{1}{\gamma}\left[(1-\beta)\frac{zf'(z)}{g(z)} + \beta\frac{(zf'(z))'}{g'(z)} - 1\right] \\ &= \frac{1}{\gamma}\left\{(1+\beta)(2a_2 - b_2)z + \left[(1+2\beta)(3a_3 - b_3) + (1+3\beta)(b_2^2 - 2a_2b_2)\right]z^2 + \cdots\right\}.\end{aligned} \tag{3.16}$$

Similar to the proof of Theorem 3.1, by substituting the power series expression of the functions $h(z), \psi(z), v(z)$ to the right of (3.15), we can get the following expression

$$h(z)[\psi(v(z)) - 1] = B_1h_0v_1z + [B_1h_1v_1 + h_0(B_1v_2 + B_2v_1^2)]z^2 + \cdots. \tag{3.17}$$

By substituting (3.16) and (3.17) into (3.15) and comparing the coefficients of the same power terms on both sides, we can get

$$a_2 = \frac{1}{2}\left(\frac{\gamma B_1h_0v_1}{1+\beta} + b_2\right) \tag{3.18}$$

and

$$a_3 = \frac{1}{3(1+2\beta)}\left\{(1+2\beta)b_3 - (1+3\beta)(b_2^2 - 2a_2b_2) + \gamma[B_1h_1v_1 + h_0(B_1v_2 + B_2v_1^2)]\right\}.$$

Further,

$$\begin{aligned}a_3 - \tau a_2^2 =& \frac{1}{3}\left(b_3 - \frac{3}{4}\tau b_2^2\right) + \frac{\gamma[2(1+3\beta) - 3\tau(1+2\beta)]}{6(1+\beta)(1+2\beta)}B_1h_0v_1b_2 \\ &+ \frac{\gamma B_1}{3(1+2\beta)}\left[h_1v_1 + h_0\left(v_2 + \frac{B_2}{B_1}v_1^2\right) - \frac{3\tau\gamma(1+2\beta)}{4(1+\beta)^2}B_1h_0^2v_1^2\right].\end{aligned} \tag{3.19}$$

Applying Lemmas 1.9 and 1.10 to (3.18) and (3.19), we obtain

$$|a_2| \leq \frac{1}{2}\left(\frac{|\gamma|B_1}{1+\beta} + |b_2|\right) \tag{3.20}$$

and

$$|a_3 - \tau a_2^2| \leq \frac{1}{3}\left|b_3 - \frac{3}{4}\tau b_2^2\right| + \frac{|\gamma[2(1+3\beta) - 3\tau(1+2\beta)]|}{6(1+\beta)(1+2\beta)}B_1|b_2| + \frac{|\gamma|B_1}{3(1+2\beta)}\left|h_1 v_1 + h_0\left(v_2 + \frac{B_2}{B_1}v_1^2\right) - \frac{3\tau\gamma(1+2\beta)}{4(1+\beta)^2}B_1 h_0^2 v_1^2\right|. \tag{3.21}$$

According to Theorem 3.1, it follows that

$$|b_2| \leq \frac{|\mu|A_1}{1+\alpha} \tag{3.22}$$

and, for any complex number τ, we have

$$|b_3 - \frac{3}{4}\tau b_2^2| \leq \frac{|\mu|A_1}{2(1+2\alpha)} \max\left\{1, \left|PA_1 - \frac{A_2}{A_1}\right|\right\}, \tag{3.23}$$

where

$$P = \frac{\mu[3\tau(1+2\alpha) - 2(1+3\alpha)]}{2(1+\alpha)^2}.$$

Similar to the proof of Theorem 3.1, we can also get the following inequality

$$\left|h_1 v_1 + h_0\left(v_2 + \frac{B_2}{B_1}v_1^2\right) - \frac{3\tau\gamma(1+2\beta)}{4(1+\beta)^2}B_1 h_0^2 v_1^2\right| \leq \max\left\{1, \left|QB_1 - \frac{B_2}{B_1}\right|\right\}, \tag{3.24}$$

where

$$Q = \frac{3\tau\gamma(1+2\beta)}{4(1+\beta)^2}.$$

By substituting (3.22) into (3.20), we get (3.12). And by substituting (3.22)–(3.24) into (3.21), we can conclude (3.14). For $\tau = 0$ in (3.14), we have (3.13).

The results of (3.12) and (3.13) are sharp for $\beta \neq 0$ if

$$f(z) = \frac{1}{\beta}\int_0^z \frac{[g(t)]^{1-\frac{1}{\beta}}}{t}\left(\int_0^t [g(\xi)]^{\frac{1}{\beta}-1} g'(\xi)[1 + \gamma(\psi(\xi) - 1)]d\xi\right)dt$$

or

$$f(z) = \frac{1}{\beta}\int_0^z \frac{[g(t)]^{1-\frac{1}{\beta}}}{t}\left(\int_0^t [g(\xi)]^{\frac{1}{\beta}-1} g'(\xi)[1 + \gamma\left(\psi(\xi^2) - 1\right)]d\xi\right)dt,$$

and the results of (3.14) and (3.15) are sharp for $\beta = 0$ if

$$f(z) = \int_0^z \frac{g(t)}{t}[1 + \gamma(\psi(t) - 1)]dt$$

or

$$f(z) = \int_0^z \frac{g(t)}{t}[1 + \gamma\left(\psi(t^2) - 1\right)]dt.$$

Thus, the proof of Theorem 3.3 is complete. □

By taking special values of parameters α,β,μ in Theorem 3.3, we can obtain coefficient estimates for functions belonging to some subclasses of the class $C_q(\alpha,\beta,\mu,\gamma;\phi,\psi)$.

Corollary 3.4. Let the function $f(z) \in \mathcal{K}_q(\alpha,\mu,\gamma;\phi,\psi)$. Then

$$|a_2| \leq \frac{|\mu|A_1}{2(1+\alpha)} + \frac{|\gamma|B_1}{2},$$

$$|a_3| \leq \frac{|\mu|A_1}{6(1+2\alpha)} \max\left\{1, \left|\frac{\mu(1+3\alpha)}{(1+\alpha)^2}A_1 + \frac{A_2}{A_1}\right|\right\} + \frac{|\gamma|B_1}{3} \max\left\{1, \frac{|B_2|}{B_1}\right\} + \frac{|\mu\gamma|}{3(1+\alpha)}A_1B_1,$$

and for any $\tau \in \mathbb{C}$,

$$|a_3 - \tau a_2^2| \leq \frac{|\mu|A_1}{6(1+2\alpha)} \max\left\{1, \left|\frac{\mu[3\tau(1+2\alpha) - 2(1+3\alpha)]}{2(1+\alpha)^2}A_1 - \frac{A_2}{A_1}\right|\right\} + \frac{|\gamma|B_1}{3} \max\left\{1, \left|\frac{3\tau\gamma}{4}B_1 - \frac{B_2}{B_1}\right|\right\} + \frac{|\mu\gamma(2-3\tau)|}{6(1+\alpha)}A_1B_1.$$

Remark 3.5. For $\alpha = \beta = 0, \mu = 1$ in Theorem 3.3 or $\alpha = 0, \mu = 1$ in Corollary 3.4, we obtain the result which is Corollary 3 in [21].

Corollary 3.6. Let the function $f(z) \in \mathcal{H}_q(\gamma;\phi,\psi)$. Then

$$|a_2| \leq \frac{A_1 + 2|\gamma|B_1}{4},$$

$$|a_3| \leq \frac{A_1}{18} \max\left\{1, \left|A_1 + \frac{A_2}{A_1}\right|\right\} + \frac{|\gamma|B_1}{3} \max\left\{1, \frac{|B_2|}{B_1}\right\} + \frac{|\gamma|}{6}A_1B_1,$$

and for any $\tau \in \mathbb{C}$

$$|a_3 - \tau a_2^2| \leq \frac{A_1}{18} \max\left\{1, \left|\frac{9\tau - 8}{8}A_1 - \frac{A_2}{A_1}\right|\right\} + \frac{|\gamma|B_1}{3} \max\left\{1, \left|\frac{3\tau\gamma}{4}B_1 - \frac{B_2}{B_1}\right|\right\} + \frac{|\gamma(2-3\tau)|}{12}A_1B_1.$$

Especially, let

$$\gamma = 1, \quad \phi(z) = \psi(z) = \frac{1+z}{1-z},$$

we can obtain the following result.

Remark 3.7. Let the function

$$f(z) \in \mathcal{H}_q(1; \frac{1+z}{1-z}, \frac{1+z}{1-z}).$$

Then

$$|a_2| \leq \frac{3}{2}, \quad |a_3| \leq \frac{5}{3},$$

and for any $\tau \in \mathbb{C}$

$$|a_3 - \tau a_2^2| \leq \frac{1}{9} \max\left\{1, \left|\frac{9\tau - 12}{4}\right|\right\} + \frac{2}{3} \max\left\{1, \left|\frac{3\tau}{2} - 1\right|\right\} + \frac{|2-3\tau|}{3}.$$

The sharpness of the estimates is demonstrated by the functions

$$f_1(z) = \frac{2z}{1-z} + \log(1-z)$$

or

$$f_2(z) = \frac{z}{1-z} + \frac{1}{2}\log(1-z^2),$$

and the graph of the functions $f_1(z)$ and $f_1(z)$ are shown as follows,

In Figures 1 and 2, the three-dimensional coordinate system, coupled with color, is used to represent complex functions. Specifically, the x-axis corresponds to the real part of the variable z, the y-axis to the imaginary part of z, the z-axis indicates the real part of the function, and the color signifies the imaginary part of the function.

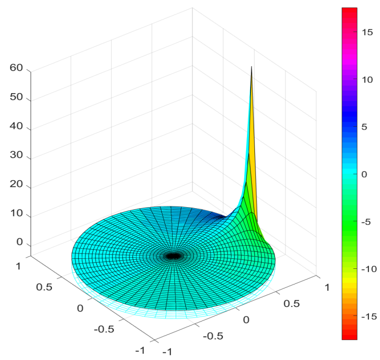

Figure 1. The image of $f_1(z)$.

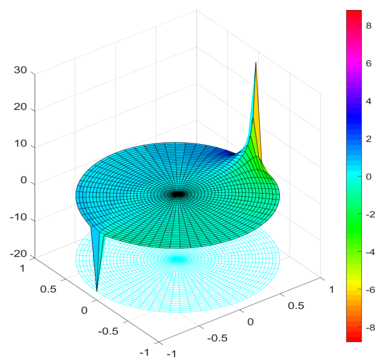

Figure 2. The image of $f_2(z)$.

Corollary 3.8. Let the function $f(z) \in C_q^*(\alpha, \mu, \gamma; \phi, \psi)$. Then

$$|a_2| \leq \frac{|\mu|A_1}{2(1+\alpha)} + \frac{|\gamma|B_1}{4},$$

$$|a_3| \leq \frac{|\mu|A_1}{6(1+2\alpha)} \max\left\{1, \left|\frac{\mu(1+3\alpha)}{(1+\alpha)^2}A_1 + \frac{A_2}{A_1}\right|\right\} + \frac{|\gamma|B_1}{9} \max\left\{1, \frac{|B_2|}{B_1}\right\} + \frac{2|\mu\gamma|}{9(1+\alpha)}A_1B_1,$$

and for any $\tau \in \mathbb{C}$,

$$|a_3 - \tau a_2^2| \leq \frac{|\mu|A_1}{6(1+2\alpha)} \max\left\{1, \left|\frac{\mu[3\tau(1+2\alpha) - 2(1+3\alpha)]}{2(1+\alpha)^2}A_1 - \frac{A_2}{A_1}\right|\right\}$$

$$+\frac{|\gamma|B_1}{9}\max\left\{1,\left|\frac{9\tau\gamma}{16}B_1-\frac{B_2}{B_1}\right|\right\}+\frac{|\mu\gamma(8-9\tau)|}{36(1+\alpha)}A_1B_1.$$

Corollary 3.9. Let the function $f(z)\in C_q^*(\gamma;\phi,\psi)$. Then

$$|a_2|\leq\frac{A_1+|\gamma|B_1}{4},$$
$$|a_3|\leq\frac{A_1}{18}\max\left\{1,\left|A_1+\frac{A_2}{A_1}\right|\right\}+\frac{|\gamma|B_1}{9}\max\left\{1,\frac{|B_2|}{B_1}\right\}+\frac{|\gamma|}{9}A_1B_1,$$

and for any $\tau\in\mathbb{C}$,

$$|a_3-\tau a_2^2|\leq\frac{A_1}{18}\max\left\{1,\left|\frac{9\tau-8}{8}A_1-\frac{A_2}{A_1}\right|\right\}+\frac{|\gamma|B_1}{9}\max\left\{1,\left|\frac{9\tau\gamma}{16}B_1-\frac{B_2}{B_1}\right|\right\}+\frac{|\gamma(8-9\tau)|}{72}A_1B_1.$$

Corollary 3.10. Let the function $f(z)\in\mathcal{L}_q(\gamma;\phi,\psi)$. Then

$$|a_2|\leq\frac{2A_1+|\gamma|B_1}{4},$$
$$|a_3|\leq\frac{A_1}{6}\max\left\{1,\left|A_1+\frac{A_2}{A_1}\right|\right\}+\frac{|\gamma|B_1}{9}\max\left\{1,\frac{|B_2|}{B_1}\right\}+\frac{2|\gamma|}{9}A_1B_1,$$

and for any $\tau\in\mathbb{C}$,

$$|a_3-\tau a_2^2|\leq\frac{A_1}{6}\max\left\{1,\left|\frac{3\tau-2}{2}A_1-\frac{A_2}{A_1}\right|\right\}+\frac{|\gamma|B_1}{9}\max\left\{1,\left|\frac{9\tau\gamma}{16}B_1-\frac{B_2}{B_1}\right|\right\}+\frac{|\gamma(8-9\tau)|}{36}A_1B_1.$$

4. Conclusions

In this paper, we introduce the new function class $C_q(\alpha,\beta,\mu,\gamma;\phi,\psi)$, which is a expanded close-to-convex functions defined by quasi-subordination. We mainly study the integral expression, the first two coefficient estimates and Fekete-Szegö problem for this class and some of its subclasses. In the future, we can consider to study other forms of coefficient estimation, such as Milin coefficient eatimate, Zalcman functional estimate, high order Hankel Determinant estimate for these classes using the concepts dealt with in the paper.

Author contributions

Aoen: conceptualization, methodology, software, investigation, writing—original draft preparation, writing—review and editing, project administration, funding acquisition, visualization; Shuhai Li: conceptualization, methodology, formal analysis, resources; Tula: validation, data curation; Shuwen Li and Hang Gao: supervision. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The present investigation was supported by the Natural Science Foundation of Inner Mongolia Autonomous Region of China under Grant (No. 2025MS01034, 2024MS01014, 2020MS01010), the National Natural Science Foundation of China (Grant No.11561001) and the Program for Young Talents of Science and Technology in Universities of Inner Mongolia Autonomous Region under Grant (No. NJYT-18-A14).

Conflicts of interest

The authors declare no conflicts of interest.

References

1. R. S. Robertson, Quasi-subordination and coefficient conjectures, *Bull. Amer. Math. Soc.*, **620** (1970), 1–9.
2. S. Y. Lee, Quasi-subordinate functions and coefficient conjectures, *J. Korean Math. Soc.*, **12** (1975), 43–50.
3. F. Y. Ren, S. Owa, S. Fukui, Some inequalities on quasi-subordinate functions, *Bull. Austr. Math. Soc.*, **43** (1991), 317–324. https://doi.org/10.1017/S0004972700029117
4. O. Altmtas, S. Owa, Majorizations and quasi-subordinations for certain analytic functions, *Proc. Jpn. Acad. Ser. A Math. Sci.*, **68** (1992), 181–85. https://doi.org/10.3792/pjaa.68.181
5. M. M. Mohd, M. Darus, Fekete-Szegö problems for quasi-subordination classes, *J. Abstr. Appl. Anal.*, **3** (2012), 1–14. https://doi.org/10.1155/2012/192956
6. P. Gurusamy, J. Sokoł, S. Sivasubramanian, The Fekete-Szegö functional associated with k-th root transformation using quasi-subordination, *C. R. Math.*, **353** (2015), 617–622. https://doi.org/10.1016/j.crma.2015.03.016
7. R. El-Ashwah, S. Kanas, Fekete-Szegö inequalities for quasi-subordination functions classes of complex order, *Kyungpook Math.J.*, **55** (2015), 679–688. https://doi.org/10.5666/KMJ.2015.55.3.679
8. C. Ramachandran, T. Soupramanien, J. Sokoł, The Fekete-Szegö functional associated with k-th root transformation using quasi-subordination, *J. Math.*, **28** (2020), 199–208. DOI:10.1007/s41478-017-0059-0
9. P. P. Vyas, S. Kant, Certain subclasses of bi-univalent functions associated with quasi-subordination, *J. Rajasthan Acad. Phys. Sci.*, **15** (2016), 315–325. https://www.researchgate.net/publication/315831303
10. Ş. Altinkaya, S. Yalçin, Quasi-subordinations for certain subclasses of bi-univalent functions, *Math. Adv. Pure Appl. Sci.*, **1** (2018), 56–64.
11. S. P. Goyal, O. Singh, R. Mukherjee, Certain results on a subclass of analytic and bi-univalent functions associated with coefficient estimates and quasi-subordination, *Palest. J. Math.*, **5** (2016), 79–85.

12. J. H. Choi, New subclass of bi-univalent functions based on quasi-subordination, *Int. J. Appl. Math.*, **36** (2023), 685–697. https://doi.org/10.12732/ijam.v36i5.8

13. S. G. A. Shah, S. Hussain, A. Rasheed, Z. Shareef, Application of quasi-subordiantion to certain classes of meromorphic fucntions, *J. Funct. Spaces*, **2020** (2020), 4581926. https://doi.org/10.1155/2020/4581926

14. Aoen, L. Shuhai, H. Tang, Siqinqimuge, Some properties of certain subclasses of p-valent meromorphic functions associated with quasi-subordination, *J. Math. Res. Appl.*, **38** (2018), 147–161. DOI:10.3770/j.issn:2095-2651.2018.02.005

15. K. R. Karthikeyan, G. Murugusundaramoorthy, N. E. Cho, Some inequalities on Bazilevič class of functions involving quasi-subordination, *AIMS Math.*, **6** (2021), 7111–7124. https://doi.org/10.3934/math.2021417

16. S. G. A. Shah, S. Al-Sa'di, S. Hussain, A. Tasleem, A. Rasheed, I. Z. Cheema, et al., Fekete-Szegö functional for a class of non-Bazilevič functions related to quasi-subordination, *Demonstr. Math.*, **56** (2023), 1–10. https://doi.org/10.1515/dema-2022-0232

17. M. Çağlar, P. Gurusamy, H. Orhan, The coefficient estimates for a class defined by hohlov operator using conic domains, *Twms J. Pure Appl.*, **11** (2020), 157–172.

18. K. Y. Taha, S. A. Rahman, Juma, Fekete-Szegö problem for certain subclass of q-difference operator using quasi-subordination, *Tikrit J. Pure Sci.*, **26** (2021), 91–94. https://doi.org/10.25130/tjps.v26i6.197

19. Aoen, L. Shuhai, T. Huo, Coefficient related problem studies for new subclass of bi-univalent functions defined by (s, t)-derivative operator and quasi-subordination, *J. Math. Res. Appl.*, **41** (2021), 579–593. DOI:10.3770/j.issn:2095-2651.2021.06.003

20. S. Gurmeet, S. Gurcharanjit, S. Gagandeep, Certain subclasses of bi-close-to-convex functions associated with quasi-subordination, *Abstr. Appl. Anal.*, **2019** (2019), 6947020. https://doi.org/10.1155/2019/6947020

21. Aoen, L. Shuhai, Some properties of generalized complex-order α type Bazilevič functions, *J. Yangzhou Univer.*, **26** (2023), 42. DOI:10.19411/j.1007-824x.2023.03.001

22. H. Arikan, H. Arikan, H. Orhan, Fekete-Szegö problem for subclasses of analytic functions associated with quasi-subordination, *Bulletin of the Transilvania University of Braşov Series III: Mathematics, Informatics, Physics*, **61** (2019), 221–232. https://doi.org/10.31926/but.mif.2019.12.61.2.3

23. M. J. Marut, M. H. Mohd, S. Supramaniam, The Fekete-Szegö inequalities for meromorphic functions associated with quasi-subordination, *AlP Conf. Proc.*, **2184** (2019), 020005. https://doi.org/10.1063/1.5136359

24. Aoen, L. Shuhai, T. Huo, Coefficient estimates and Fekete-Szegö inequality for new subclass of bi-Bazilevič functions by (s, t)-derivative operator and quasi-subordination, *J. Math. Inequal.*, **15** (2021), 923–940. https://dx.doi.org/10.7153/jmi-2021-15-64

25. l. A. R. Ahman, W. G. Atshan, Results on coefficients bounds for new subclasses by using quasi-subordination of analytic function estimates, *AlP Conf. Proc.*, **2398** (2022), 060021. https://doi.org/10.1063/5.0093380

26. P. L. Duren, *Univalent functions*, Springer-Verlag, 1983.

27. S. Srikandan, A coefficient inequality for certain classes of analytic functions, *Proc. Amer. Math. Soc.*, **20** (1969), 8–12. https://doi.org/10.20935/MHealthWellB7426

28. Z. Nehari, *Conformal mapping*, Courier Corporation, 1975.