Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(5): 12135–12148.
DOI: 10.3934/math.2025550
Received: 30 March 2025
Revised: 06 May 2025
Accepted: 13 May 2025
Published: 27 May 2025

Research article

Some new characterizations of the normality for group invertible matrices

Zhirong Guo[1] **and Qianglian Huang**[2,*]

[1] College of Mathematics, Yangzhou Vocational University, Yangzhou 225009, China
[2] School of Mathematical Sciences, Yangzhou University, Yangzhou 225002, China

* **Correspondence:** Email: huangql@yzu.edu.cn.

Abstract: Normal matrices are an important class of matrices; many authors discuss the normality of matrices. In terms of projection matrices, the inverse of matrices, one-sided X-equality, and X-idempotency of matrices, many new characterizations of normal matrices are obtained in this paper. It may be the first time that the projection, one-sided X-equality, and one-sided X-idempotency are used to characterize the normality of matrices.

Keywords: normal matrix; group invertible matrix; Moore-Penrose inverse matrix; projection
Mathematics Subject Classification: 16B99, 16W10

1. Introduction and Preliminaries

In this paper, the matrices that we consider are all $n \times n$ complex matrices, and we denote the set of all $n \times n$ matrices over the complex number field $\mathbb{C}$ by $\mathbb{C}^{n\times n}$. Let $A \in \mathbb{C}^{n\times n}$; then there always exists a matrix $B \in \mathbb{C}^{n\times n}$ such that

$$A = ABA, \quad B = BAB, \quad (AB)^H = AB \quad \text{and} \quad (BA)^H = BA,$$

where A^H denotes the conjugate transpose matrix of A. The matrix B is called the Moore−Penrose inverse matrix of A, which is uniquely determined and is denoted by $A^\dagger$ [1–3].

The matrix A is said to be group invertible if there is a matrix $B \in \mathbb{C}^{n\times n}$ such that

$$A = ABA, \quad B = BAB \quad \text{and} \quad AB = BA.$$

The matrix B is called the group inverse matrix of A. If it exists, then it is unique [4] and is always denoted by $A^\#$.

We use $G_n(\mathbb{C})$ to denote the set of all group invertible matrices in $\mathbb{C}^{n\times n}$. It is well known that $A \in G_n(\mathbb{C})$ if and only if $\text{rank}(A) = \text{rank}(A^2)$. Hence, for any $A \in \mathbb{C}^{n\times n}$, AA^H, A^HA, and $A + A^H$ are all group invertible matrices.

$A \in G_n(\mathbb{C})$ is called an EP matrix if $A^{\#} = A^{\dagger}$. It is known that A is EP if and only if $AA^{\dagger} = A^{\dagger}A$ [2]. For the study of EP matrices, we can also refer to [5]. And A is called a normal matrix if $A^H A = AA^H$. In [6], it gives some counterexamples on normal matrices. In recent years, the normality of matrices was wildly studied. For example, in [7, Theorem 5.1], it is shown that $A \in G_n(\mathbb{C})$ is normal if and only if the equation

$$AX(A^{\#})^H = A^H A(A^{\dagger})^H$$

is consistent and the general solution is given by

$$X = A^H + U - A^{\dagger}AUA^{\dagger}A, \text{ where } U \in \mathbb{C}^{n\times n}.$$

In [8, Theorem 3.1], it is proved that $A \in G_n(\mathbb{C})$ is normal if and only if the equation

$$XA^{\#}A^H = A^{\#}A^H X$$

has at least one solution in $\{A, A^{\#}, A^{\dagger}, A^H, (A^{\dagger})^H, (A^{\#})^H\}$. For the studies of normality in a ring with involution, the readers can refer to [2, 5, 9–11].

Let A, B and $C \in \mathbb{C}^{n\times n}$. Recall that A is said to be a projection if $A^2 = A = A^H$. Clearly, A is a projection if and only if $A = AA^H$ or $A = A^H A$always [12]. B and C are called left (right) A-equality if $AB = AC$ ($BA = CA$). A is called left (right) B-idempotent if $A^2 = BA$ ($A^2 = AB$). Clearly, A is left and right A-idempotent. And A is an idempotent matrix if and only if A is left $2A$-E_n-idempotent.

In [13], using the projections, many characterizations of EP elements were founded. Utilizing the generalized inverse of matrices to construct the inverse matrices of related matrices, the authors provide some characterizations of normal matrices in [7]. Inspired by these, we use the projection, the generalized inverse, one-sided X-equality, and X-idempotency of matrices to investigate the normality of the group invertible matrices. It seems to be the first time that the projection, one-sided X-equality, and one-sided X-idempotency are used to characterize the normality of matrices.

This paper is organized as follows: in terms of projections, many new and interesting characterizations of normal matrices are obtained in Section 2. In Section 3, we characterize normal matrices by constructing the inverse of the product of some matrices. In Sections 4 and 5, utilizing one-sided A-equalitivity and one-sided X-idempotency, we give some properties and characterizations of normal matrices. We conclude in Section 6.

2. Using projections to characterize normal matrices

We begin with the following lemma, which appears in [8, Lemma 2.6].

Lemma 2.1. *Let $A \in G_n(\mathbb{C})$. Then A is normal if and only if $(A^{\dagger})^H A^{\#} A^H = A^{\dagger}$.*

Lemma 2.2. *Let $A \in G_n(\mathbb{C})$. Then A is a projection if and only if $A^H = AA^{\dagger}$.*

Proof. Necessity. If A is a projection, then $A^2 = A = A^H$ and it follows that $A = A^2 = A^H A$ and $AA^{\dagger} = A^H AA^{\dagger} = A^H$.

Sufficiency. Suppose that $A^H = AA^{\dagger}$. Then $A = (AA^{\dagger})^H = AA^{\dagger} = A^H$ and so $A^2 = (AA^{\dagger})^2 = AA^{\dagger} = A$. Hence A is a projection. □

From Lemma 2.1, one knows that if A is normal, then $(A^\dagger)^H A^\# A^H A = A^\dagger A$ is a projection. Using Lemma 2.2, we can obtain

$$(A^\dagger)^H A^\# A^H A((A^\dagger)^H A^\# A^H A)^\dagger = ((A^\dagger)^H A^\# A^H A)^H.$$

This implies $((A^\dagger)^H A^\# A^H A)^\dagger = A^\#(A^\dagger)^H AA^H A^H A(A^\#)^H A^\dagger$, if A is normal. The following lemma points out exactly what $((A^\dagger)^H A^\# A^H A)^\dagger$ is for any $A \in G_n(\mathbb{C})$.

Lemma 2.3. *Let $A \in G_n(\mathbb{C})$. Then*
(1) $((A^\dagger)^H A^\# A^H A)^\dagger = A^\dagger(A^\#)^H A^\dagger A^2 A^H$.
(2) $((A^\dagger)^H A^\# A^H)^\dagger = (A^\dagger)^H AA^H$.
(3) $((A^\#)^H A^\# A^H)^\dagger = (A^\dagger)^H A^2 A^\dagger A^H A^\dagger A$.
(4) $((A^\#)^H A^\# A^H)^\# = (A^\#)^H A^\dagger A^3 A^\dagger A^H$.

Proof. It is routine, we omit the proof. □

Theorem 2.1. *Let $A \in G_n(\mathbb{C})$. Then the following statements are equivalent:*
(1) *A is normal;*
(2) $(A^\dagger)^H AA^H = A$*;*
(3) $(A^\#)^H AA^H A^\dagger$ *is a projection;*
(4) $(A^\dagger)^H AA^H A^\#$ *is a projection.*

Proof. (1)⇒(2). Since A is normal, by Lemma 2.1, $(A^\dagger)^H A^\# A^H = A^\dagger$. It follows from Lemma 2.3 that

$$A = ((A^\dagger)^H A^\# A^H)^\dagger = (A^\dagger)^H AA^H.$$

(2)⇒(3). If $A = (A^\dagger)^H AA^H$, then

$$A^2 A^\dagger = (A^\dagger)^H AA^H AA^\dagger = (A^\dagger)^H AA^H = A.$$

Hence by [5, Theorem 1.2.1], A is EP and

$$(A^\#)^H AA^H A^\dagger = (A^\dagger)^H AA^H A^\dagger = AA^\dagger$$

is a projection.

(3)⇒(4). Under the assumption, one obtains

$$(A^\#)^H AA^H A^\dagger = ((A^\#)^H AA^H A^\dagger)^H = (A^\dagger)^H AA^H A^\#.$$

Hence $(A^\dagger)^H AA^H A^\#$ is a projection.

(4)⇒(1). Assuming that $(A^\dagger)^H AA^H A^\#$ is a projection, then

$$(A^\dagger)^H AA^H A^\# = ((A^\dagger)^H AA^H A^\#)^2, \tag{2.1}$$

and

$$(A^\dagger)^H AA^H A^\# = ((A^\dagger)^H AA^H A^\#)^H = (A^\#)^H AA^H A^\dagger. \tag{2.2}$$

Multiplying (2.2) on the right by $AA^\dagger$, one has

$$(A^\dagger)^H AA^H A^\# = (A^\dagger)^H AA^H A^\# AA^\dagger. \tag{2.3}$$

Multiplying (2.3) on the left by $(A^\dagger)^H A^\# A^H$, one yields

$$A^\# = A^\# A A^\dagger.$$

Hence, by [5, Theorem 1.2.1], A is EP, which induces $A^\# A = AA^\dagger$. Now multiplying (2.1) on the left by $(A^\dagger)^H A^\# A^H$, one obtains

$$A^\# = A^\#(A^\dagger)^H AA^H A^\#$$

and

$$A = AA^\# A = A(A^\# A^\dagger)^H AA^H A^\#)A = (A^\dagger)^H AA^H.$$

It follows from Lemma 2.3 that

$$A^\dagger = ((A^\dagger)^H AA^H)^\dagger = (A^\dagger)^H A^\# A^H.$$

By Lemma 2.1, A is normal. □

Example 2.1. Let

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & i \end{pmatrix},$$

then A is normal and $A^\dagger = A^\# = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -i \end{pmatrix}$. Hence

$$(A^\dagger)^H AA^H = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & i \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & i \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -i \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & i \end{pmatrix} = A$$

and

$$(A^\#)^H AA^H A^\dagger = (A^\dagger)^H AA^H A^\# = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & i \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -i \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a projection.

Theorem 2.2. *Let $A \in G_n(\mathbb{C})$. Then the following statements are equivalent:*

(1) *A is normal;*

(2) *$(A^\dagger)^H A^\# A^H A$ is a projection;*

(3) *$A^\dagger (A^\dagger)^H AA^H$ is a projection.*

Proof. (1)⇒(2). Assume that A is normal. Then $(A^\dagger)^H A^\# A^H = A^\dagger$ by Lemma 2.1. It follows that $(A^\dagger)^H A^\# A^H A = A^\dagger A$ is a projection.

(2)⇒(3). The projectivity of $(A^\dagger)^H A^\# A^H A$ implies

$$(A^\dagger)^H A^\# A^H A = (A^\dagger)^H A^\# A^H A((A^\dagger)^H A^\# A^H A)^H = (A^\dagger)^H A^\# A^H AA^H A(A^\#)^H A^\dagger.$$

Multiplying the equality on the left by $(A^\dagger)^H AA^H$, one gets

$$A = AA^H A(A^\#)^H A^\dagger.$$

This give

$$A^2A^\dagger = AA^HA(A^\#)^HA^\dagger AA^\dagger = AA^HA(A^\#)^HA^\dagger = A.$$

Hence A is EP [5, Theorem 1.2.1], which induces

$$A^\dagger A = A^\dagger AA^HA(A^\#)^HA^\dagger = A^HA(A^\#)^HA^\dagger.$$

Applying the involution, one has

$$A^\dagger A = (A^\dagger)^HA^\#A^HA$$

and

$$A^\dagger = A^\dagger AA^\dagger = (A^\dagger)^HA^\#A^HAA^\dagger = (A^\dagger)^HA^\#A^H.$$

Therefore,

$$A^\dagger(A^\dagger)^HAA^H = ((A^\dagger)^HA^\#A^H)(A^\dagger)^HAA^H = (A^\dagger)^HA^\#A^\dagger A^2A^H = (A^\dagger)^HA^\#AA^H = AA^\dagger$$

is a projection.

(3)⇒(1). Using the projectivity of $A^\dagger(A^\dagger)^HAA^H$, one obtains

$$A^\dagger(A^\dagger)^HAA^H = (A^\dagger(A^\dagger)^HAA^H)^H(A^\dagger(A^\dagger)^HAA^H) = AA^HA^\dagger(A^\dagger)^HA^\dagger(A^\dagger)^HAA^H.$$

Multiplying the equality on the right by $(A^\dagger)^HA^\#A^H$, one gets

$$A^\dagger = AA^HA^\dagger(A^\dagger)^HA^\dagger$$

and

$$A^H = A^\dagger AA^H = AA^HA^\dagger(A^\dagger)^HA^\dagger AA^H = AA^HA^\dagger.$$

Hence A is normal by [5, Theorem 1.3.2]. □

Example 2.2. Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then A is not normal; A is group invertible with $A^\# = A$ and

$$A^\dagger = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Hence both

$$(A^\dagger)^HA^\#A^HA = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$A^\dagger(A^\dagger)^HAA^H = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

are not projections.

From Lemma 2.2, we know that A is a projection matrix if and only if A^H is a projection matrix. Hence, we can obtain the following corollary.

Corollary 2.1. *Let $A \in G_n(\mathbb{C})$. Then the following statements are equivalent:*
(1) *A is normal;*
(2) *$A^H A (A^{\#})^H A^{\dagger}$ is a projection;*
(3) *$AA^H A^{\dagger}(A^{\dagger})^H$ is a projection.*

3. The representation form of the inverses of matrices and normal matrices

According to [5, Lemmas 1.3.2 and 1.3.3], A is normal if and only if A is EP and $A^{\dagger}A^H = A^H A^{\dagger}$. Using the representation of the Moore-Penrose inverse of the product of generalized inverse elements, we can get the following conclusion.

Theorem 3.1. *Let $A \in G_n(\mathbb{C})$. Then A is a normal matrix if and only if*

$$((A^{\#})^H A^{\#} A^H)^{\dagger} = (A^{\dagger})^H A^2 A^H A^{\dagger}.$$

Proof. Necessity. Suppose that A is normal. Then A is EP and $A^{\dagger}A^H = A^H A^{\dagger}$. By Lemma 2.3,

$$((A^{\#})^H A^{\#} A^H)^{\dagger} = (A^{\dagger})^H A^2 A^{\dagger} A^H A^{\dagger} A = (A^{\dagger})^H A^2 A^H A^{\dagger} A^{\dagger} A = (A^{\dagger})^H A^2 A^H A^{\dagger}.$$

Sufficiency. Using the hypothesis and Lemma 2.3, one yields

$$(A^{\dagger})^H A^2 A^H A^{\dagger} = (A^{\dagger})^H A^2 A^{\dagger} A^H A^{\dagger} A.$$

Multiplying the equality on the left by $A^{\dagger} A^{\#} A^H$, one obtains

$$A^H A^{\dagger} = A^{\dagger} A^H A^{\dagger} A = (A^{\dagger} A^H A^{\dagger} A) A^{\dagger} A = A^H A^{\dagger} A^{\dagger} A.$$

It follows that

$$A^{\dagger} = (A^{\#})^H A^H A^{\dagger} = (A^{\#})^H A^H A^{\dagger} A^{\dagger} A = A^{\dagger} A^{\dagger} A.$$

Hence, A is EP, which infers

$$A^H A^{\dagger} = A^{\dagger} A^H A^{\dagger} A = A^{\dagger} A^H.$$

Thus, A is normal. □

Corollary 3.1. *Let $A \in G_n(\mathbb{C})$. Then A is a normal matrix if and only if*

$$((A^{\#})^H A^{\#} A^H)^{\#} = (A^{\dagger})^H A^2 A^H A^{\dagger}.$$

Proof. Necessity. Assume that A is normal. Then A is EP and $A^{\dagger}A^H = A^H A^{\dagger}$. By Lemma 2.1, $(A^{\dagger})^H A^{\#} A^H = A^{\dagger}$. Then

$$((A^{\#})^H A^{\#} A^H)^{\#} = ((A^{\dagger})^H A^{\#} A^H)^{\#} = (A^{\dagger})^{\#} = (A^{\#})^{\#} = A.$$

Hence, by Theorem 2.4,

$$((A^{\#})^H A^{\#} A^H)^{\#} = (A^{\dagger})^H A A^H = (A^{\dagger})^H A^2 A^{\dagger} A^H = (A^{\dagger})^H A^2 A^H A^{\dagger}.$$

Sufficiency. From the assumption and Lemma 2.3, one yields

$$(A^{\dagger})^H A^2 A^H A^{\dagger} = (A^{\#})^H A^{\dagger} A^3 A^{\dagger} A^H = (A^{\dagger} A (A^{\#})^H) A^{\dagger} A^3 A^{\dagger} A^H = A^{\dagger} A (A^{\dagger})^H A^2 A^H A^{\dagger}.$$

By [14, Lemma 2.11], one gets

$$(A^{\dagger})^H A^2 A^H = A^{\dagger} A (A^{\dagger})^H A^2 A^H.$$

Multiplying the equality on the right by $(A^{\dagger})^H A^{\#} A^{\#}$, one has

$$(A^{\dagger})^H = A^{\dagger} A (A^{\dagger})^H.$$

So $A^{\dagger} = A^{\dagger} A^{\dagger} A$, this implies A is EP. It follows from Lemma 2.3 that

$$(A^{\dagger})^H A^2 A^H A^{\dagger} = ((A^{\#})^H A^{\#} A^H)^{\#} = (A^{\#})^H A^{\dagger} A^3 A^{\dagger} A^H$$

$$= (A^{\dagger})^H A A^H = ((A^{\dagger})^H A^{\#} A^H)^{\dagger} = ((A^{\#})^H A^{\#} A^H)^{\dagger}.$$

By Theorem 2.4, A is normal. □

It is well known that for a group invertible matrix A, $A + E_n - AA^{\#}$ is invertible and

$$(A + E_n - AA^{\#})^{-1} = A^{\#} + E_n - AA^{\#}.$$

In [7, Theorem 4.2], it is proved that a group invertible matrix A is normal if and only if

$$(AA^H (A^{\#})^H + E_n - AA^{\dagger})^{-1} = AA^{\dagger} A^{\dagger} + E_n - AA^{\dagger}.$$

In the following, we shall characterize normal matrices by using $(AA^{\#})^H$ to construct invertible matrices. Noting that $((A^{\#})^H A^{\#} A^H)((A^{\#})^H A^{\#} A^H)^{\#} = AA^{\#}$ by Lemma 2.3, we can get

Theorem 3.2. *Let $A \in G_n(\mathbb{C})$. Then A is normal if and only if $(A^{\#})^H A^{\#} A^H + E_n - (AA^{\#})^H$ is invertible with*

$$((A^{\#})^H A^{\#} A^H + E_n - (AA^{\#})^H)^{-1} = (A^{\dagger})^H A^2 A^H A^{\dagger} + E_n - (AA^{\#})^H.$$

Corollary 3.2. *Let $A \in G_n(\mathbb{C})$. Then A is normal if and only if*

$$((A^{\#})^H A^{\#} A^H + E_n - AA^{\dagger})^{-1} = (A^{\dagger})^H A^2 A^H A^{\dagger} + E_n - AA^{\dagger}.$$

Proof. Necessity. Since A is normal, $(AA^{\#})^H = AA^{\#} = AA^{\dagger}$. By Theorem 3.3, we are done.

Sufficiency. Using the assumption, one has

$$\begin{aligned} E_n &= ((A^{\#})^H A^{\#} A^H + E_n - AA^{\dagger})((A^{\dagger})^H A^2 A^H A^{\dagger} + E_n - AA^{\dagger}) \\ &= (A^{\#})^H A^{\#} A^H (A^{\dagger})^H A^2 A^H A^{\dagger} + E_n - AA^{\dagger}. \end{aligned}$$

This gives

$$(A^{\#})^H A A^H A^{\dagger} = AA^{\dagger}.$$

By Theorem 3.4, A is normal. □

Corollary 3.3. *Let $A \in G_n(\mathbb{C})$. Then A is normal if and only if*

$$((A^{\#})^H A^{\#} A^H + E_n - A^{\dagger}A)^{-1} = (A^{\dagger})^H A^2 A^H A^{\dagger} + E_n - A^{\dagger}A.$$

Proof. Necessity. It follows from Corollary 3.4 and the fact that $AA^{\dagger} = A^{\dagger}A$.

Sufficiency. Under the assumption, one obtains

$$\begin{aligned} E_n &= ((A^{\dagger})^H A^2 A^H A^{\dagger} + E_n - AA^{\dagger})((A^{\#})^H A^{\#} A^H + E_n - AA^{\dagger}) \\ &= (A^{\dagger})^H A^2 A^H A^{\dagger} (A^{\#})^H A^{\#} A^H + E_n - A^{\dagger}A. \end{aligned}$$

This induces

$$A^{\dagger}A = (A^{\dagger})^H A^2 A^H A^{\dagger} (A^{\#})^H A^{\#} A^H = (A^{\dagger})^H A^2 A^H A^{\dagger} (A^{\#})^H A^{\#} A^H AA^{\dagger} = A^{\dagger}AAA^{\dagger}.$$

Hence A is EP, which infers $A^{\dagger}A = AA^{\dagger}$. By Corollary 3.4, A is normal. □

In [8, Theorem 4.3], it is shown that a group invertible matrix A is normal if and only if

$$(A^{\#} A^H X)^{\#} = X^{\dagger} A (A^{\dagger})^H$$

for some $X \in \{A^{\dagger}, A^H, (A^{\#})^H\}$. This inspired us to consider the Moore−Penrose inverse; we have the following theorem.

Theorem 3.3. *Let $A \in G_n(\mathbb{C})$. Then A is a normal matrix if and only if*

$$((A^{\#})^H A^{\#} X)^{\dagger} = X^{\dagger} A^2 A^H A^{\dagger}$$

for some $X \in \{A, A^{\#}, A^{\dagger}, A^H, (A^{\#})^H, (A^{\dagger})^H\}$.

Proof. Necessity. It is an immediate corollary of Theorem 3.1.

Sufficiency. (1) If $X = A$, then $((A^{\#})^H A^{\#} A)^{\dagger} = A^{\dagger} A^2 A^H A^{\dagger}$.

Noting that $((A^{\#})^H A^{\#} A)^{\dagger} = A^{\dagger} A^2 A^{\dagger} A^H A^{\dagger} A$. Then one obtains

$$A^{\dagger} A^2 A^H A^{\dagger} = A^{\dagger} A^2 A^{\dagger} A^H A^{\dagger} A.$$

Multiplying the equality on the left by $(A^{\dagger})^H A^{\#}$, one yields

$$AA^{\dagger}A^{\dagger} = (A^{\dagger})^H A^{\dagger} A^H A^{\dagger} A = ((A^{\dagger})^H A^{\dagger} A^H A^{\dagger} A) A^{\dagger} A = AA^{\dagger} A^{\dagger} A^{\dagger} A$$

and so

$$A^{\dagger}A^{\dagger} = A^{\dagger}A^{\dagger}A^{\dagger}A.$$

By [14, Corollary 2.10], $A^{\dagger} = A^{\dagger}A^{\dagger}A$. Hence A is EP, which gives

$$A^{\dagger} = AA^{\dagger}A^{\dagger} = (A^{\dagger})^H A^{\dagger} A^H A^{\dagger} A = (A^{\dagger})^H A^{\dagger} A^H$$

and

$$A^H A^{\dagger} = A^H (A^{\dagger})^H A^{\dagger} A^H = A^{\dagger} A^H.$$

Hence A is normal.

(2) If $X = A^{\#}$, then
$$((A^{\#})^H A^{\#} A^{\#})^{\dagger} = (A^{\#})^{\dagger} A^2 A^H A^{\dagger} = A^{\dagger} A^4 A^H A^{\dagger}.$$
Since $((A^{\#})^H A^{\#} A^{\#})^{\dagger} = A^{\dagger} A^4 A^{\dagger} A^H A^{\dagger} A$, one obtains
$$A^{\dagger} A^4 A^H A^{\dagger} = A^{\dagger} A^4 A^{\dagger} A^H A^{\dagger} A.$$
Multiplying the equality on the left by $(A^{\dagger})^H (A^{\#})^3$, one gets
$$AA^{\dagger} A^{\dagger} = (A^{\dagger})^H A^{\dagger} A^H A^{\dagger} A.$$
By the proof of (1), one obtains that A is normal.

(3) If $X = A^{\dagger}$, then
$$A^3 A^{\dagger} A^H A^{\dagger} A = ((A^{\#})^H A^{\#} A^{\dagger})^{\dagger} = (A^{\dagger})^{\dagger} A^2 A^H A^{\dagger} = A^3 A^H A^{\dagger}.$$
Multiplying the equality on the left by $A^{\dagger} A^{\#}$, one has
$$A^{\dagger} A^2 A^{\dagger} A^H A^{\dagger} A = A^{\dagger} A^2 A^H A.$$
By the proof of (1), A is normal.

(4) If $X = A^H$, then A is normal by Theorem 3.1.

(5) If $X = (A^{\dagger})^H$, then
$$A^H A^2 A^{\dagger} A^H A^{\dagger} A = ((A^{\#})^H A^{\#} (A^{\dagger})^H)^{\dagger} = ((A^{\dagger})^H)^{\dagger} A^2 A^H A^{\dagger} = A^H A^2 A^H A^{\dagger}.$$
Multiplying the equality on the left by $A^{\dagger}(A^{\dagger})^H$, one obtains
$$A^{\dagger} A^2 A^{\dagger} A^H A^{\dagger} A = A^{\dagger} A^2 A^H A^{\dagger}.$$
By (1), A is normal.

(6) If $X = (A^{\#})^H$, then
$$AA^{\dagger} A^H A^{\dagger} A^3 A^{\dagger} A^H A^{\dagger} A = ((A^{\#})^H A^{\#} (A^{\#})^H)^{\dagger} = ((A^{\#})^H)^{\dagger} A^2 A^H A^{\dagger} = AA^{\dagger} A^H A^{\dagger} A^3 A^H A^{\dagger}.$$
Multiplying the equality on the left by $A^{\dagger} A A^{\#} (A^{\#})^H$, one has
$$A^{\dagger} A^2 A^{\dagger} A^H A^{\dagger} A = A^{\dagger} A^2 A^H A^{\dagger}.$$
Hence A is normal by (1). □

4. Using one-sided equalities to characterize normal matrices

From Lemma 2.2, one knows that $A \in \mathbb{C}^{n\times n}$ is a projection if and only if $A^{\dagger}A$ and $(A^{\dagger})^H$ are right A-equality.

From Lemma 2.1 and Theorem 2.4, we have

Theorem 4.1. *Let $A \in G_n(\mathbb{C})$. Then the following statements are equivalent:*
(1) A is normal;
(2) $(A^\dagger)^H A^\#,\ A^\dagger(A^\dagger)^H$ are right A^H-equality;
(3) $A^\dagger(A^\dagger)^H AA^H,\ A^\dagger A$ are left A-equality.

Lemma 4.1. *Let B and C be right A-equality. Then B^2 and BC are right A-equality.*

Proof. It is clear. □

Lemma 4.2. *Let $A \in G_n(\mathbb{C})$. Then A is normal if and only if $(A^\dagger)^H A^\# = A^\dagger(A^\dagger)^H$.*

Proof. Necessity. Suppose that A is normal. Then $(A^\dagger)^H A^\# A^H = A^\dagger$ by Lemma 2.1, it follows that $(A^\dagger)^H A^\# = (A^\dagger)^H A^\# A^\dagger A = (A^\dagger)^H A^\# A^H (A^\dagger)^H = A^\dagger(A^\dagger)^H$.

Sufficiency. Applying the condition "$(A^\dagger)^H A^\# = A^\dagger(A^\dagger)^H$", one obtains

$$(A^\dagger)^H A^\# A^H = A^\dagger(A^\dagger)^H A^H = A^\dagger.$$

By Lemma 2.1, A is normal. □

Theorem 4.2. *Let $A \in G_n(\mathbb{C})$. Then A is normal if and only if both $((A^\dagger)^H A^\#)^2$ and $A^\dagger(A^\dagger)^H(A^\dagger)^H A^\#$ are right A^H-equality.*

Proof. Necessity. Under the assumption, $(A^\dagger)^H A^\#,\ A^\dagger(A^\dagger)^H$ are right A^H-equality by Theorem 4.1, and at once, $((A^\dagger)^H A^\#)^2, (A^\dagger)^H A^\# A^\dagger (A^\dagger)^H$ are right A^H-equality by Lemma 4.2. Hence $((A^\dagger)^H A^\#)^2,\ A^\dagger(A^\dagger)^H(A^\dagger)^H A^\#$ are right A^H-equality by Lemma 4.3.

Sufficiency. From the hypothesis, one gets

$$(A^\dagger)^H A^\# (A^\dagger)^H A^\# A^H = A^\dagger(A^\dagger)^H(A^\dagger)^H A^\# A^H. \tag{4.1}$$

Multiplying (4.1) on the right by $(A^\dagger)^H AA^H A$, one obtains

$$(A^\dagger)^H = A^\dagger(A^\dagger)^H A$$

and

$$(A^\dagger)^H A^\# = A^\dagger(A^\dagger)^H AA^\# = A^\dagger(A^\dagger)^H.$$

By Lemma 4.3, A is normal. □

Observing the formula (4.1), we have the following result.

Corollary 4.1. *Let $A \in G_n(\mathbb{C})$. Then A is normal if and only if both $(A^\dagger)^H A^\#$ and $A^\dagger(A^\dagger)^H$ are right $(A^\dagger)^H A^\# A^H$-equality.*

Noting that

$$A^\# A^\dagger A = A^\#;\ (A^\dagger)^H A^+ A = (A^\dagger)^H;\ AA^\#(A^\dagger)^H = (A^\dagger)^H;\ AA^\dagger(A^\dagger)^H = (A^\dagger)^H.$$

Then Corollary 4.5 induces

Corollary 4.2. *Let $A \in G_n(\mathbb{C})$. Then the following statements are equivalent:*
(1) *A is normal;*
(2) *$(A^\dagger)^H A^\# A^+$ and $A^\dagger(A^\dagger)^H A^+$ are right $A(A^\dagger)^H A^\# A^H$-equality;*
(3) *$(A^\dagger)^H A^\# A$ and $A^\dagger(A^\dagger)^H A$ are right $A^\#(A^\dagger)^H A^\# A^H$-equality;*
(4) *$(A^\dagger)^H A^\# A^\#$ and $A^\dagger(A^\dagger)^H A^\#$ are right $A(A^\dagger)^H A^\# A^H$-equality.*

5. Using one-sided idempotents to characterize normal matrices

Theorem 5.1. *Let $A \in G_n(\mathbb{C})$. Then A is normal if and only if $(A^\dagger)^H AA^H$ is right A-idempotent.*

Proof. Necessity. Since A is normal, $(A^\dagger)^H AA^H = A$ by Theorem 2.4. Hence $(A^\dagger)^H AA^H$ is right A-idempotent.

Sufficiency. Applying the assumption, one yields

$$(A^\dagger)^H AA^H (A^\dagger)^H AA^H = (A^\dagger)^H AA^H A.$$

Multiplying the equality on the left by $A^\dagger AA^\# A^H$, one has

$$A^H (A^\dagger)^H AA^H = A^H A,$$

i.e., $A^\dagger A^2 A^H = A^H A$. It follows that

$$A = (A^\dagger)^H A^H A = (A^\dagger)^H A^\dagger A^2 A^H = (A^\dagger)^H AA^H.$$

Hence, A is normal by Theorem 2.4. □

Theorem 5.2. *Let $A \in G_n(\mathbb{C})$. Then A is normal if and only if $(A^\#)^H AA^H$ is left A-idempotent.*

Proof. Necessity. Assume that A is normal. Then $A^\dagger = A^\#$ and, by Theorem 2.4, $(A^\dagger)^H AA^H = A$. It follows that $(A^\#)^H AA^H = A$. Hence $(A^\#)^H AA^H$ is left A-idempotent.

Sufficiency. Applying the condition, one obtains

$$(A^\#)^H AA^H (A^\#)^H AA^H = A(A^\#)^H AA^H.$$

Multiplying the equality on the right by $(A^\dagger)^H A^\dagger A^H A^\dagger A$, one obtains

$$(A^\#)^H AA^H A^\dagger A = A$$

and

$$A^\dagger A^2 = A^\dagger A(A^\#)^H AA^H A^\dagger A = (A^\#)^H AA^H A^\dagger A = A.$$

Hence A is EP, which leads to $A = (A^\#)^H AA^H A^\dagger A = (A^\dagger)^H AA^H$. By Theorem 2.4, A is normal. □

Corollary 5.1. *Let $A \in G_n(\mathbb{C})$. Then A is normal if and only if $A - (A^\#)^H AA^H$ is right A-idempotent.*

Proof. It follows from Theorem 5.2 and the fact: C is left D-idempotent if and only if $D - C$ is right D-idempotent. □

It is easy to see that C is right D-idempotent if and only if C, D are left C-equality. Hence Theorem 5.1 induces the following results.

Corollary 5.2. *Let $A \in G_n(\mathbb{C})$. Then A is normal if and only if both $(A^\dagger)^H AA^H$ and A are left $(A^\dagger)^H AA^H$-equality.*

Corollary 5.3. *Let $A \in G_n(\mathbb{C})$. Then A is normal if and only if both $(A^\dagger)^H AA^H + E_n - AA^\dagger$ and $A + E_n - AA^\dagger$ are left $(A^\dagger)^H AA^H$-equality.*

Theorem 5.3. *Let $A \in G_n(\mathbb{C})$. Then A is normal if and only if $(AA^\#)^H A$ is left $(A^\dagger)^H AA^H$-idempotent.*

Proof. Necessity. Assume that A is normal. Then $(A^\dagger)^H AA^H = A$ by Theorem 2.4 and

$$(AA^\#)^H A = (A^\#)^H A^H A = (A^\#)^H AA^H.$$

By Theorem 5.2, one obtains $(AA^\#)^H A$ is left $(A^\dagger)^H AA^H$-idempotent.

Sufficiency. Applying the condition, one obtains

$$(AA^\#)^H A(AA^\#)^H A = (A^\dagger)^H AA^H (AA^\#)^H A.$$

Multiplying the equality on the right by $A^\dagger A^\dagger$, one obtains

$$(AA^\#)^H = (A^\dagger)^H AA^H A^\dagger = AA^\dagger (A^\dagger)^H AA^H A^\dagger = AA^\dagger (AA^\#)^H = AA^\dagger.$$

Hence, A is EP, which leads to

$$A^H = A^H (AA^\#)^H = A^H ((A^\dagger)^H AA^H A^\dagger) = A^+ A^2 A^H A^\dagger = AA^H A^\dagger.$$

Therefore, A is normal by [5, Theorem 1.3.2]. □

6. Conclusions

In this paper, we have given many new characterizations of the normality for the group invertible matrices. These characterizations concern the projection, the Moore-Penrose inverse, one-sided X-equality and X-idempotency of matrices. To our knowledge, it is the first time that the projection, one-sided X-equality, and one-sided X-idempotency are used to characterize the normality of matrices. We shall consider the similar characterizations for the normality in C^*-algebra or a ring with involution. Moreover, we shall investigate some interesting applications of our results, for instance in some fields such as degenerate polynomial and stochastic equations [15, 16].

Author contributions

Zhirong Guo: Writing-original draft, review and editing; Qianglian Huang: Supervision, Writing-review and editing, Funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research is supported by the National Natural Science Foundation of China (12471133, 11771378).

Conflict of interest

The authors declare that there are no conflicts of interest.

References

1. A. Ben-Israel, T. Greville, *Generalized inverses: theory and applications*, Springer Science & Business Media, 2003.
2. O. Baksalary, G. Trenkler, Characterizations of EP, normal and Hermitian matrices, *Linear Multilinear A.*, **56** (2008), 299–304. https://doi.org/10.1080/03081080600872616
3. E. Boasso, On the Moore-Penrose inverse in C^*-algebras, *Extracta Math.*, **21** (2006), 93–106. https://doi.org/10.48550/arXiv.1308.3429
4. R. Bru, N. Thome, Group inverse and group involutory matrices, *Linear Multilinear A.*, **45** (1998), 207–218. https://doi.org/10.1080/03081089808818587
5. D. Mosić, Generalized inverses, *Faculty of Sciences and Mathematics*, University of Niš, 2018.
6. W. Chen, On EP elements, normal elements and paritial isometries in rings with involution, *Electron. J. Linear Al.*, **23** (2012), 553–561. https://doi.org/10.13001/1081-3810.1540
7. C. Peng, H. Zhou, J. Wei, Some new characterizations of normal matrices, *Filomat*, **38** (2024), 393–404. https://doi.org/10.2298/FIL2402393P
8. Y. Tao, X. Ji, J. Wei, Equation characterizations of normal matrices, *Georgian Math. J.*, 2025, accepted.
9. Y. Qu, J. Wei, H. Yao, Characterizations of normal elements in rings with involution, *Acta. Math. Hungar.*, **156** (2018), 459–464. https://doi.org/10.1007/s10474-018-0874-z
10. D. Mosić, D. Djordjević, New characterizations of EP, generalized normal and generalized Hermitian elements in rings. *Appl. Math. Comput.*, **218** (2012), 6702–6710. https://doi.org/10.1016/j.amc.2011.12.030
11. L. Shi, J. Wei, Some new characterizations of normal elements, *Filomat*, **33** (2019), 4115–4120. https://doi.org/10.2298/FIL1913115S
12. Y. Qu, S. Fan, J. Wei, Projections, one-sided idempotents and SEP elements in a ring with involution, *Georgian Math. J.*, 2025. https://doi.org/10.1515/gmj-2024-2080
13. B. Gadelseeda, J. Wei, One sided x-projection, one sided x-idempotent and strongly EP elements in a $*$-ring, *Filomat.*, **39** (2025), 1539–1550. http://doi.org/10.2298/FIL2505539G
14. D. Zhao, J. Wei, Strongly EP elements in rings with involution, *J. Algebra Appl.*, **21** (2022), 2250088. https://doi.org/10.1142/S0219498822500888
15. W. Ramirez, C. Cesarano, Some new classes of degenerated generalized Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials, *Carpathian Math. Publ.*, **14** (2022), 354–363. https://doi.org/10.15330/cmp.14.2.354-363

16. F. Al-Askar, C. Cesarano, W. Mohammed, Multiplicative Brownian motion stabilizes the exact stochastic solutions of the Davey-Stewartson equations, *Symmetry*, **14** (2022), 2176. https://doi.org/10.3390/sym14102176

AIMS Press