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Research article

Decay properties for the Cauchy problem of the linear JMGT-viscoelastic plate with Cattaneo heat conduction

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Abstract: In this work, we investigate the Cauchy problem for the JMGT-viscoelastic plate system coupled with Cattaneo-type heat conduction. Our focus is on deriving optimal decay rate results for both the subcritical and critical regimes. Specifically, we improve upon the results in [Commun. Pure Appl. Anal. 2023] by showing that the decay behavior exhibits no regularity loss in the subcritical case. In contrast, a regularity-loss phenomenon arises in the critical case. Furthermore, we perform an asymptotic analysis of the eigenvalues to confirm the optimality of the decay rates in both scenarios.

Keywords: Decay estimate; JMGT equation; viscoelasticity; Cattaneo heat conduction; regularity-loss type
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1. Introduction

We consider the following Cauchy problem for the JMGT-thermoviscoelastic plate with Cattaneo-type heat conduction:

$$\begin{cases} \tau\rho u_{ttt} + \rho u_{tt} = -k^*\Delta^2 u - k\Delta^2 u_t - m\Delta\theta, & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \\ \theta_t + \kappa\nabla\cdot q - m\tau\Delta u_{tt} - m\Delta u_t = 0 & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \\ \tau_0 q_t + q + \kappa\nabla\theta = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \end{cases} \tag{1.1}$$

with initial data

$$(u, u_t, u_{tt}, \theta, q)(x, 0) = (u_0, u_1, u_2, \theta_0, q_0)(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $\tau, \rho, k^*, k, m, \tau_0, \kappa$ are positive constants, and the critical parameter is given by $K := k - \tau k^*$.

The linear Jordan-Moore-Gibson-Thompson equation (JMGT) is expressed as follows:

$$\tau u_{ttt} + \delta u_{tt} + \beta A u_t + \gamma A u = 0, \tag{1.3}$$

where A is a strictly positive operator in a Hilbert space, and $\tau, \delta, \beta, \gamma$ are positive constants.

Equation (1.3) originally arises as a model for wave propagation in viscous, thermally relaxing fluids (cf. [5, 6]). A similar form of the equation appears in the standard linear solid model (cf. [4]) and in the formulation of a relaxation parameter within the Green–Naghdi type III theory (cf. [3, 10]), particularly when $A = -\Delta$. Furthermore, Eq (1.3) serves as a potential model for vertical displacements in viscoelastic plates (cf. [7]) when $A = \Delta^2$.

In recent years, there has been growing interest in the study of problem (1.3). In [2], the authors investigated the MGT-viscoelastic plate coupled with the Fourier law and type III heat conduction, proving that the corresponding semigroups are analytic in the subcritical case $K > 0$. Subsequently, [1] focused on the MGT-viscoelastic plate with Cattaneo heat conduction and established the following decay result in the subcritical case $K > 0$:

- the subcritical case $K > 0$:

$$\|\nabla^p U(t)\|^2_{L^2(\mathbb{R}^n)} \le C(1+t)^{-\frac{n+2p}{4}}\|U_0\|^2_{L^1(\mathbb{R}^n)} + C(1+t)^{-l}\|\nabla^{p+l}U_0\|^2_{L^2(\mathbb{R}^n)},$$

  where $U = (u_t + \tau u_{tt}, \Delta u_t, \Delta(u + \tau u_t), \theta, q)^T$ and $U_0 = U(x, 0)$.

In this work, we improve upon the results in [1] for the subcritical case $K > 0$ and establish the decay result for the critical case $K = 0$. Additionally, we analyze the eigenvalues to demonstrate the optimality of the decay results in both cases. The specific decay rates are as follows:

- the subcritical case $K > 0$:

$$\|\nabla^p W(t)\|^2_{L^2(\mathbb{R}^n)} \le C(1+t)^{-\frac{n}{2}-p}\|W_0\|^2_{L^1(\mathbb{R}^n)} + Ce^{-Ct}\|\nabla^p W_0\|^2_{L^2(\mathbb{R}^n)},$$

  where $W = (u_t + \tau u_{tt}, \Delta u_t, \Delta(u + \tau u_t), \theta, q)^T$ and $W_0 = W(x, 0)$.
- the critical case $K = 0$:

$$\|\nabla^p Z(t)\|^2_{L^2(\mathbb{R}^n)} \le C(1+t)^{-\frac{n}{2}-p}\|Z_0\|^2_{L^1(\mathbb{R}^n)} + C(1+t)^{-l}\|\nabla^{p+l}Z_0\|^2_{L^2(\mathbb{R}^n)},$$

  where $Z = (u_t + \tau u_{tt}, \Delta(u + \tau u_t), \theta, q)^T$ and $Z_0 = Z(x, 0)$.

The paper is organized as follows. In Section 2, we introduce some notations and present our main results. Section 3 is devoted to proving the decay estimates for the JMGT-thermoviscoelastic plate with Cattaneo heat conduction. Finally, in Section 4, we establish the optimality of the decay rates obtained.

Before closing this section, we give some notations to be used below. Let the Fourier transform of a function $f = f(x)$ be denoted by $\hat{f} = \hat{f}(\xi)$, defined as

$$\mathcal{F}[f](\xi) \equiv \hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\mathrm{d}x,$$

and let the complex conjugate of $\hat{u}$ be denoted by $\bar{\hat{u}}$.

2. Preliminaries and main results

In this section, we state our main results.

Taking the Fourier transform of system (1.1)-(1.2), we have

$$\begin{cases} \tau\rho\hat{u}_{ttt} + \rho\hat{u}_{tt} + k^*|\xi|^4\hat{u} + k|\xi|^4\hat{u}_t - m|\xi|^2\hat{\theta} = 0, \\ \hat{\theta}_t + \kappa i\xi\cdot\hat{q} + m\tau|\xi|^2\hat{u}_{tt} + m|\xi|^2\hat{u}_t = 0, \\ \tau_0\hat{q}_t + \hat{q} + \kappa i\xi\hat{\theta} = 0 \end{cases} \tag{2.1}$$

with initial data

$$(\hat{u}, \hat{u}_t, \hat{u}_{tt}, \hat{\theta}, \hat{q})(\xi, 0) = (\hat{u}_0, \hat{u}_1, \hat{u}_2, \hat{\theta}_0, \hat{q}_0)(\xi), \tag{2.2}$$

where $\xi \in \mathbb{R}^n$. By the new variables

$$\hat{\varphi} = \hat{u}_t, \quad \hat{w} = \hat{u}_{tt},$$

we obtain

$$\begin{cases} \hat{u}_t - \hat{\varphi} = 0, \\ \hat{\varphi}_t - \hat{w} = 0, \\ \hat{w}_t + \dfrac{1}{\tau}\hat{w} + \dfrac{k^*}{\tau\rho}|\xi|^4\hat{u} + \dfrac{k}{\tau\rho}|\xi|^4\hat{\varphi} - \dfrac{m}{\tau\rho}|\xi|^2\hat{\theta} = 0, \\ \hat{\theta}_t + \kappa i\xi\cdot\hat{q} + m\tau|\xi|^2\hat{w} + m|\xi|^2\hat{\varphi} = 0, \\ \hat{q}_t + \dfrac{1}{\tau_0}\hat{q} + \dfrac{\kappa}{\tau_0}i\xi\hat{\theta} = 0. \end{cases} \tag{2.3}$$

Then, we state the following pointwise estimates and decay results.

Theorem 2.1. *Let*

$$\hat{W} := (\hat{u}_t + \tau\hat{u}_{tt}, \Delta\hat{u}_t, \Delta(\hat{u} + \tau\hat{u}_t), \hat{\theta}, \hat{q})^T,$$

where $(\hat{u}(\xi,t), \hat{\theta}(\xi,t), \hat{q}(\xi,t))$ *is the Fourier image of the solution* $(u(x,t), \theta(x,t), q(x,t))$*. Assume that* $K > 0$*. Then,* $\hat{W}$ *satisfies the following pointwise estimate*

$$|\hat{W}(\xi,t)|^2 \le Ce^{-c\rho_1(\xi)t}|\hat{W}_0(\xi)|^2, \tag{2.4}$$

for any $t \ge 0$*, where* $\rho_1(\xi) := \frac{|\xi|^2}{1+|\xi|^2}$*, and where* $C, c > 0$ *are independent of* t, ξ*, and the initial data.*

Furthermore, let $W = (u_t + \tau u_{tt}, \Delta u_t, \Delta(u + \tau u_t), \theta, q)^T$*, where* $(u(x,t), \theta(x,t), q(x,t))$ *is the solution of problem* (1.1), (1.2)*, and* $W_0 = W(x,0) \in H^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$*, where s is nonnegative. Then, W satisfies the decay estimate*

$$\|\nabla^p W(t)\|^2_{L^2(\mathbb{R}^n)} \le C(1+t)^{-\frac{n}{2}-p}\|W_0\|^2_{L^1(\mathbb{R}^n)} + Ce^{-Ct}\|\nabla^p W_0\|^2_{L^2(\mathbb{R}^n)}, \tag{2.5}$$

for all $0 \le p \le s$*.*

Remark 2.2. *The decay result* (2.5) *does not exhibit the regularity-loss phenomenon. In consideration of [1], the decay estimate presented here aligns with the exponential stability of the MGT-viscoelastic plate with Cattaneo-type heat conduction in a bounded domain. At the same time, we improve the result in unbounded domain obtained in [1]. Noting the asymptotic expansion of the eigenvalues in Section 4, we find that the exponent in pointwise estimate* (2.4) *is optimal. Thus, the decay estimate* (2.5) *is optimal.*

Remark 2.3. *Note that the decay estimate* (2.5) *and the MGT-viscoelastic plate with the Gurtin-Pipkin thermal law in [11] exhibit the same decay rate when $K > 0$, despite the absence of a regularity-loss phenomenon in* (2.5).

Theorem 2.4. *Let*

$$\hat{Z} = (\hat{u}_t + \tau\hat{u}_{tt}, \Delta(\hat{u} + \tau\hat{u}_t), \hat{\theta}, \hat{q})^T,$$

where $(\hat{u}(\xi, t), \hat{\theta}(\xi, t), \hat{q}(\xi, t))$ is the Fourier image of the solution $(u(x, t), \theta(x, t), q(x, t))$. Assume that $K = 0$. Then, $\hat{Z}$ has the following pointwise estimate

$$|\hat{Z}(\xi, t)|^2 \leq Ce^{-c\rho_2(\xi)t}|\hat{Z}_0(\xi)|^2, \tag{2.6}$$

for any $t \geq 0$, where $\rho_2(\xi) := \frac{|\xi|^2}{(1+|\xi|^2)^2}$. Furthermore, let $Z = (u_t + \tau u_{tt}, \Delta(u + \tau u_t), \theta, q)^T$, where $(u(x, t), \theta(x, t), q(x, t))$ is the solution of problem (1.1)-(1.2)*, and $Z_0 = Z(x, 0) \in H^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, where s is nonnegative. Then, Z satisfies the following decay estimate*

$$\|\nabla^p Z(t)\|^2_{L^2(\mathbb{R}^n)} \leq C(1 + t)^{-\frac{n}{2}-p}\|Z_0\|^2_{L^1(\mathbb{R}^n)}+C(1 + t)^{-l}\|\nabla^{p+l} Z_0\|^2_{L^2(\mathbb{R}^n)}, \tag{2.7}$$

for all $0 \leq p + l \leq s$.

Remark 2.5. *According to [11], we find that the MGT-viscoelastic plate with Gurtin-Pipkin thermal law has the same decay result as* (2.7) *under the condition $K = 0$, including both the decay rate and regularity-loss phenomenon.*

3. Decay estimates

In this section, we consider the decay estimates of the norm related to (1.1)-(1.2).

3.1. Decay estimates–the case $K > 0$

In this subsection, we define the energy functional of system (2.3) as

$$\hat{E}(\xi, t) := |\hat{\varphi} + \tau\hat{w}|^2 + \frac{\tau}{\rho}K|\xi|^4|\hat{\varphi}|^2 + \frac{k^*}{\rho}|\xi|^4|\hat{u} + \tau\hat{\varphi}|^2 + \frac{1}{\rho}|\hat{\theta}|^2 + \frac{\tau_0}{\rho}|\hat{q}|^2, \tag{3.1}$$

which is equivalent to $|\hat{W}(\xi, t)|^2$. To derive our main result, we begin by stating and proving several lemmas.

Lemma 3.1. *Let $(\hat{u}, \hat{\varphi}, \hat{w}, \hat{\theta}, \hat{q})$ be the solution of* (2.3)*. Assume that $K > 0$. Then, $\hat{E}(\xi, t)$ satisfies*

$$\frac{d}{dt}\hat{E}(\xi, t) = -\frac{1}{\rho}K|\xi|^4|\hat{\varphi}|^2 - \frac{1}{\rho}|\hat{q}|^2.$$

Lemma 3.2. *The following inequality holds true:*

$$\frac{d}{dt}F_1(t) + \left(\frac{k^*}{\rho} - 2\varepsilon_1\right)|\xi|^4|\hat{u} + \tau\hat{\varphi}|^2 \leq |\hat{\varphi} + \tau\hat{w}|^2 + C(\varepsilon_1)|\xi|^4|\hat{\varphi}|^2 + C(\varepsilon_1)|\hat{\theta}|^2, \tag{3.2}$$

for any $\varepsilon_1 > 0$, where

$$F_1(t) := \mathrm{Re}((\hat{\varphi} + \tau\hat{w})(\bar{\hat{u}} + \tau\bar{\hat{\varphi}})).$$

Proof. We can easily obtain

$$\frac{d}{dt}F_1(t) + \frac{k^*}{\rho}|\xi|^4|\hat{u} + \tau\hat{\varphi}|^2 - |\hat{\varphi} + \tau\hat{w}|^2 = -\frac{1}{\rho}K|\xi|^4\mathrm{Re}(\hat{\varphi}(\bar{\hat{u}} + \tau\bar{\hat{\varphi}})) + \frac{m}{\rho}|\xi|^2\mathrm{Re}(\hat{\theta}(\bar{\hat{u}} + \tau\bar{\hat{\varphi}})). \tag{3.3}$$

By virtue of Young's inequality, for any $\varepsilon_1 > 0$, we have

$$-\frac{1}{\rho}K|\xi|^4\mathrm{Re}(\hat{\varphi}(\bar{\hat{u}} + \tau\bar{\hat{\varphi}})) \le \varepsilon_1|\xi|^4|\hat{u} + \tau\hat{\varphi}|^2 + C(\varepsilon_1)|\xi|^4|\hat{\varphi}|^2, \tag{3.4}$$

$$\frac{m}{\rho}|\xi|^2\mathrm{Re}(\hat{\theta}(\bar{\hat{u}} + \tau\bar{\hat{\varphi}})) \le \varepsilon_1|\xi|^4|\hat{u} + \tau\hat{\varphi}|^2 + C(\varepsilon_1)|\hat{\theta}|^2. \tag{3.5}$$

Combining (3.3)–(3.5), we obtain the desired result (3.2). □

Lemma 3.3. *The functional*

$$F_2(t) := \mathrm{Re}(\hat{\theta}(\bar{\hat{\varphi}} + \tau\bar{\hat{w}}))$$

satisfies

$$\frac{d}{dt}F_2(t) + (m - \varepsilon_2)|\xi|^2|\hat{\varphi} + \tau\hat{w}|^2 \le C(\varepsilon_2)|\hat{q}|^2 + \varepsilon_2'|\xi|^6|\hat{u} + \tau\hat{\varphi}|^2 + \varepsilon_2|\xi|^6|\hat{\varphi}|^2 + C(\varepsilon_2, \varepsilon_2')|\xi|^2|\hat{\theta}|^2, \tag{3.6}$$

for any $\varepsilon_2, \varepsilon_2' > 0$.

Proof. It is easy to obtain

$$\frac{d}{dt}F_2(t) + m|\xi|^2|\hat{\varphi} + \tau\hat{w}|^2 = -\kappa\mathrm{Re}(i\xi\hat{q}(\bar{\hat{\varphi}} + \tau\bar{\hat{w}})) - \frac{k^*}{\rho}|\xi|^4\mathrm{Re}(\hat{u}\bar{\hat{\theta}}) - \frac{k}{\rho}|\xi|^4\mathrm{Re}(\hat{\varphi}\bar{\hat{\theta}}) + \frac{m}{\rho}|\xi|^2|\hat{\theta}|^2. \tag{3.7}$$

Applying Young's inequality with $\varepsilon_2, \varepsilon_2' > 0$, we get

$$-\kappa\mathrm{Re}(i\xi\hat{q}(\bar{\hat{\varphi}} + \tau\bar{\hat{w}})) \le \varepsilon_2|\xi|^2|\hat{\varphi} + \tau\hat{w}|^2 + C(\varepsilon_2)|\hat{q}|^2, \tag{3.8}$$

$$-\frac{k^*}{\rho}|\xi|^4\mathrm{Re}(\hat{u}\bar{\hat{\theta}}) \le \varepsilon_2'|\xi|^6|\hat{u} + \tau\hat{\varphi}|^2 + C(\varepsilon_2')|\xi|^2|\hat{\theta}|^2, \tag{3.9}$$

$$-\frac{k}{\rho}|\xi|^4\mathrm{Re}(\hat{\varphi}\bar{\hat{\theta}}) \le \varepsilon_2|\xi|^6|\hat{\varphi}|^2 + C(\varepsilon_2)|\xi|^2|\hat{\theta}|^2. \tag{3.10}$$

Thanks to (3.7)–(3.10), we deduce (3.6). □

Lemma 3.4. *Define the functional*

$$F_3(t) := \mathrm{Re}\left(i\xi\tau_0\hat{q}\bar{\hat{\theta}} + im\tau\tau_0\xi^3\hat{\varphi}\bar{\hat{q}}\right).$$

Then,

$$\frac{d}{dt}F_3(t) + (k - 2\varepsilon_3)\,|\xi|^2|\hat{\theta}|^2 \le C(\varepsilon_3)(1 + |\xi|^2)|\hat{q}|^2 + C(\varepsilon_3)|\xi|^6|\hat{\varphi}|^2, \tag{3.11}$$

for any $\varepsilon_3 > 0$.

Proof. Multiplying $(2.3)_4$ and $(2.3)_5$ by $i\tau_0\xi\bar{\hat{q}}$ and $(-i\tau_0\xi\bar{\hat{\theta}})$, respectively, adding the resulting equations, and taking the real part, we have

$$\frac{d}{dt}\mathrm{Re}(i\tau_0\xi\hat{q}\bar{\hat{\theta}}) + \kappa|\xi|^2|\hat{\theta}|^2 - \kappa\tau_0|\xi|^2|\hat{q}|^2 = \mathrm{Re}(i\xi\hat{q}\bar{\hat{\theta}}) - m\tau\tau_0\mathrm{Re}(i\xi^3\hat{w}\bar{\hat{q}}) - m\tau_0\mathrm{Re}(i\xi^3\hat{\varphi}\bar{\hat{q}}). \tag{3.12}$$

To eliminate $\mathrm{Re}(i\xi^3\hat{w}\bar{\hat{q}})$, we multiply $(2.3)_2$ and $(2.3)_5$ by $im\tau\tau_0\xi^3\bar{\hat{q}}$ and $-im\tau\tau_0\xi^3\bar{\hat{\varphi}}$, respectively. Then, combining the resulting equations and taking real parts, we have

$$\frac{d}{dt}\mathrm{Re}\left(im\tau\tau_0\xi^3\hat{\varphi}\bar{\hat{q}}\right) = m\tau\tau_0\mathrm{Re}(i\xi^3\hat{w}\bar{\hat{q}}) + m\tau\mathrm{Re}(i\xi^3\hat{q}\bar{\hat{\varphi}}) - \kappa m\tau|\xi|^4\mathrm{Re}(\hat{\theta}\bar{\hat{\varphi}}). \tag{3.13}$$

Summing up (3.12) and (3.13), we arrive at

$$\frac{d}{dt}F_3(t) + \kappa|\xi|^2|\hat{\theta}|^2 - \kappa\tau_0|\xi|^2|\hat{q}|^2 = \mathrm{Re}(i\xi\hat{q}\bar{\hat{\theta}}) + (m\tau + m\tau_0)\mathrm{Re}(i\xi^3\hat{q}\bar{\hat{\varphi}}) - \kappa m\tau|\xi|^4\mathrm{Re}(\hat{\theta}\bar{\hat{\varphi}}). \tag{3.14}$$

Young's inequality yields, for any $\varepsilon_3 > 0$,

$$\mathrm{Re}(i\xi\hat{q}\bar{\hat{\theta}}) \leq \varepsilon_3|\xi|^2|\hat{\theta}|^2 + C(\varepsilon_3)|\hat{q}|^2, \tag{3.15}$$

$$(m\tau + m\tau_0)\mathrm{Re}(i\xi^3\hat{q}\bar{\hat{\varphi}}) \leq \varepsilon_3|\xi|^6|\hat{\varphi}|^2 + C(\varepsilon_3)|\hat{q}|^2, \tag{3.16}$$

$$-\kappa m\tau|\xi|^4\mathrm{Re}(\hat{\theta}\bar{\hat{\varphi}}) \leq \varepsilon_3|\xi|^2|\hat{\theta}|^2 + C(\varepsilon_3)|\xi|^6|\hat{\varphi}|^2. \tag{3.17}$$

Hence, plugging (3.15)–(3.17) into (3.14), we arrive at (3.11). □

We now proceed to prove our main result.

Proof of Theorem 2.1. We define the Lyapunov functional as follows:

$$L_1(\xi, t) := N(1 + |\xi|^2)\hat{E}(\xi, t) + |\xi|^2F_1(t) + N_2F_2(t) + N_3F_3(t),$$

where N, N_2, and N_3 are positive constants to be determined later. By utilizing the previously established lemmas, we obtain

$$\begin{aligned}
&\frac{d}{dt}L_1(\xi, t) + \left[\left(\frac{k^*}{\rho} - 2\varepsilon_1\right) - N_2\varepsilon_2'\right]|\xi|^6|\hat{u} + \tau\hat{\varphi}|^2 \\
&+ \Big[N_2(m - \varepsilon_2) - 1\Big]|\xi|^2|\hat{\varphi} + \tau\hat{w}|^2 + \Big[N_3(\kappa - 2\varepsilon_3) - C(\varepsilon_1) - N_2C(\varepsilon_2, \varepsilon_2')\Big]|\xi|^2|\hat{\theta}|^2 \\
&+ \left[\frac{NK}{\rho}|\xi|^4(1 + |\xi|^2) - C(\varepsilon_1)|\xi|^6 - N_2\varepsilon_2|\xi|^6 - N_3C(\varepsilon_3)|\xi|^6\right]|\hat{\varphi}|^2 \\
&+ \left[\frac{N}{\rho}(1 + |\xi|^2) - N_2C(\varepsilon_2) - N_3C(\varepsilon_3)(1 + |\xi|^2)\right]|\hat{q}|^2 \\
\leq& 0.
\end{aligned} \tag{3.18}$$

At this stage, we aim to determine the constants in Eq (3.18). We begin by selecting

$$\varepsilon_1 < \frac{k^*}{2\rho}, \quad \varepsilon_2 < m, \quad \varepsilon_3 < \frac{\kappa}{2}.$$

Next, we fix $N_2 > \dfrac{1}{m - \varepsilon_2}$ and choose $\varepsilon_2' < \dfrac{k^*}{\rho N_2} - \dfrac{2\varepsilon_1}{N_2}$. Then, we select N_3 such that

$$N_3 > \frac{C(\varepsilon_1) + N_2 C(\varepsilon_2, \varepsilon_2')}{\kappa - 2\varepsilon_3}.$$

Finally, we choose N sufficiently large to satisfy

$$N > \max\left\{\frac{\rho[C(\varepsilon_1) + N_2\varepsilon_2 + N_3 C(\varepsilon_3)]}{K}, \rho[N_2 C(\varepsilon_2) + N_3 C(\varepsilon_3)]\right\}.$$

Consequently, we obtain, with a positive constant C_1,

$$\frac{d}{dt}L_1(\xi, t) + C_1 M_1(t) \leq 0, \tag{3.19}$$

where

$$\begin{aligned} M_1(t) &= |\xi|^6|\hat{u} + \tau\hat{\varphi}|^2 + |\xi|^2|\hat{\varphi} + \tau\hat{w}|^2 + |\xi|^2|\hat{\theta}|^2 + |\xi|^6|\hat{\varphi}|^2 + |\xi|^2|\hat{q}|^2 \\ &= |\xi|^2\hat{E}(\xi, t). \end{aligned}$$

From the definitions of $\hat{E}(\xi, t)$ and $L_1(\xi, t)$, we know that there exist two positive constants C_2 and C_3 such that the following relation holds

$$C_2(1 + |\xi|^2)\hat{E}(\xi, t) \leq L_1(\xi, t) \leq C_3(1 + |\xi|^2)\hat{E}(\xi, t).$$

Thus, Eq (3.19) transforms into

$$\frac{d}{dt}\hat{E}(\xi, t) + C\frac{|\xi|^2}{1 + |\xi|^2}\hat{E}(\xi, t) \leq 0. \tag{3.20}$$

Finally, the estimate in (3.20) leads to the desired result (2.4), allowing us to derive the decay estimate (2.5). The proof of (2.5) is the same as the one of Theorem 3.6 in [8], so we omit it here. □

3.2. Decay estimates–the case $K = 0$

Based on Lemmas 3.1–3.3 and the condition $K = 0$, we have the following conclusion.

Lemma 3.5. *Under the condition $K = 0$, the energy functional* (3.1) *becomes*

$$\hat{\mathcal{E}}(\xi, t) := |\hat{\varphi} + \tau\hat{w}|^2 + \frac{k^*}{\rho}|\xi|^4|\hat{u} + \tau\hat{\varphi}|^2 + \frac{1}{\rho}|\hat{\theta}|^2 + \frac{\tau_0}{\rho}|\hat{q}|^2, \tag{3.21}$$

and then $\hat{\mathcal{E}}(\xi, t)$ satisfies

$$\frac{d}{dt}\hat{\mathcal{E}}(\xi, t) = -\frac{1}{\rho}|\hat{q}|^2 \tag{3.22}$$

and the following inequality holds true:

$$\frac{d}{dt}F_1(t) + \left(\frac{k^*}{\rho} - \epsilon_1\right)|\xi|^4|\hat{u} + \tau\hat{\varphi}|^2 \leq |\hat{\varphi} + \tau\hat{w}|^2 + C(\epsilon_1)|\hat{\theta}|^2, \tag{3.23}$$

$$\frac{d}{dt}F_2(t) + (m - \epsilon_2)|\xi|^2|\hat{\varphi} + \tau\hat{w}|^2 \leq C(\epsilon_2)|\hat{q}|^2 + \epsilon_2'|\xi|^6|\hat{u} + \tau\hat{\varphi}|^2 + C(\epsilon_2')|\xi|^2|\hat{\theta}|^2, \tag{3.24}$$

for any $\epsilon_1 > 0$ and $\epsilon_2, \epsilon_2' > 0$.

Proof. It is straightforward to obtain Eqs (3.22) and (3.23). It follows from (3.7) that

$$\frac{d}{dt}F_2(t) + m|\xi|^2|\hat{\varphi} + \tau\hat{w}|^2 = -\kappa\mathrm{Re}(i\xi\hat{q}(\bar{\hat{\varphi}} + \tau\bar{\hat{w}})) - \frac{k^*}{\rho}|\xi|^4\mathrm{Re}((\hat{u} + \tau\hat{\varphi})\bar{\hat{\theta}}) + \frac{m}{\rho}|\xi|^2|\hat{\theta}|^2.$$

Using Young's inequality, we get

$$\begin{aligned} &-\kappa\mathrm{Re}(i\xi\hat{q}(\bar{\hat{\varphi}} + \tau\bar{\hat{w}})) \le \epsilon_2|\xi|^2|\hat{\varphi} + \tau\hat{w}|^2 + C(\epsilon_2)|\hat{q}|^2, \\ &-\frac{k^*}{\rho}|\xi|^4\mathrm{Re}((\hat{u} + \tau\hat{\varphi})\bar{\hat{\theta}}) \le \epsilon_2'|\xi|^6|\hat{u} + \tau\hat{\varphi}|^2 + C(\epsilon_2')|\xi|^2|\hat{\theta}|^2, \end{aligned}$$

where $\epsilon_2, \epsilon_2' > 0$. Collecting the above estimates, we obtain (3.24). □

Lemma 3.6. *The functional*

$$\bar{F}_3(t) := \mathrm{Re}\left(i\tau_0\xi\hat{q}\bar{\hat{\theta}}\right)$$

satisfies

$$\frac{d}{dt}\bar{F}_3(t) + (\kappa - \epsilon_3)|\xi|^2|\hat{\theta}|^2 \le C(\epsilon_3, \epsilon_3')\left(1 + |\xi|^2 + |\xi|^4\right)|\hat{q}|^2 + \epsilon_3'|\xi|^2|\hat{\varphi} + \tau\hat{w}|^2, \tag{3.25}$$

for any $\epsilon_3, \epsilon_3' > 0$.

Proof. Taking (3.12) into account, we arrive at

$$\frac{d}{dt}\bar{F}_3(t) + \kappa|\xi|^2|\hat{\theta}|^2 - \kappa\tau_0|\xi|^2|\hat{q}|^2 = \mathrm{Re}(i\xi\hat{q}\bar{\hat{\theta}}) - m\tau_0\mathrm{Re}(i\xi^3(\hat{\varphi} + \tau\hat{w})\bar{\hat{q}}). \tag{3.26}$$

Taking advantage of Young's inequality, we obtain (3.25). The proof is complete. □

Proof of Theorem 2.4. We define the new Lyapunov functional $L_2(\xi, t)$ associated to the case $K = 0$ as follows:

$$L_2(\xi, t) := \bar{N}(1 + |\xi|^2)^2\hat{\mathcal{E}}(\xi, t) + |\xi|^2F_1(t) + \bar{N}_2F_2(t) + \bar{N}_3\bar{F}_3(t). \tag{3.27}$$

Taking the derivative of (3.27) with respect to t and making use of the above lemmas, we derive

$$\begin{aligned} &\frac{d}{dt}L_2(\xi, t) + \left[\left(\frac{k^*}{\rho} - \epsilon_1\right) - \bar{N}_2\epsilon_2'\right]|\xi|^6|\hat{u} + \tau\hat{\varphi}|^2 \\ &+ \left[\bar{N}_2(m - \epsilon_2) - 1 - \bar{N}_3\epsilon_3'\right]|\xi|^2|\hat{\varphi} + \tau\hat{w}|^2 \\ &+ \left[\bar{N}_3(\kappa - \epsilon_3) - C(\epsilon_1) - \bar{N}_2C(\epsilon_2')\right]|\xi|^2|\hat{\theta}|^2 \\ &+ \left[\frac{\bar{N}}{\rho}(1 + |\xi|^2)^2 - \bar{N}_2C(\epsilon_2) - \bar{N}_3C(\epsilon_3, \epsilon_3')(1 + |\xi|^2 + |\xi|^4)\right]|\hat{q}|^2 \\ \le& 0. \end{aligned} \tag{3.28}$$

At this point, we choose our constants carefully like before. First, we pick

$$\epsilon_1 < \frac{k^*}{\rho}, \quad \epsilon_2 < m, \quad \epsilon_3 < \kappa.$$

Next, we choose

$$\bar{N}_2 > \frac{1}{m-\epsilon_2} \quad \text{and} \quad \bar{N}_3 > \frac{C(\epsilon_1)+\bar{N}_2 C(\epsilon_2')}{\kappa-\epsilon_3}.$$

Then, we fix ϵ_3' satisfying

$$\epsilon_2' < \frac{(k^*/\rho-\epsilon_1)}{\bar{N}_2}, \quad \epsilon_3' < \frac{\bar{N}_2\,(m-\epsilon_2)-1}{\bar{N}_3}.$$

Finally, we choose N large enough such that

$$\bar{N} > \rho\bar{N}_2 C(\epsilon_1) + \rho\bar{N}_3 C(\epsilon_3, \epsilon_3').$$

Thus, we arrive at

$$\frac{d}{dt}L_2(\xi,t) + C_4 M_2(t) \le 0, \tag{3.29}$$

where

$$\begin{aligned} M_2(t) &= |\xi|^6|\hat{u}+\tau\hat{\varphi}|^2 + |\xi|^2|\hat{\varphi}+\tau\hat{w}|^2 + |\xi|^2|\hat{\theta}|^2 + |\xi|^2|\hat{q}|^2 \\ &= |\xi|^2\hat{\mathcal{E}}(\xi,t). \end{aligned}$$

From the definition of $\hat{\mathcal{E}}(\xi,t)$ and (3.27), it is obviously that $L_2(\xi,t) \sim (1+|\xi|^2)^2\hat{\mathcal{E}}(\xi,t)$. Then,

$$\frac{d}{dt}\hat{\mathcal{E}}(\xi,t) + C\frac{|\xi|^2}{(1+|\xi|^2)^2}\hat{\mathcal{E}}(\xi,t) \le 0. \tag{3.30}$$

Thus, we achieve the desired pointwise estimate (2.6), which leads to the conclusion (2.7). The proof process of (2.7) is similar to the proof of Theorem 3.1 in [9], so we omit it here. □

4. Asymptotic expansion of eigenvalues

From now on, we study the asymptotic expansion of the eigenvalues for $|\xi| \to 0$ and $|\xi| \to \infty$ to show the optimality.

Let $\hat{V} = (\hat{u},\hat{\varphi},\hat{w},\hat{\theta},\hat{q})^T$ and $\hat{V}_0 = (\hat{u}_0,\hat{\varphi}_0,\hat{w}_0,\hat{\theta}_0,\hat{q}_0)^T$. Then, we can rewrite system (2.1)-(2.2) as

$$\begin{cases} \hat{V}_t + i\xi A\hat{V} + |\xi|^2 B\hat{V} + |\xi|^4 D\hat{V} + L\hat{V} = 0, \\ \hat{V}(\xi,0) = \hat{V}_0(\xi), \end{cases} \tag{4.1}$$

where

$$A = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&\kappa\\ 0&0&0&\frac{\kappa}{\tau_0}&0 \end{pmatrix}, \quad L = \begin{pmatrix} 0&-1&0&0&0\\ 0&0&-1&0&0\\ 0&0&\frac{1}{\tau}&0&0\\ 0&0&0&0&0\\ 0&0&0&0&\frac{1}{\tau_0} \end{pmatrix},$$

$$B = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&-\frac{m}{\tau\rho}&0\\ 0&m&m\tau&0&0\\ 0&0&0&0&0 \end{pmatrix}, \quad D = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ \frac{k^*}{\tau\rho}&\frac{k}{\tau\rho}&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}.$$

For (4.1), the solution is given by

$$\hat{V}(\xi,t) = e^{t\hat{\Phi}(i\xi)}\hat{V}_0(\xi),$$

where $e^{t\hat{\Phi}(i\xi)}$ denotes the matrix exponential with

$$\hat{\Phi}(i\xi) = -(L + i\xi A + |\xi|^2 B + |\xi|^4 D).$$

Setting $\zeta = i\xi$, we get

$$\hat{\Phi}(\zeta) = -(L + \zeta A - \zeta^2 B + \zeta^4 D).$$

Let $\lambda_j(\zeta)$ denote the eigenvalues of the matrix $\hat{\Phi}(\zeta)$. By the direct calculation, we find that the characteristic polynomial of $\hat{\Phi}(\zeta)$ is

$$\begin{aligned}&\tau\rho c\det(\lambda I - \hat{\Phi}(\zeta))\\ =&\tau\rho\tau_0\lambda^5 + (\tau_0\rho + \tau\rho)\lambda^4 + [(\tau_0 k + m^2\tau\tau_0)\zeta^4 - \tau\rho\kappa^2\zeta^2 + \rho]\lambda^3\\ &+ [(\tau_0 k^* + m^2\tau + m^2\tau_0 + k)\zeta^4 - \rho\kappa^2\zeta^2]\lambda^2 + [(m^2 + k^*)\zeta^4 - \kappa^2 k\zeta^6]\lambda - \kappa^2 k^*\zeta^6.\end{aligned} \tag{4.2}$$

Lemma 4.1. *The real parts of the eigenvalues of* (2.1)-(2.2) *satisfy the following asymptotic expansion:*

$$\mathrm{Re}\lambda_j(i\xi) = \begin{cases} -\dfrac{1}{\tau} + O(|\xi|^2), & j = 1,\\ -\dfrac{1}{\tau_0} + O(|\xi|^2), & j = 2,\\ -\mathrm{Re}(\phi_j)|\xi|^2 + O(|\xi|^3), & j = 3,4,5,\end{cases} \tag{4.3}$$

for $|\xi| \to 0$.

Proof. We consider $\lambda_j(\zeta)$ the following asymptotic expansion:

$$\lambda_j(\zeta) = \sum_{h=0}^{\infty} \lambda_j^{(h)}|\zeta|^h, \tag{4.4}$$

for $|\zeta| \to 0$. Straightforward computations yield

$$\begin{aligned}&\lambda_j^{(0)} = -\frac{1}{\tau}, \quad j = 1,\\ &\lambda_j^{(0)} = -\frac{1}{\tau_0}, \quad j = 2,\\ &\lambda_j^{(0)} = \lambda_j^{(1)} = 0, \quad \lambda_j^{(2)} = \phi_j, \quad j = 3,4,5,\end{aligned}$$

where ϕ_j are the roots of equation $\rho X^3 - \rho\kappa^2 X^2 + (m^2 + k^*)X - \kappa^2 k^* = 0$. To demonstrate that $\mathrm{Re}(\phi_j) > 0$, we set

$$f(X) := \rho X^3 - \rho\kappa^2 X^2 + (m^2 + k^*)X - \kappa^2 k^*. \tag{4.5}$$

Since $f(0)f(\kappa^2) < 0$, we conclude that f has at least one real root $X = \phi_1$ in the interval $(0,\kappa^2)$. We express Eq (4.5) in the form

$$f(X) = (X - \phi_1)(\rho X^2 + d_1 X + d_0)$$

with $d_1 = -\rho\kappa^2 + \phi_1\rho < 0$ and $d_0 = \dfrac{\kappa^2 k^*}{\phi_1} > 0$. For the remaining roots ϕ_2 and ϕ_3, we find that

$$\phi_2 + \phi_3 = -\frac{d_1}{\rho} > 0, \quad \phi_2\phi_3 = \frac{d_0}{\rho} > 0.$$

This implies that if ϕ_2 and ϕ_3 are real, they are both positive; if ϕ_2 and ϕ_3 are complex conjugates, then

$$\mathrm{Re}(\phi_2) = \mathrm{Re}(\phi_3) = \frac{1}{2}(\kappa^2 - \phi_1) > 0.$$

Thus, we have arrived at the desired result in (4.3). This completes the proof. □

When $|\zeta| \to \infty$, we rewrite $\hat{\Phi}(\zeta)$ as $\hat{\Phi}(\zeta) = \zeta^2\hat{\Psi}(\zeta^{-1})$, where $\hat{\Psi}(\zeta^{-1}) = B - \zeta^{-1}A - \zeta^{-2}L - \zeta^2 D$, and consider the eigenvalues $\mu_j(\zeta^{-1})$, for $j = 1,2,3,4,5$ of the matrix $\hat{\Psi}(\zeta^{-1})$. Meanwhile, these eigenvalues $\mu_j(\zeta^{-1})$ are the solutions to the characteristic equation

$$\begin{aligned}
&\tau\rho c \det(\mu I - \hat{\Phi}(\zeta^{-1}))\\
=&\tau\rho\tau_0\mu^5 + (\tau_0\rho + \tau\rho)\zeta^{-2}\mu^4 + [\rho\zeta^{-4} - \tau\rho\kappa^2\zeta^{-2} + (\tau_0 k + m^2\tau\tau_0)]\mu^3\\
&+ [-\rho\kappa^2\zeta^{-4} + (\tau_0 k^* + m^2\tau + m^2\tau_0 + k)\zeta^{-2}]\mu^2 + [(m^2 + k^*)\zeta^{-4} - \kappa^2 k\zeta^{-2}]\mu - \kappa^2 k^*\zeta^{-4}.
\end{aligned}$$

Lemma 4.2. *When $K > 0$, the real parts of the eigenvalues of* (2.1)-(2.2) *satisfy the asymptotic expansion*

$$\mathrm{Re}\lambda_j(i\xi) = \begin{cases} -\dfrac{K\tau_0\rho}{2\tau\rho(\tau_0 k + m^2\tau\tau_0)} + O(|\xi|^{-1}), & j = 1,2,\\ -\dfrac{m^2\tau\kappa^2 k + \kappa^2 k^2 + m^2\tau_0\kappa^2 K}{2\kappa^2 k(\tau_0 k + m^2\tau\tau_0)} + O(|\xi|^{-1}), & j = 3,4,\\ -1 + O(|\xi|^{-1}), & j = 5, \end{cases} \tag{4.6}$$

for $|\xi| \to \infty$.

When $K = 0$, the real parts of the eigenvalues of (2.1)-(2.2) *satisfy the asymptotic expansion*

$$\mathrm{Re}\lambda_j(i\xi) = \begin{cases} -\dfrac{\kappa^2 m^2\tau^2\rho}{2\tau_0^2(k + m^2\tau)^2}|\xi|^{-2} + O(|\xi|^{-3}), & j = 1,2,\\ -\dfrac{m^2\tau\kappa^2 k + \kappa^2 k^2}{2\kappa^2 k(\tau_0 k + m^2\tau\tau_0)} + O(|\xi|^{-1}), & j = 3,4,\\ -1 + O(|\xi|^{-1}), & j = 5, \end{cases} \tag{4.7}$$

for $|\xi| \to \infty$.

Proof. As $|\zeta| \to \infty$, similar calculation as before yields

$$\mu_j^{(2)} = \pm\sqrt{\frac{k + m^2\tau}{\tau\rho}}i, \quad \mu_j^{(1)} = 0, \quad \mathrm{Re}\Big(\mu_j^{(0)}\Big) = -\frac{K\tau_0\rho}{2\tau\rho(\tau_0 k + m^2\tau\tau_0)},$$
$$j = 1,2, \quad \text{when} \quad K > 0;$$

$$\mu_j^{(2)} = \pm\sqrt{\frac{k+m^2\tau}{\tau\rho}}i, \quad \mu_j^{(1)} = 0, \quad \mu_j^{(0)} = \mp\frac{\tau^2\rho\kappa^2m^2}{2\sqrt{\tau\rho(k+m^2\tau)}(\tau_0k+m^2\tau\tau_0)}i,$$

$$\mu_j^{(-1)} = 0, \quad \mathrm{Re}\Big(\mu_j^{(-2)}\Big) = \frac{\kappa^2m^2\tau^2\rho}{2\tau_0^2(k+m^2\tau)^2},$$

$$j = 1,2, \quad \text{when} \quad K = 0;$$

$$\mu_j^{(2)} = 0, \quad \mu_j^{(1)} = \pm\sqrt{\frac{\kappa^2k}{\tau_0k+m^2\tau\tau_0}}, \quad \mu_j^{(0)} = -\frac{m^2\tau\kappa^2k+\kappa^2k^2+m^2\tau_0\kappa^2K}{2\kappa^2k(\tau_0k+m^2\tau\tau_0)},$$

$$j = 3,4,$$

$$\mu_j^{(2)} = 0, \quad \mu_j^{(1)} = 0, \quad \mu_j^{(0)} = -1, \qquad j = 5.$$

Consequently, our conclusion holds. □

5. Conclusions

In this work, we have investigated the Cauchy problem for the JMGT-viscoelastic plate system coupled with Cattaneo-type heat conduction, focusing on the optimal decay rates of solutions in both the subcritical and critical cases. Our main contributions can be summarized as follows:

(1) The subcritical case: We proved that the system exhibits exponential decay without regularity loss, improving upon previous results in the literature. This indicates that the dissipation mechanism in this regime preserves the initial regularity of solutions.

(2) The critical case: In contrast, we observed a regularity-loss phenomenon in the decay rates, demonstrating a fundamental difference in the long-time behavior compared to the subcritical case.

Author contributions

Danhua Wang performed the formal analysis and wrote the manuscript; Kewang Chen performed the validation and review. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they haven't used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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