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Research article

Strong convergence of the Euler-Maruyama method for the stochastic volatility jump-diffusion model and financial applications

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Abstract: This work considered strong convergence of the Euler-Maruyama (EM) method for a stochastic volatility jump-diffusion model (SVJD model, for short). In this model, the underlying asset price follows a jump-diffusion geometric Brownian motion with stochastic volatility, and the volatility process obeys a mean-reverting square root process with Poisson jumps. As preliminary results, the existence and uniqueness of nonnegative solutions for the SVJD model was shown by means of Tanaka's formula and the comparison theorem. Also, some moment properties of the solution to the SVJD model were given. In view of unavailability of an explicit solution for the SVJD model, we used the EM method to approximate the exact solution and proved strong convergence of the EM approximation in the L^2 sense. In addition, the EM approximation for the SVJD model was applied to approximately compute expected payoffs of a European option and a barrier option. Finally, simulations were presented to verify the theoretical analysis.

Keywords: jump-diffusion stochastic volatility model; Heston model; Euler-Maruyama method; strong convergence; barrier option
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1. Introduction

Stochastic differential equations (SDEs) have important applications in fields of finance, biology, statistical mechanics, and so on (see, e.g., [3, 8, 17]). The mean-reverting square root process is a special SDE, which has been widely used as a model for stochastic volatility, stochastic interest rates and financial asset pricing. It is well-known that the Black-Scholes option pricing model cannot explain the volatility smile since the volatility remains constant over time of the option's expiry (see [15, 19]). In order to overcome this issue, Bates [2], Higham and Kloeden [9], and Yang and Wang [24] proposed a jump-diffusion stochastic model, which has been an effective approach among others. In addition, in terms of model parameter estimation, Fukasawa et al. [6] proposed a log-normal fractional stochastic volatility model to test roughness ($H < 0.5$), developing a quasi-likelihood estimator using realized volatility errors and Whittle-type covariance approximation. The estimator, applied to volatility series, shows consistency in high-frequency asymptotics, validated by simulations, with empirical evidence confirming roughness. Wang et al. [20] proposed modeling and forecasting realized volatility (RV) using the fractional Ornstein-Uhlenbeck (fO-U) process with a general Hurst parameter, H, while introducing a two-stage estimation method for fO-U process parameters based on discrete-sampled observations.

It should be noted that the mean-reverting square root process is not only a nonlinear SDE, but also its diffusion coefficient does not satisfy the global Lipschitz condition, so that it does not have an explicit solution, see, e.g., [12, 13, 16]. Therefore, the numerical solution of the mean-reverting square root process will play an important role in practical applications. Early work by Higham and Mao [10] investigated convergence of Monte Carlo simulations involving the mean-reverting square root process. Mao et al. [15] used the numerical method to study the mean-reverting stochastic volatility model under regime-switching. Wu et al. [21] studied the strong convergence of the mean-reverting square root process with delay. From both theoretical and numerical points of view, the numerical solution can be used for approximating the exact solution, only if the former could converge to the latter in some probabilistic sense. In financial applications, the EM approximation, as a Monte Carlo method, is often applied to calculate the expected price of financial derivatives (see, e.g., [4] and [5]).

In view of the volatility smile and the volatility's randomness, we shall consider the following stochastic volatility jump-diffusion model, in which the volatility follows a mean-reverting square root process with jumps, namely

$$dS(t) = \mu S(t)dt + \sqrt{V(t)}S(t)dW_1(t) + JS(t^-)dN_1(t), \tag{1.1}$$

$$dV(t) = \kappa(\theta - V(t))dt + \gamma\sqrt{V(t)}dW_2(t) + \tau dN_2(t), \tag{1.2}$$

where $S(t^-)$ denotes $\lim_{r\uparrow t} S(r)$, and $\{V(t)\}_{t\geq 0}$ is the volatility process; μ, $\kappa, \theta, \gamma, \tau$ are nonnegative constants and $J > -1$; $W_1(t)$ and $W_2(t)$ are two correlated Brownian motions; and $N_1(t)$ and $N_2(t)$ are two correlated scalar Poisson processes with intensities λ_1 and λ_2, respectively. To be more precise, the stochastic volatility jump-diffusion model can be seen as the Heston model with Poisson jumps. For simplicity, we assume that the initial values $S(0)$ and $V(0)$ are both positive constants.

It should be emphasized that Alfonsi [1], Higham and Kloeden [11], and Yang and Wang [24] adopted implicit numerical methods to study the mean-reverting process. In particular, under the condition of $2\kappa\theta \geq \gamma^2$, Yang and Wang [24] provided a positivity preserving numerical scheme, called the transformed jump adapted backward Euler method, for a more general mean-reverting model. However, in this paper, we focus on the explicit EM method because of its convenient calculation and acceptable convergence rate. So we only need to ensure that $V(t) \geq 0$ for all $t \geq 0$ almost surely under weaker assumptions. On the other hand, not only do we need to approximate the solutions to (1.1) and (1.2), but we also need to approximate quantities that are path-dependent, for example, the European barrier option value.

The outline of the paper is as follows. In the next section, we will give some properties of the solution to the SVJD model, including the nonnegativity and the moment boundedness. In Section 3, we investigate the EM approximation of the SVJD model and obtain moment bounds of discrete approximate solutions. In Section 4, we prove that the EM approximate solution converges to the exact solution in the L^2 sense. In Section 5, we utilize the EM approximation for the SVJD model to compute expected payoffs of a bond, a European option, and a barrier option. Finally, Section 6 gives some simulation results to illustrate the theoretical results.

2. Properties of the solutions to the SVJD model

Throughout this paper, $\mathbb{R}_+$ denotes the family of nonnegative real numbers, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, and $\mathbb{R}^{n\times m}$ denotes the space of $n \times m$ matrices with real entries. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, namely, it is right continuous and increasing, while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. We assume that $W_1(t), W_2(t), N_1(t), N_2(t)$ are all defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. If G is a set, its indicator function is denoted by $\mathbf{1}_G$, namely $\mathbf{1}_G(x) = 1$ if $x \in G$ and 0 otherwise. Let $[x]$ denote the largest integer which does not exceed x. Moreover, for two real numbers a and b, we use $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.

Since $S(t)$ is the underlying asset price and $V(t)$ is the stochastic volatility at time t, it is necessary to guarantee the nonnegativity of the solutions to (1.1) and (1.2). To proceed, we need Itô's formula for jump-diffusion processes [7, p. 680].

Lemma 1. *Let*

$$S(t) = S(0) + \int_0^t b(r, S(r^-))dr + \int_0^t \sigma(r, S(r^-))dW(r) + \int_0^t c(r, S(r^-))dN(r), \tag{2.1}$$

where $S(0) \in \mathbb{R}^n$, $b(\cdot,\cdot)\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma(\cdot,\cdot)\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $W(t)$ *is an m-dimensional Brownian motion,* $N(t)$ *is a Poisson process with intensity* λ, *and* $c(\cdot,\cdot)\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$.

If $f(t, x) \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$, *then* $f(t, S(t))$ *is again a jump-diffusion process, and*

$$f(t, S(t)) = f(0, S(0)) + \int_0^t L_0 f(r, S(r^-))dr + \int_0^t L_1 f(r, S(r^-))dW(r) + \int_0^t L_{-1} f(r, S(r^-))dN(r),$$

where

$$\begin{aligned}
L_0 f(t, x) &= \frac{\partial f(t, x)}{\partial t} + (\frac{\partial f(t, x)}{\partial x})^\top b(t, x) + \frac{1}{2}\text{trace}\left[\sigma^\top(t, x)\frac{\partial^2 f(t, x)}{\partial x^2}\sigma(t, x)\right],\\
L_1 f(t, x) &= (\frac{\partial f(t, x)}{\partial x})^\top \sigma(t, x),\\
L_{-1} f(t, x) &= f(t, x + c(t, x)) - f(t, x),
\end{aligned}$$

and the superscript $\top$ *denotes the transpose, and* trace(A) *denotes the trace of matrix A.*

Since (1.2) is mainly used to model stochastic volatility, it's naturally required that $V(t)$ will never become negative. The following theorem shows that the solution of (1.2) is unique and nonnegative.

Theorem 1. *For any given initial value* $V(0) = V_0 > 0$, *there exists a unique solution* $V(t)$ *to* (1.2), *and*

$$\mathbb{P}(V(t) \geq 0, \forall t \geq 0) = 1. \tag{2.2}$$

Proof. It is sufficient to show that the following SDE

$$dV(t) = \kappa(\theta - V(t))dt + \gamma \sqrt{|V(t)|}dW_2(t) + \tau dN_2(t) \tag{2.3}$$

has a unique solution $V(t)$, and $\mathbb{P}(V(t) \geq 0, \forall t \geq 0) = 1$.

Uniqueness. Let $V_1(t)$ and $V_2(t)$ be two solutions of (2.3). It is known that there exists a unique strong solution to (2.3) without jumps, see, for instance, [22, p. 2642]. Since

$$V_1(t) - V_2(t) = -\kappa \int_0^t (V_1(r) - V_2(r))\, dr + \gamma \int_0^t \left(\sqrt{|V_1(r)|} - \sqrt{|V_2(r)|}\right) dW_2(r),$$

the uniqueness of the solution is obtained immediately from that of the mean-reverting square root process in [22].

Existence. Since the coefficients of (2.3) satisfy the linear growth condition, the existence of the solution can be similarly derived by extending the proof of Theorem 1.3.1 in [14]. The key point is to deal with the jump term $\int_0^t \tau d\widetilde{N}_2(r)$ as in [9], which satisfies

$$\mathbb{E}\left(\int_0^t \tau d\widetilde{N}_2(r)\right)^2 = \lambda_2 \tau^2 t$$

by the martingale isometry for the compensated Poisson process ($\widetilde{N}_2(t) := N_2(t) - \lambda_2 t$).

Nonnegativity. To prove the nonnegativity of $V(t)$, we rewrite SDE (2.3) as

$$V(t) = V_0 + \int_0^t (\kappa\theta + \lambda_2\tau - \kappa V(r))dr + \gamma \int_0^t \sqrt{|V(r)|}dW_2(r) + \int_0^t \tau d\widetilde{N}_2(r).$$

Then we introduce the following equation:

$$X(t) = \int_0^t (\lambda_2\tau - \kappa X(r))dr + \gamma \int_0^t \sqrt{|X(r)|}dW_2(r) + \int_0^t \tau d\widetilde{N}_2(r).$$

It is easy to observe that

$$\gamma^2 \left| \sqrt{|x|} - \sqrt{|y|} \right|^2 \leq \gamma^2 |x - y|$$

and $\int_{0+} u^{-1} du = \infty$. Hence, applying the Tanaka-type formula (see, e.g., Theorem 152 in [18]), we have

$$\begin{aligned} X(t)^- &= -\int_0^t \mathbf{1}_{\{X(r-)\leq 0\}} dX(r) + \int_0^t \left[(X(r-) + \tau)^- - (X(r-))^- + \mathbf{1}_{\{X(r-)\leq 0\}}\tau \right] dN_2(r) \\ &\leq -\int_0^t \mathbf{1}_{\{X(r-)\leq 0\}} dX(r) + \int_0^t \mathbf{1}_{\{X(r-)\leq 0\}} \tau dN_2(r). \end{aligned}$$

We may easily find that

$$0 \leq \mathbb{E}X(t)^- \leq \kappa \mathbb{E} \int_0^t \mathbf{1}_{\{X(r)\leq 0\}} X(r) dr \leq 0.$$

We therefore have $X(t) \geq 0$ a.s. Since κ, θ, λ_2, τ are nonnegative, applying the comparison theorem (see, for instance, Theorem 295 in [18]), we have $V(t) \geq X(t)$ due to $V_0 > X(0) = 0$. This completes the proof. □

We know that (1.1) is often used to model financial quantities, so it is natural that the solution $S(t) > 0$. The following lemma shows that the solution of (1.1) is unique and positive.

Lemma 2. *For any given initial value $S(0) > 0$ and $V(t)$ in (1.2), there exists a unique solution $S(t)$ to (1.1), and $S(t) > 0$ holds true with probability one.*

Proof. If $V(t)$ is bounded, then it is obvious that Eq (1.1) has a unique solution since its coefficients satisfy the Lipschitz condition and the linear growth condition. In view of this, for each $n \geq 1$, we define the following sequence of stopping times:

$$\tau_n = \inf\{t \geq 0 : |V(t)| \geq n\}.$$

Clearly, $\tau_n \uparrow \infty$ holds as $n \to \infty$ and $|V(t \wedge \tau_n)| \leq n$ with probability one. Then there exists a unique solution to the equation

$$S(t \wedge \tau_n) = S(0) + \mu \int_0^{t\wedge\tau_n} S(r)dr + \int_0^{t\wedge\tau_n} \sqrt{V(r)}S(r)dW_1(r) + J \int_0^{t\wedge\tau_n} S(r^-)dN_1(r).$$

Letting $n \to \infty$, we know that $S(t)$ is the unique solution to (1.1).

Now we prove $S(t) > 0$ a.s. It follows easily from Itô's formula with jumps that

$$S(t) = S(0)(1 + J)^{N_1(t)} \exp\left[\int_0^t (\mu - \tfrac{1}{2}V(r))dr + \int_0^t \sqrt{V(r)}dW_1(r)\right].$$

Noting that $S(0) > 0$ and $J > -1$, we have $S(t) > 0$ a.s. The proof is complete. □

Taking expectations on both sides of (1.1) with respect to its integral equation and using the martingale property, we may easily derive that $\mathbb{E}S(t) = S(0)\exp(\mu + \lambda_1 J)t$. Furthermore, we have $\mathbb{E}S(t) \to 0$ as $t \to \infty$, if $\mu + \lambda_1 J < 0$.

We now consider the first and the second moment properties of $V(t)$.

Lemma 3. *Assume that $V(t)$ is the solution of (1.2). Let*

$$A = V(0)^2 + (2\kappa\theta + \gamma^2 + 2\lambda_2\tau)\left[\frac{V(0)}{\kappa} - \left(\frac{\theta}{\kappa} + \frac{\lambda_2\tau}{\kappa^2}\right)\right],$$
$$B = \lambda_2\tau^2 + (2\kappa\theta + \gamma^2 + 2\lambda_2\tau)\left(\theta + \frac{\lambda_2\tau}{\kappa}\right),$$
$$C = (2\kappa\theta + \gamma^2 + 2\lambda_2\tau)\left(\frac{\theta}{\kappa} + \frac{\lambda_2\tau}{\kappa^2} - \frac{V(0)}{\kappa}\right).$$

Then

$$\mathbb{E}V(t) = V(0)e^{-\kappa t} + \left(\theta + \frac{\lambda_2\tau}{\kappa}\right)(1 - e^{-\kappa t})$$

and

$$\mathbb{E}V(t)^2 = \frac{B}{2\kappa} - Ce^{-\kappa t} + \left(A - \frac{B}{2\kappa} + 2C\right)e^{-2\kappa t}.$$

Moreover,

$$\lim_{t\to\infty} \mathbb{E}V(t) = \theta + \frac{\lambda_2\tau}{\kappa} \tag{2.4}$$

and

$$\lim_{t\to\infty} \mathbb{E}V(t)^2 = \frac{B}{2\kappa}. \tag{2.5}$$

Proof. By (1.2) and the definition of $\widetilde{N}_2$, it is easy to see that

$$\mathbb{E}V(t) = V(0) + (\kappa\theta + \lambda_2\tau)t - \kappa\int_0^t \mathbb{E}V(r)dr.$$

This implies

$$\mathbb{E}V(t) = V(0)e^{-\kappa t} + \left(\theta + \frac{\lambda_2\tau}{\kappa}\right)(1 - e^{-\kappa t}).$$

Hence

$$\lim_{t\to\infty} \mathbb{E}V(t) = \theta + \frac{\lambda_2\tau}{\kappa}$$

as required. Applying Lemma 1 to (1.2) and taking expectations, we obtain

$$\begin{aligned}\mathbb{E}V(t)^2 &= V(0)^2 + \lambda_2\tau^2 t + (2\kappa\theta + \gamma^2 + 2\lambda_2\tau)\int_0^t \mathbb{E}V(r)dr - 2\kappa\int_0^t \mathbb{E}V(r)^2 dr \\ &= A + Bt + Ce^{-\kappa t} - 2\kappa\int_0^t \mathbb{E}V(r)^2 dr.\end{aligned}$$

Hence

$$\mathbb{E}V(t)^2 = \frac{B}{2\kappa} - Ce^{-\kappa t} + \left(A - \frac{B}{2\kappa} + 2C\right)e^{-2\kappa t}.$$

The required result (2.5) follows by letting $t \to \infty$. □

For later use, we will show the following theorem, which gives an upper bound for the second moment of $\log S(t)$ in the strong sense.

Theorem 2. *Suppose that $S(t)$ satisfies (1.1). Then, for any $T > 0$, we have*

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le t\le T} |\log S(t)|^2\right) \le\ & 4(|\log S(0)| + \mu T)^2 + DT^2 + 16\sqrt{D}T \\ & + 4\lambda_1(\log(1+J))^2 T + 4\lambda_1^2(\log(1+J))^2 T^2,\end{aligned} \tag{2.6}$$

where $D = \frac{B}{2\kappa} + |C| + \left|A - \frac{B}{2\kappa} + 2C\right|$.

Proof. Applying Lemma 1 to (1.1), we obtain that, for any $0 \le t \le T$,

$$\log S(t) = \log S(0) + \int_0^t \left(\mu - \tfrac{1}{2}V(r)\right)dr + \int_0^t \sqrt{V(r)}dW_1(r) + \int_0^t \log(1+J)dN_1(r).$$

We therefore derive

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le t\le T} |\log S(t)|^2\right) \le\ & 4(|\log S(0)| + \mu T)^2 + \mathbb{E}\left[\sup_{0\le t\le T}\left(\int_0^t |V(r)|dr\right)^2\right] \\ & + 4\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t \sqrt{V(r)}dW_1(r)\right|^2\right] \\ & + 4\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t \log(1+J)dN_1(r)\right|^2\right].\end{aligned}$$

By the Cauchy-Schwarz inequality, the Doob martingale inequality, and the martingale isometry, we further get

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le t\le T}|\log S(t)|^2\right) &\le 4(|\log S(0)|+\mu T)^2 + T\mathbb{E}\left(\sup_{0\le t\le T}\int_0^t |V(r)|^2 dr\right) + 16\mathbb{E}\left|\int_0^T \sqrt{V(r)}dW_1(r)\right|^2 \\ &\quad + 4(\log(1+J))^2\mathbb{E}\left[N_1(T)\right]^2 \\ &\le 4(|\log S(0)+\mu T)^2 + T\mathbb{E}\int_0^T |V(r)|^2 dr + 16\mathbb{E}\int_0^T |V(r)|dr + 4\lambda_1(\log(1+J))^2 T \\ &\quad + 4\lambda_1^2(\log(1+J))^2 T^2.\end{aligned}$$

It is easy to see from Lemma 3 that $\mathbb{E}V(t)^2 \le D$. Hence, the required result follows. □

3. Moment bounds of the EM approximate solutions

In this section, we will give the EM approximation of the SVJD model and derive moment bounds for the EM approximation. Since $S(t)$ is positive, it follows from Lemma 1 that

$$d\log S(t) = (\mu - \tfrac{1}{2}V(t))dt + \sqrt{V(t)}dW_1(t) + \log(1+J)dN_1(t), \tag{3.1}$$

$$dV(t) = \kappa(\theta - V(t))dt + \gamma\sqrt{V(t)}dW_2(t) + \tau dN_2(t). \tag{3.2}$$

To be computationally feasible, we replace (3.1) and (3.2) with

$$d\log S(t) = (\mu - \tfrac{1}{2}V(t))dt + \sqrt{|V(t)|}dW_1(t) + \log(1+J)dN_1(t), \tag{3.3}$$

$$dV(t) = \kappa(\theta - V(t))dt + \gamma\sqrt{|V(t)|}dW_2(t) + \tau dN_2(t). \tag{3.4}$$

Now we define the discrete EM approximate solutions. Given a stepsize $\Delta t > 0$, $S(0) = S_0 = s_0$ and $V(0) = V_0 = v_0$. Applying the EM method to (3.3) and (3.4) yields

$$\log s_{i+1} = \log s_i + (\mu - \tfrac{1}{2}|v_i|)\Delta t + \sqrt{|v_i|}\Delta W_{1,i} + \log(1+J)\,\Delta N_{1,i}, \tag{3.5}$$

$$v_{i+1} = v_i(1-\kappa\Delta t) + \kappa\theta\Delta t + \gamma\sqrt{|v_i|}\Delta W_{2,i} + \tau\Delta N_{2,i}, \tag{3.6}$$

where $\Delta W_{1,i} = W_1(t_{i+1}) - W_1(t_i)$, $\Delta W_{2,i} = W_2(t_{i+1}) - W_2(t_i)$, $\Delta N_{1,i} = N_1(t_{i+1}) - N_1(t_i)$, $\Delta N_{2,i} = N_2(t_{i+1}) - N_2(t_i)$. To extend the discrete time approximation to the continuous time case, we first define the step functions $\bar{s}(t)$ and $\bar{v}(t)$ by

$$\log\bar{s}(t) := \log s_i \text{ and } \bar{v}(t) := v_i,\ t\in[t_i, t_{i+1}). \tag{3.7}$$

The continuous-time approximations of $\log s(t)$ and $v(t)$ are accordingly defined by

$$\log s(t) := \log s_0 + \int_0^t \left(\mu - \tfrac{1}{2}|\bar{v}(r)|\right)dr + \int_0^t \sqrt{|\bar{v}(r)|}dW_1(r) + \int_0^t \log(1+J)dN_1(r), \tag{3.8}$$

$$v(t) := v_0 + \int_0^t \kappa(\theta - \bar{v}(r))dr + \gamma\int_0^t \sqrt{|\bar{v}(r)|}dW_2(r) + \int_0^t \tau dN_2(r) \tag{3.9}$$

for any $t \geq 0$. Moreover, for $t \in [t_i, t_{i+1})$, the above continuous-time approximations can be rewritten in the following equivalent forms:

$$\log s(t) = \log s_i + (\mu - \tfrac{1}{2}|v_i|)(t - t_i) + \sqrt{|v_i|}(W_1(t) - W_1(t_i)) + \log(1 + J)\,(N_1(t) - N_1(t_i)), \tag{3.10}$$

$$v(t) = v_i + \kappa(\theta - v_i)(t - t_i) + \gamma\sqrt{|v_i|}(W_2(t) - W_2(t_i)) + \tau(N_2(t) - N_2(t_i)). \tag{3.11}$$

Noticing that $\log s(t)$, $\log \bar{s}(t)$, $v(t)$, and $\bar{v}(t)$ coincide with their discrete solutions at the gridpoints, we have $\log s(t_i) = \log \bar{s}(t_i) = \log s_i$, $v(t_i) = \bar{v}(t_i) = v_i$. Besides, we have the following natural relationship:

$$\sup_{0\leq t\leq T} \mathbb{E}|\bar{v}(t)|^p \leq \sup_{0\leq t\leq T} \mathbb{E}|v(t)|^p \tag{3.12}$$

for any $p > 0$ and $T > 0$. In the remainder of this section, we let $T > 0$ arbitrarily.

Now we shall give the first and the second moment bounds of the continuous-time approximations of $v(t)$ and $\log s(t)$.

Lemma 4. *For the SDE (3.9), we have*

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |v(t)|\right) \leq (v_0 + \kappa\theta T + \lambda_2\tau T + 16\gamma^2)e^{(\kappa+\frac{1}{2})T} \tag{3.13}$$

and

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |v(t)|^2\right) \leq \Bigg\{ \left[4(v_0 + \kappa\theta T)^2 + 4\lambda_2\tau^2 T\,(1 + \lambda_2 T)\right] \\ + 16\gamma^2 T\left(v_0 + \kappa\theta T + \lambda_2\tau T + 16\gamma^2\right) e^{(\kappa+\frac{1}{2})T}\Bigg\} e^{4\kappa^2 T^2}. \tag{3.14}$$

Proof. It is easy to see that for any $0 \leq t \leq T$,

$$v(t) = (v_0 + \kappa\theta t) - \kappa\int_0^t \bar{v}(r)dr + \gamma\int_0^t \sqrt{|\bar{v}(r)|}dW_2(r) + \tau\int_0^t dN_2(r). \tag{3.15}$$

For any $0 \leq t_1 \leq T$, taking supremum t over $[0, t_1]$ and expectations on both sides of (3.15), we obtain that

$$\mathbb{E}\left(\sup_{0\leq t\leq t_1} |v(t)|\right) \leq (v_0 + \kappa\theta T) + \kappa\int_0^{t_1} \mathbb{E}|\bar{v}(r)|dr + \gamma\mathbb{E}\left(\sup_{0\leq t\leq t_1}\left|\int_0^t \sqrt{|\bar{v}(r)|}dW_2(r)\right|\right) \\ + \tau\mathbb{E}\left(\sup_{0\leq t\leq t_1}\left|\int_0^t dN_2(r)\right|\right).$$

We then apply the Burkholder-Davis-Gundy inequality to derive that

$$\mathbb{E}\left(\sup_{0\leq t\leq t_1} |v(t)|\right) \leq (v_0 + \kappa\theta T) + \kappa\int_0^{t_1} \mathbb{E}|\bar{v}(r)|dr + \sqrt{32}\gamma\mathbb{E}\left(\int_0^{t_1} |\bar{v}(r)|dr\right)^{\frac{1}{2}} + \tau\mathbb{E}N_2(T) \\ \leq (v_0 + \kappa\theta T + \lambda_2\tau T) + \kappa\int_0^{t_1} \mathbb{E}|\bar{v}(r)|dr + \frac{1}{2}\left(32\gamma^2 + \int_0^{t_1} \mathbb{E}|\bar{v}(r)|dr\right)$$

$$\leq (v_0 + \kappa\theta T + \lambda_2\tau T + 16\gamma^2) + (\kappa + \frac{1}{2}) \int_0^{t_1} \mathbb{E}\left(\sup_{0\leq t\leq r} |v(t)|\right) dr.$$

The required result (3.13) follows from the Gronwall inequality. Moreover, for any $0 \leq t \leq T$, according to (3.9), we may easily observe that

$$|v(t)|^2 \leq 4(v_0 + \kappa\theta T)^2 + 4\kappa^2 \left(\int_0^t |\bar{v}(r)|dr\right)^2 + 4\gamma^2 \left(\int_0^t \sqrt{|\bar{v}(r)|}dW_2(r)\right)^2 + 4\tau^2 \left(\int_0^t dN_2(r)\right)^2.$$

So, for any $0 \leq t_1 \leq T$, we have

$$\mathbb{E}\left(\sup_{0\leq t\leq t_1} |v(t)|^2\right) \leq 4(v_0 + \kappa\theta T)^2 + 4\kappa^2 \mathbb{E}\left[\sup_{0\leq t\leq t_1} \left(\int_0^t |\bar{v}(r)|dr\right)^2\right]$$
$$+ 4\gamma^2 \mathbb{E}\left[\sup_{0\leq t\leq t_1} \left(\int_0^t \sqrt{|\bar{v}(r)|}dW_2(r)\right)^2\right] + 4\tau^2 \mathbb{E}\left[\sup_{0\leq t\leq t_1} \left(\int_0^t dN_2(r)\right)^2\right].$$

By the Cauchy-Schwarz inequality, the Doob martingale inequality, and the martingale isometry,

$$\mathbb{E}\left(\sup_{0\leq t\leq t_1} |v(t)|^2\right) \leq 4(v_0 + \kappa\theta T)^2 + 4\kappa^2 T \int_0^{t_1} \mathbb{E}|\bar{v}(r)|^2 dr + 16\gamma^2 \mathbb{E}\left(\int_0^{t_1} \sqrt{|\bar{v}(r)|}dW_2(r)\right)^2 + 4\tau^2 \mathbb{E}\left[N_2(T)\right]^2$$
$$\leq 4(v_0 + \kappa\theta T)^2 + 4\kappa^2 T \int_0^{t_1} \mathbb{E}|\bar{v}(r)|^2 dr + 16\gamma^2 \mathbb{E} \int_0^{t_1} |\bar{v}(r)|dr + 4\lambda_2\tau^2 T + 4\lambda_2^2\tau^2 T^2$$
$$\leq 4(v_0 + \kappa\theta T)^2 + 4\kappa^2 T \int_0^{t_1} \mathbb{E}\left(\sup_{0\leq t\leq r} |\bar{v}(t)|^2\right) dr + 16\gamma^2 \int_0^{t_1} \mathbb{E}\left(\sup_{0\leq t\leq r} |\bar{v}(t)|\right) dr + 4\lambda_2\tau^2 T$$
$$+ 4\lambda_2^2\tau^2 T^2.$$

Then, by (3.13), we obtain

$$\mathbb{E}\left(\sup_{0\leq t\leq t_1} |v(t)|^2\right) \leq \left[4(v_0 + \kappa\theta T)^2 + 4\lambda_2\tau^2 T\left(1 + \lambda_2 T\right)\right] + 4\kappa^2 T \int_0^{t_1} \mathbb{E}\left(\sup_{0\leq t\leq r} |\bar{v}(t)|^2\right) dr$$
$$+ 16\gamma^2 T\left(v_0 + \kappa\theta T + \lambda_2\tau T + 16\gamma^2\right) e^{(\kappa+\frac{1}{2})T}.$$

Finally, the Gronwall inequality yields the desired assertion (3.14). □

Lemma 5. *For the SDE (3.8), we have*

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |\log s(t)|^2\right) \leq 4(|\log s_0| + \mu T)^2 + [4(v_0 + \kappa\theta T)^2 + 4\lambda_2\tau^2 T(1 + \lambda_2 T)]$$
$$\times T^2 e^{4\kappa^2 T^2} + 16T^3\gamma^2(v_0 + \kappa\theta T + \lambda_2\tau T + 16\gamma^2)$$
$$\times e^{(\kappa+\frac{1}{2})T + 4\kappa^2 T^2} + 16T(v_0 + \kappa\theta T + \lambda_2\tau T + 16\gamma^2)$$
$$\times e^{(\kappa+\frac{1}{2})T} + 4\lambda_1[\log(1 + J)]^2 T + 4\lambda_1^2[\log(1 + J)]^2 T^2. \quad (3.16)$$

Proof. For any $0 \leq t \leq T$, we note that

$$|\log s(t)|^2 \leq 4(|\log s_0| + \mu T)^2 + \left(\int_0^t |\bar{v}(r)|dr\right)^2 + 4\left|\int_0^t \sqrt{|\bar{v}(r)|}dW_1(r)\right|^2 + 4\left|\int_0^t \log(1 + J)dN_1(r)\right|^2.$$

We therefore obtain

$$\mathbb{E}\Big(\sup_{0\le t\le T}|\log s(t)|^2\Big) \le 4(|\log s_0| + \mu T)^2 + \mathbb{E}\left[\sup_{0\le t\le T}\left(\int_0^t |\bar{v}(r)|dr\right)^2\right] + 4\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t \sqrt{|\bar{v}(r)|}dW_1(r)\right|^2\right]$$
$$+ 4\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t \log(1+J)dN_1(r)\right|^2\right].$$

By the Cauchy-Schwarz inequality, the Doob martingale inequality, and the martingale isometry,

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\le t\le T}|\log s(t)|^2\right) &\le 4(|\log s_0| + \mu T)^2 + T\int_0^T \mathbb{E}|\bar{v}(r)|^2 dr + 16\mathbb{E}\left|\int_0^T \sqrt{|\bar{v}(r)|}dW_1(r)\right|^2 \\
&\quad + 4[\log(1+J)]^2\mathbb{E}\,[N_1(T)]^2 \\
&\le 4(|\log s_0| + \mu T)^2 + T\int_0^T \mathbb{E}|\bar{v}(r)|^2 dr + 16\mathbb{E}\int_0^T |\bar{v}(r)|dr + 4\lambda_1(\log(1+J))^2 T \\
&\quad + 4\lambda_1^2[\log(1+J)]^2 T^2 \\
&\le 4(|\log s_0| + \mu T)^2 + T\int_0^T \mathbb{E}\left(\sup_{0\le r\le T}|\bar{v}(r)|^2\right)dr + 16\int_0^T \mathbb{E}\left(\sup_{0\le r\le T}|\bar{v}(r)|\right)dr \\
&\quad + 4\lambda_1(\log(1+J))^2 T + 4\lambda_1^2[\log(1+J)]^2 T^2.
\end{aligned}$$

Applying Lemma 4 yields the desired result. □

4. Strong convergence

In this section, we will prove the convergence of the EM approximation on the finite time interval $[0,T]$. As a bridge, we begin with considering the difference between $v(t)$ and $\bar{v}(t)$ in the L^2 sense.

Lemma 6. *If $\bar{v}(t)$ of (3.7) is given and $v(t)$ satisfies (3.9), then*

$$\begin{aligned}
\sup_{0\le t\le T}\mathbb{E}(v(t)-\bar{v}(t))^2 &\le \left[12\kappa^2(\theta^2+\widetilde{Q})\Delta t + 6\lambda_2^2\tau^2\Delta t + 3\sqrt{3}\gamma^2\widetilde{Q}^{\frac{1}{2}} + 3\lambda_2\tau^2\right]\Delta t \\
&=: L\Delta t, \qquad (4.1)
\end{aligned}$$

where $\widetilde{Q}$ is defined by the right-hand side of (3.14). Consequently

$$\lim_{\Delta t\to 0}\sup_{0\le t\le T}\mathbb{E}(v(t)-\bar{v}(t))^2 = 0. \qquad (4.2)$$

Proof. For $t\in[i\Delta t,(i+1)\Delta t)$, we have

$$\begin{aligned}
v(t)-\bar{v}(t) &= (t-t_i)\kappa(\theta-v_i) + \gamma\sqrt{|v_i|}(W_2(t)-W_2(t_i)) + \tau(N_2(t)-N_2(t_i)) \\
&= (t-t_i)[\kappa(\theta-v_i)+\lambda_2\tau] + \gamma\sqrt{|v_i|}(W_2(t)-W_2(t_i)) + \tau(\widetilde{N}_2(t)-\widetilde{N}_2(t_i)).
\end{aligned}$$

By the martingale isometry and the Hölder inequality,

$$\mathbb{E}(v(t)-\bar{v}(t))^2 \le 12\kappa^2(t-t_i)^2(\theta^2+\mathbb{E}v_i^2) + 6\lambda_2^2\tau^2(t-t_i)^2$$

$$+ 3\gamma^2\mathbb{E}\left[|v_i|(W_2(t) - W_2(t_i))^2\right] + 3\tau^2\mathbb{E}(\widetilde{N}_2(t) - \widetilde{N}_2(t_i))^2$$
$$\leq 12\kappa^2(\theta^2 + \widetilde{Q})(\Delta t)^2 + 6\lambda_2^2\tau^2(\Delta t)^2 + 3\sqrt{3}\gamma^2\widetilde{Q}^{\frac{1}{2}}\Delta t + 3\lambda_2\tau^2\Delta t$$
$$\leq \left[12\kappa^2(\theta^2 + \widetilde{Q})\Delta t + 6\lambda_2^2\tau^2\Delta t + 3\sqrt{3}\gamma^2\widetilde{Q}^{\frac{1}{2}} + 3\lambda_2\tau^2\right]\Delta t.$$

Taking the supremum t over $[0, T]$ yields the required result (4.1). This implies (4.2) follows immediately. The proof is complete. □

The above lemma shows that the difference between $v(t)$ and $\bar{v}(t)$ will tend to 0 as $\Delta t \to 0$ in the L^2 sense. The following theorem shows the L^1 and L^2 convergence properties of the continuous-time approximation.

Lemma 7. *Let $n \geq 1$ be any integer. For $V(t)$ satisfying (3.2) and $v(t)$ in (3.9), we have*

$$\sup_{0\leq t\leq T} \mathbb{E}|V(t) - v(t)| \leq e^{\kappa T}\left[\frac{1}{\sqrt{n}} + \kappa T\sqrt{L\Delta t} + \frac{\gamma^2 T}{n}\left(1 + \sqrt{n}e^n\sqrt{L\Delta t}\right)\right] \tag{4.3}$$

and

$$\mathbb{E}\left[\sup_{0\leq t\leq T}(V(t) - v(t))^2\right] \leq \left(2\kappa^2T^2L\Delta t + 8\gamma^2T\sqrt{L\Delta t} + 8\gamma^2T\sup_{0\leq t\leq T}\mathbb{E}|V(t) - v(t)|\right)e^{2\kappa^2T^2}, \tag{4.4}$$

where L is as defined in Lemma 6. Consequently

$$\lim_{\Delta t\to 0}\sup_{0\leq t\leq T}\mathbb{E}|V(t) - v(t)| = 0 \tag{4.5}$$

and

$$\lim_{\Delta t\to 0}\mathbb{E}\left[\sup_{0\leq t\leq T}(V(t) - v(t))^2\right] = 0. \tag{4.6}$$

Proof. To prove (4.3), we use a smoothing skill to approximate $|x|$ (see, e.g., [23]). Let $n \geq 1$. Clearly, $\int_{1/\sqrt{n}e^n}^{1/\sqrt{n}}\frac{dx}{x} = n$. For each $n \geq 1$, let $\psi_n(u)$ be a continuous function such that its support is contained in $[1/\sqrt{n}e^n, 1/\sqrt{n}]$, $0 \leq \psi_n(u) \leq \frac{2}{nu}$, and $\int_{1/\sqrt{n}e^n}^{1/\sqrt{n}}\psi_n(u)du = 1$. Define

$$\phi_n(x) = \int_0^{|x|}dy\int_0^y\psi_n(u)du.$$

Then we obtain $\phi_n \in C^2(\mathbb{R}, \mathbb{R})$, $|\phi_n'(x)| \leq 1$, and $|\phi_n''(x)| \leq \frac{2}{n|x|}\mathbf{1}_{[1/\sqrt{n}e^n, 1/\sqrt{n}]}(x)$. We also have $|x| \leq \phi_n(x) + \frac{1}{\sqrt{n}}$, for all $x \in \mathbb{R}$. Since

$$V(t) - v(t) = -\kappa\int_0^t(V(r) - \bar{v}(r))\,dr + \gamma\int_0^t\left(\sqrt{V(r)} - \sqrt{|\bar{v}(r)|}\right)dW_2(r),$$

we can use the Itô formula to get that

$$d\phi_n(e(r)) = \left[-\kappa\phi_n'(e(r))(V(r) - \bar{v}(r)) + \frac{\gamma^2}{2}\phi_n''(e(r))\left(\sqrt{V(r)} - \sqrt{|\bar{v}(r)|}\right)^2\right]dr$$

$$+\gamma\phi_n'(e(r))\left(\sqrt{V(r)}-\sqrt{|\bar{v}(r)|}\right)dW_2(r),$$

where $e(r)=V(r)-v(r)$. By $\phi_n(\cdot)$'s properties and Lemma 6, we have

$$\begin{aligned}
\mathbb{E}\phi_n(e(r)) \le& \kappa\mathbb{E}\int_0^t |V(r)-\bar{v}(r)|\,dr+\frac{\gamma^2}{2}\mathbb{E}\int_0^t |\phi_n''(e(r))|\left(\sqrt{V(r)}-\sqrt{|\bar{v}(r)|}\right)^2 dr\\
\le& \kappa\mathbb{E}\int_0^t |V(r)-\bar{v}(r)|\,dr+\frac{\gamma^2}{2}\mathbb{E}\int_0^t |\phi_n''(e(r))|\,|V(r)-\bar{v}(r)|\,dr\\
\le& \kappa\mathbb{E}\int_0^t |V(r)-v(r)|\,dr+\kappa\mathbb{E}\int_0^t |v(r)-\bar{v}(r)|\,dr+\frac{\gamma^2}{2}\mathbb{E}\int_0^t |\phi_n''(e(r))|\,|V(r)-v(r)|\,dr\\
&+\frac{\gamma^2}{2}\mathbb{E}\int_0^t |\phi_n''(e(r))|\,|v(r)-\bar{v}(r)|\,dr\\
\le& \kappa\mathbb{E}\int_0^t |V(r)-v(r)|\,dr+\kappa T\sqrt{L\Delta t}+\frac{\gamma^2}{2}\left(\frac{2T}{n}+\frac{2e^nT}{\sqrt{n}}\sqrt{L\Delta t}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{E}\,|V(r)-v(r)| \le& \mathbb{E}\phi_n(V(r)-v(r))+\frac{1}{\sqrt{n}}\\
\le& \kappa\mathbb{E}\int_0^t |V(r)-v(r)|\,dr+\frac{1}{\sqrt{n}}+\kappa T\sqrt{L\Delta t}+\frac{\gamma^2 T}{n}\left(1+\sqrt{n}e^n\sqrt{L\Delta t}\right).
\end{aligned}$$

An application of the Gronwall inequality yields the required result.

Now let us prove (4.4). By the Cauchy-Schwarz inequality, we have

$$|V(t)-v(t)|^2 \le 2\kappa^2 T\int_0^t |V(r)-\bar{v}(r)|^2\,dr+2\gamma^2\left(\int_0^t\left(\sqrt{V(r)}-\sqrt{|\bar{v}(r)|}\right)dW_2(r)\right)^2.$$

For any $t_1\in[0,T]$, using the Doob martingale inequality, we obtain

$$\mathbb{E}\left[\sup_{0\le t\le t_1}(V(t)-v(t))^2\right]\le 2\kappa^2T\int_0^{t_1}|V(r)-\bar{v}(r)|^2\,dr+8\gamma^2\int_0^{t_1}\left(\sqrt{V(r)}-\sqrt{|\bar{v}(r)|}\right)^2 dr.$$

By the triangle inequality, we can deduce that

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le t\le t_1}(V(t)-v(t))^2\right]\le& 2\kappa^2T\int_0^{t_1}|V(r)-v(r)|^2\,dr+2\kappa^2T\int_0^{t_1}|v(r)-\bar{v}(r)|^2\,dr\\
&+8\gamma^2\int_0^{t_1}|V(r)-\bar{v}(r)|\,dr\\
\le& 2\kappa^2T\int_0^{t_1}\mathbb{E}\left[\sup_{0\le t\le r}(V(t)-v(t))^2\right]dr+2\kappa^2T^2L\Delta t+8\gamma^2T\sqrt{L\Delta t}\\
&+8\gamma^2T\sup_{0\le t\le T}\mathbb{E}|V(t)-v(t)|.
\end{aligned}$$

The Gronwall inequality will give the required assertion (4.4). Finally, (4.5) and (4.6) follow easily from (4.3) and (4.4). □

Lemma 8. *For* $\log s(t)$ *satisfying (3.8) and* $\log \bar{s}(t)$ *in (3.7), we have*

$$\sup_{0\le t\le T} \mathbb{E}(\log s(t) - \log \bar{s}(t))^2 \le \left[6\mu^2\Delta t + \frac{3}{2}\widetilde{Q}\Delta t + 3\sqrt{3}\widetilde{Q}^{\frac{1}{2}} + 3\lambda_1(\log(1+J))^2(1+\lambda_1\Delta t)\right]\Delta t, \tag{4.7}$$

where $\widetilde{Q}$ *is the same as that in Lemma 6. Consequently*

$$\lim_{\Delta t\to 0}\sup_{0\le t\le T} \mathbb{E}(\log s(t) - \log \bar{s}(t))^2 = 0. \tag{4.8}$$

Proof. For $t \in [i\Delta t, (i+1)\Delta t)$, we obtain from (3.10) that

$$\log s(t) - \log \bar{s}(t) = (t - t_i)\left(\mu - \frac{1}{2}|v_i|\right) + \sqrt{|v_i|}(W_1(t) - W_1(t_i)) + \log(1+J)(N_1(t) - N_1(t_i)).$$

Furthermore, we can derive that

$$\begin{aligned}
\mathbb{E}(\log s(t) - \log \bar{s}(t))^2 &\le 3(\Delta t)^2\mathbb{E}\left(\mu - \frac{1}{2}|v_i|\right)^2 + 3\sqrt{3}\Delta t\sqrt{\mathbb{E}|v_i|^2} + 3(\log(1+J))^2\lambda_1\Delta t(1+\lambda_1\Delta t)\\
&\le 3(\Delta t)^2\mathbb{E}\left(2\mu^2 + \frac{1}{2}|v_i|^2\right) + 3\sqrt{3}\widetilde{Q}^{\frac{1}{2}}\Delta t + 3\lambda_1(\log(1+J))^2\Delta t(1+\lambda_1\Delta t)\\
&\le 6\mu^2(\Delta t)^2 + \frac{3}{2}\widetilde{Q}(\Delta t)^2 + 3\sqrt{3}\widetilde{Q}^{\frac{1}{2}}\Delta t + 3\lambda_1(\log(1+J))^2(1+\lambda_1\Delta t)\Delta t\\
&\le \left[6\mu^2\Delta t + \frac{3}{2}\widetilde{Q}\Delta t + 3\sqrt{3}\widetilde{Q}^{\frac{1}{2}} + 3\lambda_1(\log(1+J))^2(1+\lambda_1\Delta t)\right]\Delta t.
\end{aligned}$$

Taking the supremum t over $[0, T]$ and letting $\Delta t \to 0$ yields the required assertions. □

The above lemma shows that the difference between $\log s(t)$ and $\log \bar{s}(t)$ will tend to 0 as $\Delta t \to 0$ in the L^2 sense. In order to approximate the value of the options, especially a barrier option, we will need a stronger convergence result. The following theorem shows the strong L^2 convergence property of the continuous-time approximation.

Theorem 3. *Assume that* $\log S(t)$ *satisfies (3.3) and* $\log s(t)$ *satisfies (3.8). Then*

$$\lim_{\Delta t\to 0}\mathbb{E}\left[\sup_{0\le t\le T}(\log S(t) - \log s(t))^2\right] = 0. \tag{4.9}$$

Proof. By (3.3) and (3.8), we derive that for any $t \in [0, T]$,

$$\log S(t) - \log s(t) = -\frac{1}{2}\int_0^t (|V(r)| - |\bar{v}(r)|)dr + \int_0^t \left(\sqrt{|V(r)|} - \sqrt{|\bar{v}(r)|}\right)dW_1(r).$$

So

$$(\log S(t) - \log s(t))^2 \le \frac{1}{2}\left(\int_0^t (|V(r)| - |\bar{v}(r)|)dr\right)^2 + 2\left(\int_0^t \left(\sqrt{|V(r)|} - \sqrt{|\bar{v}(r)|}\right)dW_1(r)\right)^2.$$

The Cauchy-Schwarz inequality gives

$$(\log S(t) - \log s(t))^2 \le \frac{t}{2}\int_0^t (|V(r)| - |\bar{v}(r)|)^2 dr + 2\left(\int_0^t \left(\sqrt{|V(r)|} - \sqrt{|\bar{v}(r)|}\right) dW_1(r)\right)^2.$$

For any $t_1 \in [0, T]$, by the Doob martingale inequality and Itô isometry, we have

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le t_1}(\log S(t) - \log s(t))^2\right] &\le \frac{T}{2}\mathbb{E}\int_0^{t_1}(|V(r)| - |\bar{v}(r)|)^2 dr \\ &\quad + 2\mathbb{E}\left[\sup_{0\le t\le t_1}\left(\int_0^t \left(\sqrt{|V(r)|} - \sqrt{|\bar{v}(r)|}\right) dW_1(r)\right)^2\right] \\ &\le \frac{T}{2}\mathbb{E}\int_0^{t_1}(|V(r)| - |\bar{v}(r)|)^2 dr + 8\mathbb{E}\left(\int_0^{t_1}\left(\sqrt{|V(r)|} - \sqrt{|\bar{v}(r)|}\right) dW_1(r)\right)^2 \\ &\le \frac{T}{2}\mathbb{E}\int_0^{t_1}(|V(r)| - |\bar{v}(r)|)^2 dr + 8\mathbb{E}\int_0^{t_1}\left(\sqrt{|V(r)|} - \sqrt{|\bar{v}(r)|}\right)^2 dr.\end{aligned}$$

Noting that

$$\begin{aligned}\frac{T}{2}\mathbb{E}\int_0^{t_1}(|V(r)| - |\bar{v}(r)|)^2 dr &\le T\mathbb{E}\int_0^{t_1}(|V(r)| - |v(r)|)^2 dr + T\mathbb{E}\int_0^{t_1}(|v(r)| - |\bar{v}(r)|)^2 dr \\ &\le T\mathbb{E}\int_0^{t_1}(V(r) - v(r))^2 dr + T\mathbb{E}\int_0^{t_1}(v(r) - \bar{v}(r))^2 dr \\ &\le T^2\mathbb{E}\left[\sup_{0\le t\le T}(V(t) - v(t))^2\right] + T^2\sup_{0\le t\le T}\mathbb{E}(v(t) - \bar{v}(t))^2\end{aligned}$$

and

$$\begin{aligned}8\mathbb{E}\int_0^{t_1}\left(\sqrt{|V(r)|} - \sqrt{|\bar{v}(r)|}\right)^2 dr &\le 8\mathbb{E}\int_0^{t_1}|V(r) - \bar{v}(r)|\,dr \\ &\le 8T\sup_{0\le t\le T}\mathbb{E}\,|V(t) - \bar{v}(t)| \\ &\le 8T\sup_{0\le t\le T}\mathbb{E}\,|V(t) - v(t)| + 8T\sup_{0\le t\le T}\mathbb{E}\,|v(t) - \bar{v}(t)|,\end{aligned}$$

we can easily get the required assertion (4.9) by means of (4.2), (4.5), and (4.6). □

5. Applications to financial products

In this section, we will show that expected payoffs of a European option and a barrier option given by the EM method will converge to their respective real expected payoffs when the time step goes to zero.

5.1. A European option

We consider the expected payoff for a European put option at expiry time T:

$$\Theta := \mathbb{E}[(K - S(T))^+], \tag{5.1}$$

where $K > 0$ is the exercise price. Naturally, the expected payoff based on the numerical method (3.7) is given by

$$\overline{\Theta}_{\Delta t} := \mathbb{E}[(K - \bar{s}(T))^+]. \tag{5.2}$$

The following theorem shows that the numerical approximation (5.2) converges to (5.1).

Theorem 4. *For (5.1) and (5.2), we have*

$$\lim_{\Delta t \to 0} |\Theta - \overline{\Theta}_{\Delta t}| = 0. \tag{5.3}$$

Proof. For any given pair of positive numbers m and M, let

$$A = \{|\log S(t)| \leq m, 0 \leq t \leq T\}, B = \{|\log s(t)| \leq M, 0 \leq t \leq T\}.$$

Given any $\varepsilon > 0$, by Theorem 2, Lemma 5, and the Markov inequality, we can find sufficiently large m, M such that

$$\mathbb{P}(A^c \cup B^c) \leq \frac{\varepsilon}{4K}.$$

Compute

$$\begin{aligned}
|\Theta - \overline{\Theta}_{\Delta t}| &= \left|\mathbb{E}[(K - S(T))^+] - \mathbb{E}[(K - \bar{s}(T))^+]\right| \\
&\leq \mathbb{E}\left|(K - S(T))^+ - (K - \bar{s}(T))^+\right| \\
&\leq \mathbb{E}\left(\left|(K - S(T))^+ - (K - \bar{s}(T))^+\right| \mathbf{1}_{A \cap B}\right) + \mathbb{E}\left(\left|(K - S(T))^+ - (K - \bar{s}(T))^+\right| \mathbf{1}_{A^c \cup B^c}\right).
\end{aligned}$$

Making use of the inequality $|(K - S(T))^+ - (K - \bar{s}(T))^+| \leq |S(T) - \bar{s}(T)|$, we have

$$\begin{aligned}
|\Theta - \overline{\Theta}_{\Delta t}| &\leq \mathbb{E}\left(|S(T) - \bar{s}(T)|\, \mathbf{1}_{A \cap B}\right) + 2K\mathbb{P}(A^c \cup B^c) \\
&\leq \mathbb{E}\left(|S(T) - \bar{s}(T)|\, \mathbf{1}_{A \cap B}\right) + \frac{\varepsilon}{2} \\
&\leq \mathbb{E}\left(\left|e^{\log S(T)} - e^{\log \bar{s}(T)}\right| \mathbf{1}_{A \cap B}\right) + \frac{\varepsilon}{2} \\
&\leq \mathbb{E}\left(e^{\xi} |\log S(T) - \log \bar{s}(T)| \mathbf{1}_{A \cap B}\right) + \frac{\varepsilon}{2} \\
&\leq e^{(m+M)} \mathbb{E}\left|\log S(T) - \log \bar{s}(T)\right| + \frac{\varepsilon}{2} \\
&\leq e^{(m+M)} \left(\mathbb{E}\left|\log S(T) - \log s(T)\right| + \mathbb{E}\left|\log s(T) - \log \bar{s}(T)\right|\right) + \frac{\varepsilon}{2},
\end{aligned}$$

where ξ is between $\log S(T)$ and $\log \bar{s}(T)$. This, together with Lemma 8 and Theorem 3, yields the assertion (5.3). □

5.2. *A barrier option*

We now consider an up-and-out barrier call option at expiry time T. Let K be the exercise price and B_U the barrier. For the SVJD model and the numerical method (3.7), we define

$$\Lambda := \mathbb{E}\left[(S(T) - K)^+ \mathbf{1}_{\{0 \leq S(t) \leq B_U,\ 0 \leq t \leq T\}}\right], \tag{5.4}$$

$$\overline{\Lambda}_{\Delta t} := \mathbb{E}\left[(\bar{s}(T) - K)^+ \mathbf{1}_{\{0 \leq \bar{s}(t) \leq B_U,\ 0 \leq t \leq T\}}\right]. \tag{5.5}$$

Theorem 5. *For (5.4) and (5.5), we have*

$$\lim_{\Delta t\to 0}\left|\Lambda-\overline{\Lambda}_{\Delta t}\right|=0. \tag{5.6}$$

Proof. Let

$$A:=\{0\le S(t)\le B_U, 0\le t\le T\} \text{ and } \overline{A}_{\Delta t}:=\{0\le \bar{s}(t)\le B_U, 0\le t\le T\}.$$

We have

$$\begin{aligned}\left|\Lambda-\overline{\Lambda}_{\Delta t}\right| &= \left|\mathbb{E}\Big[(S(T)-K)^+\mathbf{1}_A\Big]-\mathbb{E}\left[(\bar{s}(T)-K)^+\mathbf{1}_{\overline{A}_{\Delta t}}\right]\right|\\ &\le \mathbb{E}\left|(S(T)-K)^+\mathbf{1}_A-(\bar{s}(T)-K)^+\mathbf{1}_{\overline{A}_{\Delta t}}\right|\\ &\le \mathbb{E}\left(|(S(T)-K)^+-(\bar{s}(T)-K)^+|\mathbf{1}_{A\cap\overline{A}_{\Delta t}}\right)\\ &\quad+\mathbb{E}\left(|(S(T)-K)^+|\mathbf{1}_{A\cap\overline{A}^c_{\Delta t}}\right)+\mathbb{E}\left(|(\bar{s}(T)-K)^+|\mathbf{1}_{A^c\cap\overline{A}_{\Delta t}}\right)\\ &\le \mathbb{E}\left(|S(T)-\bar{s}(T)|\mathbf{1}_{A\cap\overline{A}_{\Delta t}}\right)+(B_U-K)\mathbb{P}\left(A\cap\overline{A}^c_{\Delta t}\right)+(B_U-K)\mathbb{P}\left(A^c\cap\overline{A}_{\Delta t}\right).\end{aligned}$$

Clearly, (5.6) follows if we could show that

$$\lim_{\Delta t\to 0}\mathbb{E}\left(|S(T)-\bar{s}(T)|\mathbf{1}_{A\cap\overline{A}_{\Delta t}}\right)=0,\ \lim_{\Delta t\to 0}\mathbb{P}\left(A\cap\overline{A}^c_{\Delta t}\right)=0,\ \text{and}\ \lim_{\Delta t\to 0}\mathbb{P}\left(A^c\cap\overline{A}_{\Delta t}\right)=0.$$

For any $i, j, k > B_U$, let $F=\{|\log S(t)|\le i, 0\le t\le T\}$, $G=\{|v(t)|\le j, 0\le t\le T\}$, and $H=\{|\log s(t)|\le k, 0\le t\le T\}$. Given any $\varepsilon>0$, by Theorem 2, and Lemmas 4 and 5, we can find sufficiently large i, j, k such that

$$\mathbb{P}(F^c\cup G^c\cup H^c)\le\frac{\varepsilon}{8(1\vee B_U)}. \tag{5.7}$$

Define a stopping time

$$\tau_{ijk}=\inf\{t\ge 0: |\log S(t)|>i \text{ or } |v(t)|>j \text{ or } |\log s(t)|>k\}.$$

Compute

$$\begin{aligned}\mathbb{E}\left(|S(T)-\bar{s}(T)|\mathbf{1}_{A\cap\overline{A}_{\Delta t}}\right) &= \mathbb{E}\left(|S(T)-\bar{s}(T)|\mathbf{1}_{A\cap\overline{A}_{\Delta t}\cap F\cap G\cap H}\right)+\mathbb{E}\left(|S(T)-\bar{s}(T)|\mathbf{1}_{A\cap\overline{A}_{\Delta t}\cap(F^c\cup G^c\cup H^c)}\right)\\ &\le \mathbb{E}\left(|S(T)-\bar{s}(T)|\mathbf{1}_{F\cap G\cap H}\right)+B_U\mathbb{P}(F^c\cup G^c\cup H^c)\\ &\le \mathbb{E}\left(|S(T)-\bar{s}(T)|\mathbf{1}_{F\cap G\cap H}\right)+\frac{\varepsilon}{8}.\end{aligned}$$

In the same way as in the proof of Theorem 4, we can show that

$$\lim_{\Delta t\to 0}\mathbb{E}\left(|S(T)-\bar{s}(T)|\mathbf{1}_{A\cap\overline{A}_{\Delta t}}\right)=0.$$

Next we will show that $\mathbb{P}\left(A\cap\overline{A}^c_{\Delta t}\right)\to 0$ as $\Delta t\to 0$. For any sufficiently small $\delta>0$, we have

$$\begin{aligned}A &= \left\{\sup_{0\le t\le T} S(t)\le B_U\right\}\\ &= \left\{\sup_{0\le t\le T} S(t)\le B_U-\delta\right\}\cup\left\{B_U-\delta<\sup_{0\le t\le T} S(t)\le B_U\right\}\end{aligned}$$

$$
\begin{aligned}
&\subseteq \left\{ \sup_{0\le n\Delta t\le T} S(n\Delta t) \le B_U - \delta \right\} \cup \left\{ B_U - \delta < \sup_{0\le t\le T} S(t) \le B_U \right\} \\
&=: \overline{A}_1 \cup \overline{A}_2,
\end{aligned}
$$

where $n = 0, 1, \cdots, [T/\Delta t]$. Using (5.7), we derive that

$$
\mathbb{P}\left(A \cap \overline{A}^c_{\Delta t}\right) \le \mathbb{P}\left(A \cap \overline{A}^c_{\Delta t} \cap F \cap G \cap H\right) + \frac{\varepsilon}{8}. \tag{5.8}
$$

Since $\bar{s}(t_n) = s(t_n)$, we obtain

$$
\begin{aligned}
A \cap \overline{A}^c_{\Delta t} \cap F \cap G \cap H &\subseteq \left(\overline{A}_1 \cap \overline{A}^c_{\Delta t} \cap F \cap G \cap H\right) \cup \overline{A}_2 \\
&\subseteq \left(\left\{ \sup_{0\le n\Delta t\le T} |S(n\Delta t) - \bar{s}(n\Delta t)| \ge \delta \right\} \cap \left\{\tau_{ijk} > T\right\}\right) \cup \overline{A}_2 \\
&= \left(\left\{ \sup_{0\le n\Delta t\le T} |S(n\Delta t) - s(n\Delta t)| \ge \delta \right\} \cap \left\{\tau_{ijk} > T\right\}\right) \cup \overline{A}_2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathbb{P}\left(A \cap \overline{A}^c_{\Delta t} \cap F \cap G \cap H\right) &\le \mathbb{P}\left(\left\{ \sup_{0\le n\Delta t\le T} |S(n\Delta t) - s(n\Delta t)| \ge \delta \right\} \cap \left\{\tau_{ijk} > T\right\}\right) + \mathbb{P}\left(\overline{A}_2\right) \\
&\le \frac{1}{\delta}\mathbb{E}\left[\sup_{0\le n\Delta t\le T} |S(n\Delta t) - s(n\Delta t)| \cdot \mathbf{1}_{\{\tau_{ijk}>T\}} \right] + \mathbb{P}\left(\overline{A}_2\right) \\
&\le \frac{1}{\delta}\mathbb{E}\left[\sup_{0\le t\le T} |S(t) - s(t)| \cdot \mathbf{1}_{\{\tau_{ijk}>T\}} \right] + \mathbb{P}\left(\overline{A}_2\right) \\
&\le \frac{1}{\delta}\mathbb{E}\left[\sup_{0\le t\le T} \left(e^{(i+k)} |\log S(t) - \log s(t)|\right) \cdot \mathbf{1}_{\{\tau_{ijk}>T\}} \right] + \mathbb{P}\left(\overline{A}_2\right) \\
&\le \frac{1}{\delta} e^{(i+k)} \mathbb{E}\left[\sup_{0\le t\le T} |\log S(t) - \log s(t)| \right] + \mathbb{P}\left(\overline{A}_2\right).
\end{aligned} \tag{5.9}
$$

For the sake of simplicity, let $Z(t) = \int_0^t \left(\mu - \frac{1}{2}V(r)\right) dr + \int_0^t \sqrt{V(r)} dW_1(r)$. By (3.1), for any $t \in [0, T]$,

$$
\begin{aligned}
\mathbb{P}(S(t) = B_U) &= \mathbb{P}(\log S(t) = \log B_U) \\
&= \sum_{k=0}^{\infty} \mathbb{P}\left(Z(t) = \log B_U - \log S_0 - k\log(1+J)\right) \\
&= \sum_{k=0}^{\infty} \mathbb{E}\left[\mathbb{P}\left(Z(t) = \log B_U - \log S_0 - k\log(1+J) \middle| V(r), r \le t\right)\right].
\end{aligned}
$$

Noting that $Z(t)$ is normally distributed with mean $\int_0^t \left(\mu - \frac{1}{2}V(r)\right) dr$ and variance $\int_0^t V(r)dr$ once $V(r), r \le t$ are given, we thus obtain

$$
\mathbb{P}\left(Z(t) = \log B_U - \log S_0 - k\log(1+J) \middle| V(r), r \le t\right) = 0.
$$

By this, we have $\mathbb{P}(S(t) = B_U) = 0$. Since $S(t)$ is a right-continuous process in $t \in [0, T]$, it is not hard to see that

$$\begin{aligned}\mathbb{P}\left(\sup_{0\le t\le T} S(t) = B_U\right) &= \mathbb{P}\left(\sup_{r\in Q\cap[0,T]\cup\{T\}} S(r) = B_U\right)\\ &\le \mathbb{P}\left(\bigcup_{r\in Q\cap[0,T]\cup\{T\}} (S(r) = B_U)\right)\\ &\le \sum_{r\in Q\cap[0,T]\cup\{T\}} \mathbb{P}(S(r) = B_U)\\ &= 0,\end{aligned}$$

where Q is the set of rational numbers. So we may choose δ so small that

$$\mathbb{P}\left(\overline{A}_2\right) < \frac{\varepsilon}{8}.$$

Furthermore, by Theorem 3, we can choose Δt so small that

$$\frac{1}{\delta} e^{(i+k)} \mathbb{E}\left[\sup_{0\le t\le T} |\log S(t) - \log s(t)|\right] < \frac{3\varepsilon}{4},$$

and hence,

$$\mathbb{P}\left(A \cap \overline{A}^c_{\Delta t} \cap F \cap G \cap H\right) < \frac{7\varepsilon}{8}.$$

This, together with (5.8), we obtain that $\mathbb{P}(A \cap \overline{A}^c_{\Delta t}) \to 0$ as $\Delta t \to 0$. Namely

$$\lim_{\Delta t\to 0} \mathbb{P}\left(A \cap \overline{A}^c_{\Delta t}\right) = 0.$$

Finally, we show that $\mathbb{P}\left(A^c \cap \overline{A}_{\Delta t}\right) \to 0$ as $\Delta t \to 0$. For any $\delta > 0$, we have

$$\begin{aligned}A^c &= \left\{\sup_{0\le t\le T} S(t) > B_U\right\}\\ &= \left\{\sup_{0\le t\le T} S(t) > B_U + \delta\right\} \cup \left\{B_U < \sup_{0\le t\le T} S(t) \le B_U + \delta\right\}\\ &=: \overline{A}_3 \cup \overline{A}_4.\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{P}\left(A^c \cap \overline{A}_{\Delta t}\right) &\le \mathbb{P}\left(\overline{A}_3 \cap \overline{A}_{\Delta t}\right) + \mathbb{P}\left(\overline{A}_4 \cap \overline{A}_{\Delta t}\right)\\ &\le \mathbb{P}\left(\sup_{0\le t\le T} |S(t) - \bar{s}(t)| > \delta\right) + \mathbb{P}\left(\overline{A}_4\right).\end{aligned} \tag{5.10}$$

We may easily observe that

$$\left\{\sup_{0\le t\le T} |S(t) - \bar{s}(t)| > \delta\right\} \subseteq \left\{\sup_{0\le t\le T} |S(t) - s(t)| > \tfrac{1}{2}\delta\right\} \cup \left\{\sup_{0\le t\le T} |s(t) - \bar{s}(t)| > \tfrac{1}{2}\delta\right\}$$

$$=: Q_1 \cup Q_2. \tag{5.11}$$

Then in the same way as (5.9), we may choose Δt so small that

$$\begin{aligned}
\mathbb{P}(Q_1) &\le \mathbb{P}(Q_1 \cap F \cap G \cap H) + \mathbb{P}(Q_1 \cap (F^c \cup G^c \cup H^c)) \\
&\le \mathbb{P}\left(\left\{\sup_{0\le t\le T} |S(t) - s(t)| > \tfrac{1}{2}\delta\right\} \cap \{\tau_{ijk} > T\}\right) + \frac{\varepsilon}{8} \\
&\le \frac{2}{\delta} e^{(i+k)} \mathbb{E}\left[\sup_{0\le t\le T} |\log S(t) - \log s(t)|\right] + \frac{\varepsilon}{8} \\
&< \frac{\varepsilon}{4}.
\end{aligned} \tag{5.12}$$

Now we consider $\mathbb{P}(Q_2)$. Let $N = [T/\Delta t]$. Define

$$\bar{S}(t) = \sum_{n=0}^{N} S(n\Delta t)\mathbf{1}_{\{n\Delta t \le t < (n+1)\Delta t\}}, \quad 0 \le t \le T.$$

We also have

$$\begin{aligned}
\left\{\sup_{0\le t\le T} |s(t) - \bar{s}(t)| > \tfrac{1}{2}\delta\right\} &\subseteq \left\{\sup_{0\le t\le T} |\bar{S}(t) - s(t)| > \tfrac{1}{4}\delta\right\} \cup \left\{\sup_{0\le t\le T} |\bar{S}(t) - \bar{s}(t)| > \tfrac{1}{4}\delta\right\} \\
&\subseteq \left\{\sup_{0\le t\le T} |S(t) - s(t)| > \tfrac{1}{4}\delta\right\} \cup \left\{\sup_{0\le n\le N} |S(n\Delta t) - s(n\Delta t)| > \tfrac{1}{4}\delta\right\}.
\end{aligned}$$

Thus,

$$\mathbb{P}\left(\sup_{0\le t\le T} |s(t) - \bar{s}(t)| > \tfrac{1}{2}\delta\right) \le \mathbb{P}\left(\sup_{0\le t\le T} |S(t) - s(t)| > \tfrac{1}{4}\delta\right) + \mathbb{P}\left(\sup_{0\le n\le N} |S(n\Delta t) - s(n\Delta t)| > \tfrac{1}{4}\delta\right). \tag{5.13}$$

Similar to (5.12), we can choose Δt so small that

$$\mathbb{P}\left\{\sup_{0\le t\le T} |S(t) - s(t)| > \tfrac{1}{4}\delta\right\} < \frac{\varepsilon}{8}$$

and

$$\mathbb{P}\left(\sup_{0\le n\le N} |S(n\Delta t) - s(n\Delta t)| > \tfrac{1}{4}\delta\right) < \frac{\varepsilon}{8}.$$

We therefore have

$$\mathbb{P}(Q_2) = \mathbb{P}\left(\sup_{0\le t\le T} |s(t) - \bar{s}(t)| > \tfrac{1}{2}\delta\right) < \frac{\varepsilon}{4}.$$

This, together with (5.11) and (5.12), yields

$$\mathbb{P}\left(\sup_{0\le t\le T} |S(t) - \bar{s}(t)| > \delta\right) < \frac{\varepsilon}{2}. \tag{5.14}$$

Recalling the definition of $\overline{A}_4$, for any $\varepsilon > 0$, we can find a sufficiently small $\delta > 0$ such that

$$\mathbb{P}\left(\overline{A}_4\right) < \frac{\varepsilon}{2}. \tag{5.15}$$

Substituting (5.14) and (5.15) into (5.10) yields

$$\mathbb{P}\left(A^c \cap \overline{A}_{\Delta t}\right) < \varepsilon.$$

In other words, we have shown that

$$\lim_{\Delta t \to 0} \mathbb{P}\left(A^c \cap \overline{A}_{\Delta t}\right) = 0.$$

This completes the proof of the theorem. □

6. Simulations

In this section, a European put option and an up-and-out barrier call option are considered, and their simulation results are given to verify the theoretical results.

For the European put option, the initial states are set to be $S_0 = 50$, $V_0 = 0.001$. We choose parameters $K = 55$, $\mu = 0.03$, $J = 1$, $\kappa = 2$, $\theta = 0.01$, $\gamma = 0.05$, $\tau = 0.002$, $\lambda_1 = 0.50$, $\lambda_2 = 0.4$, and $T = 1$ year. Table 1 shows the corresponding simulation results. For the up-and-out barrier call option, the initial states are set to be $S_0 = 60$, $V_0 = 0.002$, and the parameters are set as $K = 50$, $B_U = 80$, $\mu = 0.05$, $J = 1$, $\kappa = 2.2$, $\theta = 0.02$, $\gamma = 0.08$, $\tau = 0.003$, $\lambda_1 = 0.60$, $\lambda_2 = 0.5$, and $T = 1$ year. Table 2 illustrates the corresponding simulation results. The simulation results clearly verify the effectiveness of the theoretical results.

It should be noted that the standard error (SE) decreases with increased simulation counts and reduced step sizes, albeit at the cost of extended computation time. As demonstrated in Tables 1 and 2, a 1% error margin can be achieved with a computation time of approximately half an hour, which remains operationally acceptable.

Table 1. Simulation results under the SVJD model for a European put option.

No. of simulation trials	No. of time steps	Standard error	Computing time (sec.)
5,000	40	0.0441	2.9
20,000	100	0.0219	29.8
80,000	320	0.0110	377.8
160,000	640	0.0078	1543.5

Table 2. Simulation results under the SVJD model for an up-and-out barrier call option.

No. of simulation trials	No. of time steps	Standard error	Computing time (sec.)
5,000	40	0.0925	4.2
20,000	100	0.0458	40.8
80,000	320	0.0230	534.5
160,000	640	0.0163	2124.5

7. Conclusions

This paper investigated the strong convergence of the Euler-Maruyama (EM) method for the SVJD model. The SVJD model describes an underlying asset price governed by a jump-diffusion geometric Brownian motion with stochastic volatility, where the volatility process follows a mean-reverting square root process incorporating Poisson jumps. By applying Tanaka's formula and the comparison theorem, we established the existence and uniqueness of nonnegative solutions for the SVJD model, along with the moment properties of the solutions. Due to the absence of an explicit solution for the SVJD model, the EM method was employed to approximate the exact solution, with its strong convergence in the L_2 sense rigorously proven. Furthermore, the EM approximation was applied to numerically estimate expected payoffs for a European option and a barrier option.

Author contributions

Weiwei Shen: Conceptualization, Methodology, Writing-original draft, Writing review and editing; Yan Zhang: Validation, Writing review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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