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*Research article*

# The velocity averaging lemma to the relativistic free transport equation

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**Abstract:** In this paper, we revisit a velocity averaging lemma for the relativistic free transport equation using a modified vector field method. After averaging with respect to the velocity of the solution by certain weight functions $\varphi$, we demonstrate that the averaged quantity $\rho_\varphi(t,x)$ belongs to the Sobolev space $W^{1,p}_x$ for $p \in [1,+\infty]$. This result reveals the regularizing effect of the velocity averaging of the solution. Furthermore, we also show the quantitative effects of both the particle mass and the speed of light. The proof relies on the key observation that the differential operator $t\nabla_x + [\nabla_v(\hat{v})]^{-1}\nabla_v$ commutes with the operator $\partial_t + \hat{v}\cdot\nabla_x$.

**Keywords:** kinetic equation; velocity averaging lemma; relativistic transport equation; regularity; vector field method
**Mathematics Subject Classification:** 35Q75, 76P05, 83C30

## 1. Introduction

In this paper, we consider the velocity averaging lemma to the following relativistic free transport equation:

$$\partial_t f + \hat{v}\cdot\nabla_x f = 0\,, \qquad (t,x,v) \in \mathbb{R}^+ \times \Omega_x \times \mathbb{R}^3_v\,, \tag{1.1}$$

$$f(0,x,v) = f_0(x,v) \geq 0\,, \qquad (x,v) \in \Omega_x \times \mathbb{R}^3_v\,, \tag{1.2}$$

where the function $f = f(t,x,v) \geq 0$ describes the gas density distribution of particles at the time $t \in \mathbb{R}^+$, the space $x = (x_1,x_2,x_3) \in \Omega_x$, with the velocity $v = (v_1,v_2,v_3) \in \mathbb{R}^3_v$. Here, the spatial domain $\Omega_x$ is either the whole space $\mathbb{R}^3_x$ or the 3-dimensional torus $\mathbb{T}^3_x$. The relativistic velocity $\hat{v}$ is given by

$$\hat{v} = \frac{cv}{\sqrt{m^2c^2 + |v|^2}}\,,$$

with the particle mass $m > 0$ and the speed of light $c > 0$.

The solution $f$ of problems (1.1) and (1.2) can be solved explicitly via the method of characteristics. Then

$$f(t,x,v) = f_0(x - \hat{v}t, v)\,, \quad (x \in \Omega_x,\ v \in \mathbb{R}^3_v,\ t \geq 0)\,.$$

In this paper, our main contribution is to provide a simple method based on the modified vector field to prove the velocity averaging lemma of the problems (1.1) and (1.2). This is twofold:

(1) We adapted the vector field that was originally introduced by Gualdani, Mischler, and Mouhot [11] (see Lemma 4.17) to the relativistic case. The key point is employ the differential operator $t\,\nabla_x + [\nabla_v(\hat{v})]^{-1}\nabla_v$ that commutes with the operator $\partial_t + \hat{v}\cdot\nabla_x$. This observation will play an important role in the proof of our result.
(2) We show the quantitative effects of the particle mass $m$ and the speed of light $c$.

The kinetic theory of gases focuses on studying the evolutionary behavior of gases in the one-particle phase space of position and velocity. The celebrated velocity averaging lemmas address regularity results of solutions to kinetic transport equations. These lemmas reveal that the combination of the transport operator and averaging over the velocity variable $v$ of the solution yields some regularity with respect to the spatial variable $x$ (see [7, 9, 10]). Such results serve as a powerful mathematical tool in kinetic theory, and they have been widely applied to study regularity, global solutions, spectral analysis, and hydrodynamic limits of kinetic equations. Numerous generalizations of velocity averaging lemmas have been developed, including extensions to phenomena such as dispersion and hypoellipticity. It is worth noting that several approaches have been employed to prove velocity averaging lemmas, including the Fourier transform, Hörmander's commutators, the commutator method, harmonic analysis, the energy method, and the real space method (see [1,4,14]). For further results, interested readers may refer to [2,5,6] and the references therein. This regularity is also very useful for the mathematical study of the Navier-Stokes equations, Maxwell's equations, and Einstein's field equations (see [12,17,18]).

If one considers the solutions $f \in L^2$ of the initial-value problem for the classical free transport equation:

$$\partial_t f + v\cdot\nabla_x f = 0\,, \tag{1.3}$$

then $\rho_\varphi \in H^{\frac{1}{2}}_x$ for any $\varphi(v) \in C^\infty_c(\mathbb{R}^3_v)$. Here $H^{\frac{1}{2}}_x$ denotes the usual fractional-order Sobolev space defined by the Fourier transform. We note that there is no regularity assumption on the initial datum. See also DiPerna, Lions, and Meyer [7] for general $L^p$ $(1 < p < +\infty)$ by applying the interpolation method. In [11], Gualdani, Mischler, and Mouhot proved that $\rho_\varphi \in W^{1,p}_x$ for $p \in [1, +\infty]$. They obtain a full derivative in the $x$ variable that is stronger than the previous half-derivative, but they assumed some extra regularity in $v$ of the initial datum.

Compared with the classical free transport Eq (1.3), the relativistic free transport Eq (1.1) has a relatively short research history. In 2017, Huang and Jiang [13] examined the average regularity of solutions to the relativistic transport equation, employing methodology similar to that in [9]. Fajman, Joudioux, and Smulevici [8] made significant progress by adapting Klainerman's vector field method. They introduced the concept of complete lifts that commute with both massive and massless relativistic transport operators, thereby establishing sharp decay estimates for velocity averages of solutions. Subsequently, Bigorgne [3] advanced this line of research by eliminating the need for complete lifts of Lorentz boost vector fields, deriving sharp decay estimates specifically for spherically symmetric small data solutions of the relativistic massless Vlasov-Poisson system. Despite these developments, it remains a question whether results analogous to those in [11] can be obtained for the relativistic free transport Eqs (1.1) and (1.2). This paper aims to solve this problem in the literature. To our knowledge, our work presents the first velocity averaging lemma specifically developed for the relativistic free transport equation, marking a contribution to this field of study.

Before stating our result, we give the following notations. Let $\langle v\rangle^k = \left(1+|v|^2\right)^{\frac{k}{2}}$, for $k \geq 0$ and $v \in \mathbb{R}^3_v$. We denote the weighted space $\|f\|_{L^q_v(\langle v\rangle^k)}$ by the weighted norm for the velocity variable $v$ :

$$\|f\|_{L^q_v(\langle v\rangle^k)} = \left(\int_{\mathbb{R}^3_v} |f|^q \langle v\rangle^{qk}\,\mathrm{d}v\right)^{\frac{1}{q}},\ q \in [1,+\infty), \quad \text{and} \quad \|f\|_{L^\infty_v(\langle v\rangle^k)} = \sup_{v\in\mathbb{R}^3_v} |f(v)|\,\langle v\rangle^k .$$

The higher-order Sobolev space $W^{\sigma,p}_x$ for $\sigma \in \mathbb{N}$ is defined by

$$\|f\|_{W^{\sigma,p}_x} = \sum_{|\alpha|\leq\sigma} \|\partial^\alpha_x f\|_{L^p_x}.$$

Furthermore, we define $L^q_v L^p_x(\langle v\rangle^k)$ with $p, q \in [1,+\infty]$ through the norm

$$\|f\|_{L^q_v L^p_x(\langle v\rangle^k)} = \left\|\|f\|_{L^p_x}\langle v\rangle^k\right\|_{L^q_v}.$$

For any fixed weighted function $\varphi(v) \in C^\infty_c(\mathbb{R}^3_v)$, let us define the average quantity $\rho_\varphi(t,x)$ as

$$\rho_\varphi(t,x) = \int_{\mathbb{R}^3_v} f(t,x,v)\,\varphi(v)\,\mathrm{d}v\,,$$

where $C^\infty_c(\mathbb{R}^3_v)$ is the space of infinitely differentiable functions with compact support.

The result of our paper can be stated as follows.

**Theorem 1.1.** *Assume that the problems* (1.1) *and* (1.2) *have a solution* $f \in L^1([0,T]; L^1_v L^p_x(\langle v\rangle^3))$ *satisfying* $f_0 \in L^1_v L^p_x(\langle v\rangle^3)$ *and* $\nabla_v f_0 \in L^1_v L^p_x(\langle v\rangle^3)$ *with* $p \in [1,+\infty]$, *then we obtain the following estimates:*

*(1) If* $mc \geq 1$, *the average quantity* $\rho_\varphi$ *satisfies:*

$$\left\|\rho_\varphi\right\|_{W^{1,p}_x} \leq \left(1+\frac{3m}{t}\right)\|\varphi\|_{W^{1,\infty}_v}\left(\|f_0\|_{L^1_v L^p_x(\langle v\rangle^3)} + \|\nabla_v f_0\|_{L^1_v L^p_x(\langle v\rangle^3)}\right),$$

*(2) If* $0 < mc < 1$, *the average quantity* $\rho_\varphi$ *satisfies:*

$$\left\|\rho_\varphi\right\|_{W^{1,p}_x} \leq \left(1+\frac{\frac{3}{m^2c^3}}{t}\right)\|\varphi\|_{W^{1,\infty}_v}\left(\|f_0\|_{L^1_v L^p_x(\langle v\rangle^3)} + \|\nabla_v f_0\|_{L^1_v L^p_x(\langle v\rangle^3)}\right).$$

Our strategy for the proof of Theorem 1.1 relies on establishing a connection between derivatives in the spatial variable $x$ and those in the velocity variable $v$. A key insight comes from the work of Gualdani, Mischler, and Mouhot [11], who demonstrated that the operator $t\nabla_x + \nabla_v$ commutes with the classical transport operator $\partial_t + v\cdot\nabla_x$. This differential operator plays a pivotal role in transferring regularity between the spatial variable $x$ and the velocity variable $v$. However, extending this idea to the relativistic case presents a challenge, as finding an analogous operator that commutes with the relativistic transport operator $\partial_t + \hat{v} \cdot \nabla_x$ is non-trivial. A breakthrough was recently achieved by Lin, Lyu, and Wu [15], who observed that $t\,\nabla_x + [\nabla_v(\hat{v})]^{-1}\nabla_v$ commutes with $\partial_t + \hat{v} \cdot \nabla_x$ under normalizing $m = 1$ and $c = 1$. Notably, this result has been extended to more general cases in [16]. This observation will be instrumental in proving Theorem 1.1. Additionally, we will analyze the quantitative effects of the particle mass $m$ and the speed of light $c$.

This paper is organized as follows: In Section 2, we first present $t\,\nabla_x + [\nabla_v(\hat{v})]^{-1}\nabla_v$ can commute with the operator $\partial_t + \hat{v} \cdot \nabla_x$ and show the quantitative effects of the particle mass $m$ and the speed of light $c$. Then we prove Theorem 1.1 by considering two cases: $mc \geq 1$ and $0 < mc < 1$.

## 2. Proof of Theorem 1.1

In order to show the quantitative effects of the particle mass $m$ and the speed of light $c$, we shall give a full proof of the following Lemma 2.1 for completeness.

**Lemma 2.1.** *Let $D_t := t\,\nabla_x + [\nabla_v(\hat{v})]^{-1}\nabla_v$ be a differential operator. Then $D_t$ commutes with $\partial_t + \hat{v} \cdot \nabla_x$.*

*Proof.* By a direct computation, we have

$$\partial_{v_i}\left(\hat{v}_j\right) = \partial_{v_i}\left(\frac{c\,v_j}{\sqrt{m^2c^2 + |v|^2}}\right) = \frac{\left(m^2c^2 + |v|^2\right)\delta_{ij} - v_iv_j}{c^2\left(m^2 + \frac{|v|^2}{c^2}\right)^{\frac{3}{2}}},$$

where $\delta_{ij}$ is unity when the indices are equal and is zero otherwise.

Since

$$\begin{aligned}\nabla_v(\hat{v}) &= (a_{ij})_{1\leq i,j\leq 3}\\ &= \frac{1}{c^2\left(m^2 + \frac{|v|^2}{c^2}\right)^{\frac{3}{2}}}\begin{pmatrix} m^2c^2 + v_2^2 + v_3^2 & -v_1v_2 & -v_1v_3 \\ -v_1v_2 & m^2c^2 + v_1^2 + v_3^2 & -v_2v_3 \\ -v_1v_3 & -v_2v_3 & m^2c^2 + v_1^2 + v_2^2 \end{pmatrix},\end{aligned}$$

then we have $\det(\nabla_v(\hat{v})) = \frac{m^2}{\left(m^2 + \frac{|v|^2}{c^2}\right)^{\frac{5}{2}}}$ and

$$\begin{aligned}[\nabla_v(\hat{v})]^{-1} &= (A_{ij})_{1\leq i,j\leq 3}\\ &= \frac{\left(m^2 + \frac{|v|^2}{c^2}\right)^{\frac{1}{2}}}{m^2c^2}\begin{pmatrix} m^2c^2 + v_1^2 & v_1v_2 & v_1v_3 \\ v_1v_2 & m^2c^2 + v_2^2 & v_2v_3 \\ v_1v_3 & v_2v_3 & m^2c^2 + v_3^2 \end{pmatrix}.\end{aligned}$$

Therefore, for any $i \in \{1, 2, 3\}$, it holds

$$\begin{aligned}&\Big(t\,\partial_{x_i} + (A_{1i}\partial_{v_1} + A_{2i}\partial_{v_2} + A_{3i}\partial_{v_3})\Big)\Big(\partial_t + \hat{v} \cdot \nabla_x\Big)f\\ &= t\,\partial_{x_i}\partial_t f + t\,\hat{v} \cdot \nabla_x(\partial_{x_i} f)\\ &\quad + A_{1i}\Big(\partial_t\partial_{v_1} f + a_{11}\partial_{x_1} f + a_{12}\partial_{x_2} f + a_{13}\partial_{x_3} f + \hat{v} \cdot \nabla_x(\partial_{v_1} f)\Big)\end{aligned}$$

$$
\begin{aligned}
&+ A_{2i}\Big(\partial_t\partial_{v_2} f + a_{21}\partial_{x_1} f + a_{22}\partial_{x_2} f + a_{23}\partial_{x_3} f + \hat{v}\cdot\nabla_x(\partial_{v_2} f)\Big)\\
&+ A_{3i}\Big(\partial_t\partial_{v_3} f + a_{31}\partial_{x_1} f + a_{32}\partial_{x_2} f + a_{33}\partial_{x_3} f + \hat{v}\cdot\nabla_x(\partial_{v_3} f)\Big)\\
=\,& t\,\partial_{x_i}\partial_t f + t\,\hat{v}\cdot\nabla_x(\partial_{x_i} f) + \partial_{x_i} f + A_{1i}\Big(\partial_t\partial_{v_1} f + \hat{v}\cdot\nabla_x(\partial_{v_1} f)\Big)\\
&+ A_{2i}\Big(\partial_t\partial_{v_2} f + \hat{v}\cdot\nabla_x(\partial_{v_2} f)\Big) + A_{3i}\Big(\partial_t\partial_{v_3} f + \hat{v}\cdot\nabla_x(\partial_{v_3} f)\Big)\\
=\,&\Big(\partial_t + \hat{v}\cdot\nabla_x\Big)\Big(t\,\partial_{x_1} + (A_{1i}\partial_{v_1} + A_{2i}\partial_{v_2} + A_{3i}\partial_{v_3})\Big) f\,.
\end{aligned}
$$

Moreover, it is straightforward to check that for each entry $(A_{ij})_{1\le i,j\le 3}$ of the matrix $[\nabla_v(\hat{v})]^{-1}$ satisfies:

$$
|A_{ij}| \le \frac{\left(m^2c^2 + |v|^2\right)^{\frac{3}{2}}}{m^2c^3} \quad \text{and} \quad |\partial_{v_i} A_{ij}| \le 3\,\frac{\left(m^2c^2 + |v|^2\right)^{\frac{3}{2}}}{m^2c^3}\langle v\rangle\,,
$$

then

(i) If $mc \ge 1$ : $|A_{ij}| \le m\,\langle v\rangle^3, \quad |\partial_{v_i} A_{ij}| \le 3\,m\,\langle v\rangle^2\,,$
(ii) If $mc < 1$ : $|A_{ij}| \le \frac{1}{m^2c^3}\,\langle v\rangle^3, \quad |\partial_{v_i} A_{ij}| \le \frac{3}{m^2c^3}\,\langle v\rangle^2\,.$

□

*Proof of Theorem 1.1.* We first claim that for any $t \ge 0$, $p, q \in [1, +\infty]$ and $k \ge 0$, the solution $f$ of the problems (1.1) and (1.2) verifies

$$
\|f(t,x,v)\|_{L^q_v L^p_x(\langle v\rangle^k)} = \|f_0(x,v)\|_{L^q_v L^p_x(\langle v\rangle^k)}\,, \tag{2.1}
$$

and

$$
\|D_t f(t,x,v)\|_{L^q_v L^p_x(\langle v\rangle^k)} = \left\|D_{\{t=0\}} f_0(x,v)\right\|_{L^q_v L^p_x(\langle v\rangle^k)} = \left\|[\nabla_v(\hat{v})]^{-1}\nabla_v f_0(x,v)\right\|_{L^q_v L^p_x(\langle v\rangle^k)}\,, \tag{2.2}
$$

where $D_t = t\,\nabla_x + [\nabla_v(\hat{v})]^{-1}\nabla_v$ and $D_{\{t=0\}} = [\nabla_v(\hat{v})]^{-1}\nabla_v$ given by Lemma 2.1.

Indeed, for any $p, q \in [1, +\infty)$, it holds

$$
\begin{aligned}
\frac{d}{dt}\|f\|_{L^q_v L^p_x(\langle v\rangle^k)} =& \frac{d}{dt}\left(\int_{\mathbb{R}^3_v}\left(\int_{\Omega_x}|f|^p\,dx\right)^{\frac{q}{p}}\langle v\rangle^{qk}\,\mathrm{d}v\right)^{\frac{1}{q}}\\
=&\|f\|^{1-q}_{L^q_v L^p_x(\langle v\rangle^k)}\int_{\mathbb{R}^3_v}\|f\|^{q-p}_{L^p_x}\Big(\int_{\Omega_x}|f|^{p-1}\,\mathrm{sign}(f)\,\partial_t f\,dx\Big)\langle v\rangle^{qk}\,\mathrm{d}v\\
=&-\|f\|^{1-q}_{L^q_v L^p_x(\langle v\rangle^k)}\int_{\mathbb{R}^3_v}\|f\|^{q-p}_{L^p_x}\Big(\int_{\Omega_x}\frac{1}{p}\,\hat{v}\cdot\nabla_x(|f|^p)\,dx\Big)\langle v\rangle^{qk}\,\mathrm{d}v\\
=&0\,.
\end{aligned} \tag{2.3}
$$

Letting the limits $p \to +\infty$ and $q \to +\infty$ in (2.3), then the cases $p = +\infty$ and $q = +\infty$ also hold true.

Consequently,

$$
\|f\|_{L^q_v L^p_x(\langle v\rangle^k)} = \|f_0\|_{L^q_v L^p_x(\langle v\rangle^k)}\,.
$$

According to Lemma 2.1, we know that $D_t = t\nabla_x + [\nabla_v(\hat{v})]^{-1}\nabla_v$ commutes with $\partial_t + \hat{v}\cdot\nabla_x$. Thus, we have

$$\partial_t(D_t f) + \hat{v}\cdot\nabla_x(D_t f) = 0\,.$$

By taking the similar arguments to (2.3), it turns out that

$$\frac{d}{dt}\,\|D_t f\|_{L^q_v L^p_x(\langle v\rangle^k)} = -\,\|D_t f\|^{1-q}_{L^q_v L^p_x(\langle v\rangle^k)} \int_{\mathbb{R}^3_v} \|D_t f\|^{q-p}_{L^p_x}\Big(\int_{\Omega_x}\frac{1}{p}\,\hat{v}\cdot\nabla_x(|D_t f|^p)\,dx\Big)\langle v\rangle^{qk}\,\mathrm{d}v = 0\,.$$

Thus,

$$\|D_t f\|_{L^q_v L^p_x(\langle v\rangle^k)} = \left\|D_{\{t=0\}} f_0\right\|_{L^q_v L^p_x(\langle v\rangle^k)} = \left\|[\nabla_v(\hat{v})]^{-1}\nabla_v f_0\right\|_{L^q_v L^p_x(\langle v\rangle^k)}\,.$$

Therefore, our claim is proved.

**Step 1.** The $L^p_x$-norm of $\rho_\varphi$. By using Minkowski's integral inequality and estimate (2.1), we obtain that, for any $1 \le p < +\infty$,

$$\begin{aligned}
\|\rho_\varphi\|_{L^p_x} &= \left(\int_{\Omega_x}\left|\int_{\mathbb{R}^3_v} f(t,x,v)\,\varphi(v)\,\mathrm{d}v\right|^p \mathrm{d}x\right)^{\frac{1}{p}} \\
&\le \int_{\mathbb{R}^3_v}\left(\int_{\Omega_x}|f(t,x,v)\,\varphi(v)|^p\,dx\right)^{\frac{1}{p}}\mathrm{d}v \\
&\le \|\varphi\|_{L^\infty_v}\|f\|_{L^1_v L^p_x} \\
&= \|\varphi\|_{L^\infty_v}\|f_0\|_{L^1_v L^p_x}\,.
\end{aligned} \tag{2.4}$$

The case $p = +\infty$ can be proved by the straightforward calculations as the reasoning above.

**Step 2.** The derivatives in $x$. For clarity, we shall divide the argument about $mc$ into the following two cases: $mc \ge 1$ and $0 < mc < 1$.

On the one hand, for the case $mc \ge 1$. We first compute the $x$-derivatives of $\rho_\varphi$ as follows: For any $i \in \{1,2,3\}$,

$$\begin{aligned}
\partial_{x_i}\rho_\varphi &= \int_{\mathbb{R}^3_v}\partial_{x_i} f\,\varphi(v)\,\mathrm{d}v \\
&= \int_{\mathbb{R}^3_v}\frac{1}{t}\Big[D_{t_i} - (A_{1i}\partial_{v_1} + A_{2i}\partial_{v_2} + A_{3i}\partial_{v_3})\Big] f\,\varphi(v)\,\mathrm{d}v \\
&= \frac{1}{t}\int_{\mathbb{R}^3_v} D_{t_i} f\,\varphi(v)\,dv + \frac{1}{t}\int_{\mathbb{R}^3_v} f\Big[\partial_{v_1}(A_{1i}\varphi(v)) + \partial_{v_2}(A_{2i}\varphi(v)) + \partial_{v_3}(A_{3i}\varphi(v))\Big]\mathrm{d}v \\
&\le \frac{1}{t}\int_{\mathbb{R}^3_v}|D_{t_i} f|\,|\varphi(v)|\,\mathrm{d}v \\
&\quad + \frac{1}{t}\int_{\mathbb{R}^3_v}|f|\Big[m\,\langle v\rangle^3(|\partial_{v_1}\varphi(v)| + |\partial_{v_2}\varphi(v)| + |\partial_{v_3}\varphi(v)|) + 3m\,\langle v\rangle^2|\varphi(v)|\Big]\mathrm{d}v \\
&\le \frac{1}{t}\,\|\varphi(v)\|_{L^\infty_v}\|D_{t_i} f\|_{L^1_v} + \frac{3m}{t}\,\|\varphi(v)\|_{W^{1,\infty}_v}\|f\|_{L^1_v(\langle v\rangle^3)}\,,
\end{aligned} \tag{2.5}$$

where $D_{t_i} f = \partial_{x_i} f + \Big[(A_{1i}\partial_{v_1} + A_{2i}\partial_{v_2} + A_{3i}\partial_{v_3})\Big] f$ is the i-th component of the vector $D_t f$.

Next, by the Minkowski's integral inequality and equality (2.2), it holds

$$
\begin{aligned}
\|\partial_{x_i}\rho_\varphi\|_{L^p_x} \leq & \frac{1}{t}\, \|\varphi(v)\|_{L^\infty_v} \left(\int_{\Omega_x} \|D_{t_i} f\|^p_{L^1_v}\, \mathrm{d}x\right)^{\frac{1}{p}} + \frac{3m}{t}\, \|\varphi(v)\|_{W^{1,\infty}_v} \left(\int_{\Omega_x} \|f\|^p_{L^1_v(\langle v\rangle^3)}\, \mathrm{d}x\right)^{\frac{1}{p}} \\
\leq & \frac{1}{t}\, \|\varphi(v)\|_{L^\infty_v} \|D_{t_i} f\|_{L^1_v L^p_x} + \frac{3m}{t}\, \|\varphi(v)\|_{W^{1,\infty}_v} \|f\|_{L^1_v L^p_x(\langle v\rangle^3)} \\
= & \frac{1}{t}\, \|\varphi(v)\|_{L^\infty_v} \left\|(A_{1i}\partial_{v_1} + A_{2i}\partial_{v_2} + A_{3i}\partial_{v_3}) f_0\right\|_{L^1_v L^p_x} + \frac{3m}{t}\, \|\varphi(v)\|_{W^{1,\infty}_v} \|f_0\|_{L^1_v L^p_x(\langle v\rangle^3)} \\
\leq & \frac{m}{t}\, \|\varphi(v)\|_{L^\infty_v} \|\nabla_v f_0\|_{L^1_v L^p_x} + \frac{3m}{t}\, \|\varphi(v)\|_{W^{1,\infty}_v} \|f_0\|_{L^1_v L^p_x(\langle v\rangle^3)} \\
\leq & \frac{3m}{t}\, \|\varphi(v)\|_{W^{1,\infty}_v} \left(\|f_0\|_{L^1_v L^p_x(\langle v\rangle^3)} + \|\nabla_v f_0\|_{L^1_v L^p_x(\langle v\rangle^3)}\right). \qquad (2.6)
\end{aligned}
$$

We now gather (2.4) and (2.6) together to deduce that, for any $p \in [1, +\infty]$,

$$
\begin{aligned}
\|\rho_\varphi\|_{W^{1,p}_x} = & \|\rho_\varphi\|_{L^p_x} + \sum_{i=1}^{3} \|\partial_{x_i}\rho_\varphi\|_{L^p_x} \\
\leq & \left(1 + \frac{3m}{t}\right) \|\varphi(v)\|_{W^{1,\infty}_v} \left(\|f_0\|_{L^1_v L^p_x(\langle v\rangle^3)} + \|\nabla_v f_0\|_{L^1_v L^p_x(\langle v\rangle^3)}\right).
\end{aligned}
$$

On the other hand, for the case $0 < mc < 1$. Applying the same procedure as in the proof of (2.5) and (2.6), we find that

$$
\begin{aligned}
\|\partial_{x_i}\rho_\varphi\|_{L^p_x} \leq & \frac{1}{t}\, \|\varphi(v)\|_{L^\infty_v} \|(A_{1i}\partial_{v_1} + A_{2i}\partial_{v_2} + A_{3i}\partial_{v_3}) f_0\|_{L^1_v L^p_x} + \frac{\frac{3}{m^2c^3}}{t}\, \|\varphi(v)\|_{W^{1,\infty}_v} \|f_0\|_{L^1_v L^p_x(\langle v\rangle^3)} \\
\leq & \frac{\frac{1}{m^2c^3}}{t}\, \|\varphi(v)\|_{L^\infty_v} \|\nabla_v f_0\|_{L^1_v L^p_x} + \frac{\frac{3}{m^2c^3}}{t}\, \|\varphi(v)\|_{W^{1,\infty}_v} \|f_0\|_{L^1_v L^p_x(\langle v\rangle^3)} \\
\leq & \frac{\frac{3}{m^2c^3}}{t}\, \|\varphi(v)\|_{W^{1,\infty}_v} \left(\|f_0\|_{L^1_v L^p_x(\langle v\rangle^3)} + \|\nabla_v f_0\|_{L^1_v L^p_x(\langle v\rangle^3)}\right). \qquad (2.7)
\end{aligned}
$$

Combining (2.4) with (2.7) yields that, for any $p \in [1, +\infty]$,

$$
\|\rho_\varphi\|_{W^{1,p}_x} \leq \left(1 + \frac{\frac{3}{m^2c^3}}{t}\right) \|\varphi\|_{W^{1,\infty}_v} \left(\|f_0\|_{L^1_v L^p_x(\langle v\rangle^3)} + \|\nabla_v f_0\|_{L^1_v L^p_x(\langle v\rangle^3)}\right).
$$

Hence the proof of Theorem 1.1 is finished. □

## 3. Conclusions

This paper revisits a velocity averaging lemma for the relativistic free transport equation by employing a modified vector field method. We prove that the averaged quantity $\rho_\varphi(t, x) \in W^{1,p}_x$ for $p \in [1, +\infty]$. The key contributions of this work are twofold: (1) Relativistic adaptation of the vector field method: Building upon the framework introduced by Gualdani, Mischler, and Mouhot [11], we extend their approach to the relativistic case. A crucial observation is the use of the differential operator $t\,\nabla_x + [\nabla_v(\hat{v})]^{-1}\nabla_v$ which commutes with the relativistic free transport operator $\partial_t + \hat{v}\cdot\nabla_x$. This property plays an important role in our analysis. (2) Quantitative dependence on physical parameters: We rigorously analyze the quantitative effects of the particle mass $m$ and the speed of light $c$, providing explicit estimates that highlight their roles in the averaging process. These regularity results have direct applications in establishing existence, uniqueness, regularity, and asymptotic behavior for some relativistic kinetic equations. As an extension of this work, the study of the Cauchy problem on the relativistic Boltzmann equation is currently underway.

## Author contributions

Baoyan Sun: Methodology, Conceptualization, Writing-original draft preparation, Supervision, Writing, Resources; Man Wu: Supervision, Conceptualization, Validation, Reviewing, Formal analysis, Editing. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare no conflict of interest.

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