AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(4): 9342–9368.
DOI: 10.3934/math.2025432
Received: 01 February 2025
Revised: 08 April 2025
Accepted: 18 April 2025
Published: 23 April 2025

Research article

AI-driven controllability analysis of fractional impulsive neutral Volterra-Fredholm integro-differential equations with state-dependent delay

Prabakaran Raghavendran[1], **Tharmalingam Gunasekar**[1,2], **Irshad Ayoob**[3] **and Nabil Mlaiki**[3,*]

[1] Department of Mathematics, Vel Tech Rangarajan Dr. Sagunthala R&D Institute of Science and Technology, Chennai 600062, Tamil Nadu, India

[2] Department of Mathematics, Institute of Engineering and Technology, Srinivas University, Mukka, Mangaluru, Karnataka 574146, India

[3] Department of Mathematics and Sciences, Prince Sultan University, 11586 Riyadh, Saudi Arabia

* **Correspondence:** Email: nmlaiki@psu.edu.sa, nmlaiki2012@gmail.com.

Abstract: This paper examined the controllability of fractional impulsive neutral Volterra-Fredholm integro-differential equations with state-dependent delay, employing the Caputo fractional derivative and a semigroup of compact and analytic operators. Controllability results were established by using Schauder's fixed point theorem, addressing the complexities arising from fractional dynamics combined with state-dependent delays. The theoretical findings were further confirmed through a detailed example and numerical simulations that show convergence of the solutions. Also, the role of artificial intelligence in the analysis and control of such systems governed by these equations was investigated, thereby opening up opportunities for machine learning to be coupled with fractional calculus for a better predictive solution and better control of systems. These results provide some insight into stability as well as controllability of systems governed by fractional differential equations with impulsive and state-dependent behaviors.

Keywords: Caputo fractional derivative; Schauder's fixed point theorem; controllability; artificial intelligence; system control
Mathematics Subject Classification: 34A08, 45J05, 65L20, 93C23, 68T07

1. Introduction

The fractional Volterra-Fredholm integro-differential equations (VFIDEs) with a state-dependent delay (SDD) have become highly prominent in study due to the wide range of applications in natural sciences and engineering. This class of equations is characterized by the fractional derivative, which better models the nature of these memory-sensitive and hereditary systems. A crucial source to study fractional differential equations is the work by Zhou, which introduces readers to a deep insight of the fundamental theory of the said equations along with key ideas and mathematical representations [1]. Extensive works done by Kilbas, Srivastava, and Trujillo elaborate more on the theory and application of such differential equations that can be used as the cornerstone of study of fractional calculus [2, 3]. In fractional integro-differential equations with SDD, Agarwal and Andrade explain the different challenges that this system of equations provides. Their contribution highlights the intricacy of modeling systems where the future behavior is dependent on the current state as well as the previous states [4]. Similarly, Benchohra and Berhoun's work further expands this knowledge by investigating impulsive fractional differential equations with SDD, which contributes much to the study of dynamic systems with sudden jumps and memory [5]. Guendouzi and Bousmaha's work on the approximate controllability of fractional neutral stochastic functional integro-differential inclusions with infinite delay shows that controllability is crucial in fractional systems, especially in uncertain systems and time-delayed systems [6]. Liu and Bin's work on impulsive Riemann-Liouville fractional differential inclusions provides an essential perspective on how impulsive effects interact with fractional dynamics, adding complexity to the solution methods [7].

Balasubramaniam and Tamilalagan's study on fractional neutral stochastic integro-differential inclusions contributes to the literature by exploring approximate controllability through Mainardi's function, an important tool for addressing delays and stochasticity in fractional systems [8]. Podlubny's influential book on fractional differential equations presents a detailed treatment of the theory and applications of fractional calculus, making it an essential reference for understanding the foundational concepts used throughout this study [9]. Mainardi, Paradisi, and Gorenflo's work on probability distributions generated by fractional diffusion equations explores the connection between fractional derivatives and diffusion processes, providing valuable insights into the behavior of fractional systems [10]. In more recent studies, the existence and controllability of neutral fractional VFIDEs are carried out by Gunasekar et al. while shedding light over the complexities presented by these kinds of systems to present new results for their solutions and control [11]. Similarly, the application of the Mohand transform to solve fractional integro-differential equations by Gunasekar and Raghavendran provides valuable contributions regarding the dynamics represented by this kind of equation [12].

Hamoud's work on the existence and uniqueness of solutions for fractional neutral VFIDEs provides key results in the theory of such systems, which ultimately guarantees well-posedness of these equations [13]. The further existence and uniqueness results for VFIDEs by Hamoud, Mohammed, and Ghadle address the essential technical contributions in this field [14]. For instance, Raghavendran et al. introduced the Aboodh transform to solve fractional integro-differential equations [15]. This work provided a new approach to obtaining solutions, advancing the methodology used to handle such complex systems. Gunasekar et al. also contributed to this line of research by applying Laplace transforms on fractional integro-differential equations, further expanding the toolbox of techniques used to solve these equations as well as shedding light on various real applications of this technique [16]. Gunasekar and colleagues' recent analysis of existence, uniqueness, and stability of neutral fractional VFIDEs presents important results that contribute toward the theoretical foundations needed to be able to understand the stability of such complex systems [17]. Columbu, Frassu, and Viglialoro introduced refined criteria of boundedness related to chemotaxis systems relevant in the context of fractional nonlinear dynamics systems studies [18]. Hamoud and Ghadle's latest results regarding the uniqueness of solutions for fractional VFIDEs bring in new perspectives regarding the structure and properties of such equations and the extension of understanding solutions to such equations [19]. The work done by Ndiaye and Mansal on the existence and uniqueness of results for VFIDEs through the use of the Caputo fractional derivative has been critically useful in analyzing these types of equations [20].

Dahmani's work on high-dimensional fractional differential systems has introduced new results on existence and uniqueness that help understand the more complex systems with fractional order [21]. HamaRashid et al. investigated the existence of Volterra-Fredholm integral equations of nonlinear boundary type with new numerical results that expand the understanding of boundary conditions in fractional systems [22]. This contributes to the impulsive fractional neutral stochastic integro-differential equation and in understanding such systems that produce both impulsive effects and stochastic behavior [23]. Lastly, the work by Hernandez, Prokopczyk, and Ladeira on partial functional differential equations with SDD gives additional theoretical basis in considering systems with delay and memory to further analyze important dynamics in those systems [24]. Through this body of work, this paper will contribute to the growing understanding of fractional VFIDEs, especially with regard to controllability and SDDs. The research presented here extends the foundation of previous studies and introduces a new approach for solving these systems, opening new perspectives for future exploration.

Raghavendran et al. [25] dealt with the existence, uniqueness, and stability of fractional neutral VFIDEs with state-dependent delays. The paper brings to the reader a few significant mathematical results on the types of systems discussed above as well as an insight into the behavior of these systems under particular conditions of state-dependent delays. These are critical to understand in systems governed by equivalent fractional equations when taken along with the control theory perspective. In this paper, we extend their findings by focusing on the controllability of systems described by such equations. More specifically, we investigate how state-dependent delays affect the ability to reach desired states when applying certain control strategies, utilizing the mathematical framework developed by Raghavendran et al. [25] to derive conditions for the controllability of the fractional systems under study. In the present study, we have chosen the Caputo fractional derivative to analyze the controllability of fractional impulsive neutral Volterra-Fredholm integro-differential equations with state-dependent delays. That is considered in numerical, mathematical, and practical terms, especially compared with the other fractional derivatives—the conformable and Hilfe derivatives used, respectively, in [26, 27]—that have been employed. The main reason for the Caputo derivative is that it is compatible with classical initial conditions because it does not require initial fractional-order derivations as for the Riemann–Liouville and Hilfer derivatives. Instead, the Caputo derivative employs initial conditions of the kind $x(0) = x_0$. This property becomes especially important for physical and engineering systems that are usually given underlying, well-defined, and measurable initial conditions. While the conformable derivative in [26] eases the formulation of equations mathematically but weakens memory endorsement in the model, the Caputo derivative preserves the nonlocal integral structure. This is vital since it allows for proper modeling of the systems, where the present state depends on the past states as well as the present. Memory effects are important in fractional dynamical systems, which respond impulsively with the present state dependent on the earlier dynamics in a non-trivial way.

The Caputo derivative complements semigroup theory and fixed point theorems like Schauder's fixed point theorem. We have used this theorem to obtain the controllability results. The integral formulation is also naturally in line with the structure of Volterra-Fredholm integral terms, which allows faster, clearer, and coherent analytical treatment. An additional reason for preferring the Caputo derivative is that it is synergistic with AI-driven frameworks, as machine learning models that deal with time series and memory-based prediction (e.g., recurrent neural networks and attention mechanisms) are best suited to a Caputo-type system because of its nonlocal, history-dependent conceptual nature. The convolutional nature of the Caputo derivative reinforces the effectiveness of training models on systems with historical dependencies. In [26], the authors analyzed nonlocal controllability using conformable fractional stochastic systems. While such systems are computationally simpler, they do not have strong theoretical underpinning to model memory. On the other hand, [27] dealt with Hilfer fractional systems—also generalizing both Riemann–Liouville and Caputo types but leading to complicated mixed-type initial conditions that are best avoided for practical implementation. The Caputo derivative is balanced well mathematically, easy to implement, and interpretable physically.

In this study, we explore the controllability outcomes of mild solutions for a class of fractional impulsive neutral VFIDEs with SDD, represented by the following system:

$$^{c}D^{\eta}(\varrho(\epsilon) - g_1(\epsilon, \varrho_{\rho(\epsilon,\varrho_\epsilon)})) = \mathcal{A}\varrho(\epsilon) + g_2(\epsilon, \varrho_{\rho(\epsilon,\varrho_\epsilon)}) + \int_{\epsilon_0}^{\epsilon} Z_1(\epsilon, \xi, \varrho_{\rho(\xi,\varrho_s)})\, d\xi + \int_{\epsilon_0}^{b} Z_2(\epsilon, \xi, \varrho_{\rho(\xi,\varrho_s)})\, d\xi + Bu(\epsilon), \quad \epsilon \in J = [\epsilon_0, b],\ 0 < \eta < 1, \tag{1.1}$$

$$\Delta\varrho(\epsilon_k) = I_k(\varrho(\epsilon_k^-)), \ \ k = 1, 2, \cdots, n, \tag{1.2}$$

$$\varrho(\epsilon_0) = \varrho_0 = \varphi(\epsilon) \in \mathcal{B}, \quad \epsilon \in (-\infty, 0]. \tag{1.3}$$

The function $\varrho(\cdot)$ is an unknown function that maps values into a Banach space X, equipped with the norm $\|\cdot\|$. The Caputo fractional derivative $^{c}D^{\eta}$, of order $0 < \eta < 1$, is applied in the system. The operator $\mathcal{A}$ serves as the infinitesimal generator of a compact, analytic semigroup $\{T(\epsilon) : \epsilon \geq 0\}$, consisting of uniformly bounded linear operators acting on X. Let J represent the time interval, and define $D = \{(\epsilon, \xi) \in J \times J : \epsilon_0 \leq \xi \leq \epsilon \leq b\}$. The solution exhibits jumps at impulsive points ϵ_k $(0 < \epsilon_1 < \epsilon_2 < \cdots < \epsilon_n < b)$, expressed as $\Delta\varrho(\epsilon_k) = \varrho(\epsilon_k^+) - \varrho(\epsilon_k^-)$, where $k = 1, 2, \ldots, n$.

The functions $g_i : J \times \mathcal{B} \to X$ and $Z_i : D \times \mathcal{B} \to X$ (for $i = 1, 2$) are defined appropriately, while the function $\rho : J \times \mathcal{B} \to (-\infty, b]$ is also suitably specified. The initial condition $\varphi(0) = 0$ holds, where $\varphi \in \mathcal{B}$, and $\mathcal{B}$ is the phase space, as defined in the preliminary section.

The control function $u(\cdot)$ is an element of the Banach space $L^2(J, U)$, which includes admissible control functions, with U being another Banach space. The bounded linear operator $B : U \to X$ maps these control functions into the Banach space X. For any continuous function ϱ defined on $(-\infty, b]$ and any $\epsilon \geq 0$, the element ϱ_ϵ in $\mathcal{B}$ is defined by $\varrho_\epsilon(\theta) = \varrho(\epsilon + \theta)$ for $\theta \leq 0$, representing the state history from time $\theta \in (-\infty, 0]$ up to the current time ϵ.

The novelty of this study is in the combination of fractional calculus with impulsive neutral VFIDEs, featuring SDD. Both fractional dynamics and SDDs have been explored in control theory separately, but their simultaneous application in the context of impulsive systems is not much explored. Moreover, this work extends the classical controllability results by making use of the Caputo fractional derivative and a semigroup of compact and analytic operators, giving a new perspective on how to deal with the complexity both fractional differentiation and impulsive effects introduce. The use of fixed point theorems due to Schauder and the confirmation of theoretical results from numerical simulation prove this paper important for this kind of system in the given challenges. It is also worth noting that the ability of AI-based algorithms to predict control of this class of system should open further venues for applications involving modern computational toolboxes within the fractional control arena.

As fractional-order dynamics integrate more systems controlled by AI in recent years, AI turns out to be indispensable in modeling, predicting, and controlling systems where conventional mathematics applies less. Machine learning models such as neural networks and recurrent architectures seem terribly good at modeling memory-dependent and nonlinearity in the behavior of such fractional systems, especially in the case of Caputo derivative governing equations. AI models can be trained to understand the dynamics of a system, predict state transitions based on the past, and adaptively direct controls. Further, AI simulation can supplement the analysis worked through clear theoretical principles such as compactness and boundedness. One may hybridize some classical methods like fixed point theory with AI applications. That will ensure meaningful balance on analytical rigor to provide programmability. It opens interesting possibilities of learning and better practical control of complex systems exhibiting impulses, delays, and fractional dynamics.

The current work differs notably from the contributions in [28, 29], both of which investigate approximate controllability of stochastic fractional systems with non-instantaneous impulses and driven by fractional Brownian motion. In this case, the authors work with Sadovskii's fixed point theorem in a stochastic framework involving Hilfer fractional derivatives, while the latter uses the concept of a resolvent family within the Hilbert space framework. In contrast, our study is on exact controllability—not approximate—for a deterministic fractional impulsive neutral Volterra-Fredholm integrodifferential system with state-dependent delays and Caputo derivatives, making use of Schauder's fixed point theorem itself, as in our case, the compactness and continuity conditions rendered by the integral operator, and the delay terms are intrinsically embedded in the formulation it provides. Another distinction, both in impulse and delay, is that both [28] and [29] considered non-instantaneous impulses, while we consider instantaneous ones along with neutral, state-dependent effects that raise even more analytical challenges. The comparison shows that the present work fills an important gap by addressing an entirely new class of fractional systems with general and realistic assumptions and, thereby, extends the coverage of controllability analysis for dynamic systems to the future.

In order to familiarize readers who are less familiar with fractional calculus, we briefly sketch the intuitive understanding underlying the mathematical instruments used in this study. The Caputo model, for example, is particularly suited for systems with memory in which the present state depends on previous states of all inputs. Thus, it is handy when modeling viscoelasticity and its manifestations in published results (anomalous diffusion) and in biological processes. In addition, Caputo formulations realize enforcement of classical initial conditions furthering the practical applicability of these mathematical modelings to real settings. Correspondingly, the Riemann-Liouville fractional integral extends the concept of classical integer-order integration and serves as a direct construction for fractional derivatives. The establishment of the existence of solutions is achieved through Schauder's fixed point theorem, which is independent of the continuity and compactness of the associated operator. The Arzelà-Ascoli theorem guarantees the compactness by suggesting that the operator is uniformly bounded and equicontinuous. All of these tools together give an academic and consistent framework to analyze the given class of fractional integro-differential equations.

2. Preliminaries

In this section, we focus on the commonly used definitions in fractional calculus, including the Riemann-Liouville fractional derivative and the Caputo derivative, as discussed in various academic studies [6, 12, 17, 24]. The Banach space $C(J, X)$, where $J = [\epsilon_0, \mathsf{b}]$, is endowed with the supremum norm. For any $\varrho \in C(J, X)$, this norm is expressed as $\|\varrho\|_\infty = \sup\{|\varrho(x)| : x \in J\}$.

Definition 2.1. *[17, 24] Let $\eta > 0$ denote the order of integration, and let φ be a given function. The fractional integral of φ based on the Riemann-Liouville definition is expressed as*

$$J^\eta \varphi(\tau) = \frac{1}{\Gamma(\eta)} \int_0^\tau (\tau - \epsilon)^{\eta-1} \varphi(\epsilon)\, dt, \quad \text{for } \tau > 0 \text{ and } \eta \in \mathbb{R}^+,$$

where $\Gamma(\cdot)$ denotes the Gamma function, and $\mathbb{R}^+$ represents the set of positive real numbers. By convention, $J^0\varphi(\tau) = \varphi(\tau)$.

Definition 2.2. *[17, 24] The Caputo derivative of a function $\varphi : [0, 1) \to \mathbb{R}$, of order η within $0 < \eta < 1$, is defined as:*

$$D^\eta \varphi(\tau) = \frac{1}{\Gamma(1-\eta)} \int_0^\tau \frac{\varphi^{(0)}(\epsilon)}{(\tau - \epsilon)^\eta}\, dt, \quad \tau > 0.$$

Here, $\Gamma(1-\eta)$ denotes the Gamma function evaluated at $1-\eta$, and $\varphi^{(0)}(\epsilon)$ refers to the zeroth derivative (or the function φ itself).

Definition 2.3. *[17, 24] For a function $\varphi(\tau)$, the Caputo fractional derivative is specified for an order η between $n - 1$ and n, where $n \in \mathbb{N}$. It is given by:*

$${}^cD^\eta \varphi(\tau) = \frac{1}{\Gamma(n-\eta)} \int_0^\tau (\tau - \epsilon)^{n-\eta-1} \frac{d^n \varphi(\epsilon)}{dt^n}\, dt, \quad n - 1 < \eta < n.$$

When $\eta = n$, the fractional derivative reduces to the standard n-th order derivative:

$${}^cD^\eta \varphi(\tau) = \frac{d^n \varphi(\tau)}{d\tau^n}.$$

The order η in this context can be real or even complex, representing the derivative's fractional order.

Theorem 2.1. *(Schauder fixed point theorem, see [11]) If a continuous mapping $N : B \to B$ has a relatively compact image in the Banach space E, then there exists at least one fixed point in the closed and convex subset B.*

Theorem 2.2. *(Arzelà-Ascoli theorem, see [17]) On a closed and bounded interval $[a, b]$, any sequence of functions that is equicontinuous and bounded must possess a subsequence that converges uniformly.*

Remark 2.1. *In this paper, we use the fixed point theorem of Schauder because of the compactness and continuity of the corresponding operator which arise naturally from the integrative structure and state-dependent delay of our model. Unlike Banach's method, which requires the assumption of a contraction, which may not hold in the current setting, or even Krasnoselskii's fixed point theorem where a decomposition in terms of contraction and complete continuity is needed, the assumptions of Schauder's theorem are more naturally satisfied. Hence, it is sufficiently flexible to prove existence results in nonlinear systems that incorporate fractional dynamics and impulsive effects.*

This work incorporates an axiomatic definition for the phase space $\mathcal{B}$, similar to the frameworks described in [1]. The phase space $\mathcal{B}$ is defined as a linear space consisting of all mappings from $(-\infty, 0]$ to X. It is equipped with a seminorm $\|\cdot\|_{\mathcal{B}}$, satisfying the following axioms:

(A1) Let $\varrho : (-\infty, a] \to X$ be a function, where $a > 0$ and continuous on J, and $\varrho_0 \in \mathcal{B}$. For every $\epsilon \in J$, the following properties hold:

(i) The function ϱ_ϵ belongs to $\mathcal{B}$,

(ii) $\|\varrho(\epsilon)\| \leq H\|\varrho_\epsilon\|_{\mathcal{B}}$,

(iii) $\|\varrho_\epsilon\|_{\mathcal{B}} \leq K(\epsilon)\sup\{\|\varrho(\xi)\| : 0 \leq \xi \leq \epsilon\} + M(\epsilon)\|\varrho_0\|_{\mathcal{B}}$,

where $H > 0$ is a constant, $K : [0, \infty) \to [1, \infty)$ is a continuous function, $M : [0, \infty) \to [1, \infty)$ is locally bounded, and the parameters H, K, and M are independent of $\varrho(\cdot)$.

(A2) For $\varrho(\cdot)$ defined in (A1), the function ϱ_ϵ is continuous and takes values in $\mathcal{B}$ on J.

(A3) The space $\mathcal{B}$ is complete.

3. Controllability results for fractional neutral VFIDEs with SDD

In this section, we analyze the controllability of mild solutions for fractional neutral VFIDEs with SDD described by Eqs (1.1)–(1.3). Building upon the earlier discussion, we define the concept of a mild solution for these equations.

Definition 3.4. *A function $\varrho : (-\infty, b] \to X$ is considered a mild solution of Eqs* (1.1)–(1.3) *if $\varrho_0 : \varphi \in \mathcal{B}$, and for each $\xi, \epsilon \in J$, it satisfies the following equation:*

$$\begin{aligned}
\varrho(\epsilon) &= S_\eta(\epsilon)(\varrho_0 - g_1(\epsilon_0, \varrho_0)) + g_1(\epsilon, \varrho_{\rho(\epsilon,\varrho_\epsilon)}) + \int_{\epsilon_0}^{\epsilon} T_\eta(\epsilon - \xi)(\epsilon - \xi)^{\eta-1} g_2(\xi, \varrho_{\rho(\epsilon,\varrho_s)})\, d\xi \\
&+ \int_{\epsilon_0}^{\epsilon} T_\eta(\epsilon - \xi)(\epsilon - \xi)^{\eta-1} \int_{\xi}^{\epsilon} Z_1(\tau, \xi, \varrho_{\rho(\epsilon,\varrho_s)})\, d\tau\, d\xi \\
&+ \int_{\epsilon_0}^{\epsilon} T_\eta(\epsilon - \xi)(\epsilon - \xi)^{\eta-1} \int_{\xi}^{b} Z_2(\tau, \xi, \varrho_{\rho(\epsilon,\varrho_s)})\, d\tau\, d\xi \\
&+ \int_{\epsilon_0}^{\epsilon} T_\eta(\epsilon - \xi)(\epsilon - \xi)^{\eta-1} B(\xi)u(\xi) d\xi + \sum_{0<\epsilon_k<\epsilon} S_\eta(\epsilon - \epsilon_k) I_k(\varrho(\epsilon_k^-)),
\end{aligned} \tag{3.1}$$

where

$$S_\eta(\epsilon) = \int_0^\infty \phi_\eta(\theta) T(\epsilon^\eta \theta)\, d\theta, \qquad T_\eta(\epsilon) = \eta \int_0^\infty \theta \phi_\eta(\theta) T(\epsilon^\eta \theta)\, d\theta,$$

$$\phi_\eta(\theta) = \frac{1}{\eta}\theta^{-1-\frac{1}{\eta}} \psi_\eta(\theta^{-\frac{1}{\eta}}), \qquad \psi_\eta(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-\eta n-1}\frac{\Gamma(n\eta + 1)}{n!}\sin(n\pi\eta), \quad \theta \in (0, \infty),$$

and ϕ_η is a probability density function defined on $(0, \infty)$*, satisfying*

$$\phi_\eta(\theta) \geq 0, \quad \theta \in (0, \infty), \quad \textit{and} \quad \int_0^\infty \phi_\eta(\theta)\, d\theta = 1.$$

Lemma 3.1. *[25] The operators $S_\eta(\epsilon)$ and $T_\eta(\epsilon)$ exhibit the following properties for any $\epsilon \geq 0$:*

(a) *The operators S_η and T_η are linear and bounded for every fixed $\epsilon \geq 0$. Specifically, for any $\varrho \in X$, the following bounds apply:*

$$\|S_\eta(\epsilon)\varrho\| \leq M\|\varrho\|, \quad \|T_\eta(\epsilon)\varrho\| \leq \frac{\eta M}{\Gamma(1+\eta)}\|\varrho\|.$$

(b) *Both families $\{S_\eta(\epsilon) : \epsilon \geq 0\}$ and $\{T_\eta(\epsilon) : \epsilon \geq 0\}$ are strongly continuous.*

(c) *For any $\epsilon > 0$, the operators $S_\eta(\epsilon)$ and $T_\eta(\epsilon)$ are compact.*

Definition 3.5. *For any initial states ϱ_0 and ϱ_1 in the Banach space X, the system defined by Eqs* (1.1)–(1.3) *is considered controllable over the interval J if a control function $\delta(\epsilon)$ can be found within the space $L^2(J,U)$. The mild solution $\varrho(\epsilon)$ is ensured to satisfy the conditions $\varrho(\epsilon_0) = \varrho_0$ and $\varrho(b) = \varrho_1$ by this control function.*

To derive our results, we assume the following conditions for the continuous function $\rho : J \times \mathcal{B} \to (-\infty, b]$:

(H1) For each $\epsilon > 0$, the semigroup $T(\epsilon)$ is compact.

(H2) The mappings $g_i : J \times \mathcal{B} \to X$ are continuous for $i = 1, 2$, and the following conditions hold: For each $\epsilon \in J$ and any pair $(\vartheta, \vartheta_1) \in \mathcal{B}^2$, the following inequalities are satisfied:

$$\|g_1(\epsilon,\vartheta) - g_1(\epsilon,\vartheta_1)\|_X \leq L_{g_1}\|\vartheta - \vartheta_1\|_{\mathcal{B}},$$

$$\|g_2(\epsilon,\vartheta) - g_2(\epsilon,\vartheta_1)\|_X \leq L_{g_2}\|\vartheta - \vartheta_1\|_{\mathcal{B}}.$$

There exist constants $L^*_{g_1}$ and $L^*_{g_2}$ such that

$$L^*_{g_1} = \max_{\epsilon\in J}\|g_1(\epsilon, 0)\|_X, \quad L^*_{g_2} = \max_{\epsilon\in J}\|g_2(\epsilon, 0)\|_X.$$

(H3) The functions $Z_i : D\times\mathcal{B} \to X$ (for $i = 1, 2$) satisfy the following conditions: Continuity is ensured for $(\epsilon, \xi) \in D$ and $(\vartheta, \vartheta_1) \in \mathcal{B}^2$, with the following inequalities holding:

$$\|Z_1(\epsilon,\xi,\vartheta) - Z_1(\epsilon,\xi,\vartheta_1)\|_X \leq L_{Z_1}\|\vartheta - \vartheta_1\|_{\mathcal{B}},$$

$$\|Z_2(\epsilon,\xi,\vartheta) - Z_2(\epsilon,\xi,\vartheta_1)\|_X \leq L_{Z_2}\|\vartheta - \vartheta_1\|_{\mathcal{B}}.$$

There exist constants $L^*_{Z_1}$ and $L^*_{Z_2}$ such that:

$$L^*_{Z_1} = \max_{\epsilon,\xi\in D}\|Z_1(\epsilon,\xi,0)\|_X, \quad L^*_{Z_2} = \max_{\epsilon,\xi\in D}\|Z_2(\epsilon,\xi,0)\|_X.$$

(H4) The mapping $\epsilon \to \varphi_\epsilon$ is well-defined and continuous from $R(\rho^-) = \{\rho(\xi,\vartheta) : (\xi,\vartheta) \in J \times \mathcal{B}, \rho(\xi,\vartheta) \leq 0\}$ into $\mathcal{B}$. Additionally, there exists a bounded, continuous function $J^\varphi : R(\rho^-) \to (0,\infty)$ such that for every $\epsilon \in R(\rho^-)$,

$$\|\varphi_\epsilon\|_{\mathcal{B}} \leq J^\varphi(\epsilon)\|\varphi\|_{\mathcal{B}}.$$

(H5) The bounded linear operator $W : L^2(J,U) \to X$ is defined as

$$Wx = \frac{1}{\Gamma(\eta)}\int_{\epsilon_0}^{b}(b-\epsilon)^{\eta-1}Bu(\epsilon)\,dt,$$

where B satisfies $|B| \leq M_1$. Additionally, the operator W has an induced inverse W^{-1} acting within $\frac{L^2(J,U)}{\ker W}$, with a constant $M_2 > 0$ such that $|W^{-1}| \leq M_2$.

(H6) The functions $I_k : \mathcal{B} \to X$ for $k = 1, 2, \ldots, n$ are continuous. There exist constants $L_1 > 0$ and $L_2 > 0$ such that for all $\upsilon, \upsilon_1 \in \mathcal{B}$ and $k = 1, 2, \ldots, n$, the following conditions hold:

$$\|I_k(\upsilon) - I_k(\upsilon_1)\| \leq L_1 \|\upsilon - \upsilon_1\|_{\mathcal{B}}$$

and

$$\|I_k(\upsilon)\| \leq L_2.$$

Theorem 3.1. *Let the assumptions* $(H1)$–$(H6)$ *hold. Then, the system represented by Eqs* (1.1) *and* (1.3) *is controllable on the interval* $[\epsilon_0, b]$.

Proof. Consider the collection of functions ϖ_l, defined as the set of all continuous functions ϱ mapping the interval J to the real numbers, such that $\|\varrho\|_\infty \leq l$. Utilizing the assumptions stated in hypothesis $(H5)$, a control can be derived by leveraging the properties of an arbitrary function $\varrho(\cdot)$, given as follows:

$$\begin{aligned}\mu(\epsilon) = W^{-1}\Bigg[&\varrho_1 - S_\eta(\epsilon)(\varrho_0 - g_1(\epsilon_0, \varrho_0)) - g_1(\epsilon, \varrho_{\rho(\epsilon,\varrho_\epsilon)}) \\ &- \int_{\epsilon_0}^{\epsilon} T_\eta(\epsilon - \xi)(\epsilon - \xi)^{\eta-1}\Big[g_2(\xi, \varrho_{\rho(\epsilon,\varrho_s)}) + \int_{\xi}^{\epsilon} Z_1(\tau, \xi, \varrho_{\rho(\epsilon,\varrho_s)})\, d\tau \\ &+ \int_{\xi}^{b} Z_2(\tau, \xi, \varrho_{\rho(\epsilon,\varrho_s)})\, d\tau\Big] d\xi - \sum_{0<\epsilon_k<\epsilon} S_\eta(\epsilon - \epsilon_k) I_k(\varrho(\epsilon_k^-))\Bigg](\epsilon).\end{aligned} \tag{3.2}$$

Using the defined control, we will demonstrate that the operator Φ, which maps the set ϖ_l to itself, is given by:

$$\begin{aligned}\Phi(\varrho)(\epsilon) = {} & S_\eta(\epsilon)(\varrho_0 - g_1(\epsilon_0, \varrho_0)) + g_1(\epsilon, \varrho_{\rho(\epsilon,\varrho_\epsilon)}) + \int_{\epsilon_0}^{\epsilon} T_\eta(\epsilon - \xi)(\epsilon - \xi)^{\eta-1} g_2(\xi, \varrho_{\rho(\epsilon,\varrho_s)})\, d\xi \\ &+ \int_{\epsilon_0}^{\epsilon} T_\eta(\epsilon - \xi)(\epsilon - \xi)^{\eta-1} \int_{\xi}^{\epsilon} Z_1(\tau, \xi, \varrho_{\rho(\epsilon,\varrho_s)})\, d\tau\, d\xi \\ &+ \int_{\epsilon_0}^{\epsilon} T_\eta(\epsilon - \xi)(\epsilon - \xi)^{\eta-1} \int_{\xi}^{b} Z_2(\tau, \xi, \varrho_{\rho(\epsilon,\varrho_s)})\, d\tau\, d\xi \\ &+ \int_{\epsilon_0}^{\epsilon} T_\eta(\epsilon - \xi)(\epsilon - \xi)^{\eta-1} B\mu(\xi)\, d\xi + \sum_{0<\epsilon_k<\epsilon} S_\eta(\epsilon - \epsilon_k) I_k(\varrho(\epsilon_k^-)).\end{aligned} \tag{3.3}$$

We can deduce that a fixed point exists for the operator Φ, where $\mu(\epsilon)$ is defined as per Eq (3.3). This fixed point corresponds to the mild solution of the control problem described by Eqs (1.1) and (1.3). Specifically, it is evident that $\Phi\ \varrho(b) = \varrho_1$, which indicates that the system represented by Eqs (1.1) and (1.3) is controllable over the interval $[\epsilon_0, b]$.

Since all the functions involved in the definition of the operator are continuous, we can conclude that the operator Φ is continuous. Expanding upon Eq (3.1), for any function $\varrho \in \varpi_l$ and for all values of ϵ within the interval $[\epsilon_0, b]$, the following relationship holds:

$$\begin{aligned}
\mu(\epsilon) \leq\ & \|W^{-1}\|\Big[\|\varrho_1\| - \|S_\eta(\epsilon)\|\|\varrho_0\|\| - g_1(\epsilon_0,\varrho_0)\| + \|g_1(\epsilon,\varrho_{\rho(\epsilon,\varrho_\epsilon)})\| \\
& - \int_{\epsilon_0}^{\epsilon} T_\eta(\epsilon-\xi)\,(\epsilon-\xi)^{\eta-1}\Big[\|g_2(\xi,\varrho_{\rho(\epsilon,\varrho_s)})\| + \int_{\xi}^{\epsilon} \|Z_1(\tau,\xi,\varrho_{\rho(\epsilon,\varrho_s)})\|\,d\tau \\
& + \int_{\xi}^{b} \|Z_2(\tau,\xi,\varrho_{\rho(\epsilon,\varrho_s)})\|\,d\tau\Big]d\xi + \|\sum_{0<\epsilon_k<\epsilon} S_\eta(\epsilon-\epsilon_k)I_k(\varrho(\epsilon_k^-))\|\Big] \\
\leq\ & M_2\Big[\|\varrho_1\| + M\|\varrho_0\| + \|g_1(\epsilon,\varrho_{\rho(\epsilon,\varrho_\epsilon)})\| + \|g_1(\epsilon,\varrho_{\rho(\epsilon,\varrho_\epsilon)})\| \\
& + \frac{\eta M}{\Gamma(1+\eta)}\int_{\epsilon_0}^{\epsilon}(\epsilon-\xi)^{\eta-1}\|g_2(\xi,\varrho_{\rho(\epsilon,\varrho_s)})\|\,d\xi \\
& + \frac{\eta M}{\Gamma(1+\eta)}\int_{\epsilon_0}^{\epsilon}(\epsilon-\xi)^{\eta-1}\int_{\xi}^{\epsilon}\|Z_1(\tau,\xi,\varrho_{\rho(\epsilon,\varrho_s)})\|\,d\tau\,d\xi \\
& + \frac{\eta M}{\Gamma(1+\eta)}\int_{\epsilon_0}^{\epsilon}(\epsilon-\xi)^{\eta-1}\int_{\xi}^{b}\|Z_2(\tau,\xi,\varrho_{\rho(\epsilon,\varrho_s)})\|\,d\tau\,d\xi + M\sum_{0<\epsilon_k<\epsilon}\|I_k(\varrho(\epsilon_k^-))\|\Big] \\
\leq\ & M_2\Big[\|\varrho_1\| + M\|\varrho_0\| + \|g_1(\epsilon,\varrho_{\rho(\epsilon,\varrho_\epsilon)})\| + L_{g_1}\|\varrho_{\rho(\xi,\bar{\varrho}_s)}\|_{\mathcal{B}} + L_{g_1}^* \\
& + \frac{\eta M}{\Gamma(1+\eta)}\int_{\epsilon_0}^{\epsilon}(\epsilon-\xi)^{\eta-1}\Big[L_{g_2}\|\varrho_{\rho(\xi,\bar{\varrho}_s)}\|_{\mathcal{B}} + L_{g_2}^* + \int_{\xi}^{\epsilon}(L_{Z_1}\|\varrho_{\rho(\xi,\bar{\varrho}_s)}\|_{\mathcal{B}} + L_{Z_1}^*)d\tau \\
& + \int_{\xi}^{b}(L_{Z_2}\|\varrho_{\rho(\xi,\bar{\varrho}_s)}\|_{\mathcal{B}} + L_{Z_2}^*)d\tau\Big]d\xi\Big] + ML_2 \\
\leq\ & M_2\Big[\|\varrho_1\| + M\|\varrho_0\| + \|g_1(\epsilon,\varrho_{\rho(\epsilon,\varrho_\epsilon)})\| + L_{g_1}r^* + L_{g_1}^* + \frac{Me^\eta}{\Gamma(1+\eta)}(L_{g_2}r^* + L_{g_2}^*) \\
& + \frac{Me^{\eta+1}}{(\eta+1)\Gamma(\eta)}(L_{Z_1}r^* + L_{Z_1}^*) + \frac{Me^{\eta+1}}{(\eta+1)\Gamma(\eta)}(L_{Z_2}r^* + L_{Z_2}^*)\Big] + ML_2 \\
\leq\ & M_2\Big[\|\varrho_1\| + M\|\varrho_0\| + \|g_1(\epsilon,\varrho_{\rho(\epsilon,\varrho_\epsilon)})\| + L_{g_1}r^* + L_{g_1}^* + Me^\eta\Big[\frac{L_{g_2}}{\Gamma(\eta+1)} + \frac{e(L_{Z_1}+L_{Z_2})}{(\eta+1)\Gamma(\eta)}\Big]r^* \\
& + ML_2 + Me^\eta\Big[\frac{L_{g_2}^*}{\Gamma(\eta+1)} + \frac{e(L_{Z_1}^*+L_{Z_2}^*)}{(\eta+1)\Gamma(\eta)}\Big]\Big].
\end{aligned}$$

Applying the Eqs (3.2) and (3.3), we can derive the following result:

$$\begin{aligned}
\|\Phi(\varrho)(\epsilon)\| \leq\ & M\|\varrho_0\| + \|g_1(\epsilon,\varrho_{\rho(\epsilon,\varrho_\epsilon)})\| + \|g_1(\epsilon,\varrho_{\rho(\epsilon,\varrho_\epsilon)})\| + \frac{\eta M}{\Gamma(1+\eta)}\int_{\epsilon_0}^{\epsilon}(\epsilon-\xi)^{\eta-1}\|g_2(\xi,\varrho_{\rho(\epsilon,\varrho_s)})\|\,d\xi \\
& + \frac{\eta M}{\Gamma(1+\eta)}\int_{\epsilon_0}^{\epsilon}(\epsilon-\xi)^{\eta-1}\int_{\xi}^{\epsilon}\|Z_1(\tau,\xi,\varrho_{\rho(\epsilon,\varrho_s)})\|\,d\tau\,d\xi \\
& + \frac{\eta M}{\Gamma(1+\eta)}\int_{\epsilon_0}^{\epsilon}(\epsilon-\xi)^{\eta-1}\int_{\xi}^{b}\|Z_2(\tau,\xi,\varrho_{\rho(\epsilon,\varrho_s)})\|\,d\tau\,d\xi
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\eta M}{\Gamma(1+\eta)} \int_{\epsilon_0}^{\epsilon} (\epsilon-\xi)^{\eta-1} \|B\| \|\mu(\epsilon)\| + \| \sum_{0<\epsilon_k<\epsilon} S_\eta(\epsilon-\epsilon_k) I_k(\varrho(\epsilon_k^-))\| \\
&\leq M\|\varrho_0\| + \|g_1(\epsilon, \varrho_{\rho(\epsilon,\varrho_\epsilon)})\| + L_{g_1} \|\varrho_{\rho(\xi,\bar{\varrho}_s)}\|_{\mathcal{B}} + L_{g_1}^* \\
&\quad + \frac{\eta M}{\Gamma(1+\eta)} \int_{\epsilon_0}^{\epsilon} (\epsilon-\xi)^{\eta-1} \Big[L_{g_2} \|\varrho_{\rho(\xi,\bar{\varrho}_s)}\|_{\mathcal{B}} + L_{g_2}^* + \int_{\xi}^{\epsilon} (L_{Z_1} \|\varrho_{\rho(\xi,\bar{\varrho}_s)}\|_{\mathcal{B}} + L_{Z_1}^*) d\tau \\
&\quad + \int_{\xi}^{b} (L_{Z_2} \|\varrho_{\rho(\xi,\bar{\varrho}_s)}\|_{\mathcal{B}} + L_{Z_2}^*) d\tau \Big] d\xi + \frac{\eta M}{\Gamma(1+\eta)} \int_{\epsilon_0}^{\epsilon} (\epsilon-\xi)^{\eta-1} M_1 \|\mu(\epsilon)\| \\
&\quad + M\|I_k(\varrho(\epsilon_k^-))\| \\
&\leq M\|\varrho_0\| + \|g_1(\epsilon, \varrho_{\rho(\epsilon,\varrho_\epsilon)})\| + L_{g_1} r^* + L_{g_1}^* \\
&\quad + \frac{M e^\eta}{\Gamma(1+\eta)} (L_{g_2} r^* + L_{g_2}^*) + \frac{M e^{\eta+1}}{(\eta+1)\Gamma(\eta)} (L_{Z_1} r^* + L_{Z_1}^*) \\
&\quad + \frac{M e^{\eta+1}}{(\eta+1)\Gamma(\eta)} (L_{Z_2} r^* + L_{Z_2}^*) + M L_2 + \frac{M e^\eta}{\Gamma(1+\eta)} M_1 M_2 \Big[\|\varrho_1\| - M\|\varrho_0\| \\
&\quad + \|g_1(\epsilon, \varrho_{\rho(\epsilon,\varrho_\epsilon)})\| + L_{g_1} r^* + L_{g_1}^* + \frac{M e^\eta}{\Gamma(1+\eta)} (L_{g_2} r^* + L_{g_2}^*) + \frac{M e^{\eta+1}}{(\eta+1)\Gamma(\eta)} (L_{Z_1} r^* \\
&\quad + L_{Z_1}^*) + \frac{M e^{\eta+1}}{(\eta+1)\Gamma(\eta)} (L_{Z_2} r^* + L_{Z_2}^*) + M L_2 \Big] \\
&\leq M\|\varrho_0\| + \|g_1(\epsilon, \varrho_{\rho(\epsilon,\varrho_\epsilon)})\| + L_{g_1} r^* + L_{g_1}^* + M L_2 \\
&\quad + M e^\eta \Big[\frac{L_{g_2}}{\Gamma(\eta+1)} + \frac{e(L_{Z_1} + L_{Z_2})}{(\eta+1)\Gamma(\eta)} \Big] r^* + M e^\eta \Big[\frac{L_{g_2}^*}{\Gamma(\eta+1)} + \frac{e(L_{Z_1}^* + L_{Z_2}^*)}{(\eta+1)\Gamma(\eta)} \Big] \\
&\quad + \frac{M e^\eta}{\Gamma(1+\eta)} M_1 M_2 \Big[\|\varrho_1\| + M\|\varrho_0\| + \|g_1(\epsilon, \varrho_{\rho(\epsilon,\varrho_\epsilon)})\| + L_{g_1} r^* + L_{g_1}^* \\
&\quad + M e^\eta \Big[\frac{L_{g_2}}{\Gamma(\eta+1)} + \frac{e(L_{Z_1} + L_{Z_2})}{(\eta+1)\Gamma(\eta)} \Big] r^* + M e^\eta \Big[\frac{L_{g_2}^*}{\Gamma(\eta+1)} + \frac{e(L_{Z_1}^* + L_{Z_2}^*)}{(\eta+1)\Gamma(\eta)} \Big] + M L_2 \Big] \\
&\leq \frac{M e^\eta}{\Gamma(1+\eta)} M_1 M_2 \|\varrho_1\| + M L_2 + \Big(1 + \frac{M e^\eta}{\Gamma(1+\eta)} M_1 M_2\Big) \Big[M\|\varrho_0\| + \|g_1(\epsilon, \varrho_{\rho(\epsilon,\varrho_\epsilon)})\| + L_{g_1} r^* \\
&\quad + L_{g_1}^* + M L_2 + M e^\eta \Big[\frac{L_{g_2}}{\Gamma(\eta+1)} + \frac{e(L_{Z_1} + L_{Z_2})}{(\eta+1)\Gamma(\eta)} \Big] r^* + M e^\eta \Big[\frac{L_{g_2}^*}{\Gamma(\eta+1)} + \frac{e(L_{Z_1}^* + L_{Z_2}^*)}{(\eta+1)\Gamma(\eta)} \Big] \Big].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|\Phi(\varrho)\|_\infty \leq & \frac{M e^\eta}{\Gamma(1+\eta)} M_1 M_2 \|\varrho_1\| + M L_2 + \Big(1 + \frac{M e^\eta}{\Gamma(1+\eta)} M_1 M_2\Big) \Big[M\|\varrho_0\| + \|g_1(\epsilon, \varrho_{\rho(\epsilon,\varrho_\epsilon)})\| \\
& + L_{g_1} r^* + L_{g_1}^* + M e^\eta \Big[\frac{L_{g_2}}{\Gamma(\eta+1)} + \frac{e(L_{Z_1} + L_{Z_2})}{(\eta+1)\Gamma(\eta)} \Big] r^* \\
& + M L_2 + M e^\eta \Big[\frac{L_{g_2}^*}{\Gamma(\eta+1)} + \frac{e(L_{Z_1}^* + L_{Z_2}^*)}{(\eta+1)\Gamma(\eta)} \Big] \Big] := l.
\end{aligned}
$$

We conclude that $\|\Phi\varrho\| \leq l$, which implies that $\Phi\varrho \in \varpi_l$. Therefore, it follows that $\Phi\varpi_l \subset \varpi_l$. This establishes that the operator Φ maps the set $\varpi_l = \{\varrho \in C(J, X) : \|\varrho\|_\infty \leq l\}$ onto itself. Next, we will demonstrate that the operator $\Phi : \varpi_l \to \varpi_l$ satisfies all the conditions of (2.1). The proof will be carried out in several steps.

Step 1: The operator Φ is continuous. Let $\{\varrho_n\}$ be a sequence such that $\varrho_n \to \varrho$ in ϖ_l,

$$\begin{aligned}
\|\Phi\varrho_n(\epsilon) - \Phi\varrho(\epsilon)\| \leq{}& \|g_1(\xi, \varrho_{n\rho(\epsilon,\varrho_s)}) - g_1(\xi, \varrho_{\rho(\epsilon,\varrho_s)})\| \\
&+ \int_{\epsilon_0}^{\epsilon} T_\eta(\epsilon-\xi)(\epsilon-\xi)^{\eta-1}\|g_2(\xi, \varrho_{n\rho(\epsilon,\varrho_s)}) - g_2(\xi, \varrho_{\rho(\epsilon,\varrho_s)})\|\, d\xi \\
&+ \int_{\epsilon_0}^{\epsilon} T_\eta(\epsilon-\xi)(\epsilon-\xi)^{\eta-1}\int_{\xi}^{\epsilon} \|Z_1(\tau,\xi, \varrho_{n\rho(\epsilon,\varrho_s)}) - Z_1(\tau,\xi, \varrho_{\rho(\epsilon,\varrho_s)})\|\, d\tau\, d\xi \\
&+ \int_{\epsilon_0}^{\epsilon} T_\eta(\epsilon-\xi)(\epsilon-\xi)^{\eta-1}\int_{\xi}^{b} \|Z_2(\tau,\xi, \varrho_{n\rho(\epsilon,\varrho_s)}) - Z_2(\tau,\xi, \varrho_{\rho(\epsilon,\varrho_s)})\|\, d\tau\, d\xi \\
&+ \sum_{0<\epsilon_k<\epsilon} S_\eta(\epsilon-\epsilon_k)\|I_k(\varrho(\epsilon_k^-)) - I_k(\varrho_n(\epsilon_k^-))\| \\
&+ \int_{\epsilon_0}^{\epsilon} T_\eta(\epsilon-\xi)(\epsilon-\xi)^{\eta-1} BW^{-1}\Bigg[\|g_1(\xi, \varrho_{n\rho(\epsilon,\varrho_s)}) - g_1(\xi, \varrho_{\rho(\epsilon,\varrho_s)})\| \\
&+ \int_{\epsilon_0}^{\epsilon} T_\eta(\epsilon-\xi)(\epsilon-\xi)^{\eta-1}\|g_2(\xi, \varrho_{n\rho(\epsilon,\varrho_s)}) \\
&- g_2(\xi, \varrho_{\rho(\epsilon,\varrho_s)})\|\, d\xi + \int_{\epsilon_0}^{\epsilon} T_\eta(\epsilon-\xi)(\epsilon-\xi)^{\eta-1}\int_{\xi}^{\epsilon} \|Z_1(\tau,\xi, \varrho_{n\rho(\epsilon,\varrho_s)}) \\
&- Z_1(\tau,\xi, \varrho_{\rho(\epsilon,\varrho_s)})\|\, d\tau\, d\xi + \int_{\epsilon_0}^{\epsilon} T_\eta(\epsilon-\xi)(\epsilon-\xi)^{\eta-1}\int_{\xi}^{b} \|Z_2(\tau,\xi, \varrho_{n\rho(\epsilon,\varrho_s)}) \\
&- Z_2(\tau,\xi, \varrho_{\rho(\epsilon,\varrho_s)})\|\, d\tau\, d\xi\Bigg] d\xi.
\end{aligned}$$

Due to the continuity of g, Z_1, and Z_2, it follows that $\|\Phi\varrho_n(\epsilon) - \Phi\varrho(\epsilon)\| \to 0$ as $n \to \infty$. Hence, the operator Φ is continuous on the set ϖ_l.

Step 2: The set $\Phi(\varpi_l)$ is uniformly bounded. This is clear because $\Phi(\varpi_l) \subset \varpi_l$, implying that $\Phi(\varpi_l)$ is bounded.

Step 3: We now demonstrate that $\Phi(\varpi_l)$ is equicontinuous.

Consider ϵ_1 and ϵ_2 in the bounded set $[\epsilon_0, b] \subset C(J, X)$, as described in Step 2, along with $\varrho \in \varpi_l$ and $\epsilon_1 < \epsilon_2$. In this context, we have:

$$\begin{aligned}
\|(\Phi\varrho)(\epsilon_2) - (\Phi\varrho)(\epsilon_1)\| ={}& \Bigg\| S_\eta(\epsilon_2)\,(\varrho_0 - g_1(\epsilon_0, \varrho_0)) + g_1(\epsilon, \varrho_{\rho(\epsilon,\varrho_\epsilon)}) \\
&+ \int_{\epsilon_0}^{\epsilon_2} T_\eta(\epsilon_2-\xi)(\epsilon_2-\xi)^{\eta-1} g_2(\xi, \varrho_{\rho(\xi,\varrho_s)})\, d\xi \\
&+ \int_{\epsilon_0}^{\epsilon_2} (\epsilon_2-\xi)^{\eta-1} T_\eta(\epsilon_2-\xi)\int_{\xi}^{\epsilon_2} Z_1(\tau,\xi, \varrho_{\rho(\xi,\varrho_s)}) d\tau d\xi \\
&+ \int_{\epsilon_0}^{\epsilon_2} (\epsilon_2-\xi)^{\eta-1} T_\eta(\epsilon_2-\xi)\int_{\xi}^{b} Z_2(\tau,\xi, \varrho_{\rho(\xi,\varrho_s)}) d\tau d\xi \\
&+ \sum_{0<\epsilon_k<\epsilon_2} S_\eta(\epsilon_2-\epsilon_k)\|I_k(\varrho(\epsilon_k^-))\|
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\epsilon_0}^{\epsilon_2}(\epsilon_2-\xi)^{\eta-1}T_\eta(\epsilon_2-\xi)BW^{-1}\Big[\varrho_1-S_\eta(\epsilon_2)\,(\varrho_0-g_1(\epsilon_0,\varrho_0))\\
&+g_1(\epsilon,\varrho_{\rho(\epsilon,\varrho_\epsilon)})-\int_{\epsilon_0}^{\epsilon_2}T_\eta(\epsilon_2-\xi)\,(\epsilon_2-\xi)^{\eta-1}\Big[g_2(\xi,\varrho_{\rho(\xi,\varrho_s)})\\
&+\int_{\xi}^{\epsilon_2}Z_1(\tau,\xi,\varrho_{\rho(\epsilon_2,\varrho_s)})\,d\tau+\int_{\xi}^{b}Z_2(\tau,\xi,\varrho_{\rho(\epsilon_2,\varrho_s)})\,d\tau\Big]d\xi\\
&-\sum_{0<\epsilon_k<\epsilon_2}S_\eta(\epsilon_2-\epsilon_k)\|I_k(\varrho(\epsilon_k^-))\|\Big]\\
&-S_\eta(\epsilon_1)\,(\varrho_0-g_1(\epsilon_0,\varrho_0))\\
&+g_1(\epsilon,\varrho_{\rho(\epsilon,\varrho_\epsilon)})-\int_{\epsilon_0}^{\epsilon_1}T_\eta(\epsilon_1-\xi)(\epsilon_1-\xi)^{\eta-1}g_2(\xi,\varrho_{\rho(\xi,\varrho_s)})\,d\xi\\
&-\int_{\epsilon_0}^{\epsilon_1}(\epsilon_1-\xi)^{\eta-1}T_\eta(\epsilon_1-\xi)\int_{\xi}^{\epsilon_1}Z_1(\tau,\xi,\varrho_{\rho(\xi,\varrho_s)})d\tau d\xi\\
&-\int_{\epsilon_0}^{\epsilon_1}(\epsilon_1-\xi)^{\eta-1}T_\eta(\epsilon_1-\xi)\int_{\xi}^{b}Z_2(\tau,\xi,\varrho_{\rho(\xi,\varrho_s)})d\tau d\xi\\
&+\sum_{0<\epsilon_k<\epsilon_1}S_\eta(\epsilon_1-\epsilon_k)\|I_k(\varrho(\epsilon_k^-))\|\\
&-\int_{\epsilon_0}^{\epsilon_1}(\epsilon_1-\xi)^{\eta-1}T_\eta(\epsilon_1-\xi)BW^{-1}\Big[\varrho_1-S_\eta(\epsilon_1)\,(\varrho_0-g_1(\epsilon_0,\varrho_0))+g_1(\epsilon,\varrho_{\rho(\epsilon,\varrho_\epsilon)})\\
&-\int_{\epsilon_0}^{\epsilon_1}T_\eta(\epsilon_1-\xi)\,(\epsilon_1-\xi)^{\eta-1}\Big[g_2(\xi,\varrho_{\rho(\xi,\varrho_s)})+\int_{\xi}^{\epsilon_1}Z_1(\tau,\xi,\varrho_{\rho(\epsilon_1,\varrho_s)})\,d\tau\\
&+\int_{\xi}^{b}Z_2(\tau,\xi,\varrho_{\rho(\epsilon_1,\varrho_s)})\,d\tau\Big]d\xi+\sum_{0<\epsilon_k<\epsilon_1}S_\eta(\epsilon_1-\epsilon_k)\|I_k(\varrho(\epsilon_k^-))\|\Big]\Big\|\\
&\le\frac{M\eta}{\Gamma(1+\eta)}\Big\|g_1(\epsilon,\varrho_{\rho(\epsilon,\varrho_\epsilon)})+\int_{\epsilon_1}^{\epsilon_2}(\epsilon_2-\xi)^{\eta-1}g_2(\xi,\varrho_{\rho(\xi,\varrho_s)})d\xi\\
&+\int_{\epsilon_1}^{\epsilon_2}(\epsilon_2-\xi)^{\eta-1}\int_{\xi}^{\epsilon_2}Z_1(\tau,\xi,\varrho_{\rho(\xi,\varrho_s)})d\tau d\xi\\
&+\int_{\epsilon_1}^{\epsilon_2}(\epsilon_2-\xi)^{\eta-1}\int_{\xi}^{b}Z_2(\tau,\xi,\varrho_{\rho(\xi,\varrho_s)})d\tau d\xi\\
&-\int_{\epsilon_0}^{\epsilon_1}(\epsilon_1-\xi)^{\eta-1}g_2(\xi,\varrho_{\rho(\xi,\varrho_s)})d\xi\\
&-\int_{\epsilon_0}^{\epsilon_1}(\epsilon_1-\xi)^{\eta-1}\int_{\xi}^{\epsilon_1}Z_1(\tau,\xi,\varrho_{\rho(\xi,\varrho_s)})d\tau d\xi\\
&-\int_{\epsilon_0}^{\epsilon_1}(\epsilon_1-\xi)^{\eta-1}\int_{\xi}^{b}Z_2(\tau,\xi,\varrho_{\rho(\xi,\varrho_s)})d\tau d\xi\\
&+\int_{\epsilon_0}^{\epsilon_1}(\epsilon_2-\xi)^{\eta-1}g_2(\xi,\varrho_{\rho(\xi,\varrho_s)})d\xi\\
&+\int_{\epsilon_0}^{\epsilon_1}(\epsilon_2-\xi)^{\eta-1}\int_{\xi}^{\epsilon_2}Z_1(\tau,\xi,\varrho_{\rho(\xi,\varrho_s)})d\tau d\xi
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\epsilon_0}^{\epsilon_1}(\epsilon_2-\xi)^{\eta-1}\int_{\xi}^{b}Z_2(\tau,\xi,\varrho_{\rho(\xi,\varrho_s)})d\tau d\xi \\
&+\sum_{0<\epsilon_k<\epsilon_2}S_{\eta}(\epsilon_2-\epsilon_k)\|I_k(\varrho(\epsilon_k^-))\| \\
&+BW^{-1}\Big[g_1(\epsilon,\varrho_{\rho(\epsilon,\varrho_\epsilon)})+\int_{\epsilon_1}^{\epsilon_2}(\epsilon_2-\xi)^{\eta-1}g_2(\xi,\varrho_{\rho(\xi,\varrho_s)})d\xi \\
&+\int_{\epsilon_1}^{\epsilon_2}(\epsilon_2-\xi)^{\eta-1}\int_{\xi}^{\epsilon_2}Z_1(\tau,\xi,\varrho_{\rho(\xi,\varrho_s)})d\tau d\xi \\
&+\int_{\epsilon_1}^{\epsilon_2}(\epsilon_2-\xi)^{\eta-1}\int_{\xi}^{b}Z_2(\tau,\xi,\varrho_{\rho(\xi,\varrho_s)})d\tau d\xi \\
&-\int_{\epsilon_0}^{\epsilon_1}(\epsilon_1-\xi)^{\eta-1}g_2(\xi,\varrho_{\rho(\xi,\varrho_s)})d\xi \\
&-\int_{\epsilon_0}^{\epsilon_1}(\epsilon_1-\xi)^{\eta-1}\int_{\xi}^{\epsilon_1}Z_1(\tau,\xi,\varrho_{\rho(\xi,\varrho_s)})d\tau d\xi \\
&-\int_{\epsilon_0}^{\epsilon_1}(\epsilon_1-\xi)^{\eta-1}\int_{\xi}^{b}Z_2(\tau,\xi,\varrho_{\rho(\xi,\varrho_s)})d\tau d\xi \\
&+\int_{\epsilon_0}^{\epsilon_1}(\epsilon_2-\xi)^{\eta-1}g_2(\xi,\varrho_{\rho(\xi,\varrho_s)})d\xi \\
&+\int_{\epsilon_0}^{\epsilon_1}(\epsilon_2-\xi)^{\eta-1}\int_{\xi}^{\epsilon_2}Z_1(\tau,\xi,\varrho_{\rho(\xi,\varrho_s)})d\tau d\xi \\
&+\int_{\epsilon_0}^{\epsilon_1}(\epsilon_2-\xi)^{\eta-1}\int_{\xi}^{b}Z_2(\tau,\xi,\varrho_{\rho(\xi,\varrho_s)})d\tau d\xi \\
&+\sum_{0<\epsilon_k<\epsilon_1}S_{\eta}(\epsilon_1-\epsilon_k)\|I_k(\varrho(\epsilon_k^-))\|\Big]\Big\| \\
&\le\frac{M\eta}{\Gamma(1+\eta)}\Big\|g_1(\epsilon,\varrho_{\rho(\epsilon,\varrho_\epsilon)})+\int_{\epsilon_1}^{\epsilon_2}(\epsilon_2-\xi)^{\eta-1}\Big[g_2(\xi,\varrho_{\rho(\xi,\varrho_s)})+\int_{\xi}^{\epsilon_2}Z_1(\tau,\xi,\varrho_{\rho(\xi,\varrho_s)})d\tau \\
&+\int_{\xi}^{b}Z_2(\tau,\xi,\varrho_{\rho(\xi,\varrho_s)})d\tau\Big]d\xi \\
&-\int_{\epsilon_0}^{\epsilon_1}(\epsilon_1-\xi)^{\eta-1}g_2(\xi,\varrho_{\rho(\xi,\varrho_s)})d\xi+\int_{\epsilon_0}^{\epsilon_1}(\epsilon_2-\xi)^{\eta-1}g_2(\xi,\varrho_{\rho(\xi,\varrho_s)})d\xi \\
&-\int_{\epsilon_0}^{\epsilon_1}\Big[(\epsilon_1-\xi)^{\eta-1}\int_{\xi}^{\epsilon_1}Z_1(\tau,\xi,\varrho_{\rho(\xi,\varrho_s)})d\tau \\
&-(\epsilon_2-\xi)^{\eta-1}\int_{\xi}^{\epsilon_2}Z_1(\tau,\xi,\varrho_{\rho(\xi,\varrho_s)})d\tau\Big]d\xi-\int_{\epsilon_0}^{\epsilon_1}\Big[(\epsilon_1-\xi)^{\eta-1} \\
&\times\int_{\xi}^{b}Z_2(\tau,\xi,\varrho_{\rho(\xi,\varrho_s)})d\tau-(\epsilon_2-\xi)^{\eta-1}\int_{\xi}^{b}Z_2(\tau,\xi,\varrho_{\rho(\xi,\varrho_s)})d\tau\Big]d\xi \\
&+\sum_{0<\epsilon_k<\epsilon_2}S_{\eta}(\epsilon_2-\epsilon_k)\|I_k(\varrho(\epsilon_k^-))\|
\end{aligned}
$$

$$+ BW^{-1}\Big[g_1(\epsilon, \varrho_{\rho(\epsilon,\varrho_\epsilon)}) + \int_{\epsilon_1}^{\epsilon_2}(\epsilon_2-\xi)^{\eta-1}\Big[g_2(\xi, \varrho_{\rho(\xi,\varrho_s)}) + \int_{\xi}^{\epsilon_2} Z_1(\tau, \xi, \varrho_{\rho(\xi,\varrho_s)})d\tau$$
$$+ \int_{\xi}^{b} Z_2(\tau, \xi, \varrho_{\rho(\xi,\varrho_s)})d\tau\Big]d\xi$$
$$- \int_{\epsilon_0}^{\epsilon_1}(\epsilon_1-\xi)^{\eta-1}g_2(\xi, \varrho_{\rho(\xi,\varrho_s)})d\xi + \int_{\epsilon_0}^{\epsilon_1}(\epsilon_2-\xi)^{\eta-1}g_2(\xi, \varrho_{\rho(\xi,\varrho_s)})d\xi$$
$$- \int_{\epsilon_0}^{\epsilon_1}\Big[(\epsilon_1-\xi)^{\eta-1}\int_{\xi}^{\epsilon_1} Z_1(\tau, \xi, \varrho_{\rho(\xi,\varrho_s)})d\tau$$
$$- (\epsilon_2-\xi)^{\eta-1}\int_{\xi}^{\epsilon_2} Z_1(\tau, \xi, \varrho_{\rho(\xi,\varrho_s)})d\tau\Big]d\xi - \int_{\epsilon_0}^{\epsilon_1}\Big[(\epsilon_1-\xi)^{\eta-1}$$
$$\times \int_{\xi}^{b} Z_2(\tau, \xi, \varrho_{\rho(\xi,\varrho_s)})d\tau - (\epsilon_2-\xi)^{\eta-1}\int_{\xi}^{b} Z_2(\tau, \xi, \varrho_{\rho(\xi,\varrho_s)})d\tau\Big]d\xi$$
$$+ \sum_{0<\epsilon_k<\epsilon_1} S_\eta(\epsilon_1-\epsilon_k)\|I_k(\varrho(\epsilon_k^-))\|\Big]\Big\|.$$

As $\epsilon_2 - \epsilon_1 \to 0$, the right-hand side tends to zero. Since $T_\eta(\epsilon)$ forms a strongly continuous semigroup for $\epsilon \geq 0$ and is compact for $\epsilon > 0$, it follows that $T_\eta(\epsilon)$ is continuous in the uniform operator topology for $\epsilon > 0$. Consequently, the equicontinuity for the other cases, such as $\epsilon_1 < \epsilon_2 \leq 0$ or $\epsilon_1 \leq 0 \leq \epsilon_2 \leq b$, can be deduced without difficulty.

By combining the results of Steps 1–3 and applying Theorem 2.2, we establish that the operator Φ is both continuous and compact. Therefore, utilizing Theorem 2.1, we conclude that a fixed point ϱ exists, which serves as a solution to the problem defined by Eqs (1.1) and (1.3). Hence, the system described by these equations is controllable on the interval $J = [\epsilon_0, b]$. This completes the proof of the theorem. □

4. Example

As an illustrative example, we examine a control system described by the fractional impulsive neutral VFIDE with SDD as follows:

$$\begin{aligned}
{}^cD^\eta\Big[x(\epsilon) - \int_{-\infty}^{\epsilon}\frac{e^{2(\xi-\epsilon)}x(\xi-\rho_1(\xi)\rho_2(\|x(\xi)\|))}{25}\,d\xi\Big] &= \frac{\partial^2}{\partial\varrho^2}\Big[x(\epsilon)+\mu(\epsilon)\\
&+ \int_{-\infty}^{\epsilon}\frac{e^{2(\xi-\epsilon)}x(\xi-\rho_1(\xi)\rho_2(\|x(\xi)\|))}{64}\,d\xi\\
&+ \int_{-\infty}^{\epsilon}\frac{e^{2(\tau-\xi)}x(\tau-\rho_1(\tau)\rho_2(\|x(\tau)\|))}{16}\,d\tau\,d\xi\\
&+ \int_{0}^{\epsilon}\sin(\epsilon-\xi)\int_{-\infty}^{\xi}\frac{e^{2(\tau-\xi)}x(\tau-\rho_1(\tau)\rho_2(\|x(\tau)\|))}{36}\,d\tau\,d\xi\\
&+ \int_{0}^{\epsilon}\sin(\epsilon-\xi)\int_{-\infty}^{\xi}\frac{e^{2(\tau-\xi)}x(\tau-\rho_1(\tau)\rho_2(\|x(\tau)\|))}{36}\,d\tau\,d\xi\Big],
\end{aligned} \tag{4.1}$$

with boundary conditions

$$x(\epsilon, 0) = 0 = x(\epsilon, \pi), \quad \epsilon \in [0, b], \tag{4.2}$$

and initial conditions

$$x(\epsilon) = \varphi(\epsilon), \quad \epsilon \le 0, \quad \varrho \in [0, \pi]. \tag{4.3}$$

$$\Delta x(\epsilon_k) = \int_{-\infty}^{\epsilon_k} \eta_k(\sigma - \epsilon_k) x(\sigma), \quad k = 1, 2, \ldots, n. \tag{4.4}$$

Consider the operator $\mathcal{A} : D(\mathcal{A}) \subset X \to X$, where $X = L^2[0, \pi]$ with the L^2-norm $\|\cdot\|_{L^2}$, and let ${}^cD_\epsilon^\eta$ represent the Caputo fractional derivative of order $\eta \in (0, 1)$. Suppose $\varphi \in \mathcal{B}$. The action of the operator $\mathcal{A}$ is defined as $\mathcal{A}\sigma = \sigma''$, where the domain $D(\mathcal{A})$ consists of functions $\sigma \in X$ such that σ and σ' are absolutely continuous, $\sigma'' \in X$, and the boundary conditions $\sigma(0) = \sigma(\pi) = 0$ are satisfied.

We can express $\mathcal{A}\sigma$ as:

$$\mathcal{A}\sigma = \sum_{n=1}^{\infty} n^2 \langle \sigma, \sigma_n \rangle \sigma_n, \quad \text{for } \sigma \in D(\mathcal{A}),$$

where the functions $\sigma_n(\xi) = \sqrt{\frac{2}{\pi}} \sin(ns)$ for $n = 1, 2, \ldots$ form an orthogonal set of eigenvectors for the operator $\mathcal{A}$. It is well-established that $\mathcal{A}$ generates an analytic semigroup $\{T(\epsilon)\}_{\epsilon \ge 0}$ in the space X, which is expressed as:

$$T(\epsilon)\sigma = \sum_{n=1}^{\infty} e^{-n^2\epsilon} \langle \sigma, \sigma_n \rangle \sigma_n, \quad \text{for every } \sigma \in X \text{ and } \epsilon > 0.$$

Since the semigroup $\{T(\epsilon)\}_{\epsilon \ge 0}$ is analytic and compact, there exists a constant $M > 0$ such that $\|T(\epsilon)\|_{L(X)} \le M$. For the phase space, we select $x = e^{2s}$ for $\xi < 0$, which yields $l = \int_{-\infty}^{0} x(\xi)\, d\xi = \frac{1}{2} < \infty$ for $\epsilon \le 0$. Additionally, we define the norm:

$$\|\sigma\|_{\mathcal{B}} = \int_{-\infty}^{0} x(\xi) \sup_{\theta \in [\xi, 0]} \|\sigma(\theta)\|_{L^2}\, d\xi.$$

Thus, for $(\epsilon, \varphi) \in [0, b] \times \mathcal{B}$, where $\varphi(\theta)(\varrho) = \varphi(\theta)$ for $(\theta) \in (-\infty, 0] \times [0, \pi]$, we define $x(\epsilon)(\varrho) = x(\epsilon)$ and $\rho(\epsilon, \varphi) = \rho_1(\epsilon)\rho_2(\|\varphi(0)\|)$. Consequently, we have:

$$g_1(\epsilon, \varphi)(\varrho) = \int_{-\infty}^{0} e^{2(\xi)} \frac{\varphi}{25} d\xi,$$

$$g_2(\epsilon, \varphi)(\varrho) = \int_{-\infty}^{0} e^{2(\xi)} \frac{\varphi}{64} d\xi,$$

$$\int_0^{\epsilon} Z_1(\epsilon, \xi, \varphi)(\varrho) d\xi = \int_0^{\epsilon} sin(\epsilon - \xi) \int_{-\infty}^{0} e^{2(\xi)} \frac{\varphi}{36} d\xi,$$

$$\int_0^{\epsilon} Z_2(\epsilon, \xi, \varphi)(\varrho) d\xi = \int_0^{\epsilon} sin(\epsilon - \xi) \int_{-\infty}^{0} e^{2(\xi)} \frac{\varphi}{36} d\xi.$$

To analyze the system defined in Eqs (4.1)–(4.4), we assume that the functions $\rho_i : [0,\infty) \to [0,\infty)$, for $i = 1, 2$, are continuous. By applying these configurations, the system can then be rewritten in the theoretical form of the design given by Eqs (1.1)–(1.3). As a result, for $\epsilon \in [0, T]$ and $\varphi, \bar{\varphi} \in \mathcal{B}$, we obtain the following:

$$\begin{aligned}
\|g_1(\epsilon,\varphi) - g_1(\epsilon,\bar{\varphi})\|_X &\leq \left(\int_0^\pi \left(\int_{-\infty}^0 e^{2(\xi)} \left\| \frac{\varphi}{25} - \frac{\bar{\varphi}}{25} \right\| d\xi \right)^2 \right)^{\frac{1}{2}} \\
&\leq \left(\int_0^\pi \left(\frac{1}{25} \int_{-\infty}^0 e^{2(\xi)} \sup \|\varphi - \bar{\varphi}\| d\xi \right)^2 \right)^{\frac{1}{2}} \\
&\leq \frac{\sqrt{\pi}}{25} \|\varphi - \bar{\varphi}\|_{\mathcal{B}} \\
&\leq L_g \|\varphi - \bar{\varphi}\|_{\mathcal{B}},
\end{aligned}$$

$$\begin{aligned}
\|g_2(\epsilon,\varphi) - g_2(\epsilon,\bar{\varphi})\|_X &\leq \left(\int_0^\pi \left(\int_{-\infty}^0 e^{2(\xi)} \left\| \frac{\varphi}{16} - \frac{\bar{\varphi}}{16} \right\| d\xi \right)^2 \right)^{\frac{1}{2}} \\
&\leq \left(\int_0^\pi \left(\frac{1}{16} \int_{-\infty}^0 e^{2(\xi)} \sup \|\varphi - \bar{\varphi}\| d\xi \right)^2 \right)^{\frac{1}{2}} \\
&\leq \frac{\sqrt{\pi}}{16} \|\varphi - \bar{\varphi}\|_{\mathcal{B}} \\
&\leq L_g \|\varphi - \bar{\varphi}\|_{\mathcal{B}},
\end{aligned}$$

$$\begin{aligned}
\|Z_1(\epsilon,\xi,\varphi) - Z_1(\epsilon,\xi,\bar{\varphi})\|_X &\leq \left(\int_0^\pi \left(\int_{-\infty}^0 e^{2(\xi)} \left\| \frac{\varphi}{36} - \frac{\bar{\varphi}}{36} \right\| d\xi \right)^2 \right)^{\frac{1}{2}} \\
&\leq \left(\int_0^\pi \left(\frac{1}{36} \int_{-\infty}^0 e^{2(\xi)} \sup \|\varphi - \bar{\varphi}\| d\xi \right)^2 \right)^{\frac{1}{2}} \\
&\leq \frac{\sqrt{\pi}}{36} \|\varphi - \bar{\varphi}\|_{\mathcal{B}} \\
&\leq L_{Z_1} \|\varphi - \bar{\varphi}\|_{\mathcal{B}}.
\end{aligned}$$

Similarly, we can deduce that

$$\begin{aligned}
\|Z_2(\epsilon,\xi,\varphi) - Z_2(\epsilon,\xi,\bar{\varphi})\|_X &\leq \left(\int_0^\pi \left(\int_{-\infty}^0 e^{2(\xi)} \left\| \frac{\varphi}{36} - \frac{\bar{\varphi}}{36} \right\| d\xi \right)^2 \right)^{\frac{1}{2}} \\
&\leq \left(\int_0^\pi \left(\frac{1}{36} \int_{-\infty}^0 e^{2(\xi)} \sup \|\varphi - \bar{\varphi}\| d\xi \right)^2 \right)^{\frac{1}{2}} \\
&\leq \frac{\sqrt{\pi}}{36} \|\varphi - \bar{\varphi}\|_{\mathcal{B}} \\
&\leq L_{Z_2} \|\varphi - \bar{\varphi}\|_{\mathcal{B}}.
\end{aligned}$$

Let $B : U \to X$ be defined by $Bu(\epsilon) = \mu(\epsilon, \varrho)$, for $0 \leq \varrho \leq \pi$, where $\mu : [0, T] \times [0, \pi] \to X$ is a continuous function.

Thus, the conditions (H1)–(H6) are satisfied. Furthermore, suppose the following values are assumed: $M = 1$, $e = 1$, $r^* = \frac{1}{2}$, $L^*_{g_1}, L^*_{g_2}, L^*_{Z_1}, L^*_{Z_2} = 0.3$, $L_2 = 0.1$, and $\eta = \frac{1}{2}$. In this case, we have the following calculation:

$$Me^{\eta}\left[L_{g_1}+\frac{L_{g2}}{\Gamma(\eta+1)}+\frac{e(L_{Z_1}+L_{Z_2})}{(\eta+1)\Gamma(\eta)}\right]r^*=0.04+\frac{1}{2}\left(\frac{0.6611}{0.8655}+\frac{2(0.053+0.22)}{3}\right)+\left(\frac{0.5}{0.8655}+\frac{2}{3}+0.1\right)$$

$$=0.04+\frac{1}{2}(0.3154+0.2110)+0.6458+0.75=0.86<1.$$

Thus, by Theorem 3.1, we conclude that the system defined by Eqs (4.1)–(4.4) has a mild solution on the interval [0, 1].

Remark 4.2. *The system defined by Eqs* (1.1)–(1.3) *integrates several real-world features such as impulsive effects, memory through fractional derivatives, neutral dependence, and state-dependent delays. These characteristics are essential for accurately modeling complex phenomena in biological, engineering, and control systems. The example provided in Eqs* (4.1)–(4.4) *illustrates the applicability of the general theory by representing a system where history-dependent and impulsive dynamics are significant.*

4.1. Graphical interpretation and comprehensive analysis

The graphical illustrations are essential to verify the obtained theoretical results and analyze the fractional impulsive neutral VFIDEs with SDD dynamics. Thus, the given graphical illustrations, in addition to confirming the solution stability, boundedness, and positivity, describe the sensitivity of the system on the main parameters.

Figure 1 provides the numerical solution $x(\epsilon)$ over the interval $\epsilon \in [\epsilon_0, \epsilon_f]$. The graph displays a smooth and bounded trajectory of $x(\epsilon)$, indicating that the solution is stable under the action of the given initial conditions and the fractional order η. The solution converges with time, and the lack of divergence points to the well-posedness of the problem. This stability is fundamental to the credibility of the approach in solving VFIDEs with SDD.

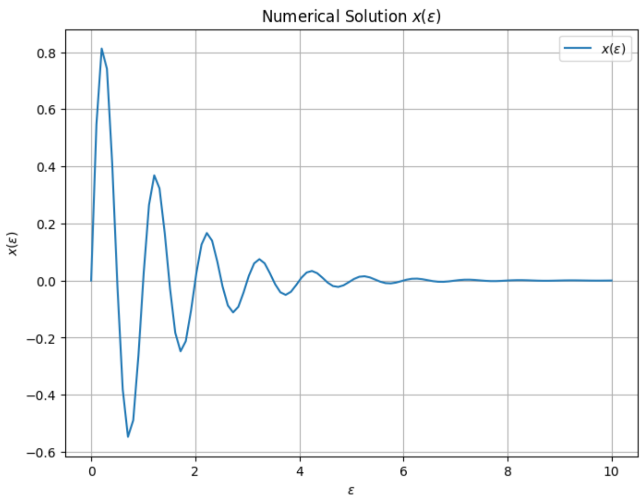

Figure 1. Numerical solution $x(\epsilon)$ over time for the VFIDE with SDD, showing smooth convergence and stability.

To further discuss the solution, Figure 2 displays the surface plot of $x(\epsilon)$ in terms of both ϵ and η. From the surface plot, it is clear that η affects the time behavior of the solution. As η is changed, the solution clearly shows different features; lower values of η are related to slower decay with high oscillatory behavior while large η values indicate fast convergence. This 3D plot gives a clear intuitive explanation of how fractional order affects the memory effects in the system and, therefore, exemplifies how fractional modeling accommodates highly complex dynamics within the phenomenon.

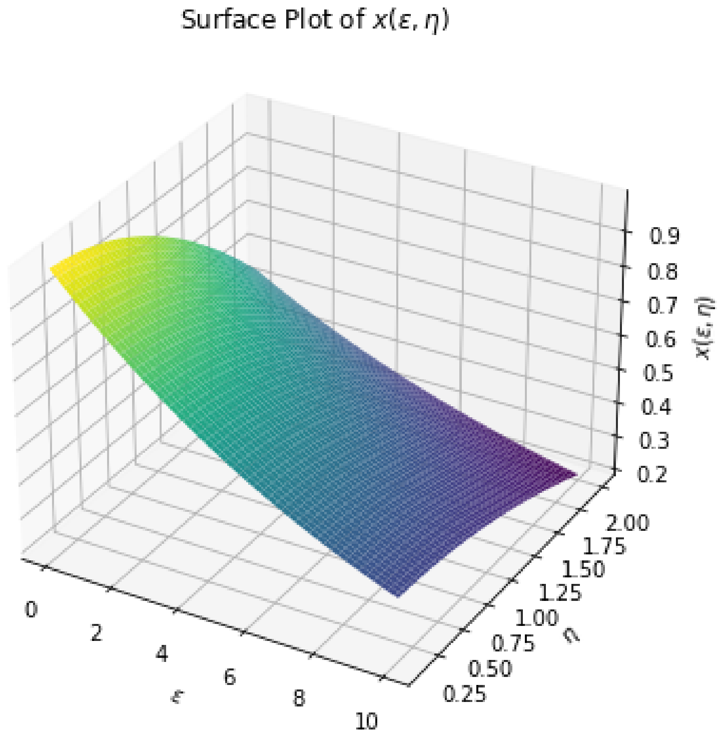

Figure 2. Surface plot of $x(\epsilon)$ as a function of both ϵ and fractional-order η, illustrating the interaction between time evolution and memory effects.

The decay of the analytic semigroup $T(\epsilon)$ is illustrated in Figure 3. This graph provides a detailed view of how $T(\epsilon)$ diminishes exponentially as ϵ increases, ensuring that the effect of initial perturbations fades over time. The rate of decay is influenced by the eigenfunction indices n, with higher indices exhibiting a faster decay. The confirmed results of these calculations confirm theoretically acquired findings where semigroup exponential decay confirms that the considered system is a stable one. Indeed, $T(\varepsilon)$ decreases very sharply further justifying this point that our scheme is strongly resistant to tiny deviations in an initial condition of a system under investigation.

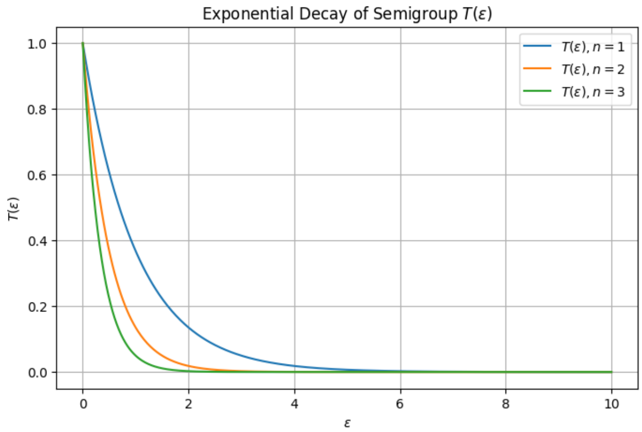

Figure 3. Exponential decay of the analytic semigroup $T(\epsilon)$ for various eigenfunction indices n, demonstrating stability.

Understanding fractional parameters influencing system behavior may require sensitivity analysis. Figure 4 depicts the effect of changing the fractional-order η on the solution $x(\epsilon)$. Decreasing η results in a slower decaying solution and stronger oscillations, which reflect a higher degree of memory in the system. In contrast, larger values of η give rise to faster convergence and a smoother trajectory, reflecting reduced memory. This observation reflects the critical importance of the order of the fraction in the way the system might be tuned by choosing an appropriate η, thus making possible the adaptation of the model for some applications.

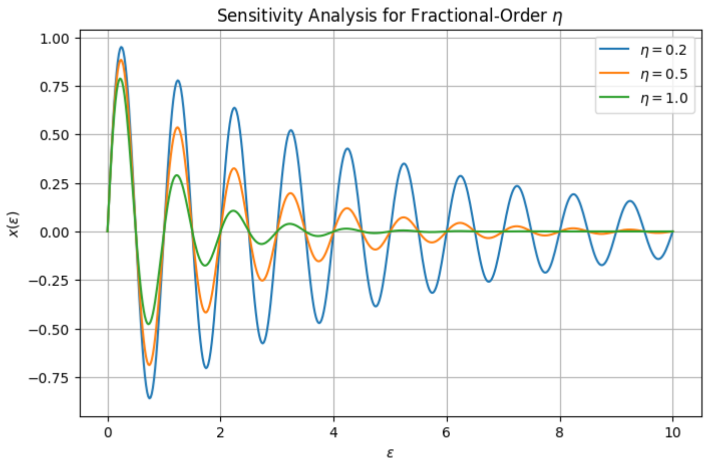

Figure 4. Effect of varying fractional-order η on the solution $x(\epsilon)$, illustrating the influence of memory effects.

Lastly, the overall understanding of fractional VFIDEs with SDD is achieved via the graphical representation of the ideas in Figures 1 and 2, among others. The smooth convergence of the solution, the multidimensional behavior captured in the surface plot, the rapid decay of the semigroup, the bounded phase space trajectory, and the sensitivity to fractional order collectively validate the theoretical results. These visualizations emphasize the applicability and robustness of the proposed method, paving the way for its application in more complex systems and real-world problems.

5. Application of reinforcement learning (RL) for control optimization in fractional impulsive neutral VFIDEs

In this study, we discuss applying RL to improve the control function $u(\epsilon)$ of systems, governed by the fractional impulsive neutral VFIDEs with SDD, given by Eqs (1.1) and (1.3). This section shows how AI, particularly RL, can be applied to optimize the control function in order to achieve improved system performance in the presence of fractional derivatives, impulsive effects, and delays. Reinforcement learning is a subfield of machine learning where an agent acts in an environment by taking actions and receiving feedback in the form of rewards or penalties. The goal of the agent is to learn a policy that maximizes the cumulative reward over time. In our system, the RL agent adjusts the control function $u(\epsilon)$ at each time step to bring the system closer to the desired behavior while minimizing the impact of impulsive jumps and fractional dynamics.

To apply RL to the system described in Eqs (1.1) and (1.3), we define the key components of the RL framework: the state, action, reward function, and policy. The state at any time ϵ is given by a mapping $\varrho(\epsilon) \in X$. This is a function in the Banach space X, in which both the present value and all preceding history of the system are contained due to fractional and delayed effects. This is, therefore, why the state also contains the present value of $\varrho(\epsilon)$ together with its previous evolution through the term ϱ_ϵ, which represents all that has occurred in the system up to time ϵ.

The action undertaken by the RL agent is the control input $u(\epsilon)$ representing how the system behavior should change at each step. The RL agent is trying to make the action choice such that it stabilizes the system by reducing the deviation from the desired state. The reward function is used by the RL agent to determine how good or bad an action it is making. The most popular reward function choice is the negative squared error between the current state and the desired state, encouraging the agent to minimize the difference. In addition, a penalty term for large impulsive jumps may be added to discourage unstable behavior at impulsive points. The reward function can be written as:

$$R(\epsilon) = -\left(\varrho(\epsilon) - \varrho_{\text{desired}}(\epsilon)\right)^2 - \alpha\left|\Delta\varrho(\epsilon)\right|,$$

where α is a constant that penalizes large jumps at the impulsive points.

The policy $\pi(\epsilon)$ is the strategy that the RL agent uses to select the next action $u(\epsilon)$ based on the current state. The policy is learned iteratively through interaction with the system, using techniques such as Q-learning or proximal policy optimization (PPO).

The RL agent learns by interaction with the system and over time modifies the control input $u(\epsilon)$ to optimize its policy. This is cast as an Markov Decision Process (MDP), where the goal of the agent is to choose actions that minimize the long-term cost or maximize the cumulative reward. The RL algorithm makes use of updates like:

$$Q(\epsilon, u(\epsilon)) \leftarrow Q(\epsilon, u(\epsilon)) + \alpha\left[R(\epsilon) + \gamma \max_{u'} Q(\epsilon', u') - Q(\epsilon, u(\epsilon))\right],$$

where α is the learning rate, γ is the discount factor, $R(\epsilon)$ is the immediate reward, and ϵ' represents the next state after taking action $u(\epsilon)$. RL control learns in an iterative method to stabilize a system by producing optimal control action at every instance of time steps. The control approach gives an RL data-driven solution to solving the problem to adapt to various complexities in a system, be it fractional derivatives, impulsive jumps, and delays.

It has the benefits of using reinforcement learning in fractional impulsive delay systems. First, RL is adaptive to complex dynamics; that is, the RL agent is able to adjust to the nonlinear and time-varying dynamics of the system, including the effects of fractional derivatives and impulsive jumps. Second, RL is a data-driven approach; it does not require a precise analytical model of the system. The agent learns from real-time data and can adjust its policy based on the system's behavior. Finally, uncertainty in the system is naturally handled by RL, for instance, the state-dependent delays and impulsive effects, because of its capability of refining the control policy continuously according to feedback from the system.

We now present an example where RL is applied to control a system similar to the one described by Eqs (1.1) and (1.3). The system's state $\varrho(\epsilon)$ evolves over time, and the RL agent must determine the optimal control function $u(\epsilon)$ to stabilize the system and minimize the deviation from a desired state. The control optimization process is illustrated in the following figures.

In this study, the RL agent is trained on simulated data generated from the mathematical model described by Eqs (1.1) and (1.3). The simulated environment reflects the behavior of the fractional impulsive neutral system with state-dependent delays, providing a controlled and accurate setup for testing the control strategy. This simulation-based approach is essential for validating the feasibility of RL in such complex systems before extending it to real-world scenarios.

Figure 5 shows the trajectory of learning an RL agent while illustrating how rewards improve with more training. Gradually, throughout time, an agent improves the policy and reduces error in control performance to reach the expected system behavior. This plot demonstrates a capacity of adapting the agent into the complex dynamics of the system to develop an appropriate control strategy. Episodes represent the number of iterations or training steps the RL agent completed. The reward in this simulation is presented as a linearly increasing function with added noise; this simulates the process of how the agent learns. The plot displays the improvement from the agent's performance over these episodes.

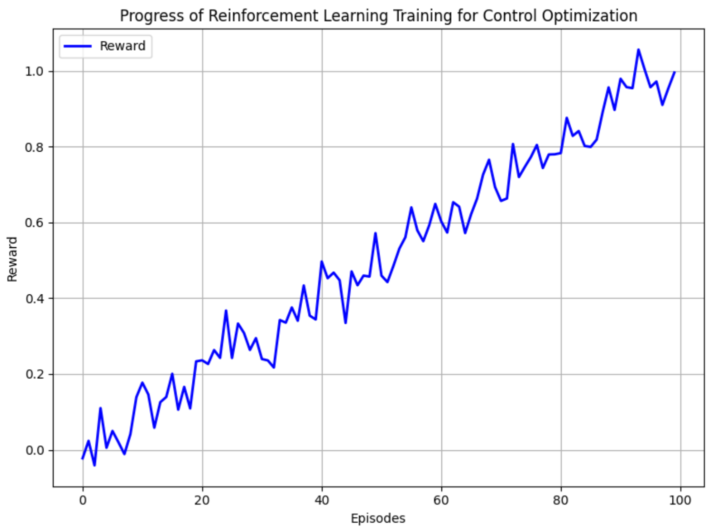

Figure 5. Progress of reinforcement learning training for control optimization. The plot shows the convergence of the RL agent's reward over time as it learns the optimal control strategy.

Figure 6 shows the optimal control input $u(\epsilon)$ obtained by the RL agent. This control input is calculated at each time step to minimize the error between the system's state $\varrho(\epsilon)$ and the desired state $\varrho_{\text{desired}}(\epsilon)$. The RL agent effectively adjusts $u(\epsilon)$ over time, showing the agent's ability to handle fractional dynamics and impulsive jumps.

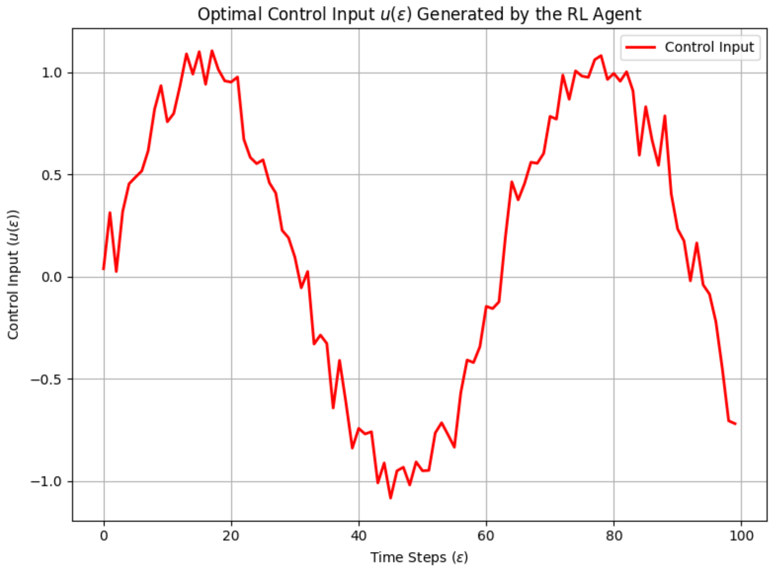

Figure 6. Optimal control input $u(\epsilon)$ generated by the RL agent. The plot shows the control input at each time step, which stabilizes the system and minimizes the error.

Therefore, reinforcement learning presents an effective optimization approach for the control function $u(\epsilon)$ of systems described by fractional impulsive neutral VFIDEs with state-dependent delays. Through RL, it is possible to iteratively optimize the control of the system so that it adapts to the complicated dynamics of the system. The RL agent learns to stabilize the system while reducing the effects of impulsive jumps and fractional effects. This application reveals the capabilities of AI, more specifically of RL, for solving the problems of optimal control in complex systems of fractional and impulsive dynamics.

The memory effects that associate the present state with both the present input as well as the whole history of the system represent one more defining aspect of fractional differential equations. Such memory effects in our RL-based control strategy are addressed by state representation design to cover both the present state and a history buffer, or previously stored past values of $\varrho(\epsilon)$, thus enabling the RL agent to recognize trends over time and alter control actions accordingly. From the computational point of view, the incorporation of memory effects creates issues in the form of an increase in dimensionality and requirement of storage. Reinforcement learning-based AI models require a high computational cost, as they involve simulating over long time horizons with historical data. This challenge can be brushed off by employing methods like memory truncation, RNNs, and efficient memory-management strategies.

6. Conclusions

In this paper, we discussed a complete investigation of the state-dependent fractional impulsive neutral VFIDE with focus on applying the Caputo fractional derivative to achieve fractional differentiation. We then made use of Schauder's fixed point theorem in order to determine the results regarding controllability under some assumptions, guaranteeing the existence and uniqueness of the solutions of the given system. We also used a detailed example to validate the theoretical findings with practical application in the behavior of the system. Numerical simulations were also performed to depict the convergence of the actual solution, exhibiting the dynamic development of the system over time. These simulations were also meant to emphasize the robustness of the proposed method.

Graphical analysis was done for the purpose of visual interpretation of the behavior of the system with respect to variation in conditions like different initial values and changes in parameters. It is clear that the solution depends sensitively on the initial conditions, thus indicating the stability as well as control mechanisms of the system. Additionally, we discussed its potential applications to control theory and machine learning problems, which highlighted its relevance to real-world systems and its application for optimizing system behaviors in the light of AI. This opens up new avenues for more study about the application of fractional calculus to complex dynamical systems with impulsive effects and state-dependent variables, as discussed in the application section.

On a computational level, the proposed method integrated Caputo fractional derivatives with Schauder's fixed point framework, which adds extra complication than classical control methods on the account memory effects appropriate for fractional systems along with the iterative nature of fixed point calculations. Whereas classical integer-order systems can typically be solved in closed form, or at least in part by linearization, the fractional-order context demands handling history-dependent operators and state delays, making the calculations even more expensive. However, such addition to the complexity is earnestly compensated by the improvement in accuracy of models and the flexibility of handling impulsive and state-dependent dynamics. Numerical methods adapted to such practical implementation, along with efficient memory management, can cut down the total cost of computation.

Author contributions

All authors wrote the main manuscript text. All authors reviewed the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgements

The authors I. Ayoob and N. Mlaiki would like to thank Prince Sultan University for paying the APC and for the support through the TAS research lab.

Conflict of interest

The authors declare no conflicts of interest.

References

1. Y. Zhou, *Basic theory of fractional differential equations*, Singapore: World Scientific, 2014.
2. T. Abdeljawad, R. P. Agarwal, E. Karapinar, P. S. Kumari, Solutions of the nonlinear integral equation and fractional differential equation using the technique of a fixed point with a numerical experiment in extended b-metric space, *Symmetry*, **11** (2019), 686. https://doi.org/10.3390/sym11050686
3. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam: Elsevier, 2006.
4. R. P. Agarwal, B. D. Andrade, On fractional integro-differential equations with state-dependent delay, *Comput. Math. Appl.*, **62** (2011), 1143–1149. https://doi.org/10.1016/j.camwa.2011.02.033
5. M. Benchohra, F. Berhoun, Impulsive fractional differential equations with state-dependent delay, *Commun. Appl. Anal.*, **14** (2010), 213–224.
6. T. Guendouzi, L. Bousmaha, Approximate controllability of fractional neutral stochastic functional integro-differential inclusions with infinite delay, *Qual. Theory Dyn. Syst.*, **13** (2014), 89–119.
7. Z. Liu, M. Bin, Approximate controllability for impulsive Riemann-Liouville fractional differential inclusions, *Abstr. Appl. Anal.*, **2013** (2013), 17. https://doi.org/10.1155/2013/639492
8. P. Balasubramaniam, P. Tamilalagan, Approximate controllability of a class of fractional neutral stochastic integro-differential inclusions with infinite delay by using Mainardi's function, *Appl. Math. Comput.*, **256** (2015), 232–246. https://doi.org/10.1016/j.amc.2015.01.035
9. I. Podlubny, *Fractional differential equations*, New York: Academic Press, 1998.
10. F. Mainardi, P. Paradisi, R. Gorenflo, Probability distributions generated by fractional diffusion equations, *arXiv Press*, 2007. https://doi.org/10.48550/arXiv.0704.0320
11. T. Gunasekar, P. Raghavendran, S. S. Santra, M. Sajid, Existence and controllability results for neutral fractional Volterra-Fredholm integro-differential equations, *J. Math. Comput. Sci.*, **34** (2024), 361–380.
12. T. Gunasekar, P. Raghavendran, The Mohand transform approach to fractional integro-differential equations, *J. Comput. Anal. Appl.*, **33** (2024), 358–371.
13. A. Hamoud, Existence and uniqueness of solutions for fractional neutral Volterra-Fredholm integro differential equations, *Adv. Theory Nonlinear Anal. Appl.*, **4** (2020), 321–331. https://doi.org/10.31197/atnaa.799854
14. A. Hamoud, N. Mohammed, K. Ghadle, Existence and uniqueness results for Volterra-Fredholm integro differential equations, *Adv. Theory Nonlinear Anal. Appl.*, **4** (2020), 361–372. https://doi.org/10.31197/atnaa.703984

15. P. Raghavendran, T. Gunasekar, H. Balasundaram, S. S. Santra, D. Majumder, D. Baleanu, Solving fractional integro-differential equations by Aboodh transform, *J. Math. Comput. Sci.*, **32** (2024), 229–240.

16. T. Gunasekar, P. Raghavendran, S. S. Santra, D. Majumder, D. Baleanu, H. Balasundaram, Application of Laplace transform to solve fractional integro-differential equations, *J. Math. Comput. Sci.*, **33** (2024), 225–237.

17. T. Gunasekar, P. Raghavendran, S. S. Santra, M. Sajid, Analyzing existence, uniqueness, and stability of neutral fractional Volterra-Fredholm integro-differential equations, *J. Math. Comput. Sci.*, **33** (2024), 390–407. https://doi.org/10.22436/jmcs.033.04.06

18. A. Columbu, S. Frassu, G. Viglialoro, Refined criteria toward boundedness in an attraction-repulsion chemotaxis system with nonlinear productions, *Appl. Anal.*, **103** (2024), 415–431. https://doi.org/10.1080/00036811.2023.2187789

19. A. Hamoud, K. Ghadle, Some new uniqueness results of solutions for fractional Volterra-Fredholm integro-differential equations, *Iran. J. Math. Sci. Inf.*, **17** (2022), 135–144.

20. A. Ndiaye, F. Mansal, Existence and uniqueness results of Volterra-Fredholm integro-differential equations via Caputo fractional derivative, *J. Math.*, **2021** (2021), 1–8. https://doi.org/10.1155/2021/5623388

21. Z. Dahmani, New existence and uniqueness results for high dimensional fractional differential systems, *Facta Univ., Ser.: Math. Inf.*, **30** (2015), 281–293.

22. H. HamaRashid, H. M. Srivastava, M. Hama, P. O. Mohammed, E. Al-Sarairah, M. Y. Almusawa, New numerical results on existence of Volterra-Fredholm integral equation of nonlinear boundary integro-differential type, *Symmetry*, **15** (2023), 1144. https://doi.org/10.3390/sym15061144

23. P. Kalamani, D. Baleanu, S. Selvarasu, M. Mallika Arjunan, On existence results for impulsive fractional neutral stochastic integro-differential equations with nonlocal and state-dependent delay conditions, *Adv. Differ. Equ.*, **2016** (2016), 1–36. https://doi.org/10.1186/s13662-016-0885-4

24. E. Hernandez, A. Prokopczyk, L. Ladeira, A note on partial functional differential equations with state-dependent delay, *Nonlinear Anal.*, **7** (2006), 510–519. https://doi.org/10.1016/j.nonrwa.2005.03.014

25. P. Raghavendran, T. Gunasekar, J. Ahmad, W. Emam, A study on the existence, uniqueness, and stability of fractional neutral Volterra-Fredholm integro-differential equations with state-dependent delay, *Fractal Fract.*, **9** (2024), 1–20. https://doi.org/10.3390/fractalfract9010020

26. H. M. Ahmed, M. A. Ragusa, Nonlocal controllability of Sobolev-type conformable fractional stochastic evolution inclusions with Clarke subdifferential, *Bull. Malays. Math. Sci. Soc.*, **45** (2022), 3239–3253. https://doi.org/10.1007/s40840-022-01377-y

27. J. Wang, H. M. Ahmed, Null controllability of nonlocal Hilfer fractional stochastic differential equations, *Miskolc Math. Notes*, **18** (2017), 1073–1083. https://doi.org/10.18514/MMN.2017.2396

28. H. M. Ahmed, M. M. El-Borai, A. O. El Bab, M. E. Ramadan, Approximate controllability of noninstantaneous impulsive Hilfer fractional integrodifferential equations with fractional Brownian motion, *Boundary Value Probl.*, **2020** (2020), 1–25. https://doi.org/10.1186/s13661-020-01418-0

29. R. Dhayal, M. Malik, S. Abbas, Approximate controllability for a class of non-instantaneous impulsive stochastic fractional differential equation driven by fractional Brownian motion, *Differ. Equ. Dyn. Syst.*, **29** (2021), 175–191. https://doi.org/10.1007/s12591-019-00463-1