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Research article

Several co-associative laws and pre-B-algebras

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Abstract: In this paper, we introduce several co-associative laws and the notion of a pre-B-algebra. We show that every B-algebra is both a pre-B-algebra and a $\perp$-algebra. We apply the notions of a post groupoid and a pre-semigroup of a groupoid to the set $(\mathbb{N}, +)$ of all nonnegative integers, and we prove that the groupoid $(\mathbb{N}, +)$ cannot be a post groupoid of a B-algebra or an edge d-algebra.

Keywords: pre-B-algebra; B-algebra; d-algebra; co-associative; $\perp$-algebra; anti-commutative; post groupoid; pre-semigroup
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1. Introduction

Neggers and Kim [1] introduced the notion of a B-algebra, and Cho and Kim [2] discussed B-algebras related to the quasigroup. Walendziak [3] and Kim and Kim [4] obtained another axiomatization of a B-algebra. Kim and Kim [4] introduced the notion of a BA-algebra and showed that the class of BA-algebras is equivalent to the class of B-algebras. Abdullah and Atshan [5] discussed several types of ideals in B-algebras and obtained some relations between those ideals. Belleza and Vilela [6] introduced the notion of a dual B-algebra and discussed some relationship between a dual B-algebra and a BCK-algebra. Al-Shehrie [7] introduced the notion of left-right derivation of B-algebras and discussed derivations of 0-commutative B-algebras. Naingue and Vilela [8] introduced the notion of a companion B-algebra and discussed a $\odot$-subalgebra and a $\odot$-ideal of a companion B-algebra.

As an application of B-algebras to fuzzy set theory, Jun et al. [9] discussed the fuzzification of (normal) B-subalgebras and characterized fuzzy B-algebras. Senapati et al. [10] introduced the notions of a cubic ideal to B-algebras and obtained several relations among cubic subalgebras with cubic ideals and cubic closed ideals of B-algebras. Gonzaga and Vilela [11] discussed a fuzzy order in fuzzy

B-algebras, and obtained useful results on the relations between the fuzzy order and the order in B-algebras. Recently, Borzooei et al. [12] discussed the notion of m-polar fuzzy (normal) subalgebras in B-algebras, and they characterized the m-polar intuitionistic fuzzy (normal) subalgebra.

As a generalization of a BCK-algebra, Neggers and Kim [13] introduced the notion of a d-algebra just deleting two complicated axioms from the BCK-algebra. They investigated some relations between d-algebras and BCK-algebras. Neggers et al. [14] introduced notions of d-subalgebras, d-ideal, $d^\sharp$-ideal, and d^*-ideal in d-algebras and discussed some relations among them.

In this paper, we introduce several co-associative laws and we select two co-associative laws for the investigation of B-algebras. We show that every B-algebra is a pre-B-algebra, but the converse does not hold in general by giving a counter-example. We introduce the notion of a $\perp$-algebra and we prove that every B-algebra is a $\perp$-algebra. We find some conditions for a $\perp$-algebra to be a semigroup. We introduce the notion of a post groupoid and a pre-semigroup of a groupoid. We apply these concepts to the set $\mathbb{N}$ of all nonnegative integers and obtain their several related properties. Finally, we prove that the groupoid $(\mathbb{N}, +)$ cannot be a post groupoid of a B-algebra or an edge d-algebra.

We introduce some definitions and theorems which are necessary to develop the theory in Section 2. We introduce several co-associative laws in Section 3. Among them we choose two axioms, and make a notion of a pre-B-algebra. We discuss some relations between pre-B-algebras, B-algebras, and $\perp$-algebras in Section 3. In Section 4, we introduce the notions of a post groupoid, a pre-semigroup, and a primitive semigroup, and discuss their roles in the groupoid $(\mathbb{N}, +)$.

2. Preliminaries

In this section, we give some definitions and theorems which are useful to investigate the theory as follows:

A nonempty set P with a constant 0 and a binary operation "$*$" is said to be a *B-algebra* [1] if

(I) $p * p = 0$,
(II) $p * 0 = p$,
(III) $(p * q) * r = p * (r * (0 * q))$

for all p, q, r in P.

Proposition 2.1. [1] *If* $(P, *, 0)$ *is a B-algebra, then*

(i) $p = 0 * (0 * p)$,
(ii) $p * (q * r) = (p * (0 * r)) * q$,
(iii) $p * q = 0 * (q * p)$

for all $p, q, r \in P$.

A nonempty set P with a constant 0 and a binary operation "$*$" is called a *d-algebra* [13] if

(I) $p * p = 0$,
(IV) $0 * p = 0$,
(V) $p * q = 0$ and $q * p = 0$ imply $p = q$

for all $p, q \in P$.

A d-algebra $(P, *, 0)$ is said to be an *edge* [13] if $p * 0 = p$ for all $p \in P$. A groupoid $(P, *)$ is called a *left zero semigroup* [15] if $p * q = p$ for any $p, q \in P$, and a groupoid $(P, *)$ is called a *right zero semigroup* [15] if $p * q = q$ for any $p, q \in P$.

3. Co-associative laws

In this section, we suggest several generalized associative laws as below, and we select two axioms for defining an algebra, called a pre-B-algebra. We discuss some relations between a pre-B-algebra and a B-algebra. Moreover, we introduce the notion of a $\perp$-algebra, and find some relations with a pre-B-algebra.

Given a groupoid $(P, *)$, i.e., a nonempty set P and a binary operation "$*$" on P, we consider various co-associative laws: for all $p, q, r, u \in P$,

- *middle co-associative*: $(p * q) * r = p * (r * ((q * q) * q))$,
- *final co-associative*: $(p * q) * r = p * (r * ((r * r) * q))$,
- *universal co-associative*: $(p * q) * r = p * (r * ((u * u) * q))$,
- *middle co′-associative*: $p * (q * r) = (p * ((q * q) * r)) * q$,
- *final co′-associative*: $p * (q * r) = (p * ((r * r) * r)) * q$,
- *universal co′-associative*: $p * (q * r) = (p * ((u * u) * r)) * q$.

Proposition 3.1. *Let $(P, \cdot, e)$ be a group. If we define a binary operation "$*$" on P by $p * q := p \cdot q^{-1}$ for all $p, q \in P$, then $(P, *)$ is both universal co-associative and universal co′-associative.*

Proof. Let $(P, \cdot, e)$ be a group. If we define a binary operation "$*$" on P by $p * q := p \cdot q^{-1}$ for all $p, q \in P$, then, for any $p, q, r \in P$, we have $(p * q) * r = (p \cdot q^{-1}) * r = (p \cdot q^{-1}) \cdot r^{-1} = p \cdot (q^{-1} \cdot r^{-1})$. We compute $p * (r * ((u * u) * q))$ as follows: for all $p, q, r, u \in P$,

$$\begin{aligned} p * (r * ((u * u) * q)) &= p \cdot (r * ((u * u) * q))^{-1} \\ &= p \cdot (r \cdot ((u \cdot u^{-1}) * q)^{-1})^{-1} \\ &= p \cdot (r \cdot (q^{-1})^{-1})^{-1} \\ &= p \cdot (r \cdot q)^{-1} \\ &= p \cdot (q^{-1} \cdot r^{-1}). \end{aligned}$$

Since $(P, *, e)$ is a group, the universal co-associative law holds.

Similarly, we can prove the universal co′-associative law. □

A groupoid $(P, *)$ is said to be a *pre-B-algebra* if it is both universal co-associative and universal co′-associative. By Proposition 3.1, we can obtain many pre-B-algebras from groups. Note that we may define other algebraic structures from the above generalized associative laws, e.g., pre-B'-algebra, pre-B''-algebra, etc., by choosing suitable axioms.

In this paper, we focus on the story of pre-B-algebras.

Theorem 3.2. *Every B-algebra is a pre-B-algebra.*

Proof. Let $(P, *, 0)$ be a B-algebra. Given $p, q, r, u \in P$, by (I), (III), we obtain

$$p * (r * ((u * u) * q)) = p * (r * (0 * q)) = (p * q) * r.$$

Hence $(P,*)$ is universal co-associative. Moreover, by Proposition 2.1(ii), we obtain

$$(p*((u*u)*r))*q = (p*(0*r))*q = p*(q*r).$$

Hence $(P,*)$ is universal co′-associative. □

The converse of Theorem 3.2 need not be true in general. See the following example.

Example 3.3. Let $(P,*)$ be a left zero semigroup and $|P| \geq 2$. Then $p*q = p$ for all $p,q \in P$. It follows that $(p*q)*r = p = p*(r*((u*u)*q))$ and $p*(q*r) = p = (p*((u*u)*r))*q$, and hence $(P,*)$ is a pre-B-algebra. But it is not a B-algebra. In fact, if it is a B-algebra, then $p = p*p = 0$ for all $p \in P$, which leads to $P = \{0\}$, which is a contradiction.

Given a groupoid $(P,*)$ and $p \in P$, we define $p^{\perp} := (p*p)*p$. We call $p^{\perp}$ a *perp* of p in $(P,*)$.

If we take $q := p$ and $r := p$ in various co-associative laws, then we obtain more simple forms with the aid of a pert $p^{\perp}$ of p as follows:

Proposition 3.4. *Let $(P,*)$ be a groupoid. Then the following holds:*

(i) *if $(P,*)$ is middle co-associative, then $p^{\perp} = p*(p*p^{\perp})$,*
(ii) *if $(P,*)$ is final co-associative, then $p^{\perp} = p*(p*p^{\perp})$,*
(iii) *if $(P,*)$ is middle co′-associative, then $p*(p*p) = (p*p^{\perp})*p$,*
(iv) *if $(P,*)$ is final co′-associative, then $p*(p*p) = (p*p^{\perp})*p$*

for all $p,q,r \in P$.

Proof. The proofs are easy, and we omit their proofs. □

Note that the results in Proposition 3.4 are analogues of the power associative law in the semigroup theory.

Since every pre-B-algebra is both universal co-associative and universal co′-associative, by Proposition 3.4, we obtain the following corollary.

Corollary 3.5. *If $(P,*)$ is a pre-B-algebra, then $p^{\perp} = p*(p*p^{\perp})$ and $(p*p^{\perp})*p = p*(p*p)$ for all $p \in P$.*

A groupoid $(P,*)$ is said to be *power co-associative* if $p^{\perp} = p*(p*p^{\perp})$ for all $p \in P$, and a groupoid $(P,*)$ is said to be *power co′-associative* if $(p*p^{\perp})*p = p*(p*p)$ for all $p \in P$.

By Corollary 3.5, pre-B-algebras are among those algebras which have both power co-associative law and power co′-associative law.

Proposition 3.6. *If $(P,*,0)$ is a B-algebra, then*

(i) *$0*p^{\perp} = p$,*
(ii) *$(p^{\perp})^{\perp} = p$*

for all $p \in P$.

Proof. (i) By Proposition 2.1(i), we have $p = 0*(0*p)$ for all $p \in P$. It follows that $p = 0*(0*p) = 0*[(p*p)*p] = 0*p^{\perp}$ for all $p \in P$.
(ii) By (I) and Proposition 3.6(i), we have $(p^{\perp})^{\perp} = (p^{\perp}*p^{\perp})*p^{\perp} = 0*p^{\perp} = p$. □

Proposition 3.7. *If* $(P, *)$ *is a left zero semigroup, then* $(p^{\perp})^{\perp} = p$ *for all* $p \in P$.

Proof. If $(P, *)$ is a left zero semigroup, then $p^{\perp} = (p * p) * p = p * p = p$ for all $p \in P$. Hence $(p^{\perp})^{\perp} = p^{\perp} = p$ for all $p \in P$. □

A groupoid $(P, *)$ is said to be a $\perp$-*algebra* (or a *perp-algebra*) if $(p^{\perp})^{\perp} = p$ for all $p \in P$.

Note that, by Proposition 3.6(ii), every B-algebra is a $\perp$-algebra, but the converse need not be true, since every left zero semigroup is a $\perp$-algebra. Example 3.3 shows that a nontrivial left zero semigroup $(P, *)$, $|P| \geq 2$, is a $\perp$-algebra, but not a B-algebra.

We give an example of a $\perp$-algebra which is not a pre-B-algebra as follows:

Example 3.8. Let $X := \{0, a, b, c\}$ with the following table:

*	0	a	b	c
0	0	$\surd_1$	$\surd_2$	c
a	$\surd_3$	b	b	b
b	$\surd_4$	a	a	b
c	$\surd_5$	$\surd_6$	$\surd_7$	c

where $\surd_i \in P$, $(i = 1, 2, \cdots, 7)$. Then it is easy to see that $(P, *)$ is a $\perp$-algebra, but not a pre-B-algebra, since $(a * c) * b = a$ and $a * (b * ((0 * 0) * c)) = b$.

Note that it is not known yet that there are pre-B-algebras which are neither a B-algebra nor a $\perp$-algebra. We draw a diagram for these relations as Figure 1:

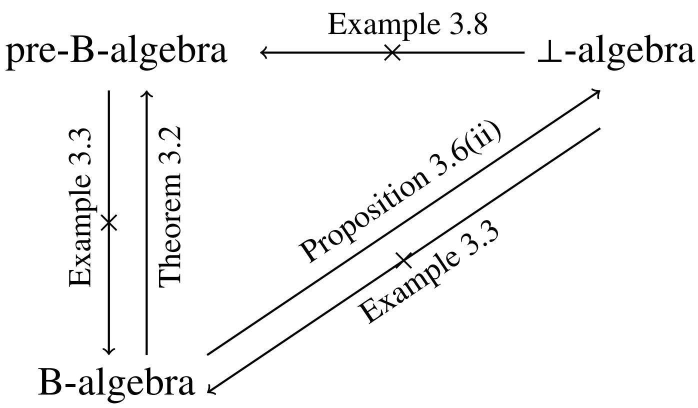

Figure 1. Relations on pre-B-algebras.

Proposition 3.9. *Let* $(P, *, 0)$ *be a B-algebra. Then*

$$(p * q)^{\perp} = q * p \tag{C}$$

for all $p, q \in P$.

Proof. Given $p, q \in P$, by (I) and Proposition 2.1(iii), we obtain

$$\begin{aligned}(p * q)^{\perp} &= [(p * q) * (p * q)] * (p * q) \\ &= 0 * (p * q) \\ &= q * p,\end{aligned}$$

which proves the proposition. □

Proposition 3.10. *Let $(P, *, 0)$ be a left zero semigroup. Then*

$$(p * q)^{\perp} = p$$

for all $p, q \in P$.

Proof. Given $p, q \in P$, we have $p = (p * p) * p = p^{\perp} = (p * q)^{\perp}$. □

Note that every left zero semigroup $(P, *)$ with $|P| \geq 2$ does not satisfy the condition (C).

A groupoid $(P, *)$ is said to be *anti-commutative* if, for all $p, q \in P$, $(p * q)^{\perp} = q * p$.

Theorem 3.11. *If a groupoid $(P, *)$ is commutative and anti-commutative and $P * P = P$, then $(P, *)$ is a $\perp$-algebra.*

Proof. If $(P, *)$ is commutative and anti-commutative, then $(p * q)^{\perp} = q * p = p * q$ for all $p, q \in P$. Since $P * P = P$, there exist $u, v \in P$ such that $p = u * v$ for any $p \in P$. It follows that $p^{\perp} = (u * v)^{\perp} = v * u = u * v = p$, which shows that $(P, *)$ is a $\perp$-algebra. □

Theorem 3.12. *Let $(P, *)$ be an anti-commutative and middle co-associative $\perp$-algebra. If we define a binary operation "$\bullet$" on P by $p \bullet q := p * q^{\perp}$ for all $p, q \in P$, then $(P, \bullet)$ is a semigroup.*

Proof. Given $p, q, r \in P$, we have

$$\begin{aligned}(p \bullet q) \bullet r &= (p * q^{\perp}) * r^{\perp} \\ &= p * (r^{\perp} * ((q^{\perp} * q^{\perp}) * q^{\perp})) \\ &= p * (r^{\perp} * (q^{\perp})^{\perp}) \\ &= p * ((q^{\perp})^{\perp} * r^{\perp})^{\perp} \\ &= p \bullet ((q^{\perp})^{\perp} \bullet r) \\ &= p \bullet (q \bullet r),\end{aligned}$$

which shows that $(P, \bullet)$ is a semigroup. □

4. Pre-semigroups and $\perp$-algebras

In this section, we discuss some relations between pre-semigroups and $\perp$-algebra, and we show that a groupoid $(\mathbb{N}, +)$ cannot be a post groupoid of a B-algebras $(\mathbb{N}, *, 0)$.

A groupoid $(P, \bullet)$ is said to be a *post groupoid* of a groupoid $(P, *)$ if $p \bullet q := p * q^{\perp}$ for all $p, q \in P$. In this case, we denote it by $(P, *) \models (P, \bullet)$.

Example 4.1. Let $\mathbb{N} := \{0, 1, 2, 3, \cdots\}$ and let "+" be the usual addition on $\mathbb{Z}$. Let $(\mathbb{N}, \bullet)$ be a post groupoid of $(\mathbb{N}, +)$. Then $p^{\perp} = (p + p) + p = 3p$ for all $p \in \mathbb{N}$. It follows that $p \bullet q = p + q^{\perp} = p + 3q$ for all $p, q \in \mathbb{N}$.

A groupoid $(P, *)$ is said to be a *pre-semigroup* of a groupoid $(P, \bullet)$ if (i) $(P, *) \models (P, \bullet)$ and (ii) $(P, \bullet)$ is a semigroup. Note that $(\mathbb{N}, +)$ is not a pre-semigroup. If we assume that there exists a semigroup $(\mathbb{N}, \bullet)$ such that $(\mathbb{N}, +) \models (\mathbb{N}, \bullet)$, then, by Example 4.1, we have $p \bullet q = p + 3q$ for any $p, q \in \mathbb{N}$. It follows that $(1 \bullet 2) \bullet 3 = (1 + 3 \cdot 2) \bullet 3 = 7 + 3 \cdot 3 = 16$, but $1 \bullet (2 \bullet 3) = 1 + 3(2 + 3 \cdot 3) = 34$, which leads to a contradiction.

Proposition 4.2. *Let $(P, *)$ be a right zero semigroup. If $(P, \bullet)$ is a groupoid with $(P, *) \models (P, \bullet)$, then $(P, *)$ is a pre-semigroup.*

Proof. Given $p, q \in P$, we have $p \bullet q = p * q^{\perp} = p * ((q * q) * q) = p * q = q$, which shows that $(P, \bullet)$ is a right zero semigroup, i.e., a semigroup. This proves that $(P, *)$ is a pre-semigroup. □

Let $(P, *)$ be a groupoid and $p, q \in P$. We define $p *^n q$ by

$$\begin{aligned} p *^1 q &:= \quad p * q, \\ p *^2 q &:= \quad (p *^1 q) * q, \\ p *^3 q &:= \quad (p *^2 q) * q, \\ \cdots\cdots & \quad\ \cdots\cdots \\ p *^n q &:= \quad (\cdots((p * \underbrace{q) * q) * q) * \cdots) * q}_{n}. \end{aligned}$$

Proposition 4.3. *Let $\mathbb{N} := \{0, 1, 2, 3, \cdots\}$ and "+" be the usual addition on $\mathbb{N}$. Let $(\mathbb{N}, *)$ be a groupoid with $(\mathbb{N}, *) \models (\mathbb{N}, +)$. Then*

(i) $p * 0^{\perp} = p = 0 * p^{\perp}$,
(ii) $p = \frac{1}{n}[p *^{n-1} p^{\perp}]$ *where* $n \geq 2$,

for all $p \in \mathbb{N}$.

Proof. (i) Since $(\mathbb{N}, *) \models (\mathbb{N}, +)$, we have $p + q = p * q^{\perp} = p * [(q * q) * q]$ for all $p, q \in \mathbb{N}$. It follows that $p = p + 0 = p * [(0 * 0) * 0] = p * 0^{\perp}$. Since $p = 0 + p$, we obtain $p = 0 * p^{\perp}$.
(ii) Given $p \in \mathbb{N}$, we have $2p = p + p = p * p^{\perp}$, and hence $p = \frac{1}{2}[p * p^{\perp}]$ and $3p = 2p + p = 2p * p^{\perp} = (p * p^{\perp}) * p^{\perp}$. It follows that $p = \frac{1}{3}[(p * p^{\perp}) * p^{\perp}] = \frac{1}{3}[p *^2 p^{\perp}]$.

Similarly, we obtain $4p = 3p + p = 3p * p^{\perp} = [p *^2 p^{\perp}] * p^{\perp} = p *^3 p^{\perp}$, and hence $p = \frac{1}{4}[p *^3 p^{\perp}]$. In this fashion, by induction, we obtain $p = \frac{1}{n}[p *^{n-1} p^{\perp}]$ for all $p \in \mathbb{N}$, where $n \geq 2$. □

Proposition 4.4. *Let $(\mathbb{N}, *)$ be a $\perp$-algebra and $(\mathbb{N}, *) \models (\mathbb{N}, +)$. Then*

(i) $p^{\perp} = 0 * p$,
(ii) $p = 0 * (0 * p)$,
(iii) $a * p = b * p$ *implies* $a = b$,
(iv) $p * p = 0$,
(v) $p * 0 = p$

for all $p, a, b \in \mathbb{N}$.

Proof. (i) Given $p, q \in \mathbb{N}$, since $(\mathbb{N}, *) \models (\mathbb{N}, +)$, we have $p + q = p * q^{\perp}$. It follows that $p^{\perp} = 0 + p^{\perp} = 0 * (p^{\perp})^{\perp}$ for all $p \in \mathbb{N}$. Since $(\mathbb{N}, *)$ is a $\perp$-algebra, we obtain $p^{\perp} = 0 * (p^{\perp})^{\perp} = 0 * p$.
(ii) By (i), we have $p = 0 + p = 0 * p^{\perp} = 0 * (0 * p)$ for all $p \in \mathbb{N}$.
(iii) Suppose $a * p = b * p$. Since $(\mathbb{N}, *)$ is a $\perp$-algebra, we obtain $a * (p^{\perp})^{\perp} = b * (p^{\perp})^{\perp}$, and hence $a + p^{\perp} = b + p^{\perp}$, which shows that $a = b$.
(iv) By (i), we have $(p * p) * p = p^{\perp} = 0 * p$ for all $p \in \mathbb{N}$. By applying (iii), we obtain $p * p = 0$ for all $p \in \mathbb{N}$.
(v) $p * 0 = p * (0 * 0) = p * 0^{\perp} = p + 0 = p$ for all $p \in \mathbb{N}$. □

Theorem 4.5. *A groupoid* $(\mathbb{N}, +)$ *cannot be a post groupoid of a B-algebra* $(\mathbb{N}, *, 0)$.

Proof. Assume that $(\mathbb{N}, +)$ is a post groupoid of a B-algebra $(\mathbb{N}, *, 0)$, i.e., $(\mathbb{N}, *) \models (\mathbb{N}, +)$. By Proposition 3.6, every B-algebra is a $\perp$-algebra. It follows that

$$(p * q) * r = (p * (q^{\perp})^{\perp}) * (r^{\perp})^{\perp} = p + q^{\perp} + r^{\perp}.$$

By Proposition 4.4(i), we obtain

$$p * r * (0 * q)) = p * (r * q^{\perp}) = p * (r + q) = p + (r + q)^{\perp}.$$

Since $(\mathbb{N}, *)$ is a B-algebra, we obtain $p + q^{\perp} + r^{\perp} = p + (r + q)^{\perp}$. Hence $q^{\perp} + r^{\perp} = (r + q)^{\perp}$ for all $q, r \in \mathbb{N}$. It follows that $2^{\perp} = (1 + 1)^{\perp} = 1^{\perp} + 1^{\perp} = 2(1^{\perp})$ and $3^{\perp} = (2 + 1)^{\perp} = 2^{\perp} + 1^{\perp} = 2(1^{\perp}) + 1^{\perp} = 3(1^{\perp})$.

In this fashion, we obtain $n^{\perp} = n(1^{\perp})$ for all $n \in \mathbb{N}$. Now, we need to calculate $1^{\perp}$. Since $(\mathbb{N}, *)$ is a $\perp$-algebra, we obtain $n = (n^{\perp})^{\perp} = [n(1^{\perp})]^{\perp} = [n(1^{\perp})]1^{\perp} = n(1^{\perp})^2$ for all $n \in \mathbb{N}$. It follows that $1^{\perp} = 1$, and hence $n^{\perp} = n(1^{\perp}) = n$ for all $n \in \mathbb{N}$. Hence $m * n = m * n^{\perp} = m + n$ for all $m, n \in \mathbb{N}$. By definition of the $\perp$, we obtain $n^{\perp} = (n * n) * n = (n + n) + n = 3n$, and hence $n = (n^{\perp})^{\perp} = (3n)^{\perp} = 3(3n) = 9n$ for all $n \in \mathbb{N}$, which is a contradiction. □

Proposition 4.6. *A groupoid* $(\mathbb{N}, +)$ *cannot be a post groupoid of an edge d-algebra* $(\mathbb{N}, *, 0)$.

Proof. Assume that $(\mathbb{N}, +)$ is a post groupoid of an edge d-algebra $(\mathbb{N}, *, 0)$, i.e., $(\mathbb{N}, *) \models (\mathbb{N}, +)$. Then $p^{\perp} = (p * p) * p = 0 * p = 0$ for all $p \in \mathbb{N}$, since $(\mathbb{N}, *, 0)$ is a d-algebra. Since $(\mathbb{N}, *)$ is an edge d-algebra and $(\mathbb{N}, *) \models (\mathbb{N}, +)$, we obtain $p + q = p * q^{\perp} = p * 0 = p$ for all $p, q \in \mathbb{N}$, i.e., $p + q = p$, which leads to $q = 0$, which is a contradiction. □

A semigroup $(P, \bullet)$ is said to be a *primitive semigroup* if there is no groupoid $(P, *)$ such that $(P, *) \models (P, \bullet)$. We consider the groupoid $(\mathbb{N}, +)$. If we assume that $(\mathbb{N}, +)$ is not a primitive semigroup, then there exists a groupoid $(\mathbb{N}, \star)$ such that $(\mathbb{N}, \star) \models (\mathbb{N}, +)$. It follows that

$$p + q = p \star q^{\perp} = p \star ((q \star q) \star q) \tag{D}$$

for all $p, q \in \mathbb{N}$. It is not sufficient to find what $p \star q$ is by using this condition only, i.e., we need additional axioms or conditions for satisfying the condition (D). For example, if we define a binary operation $\star$ on $\mathbb{N}$ by $p \star q := p - q$, then $p \star q^{\perp} = p \star ((q \star q) \star q) = p - ((q - q) - q) = p + q$ for all $p, q \in \mathbb{N}$. But $(\mathbb{N}, -)$ is not a groupoid, since $3 - 5 = -2 \notin \mathbb{N}$. It is an interesting topic for finding a groupoid $(\mathbb{N}, \star)$ satisfying the condition (D). On the basis of this reason, we may have a conjecture as follows:

Conjecture. $(\mathbb{N}, +)$ *is a primitive semigroup.*

5. Conclusions and future works

We introduced several co-associative laws, and we selected two co-associative laws for the investigation of B-algebras named it a pre-B-algebra. We showed that every B-algebra is a pre-B-algebra, but the converse does not hold in general by giving a counter-example. We introduced the notion of a $\perp$-algebra, and we proved that every B-algebra is a $\perp$-algebra. We found some conditions for a $\perp$-algebra to be a semigroup. We introduced the notion of a post groupoid and a pre-semigroup of a groupoid. We applied these concepts to the set $\mathbb{N}$ of all nonnegative integers, and we obtained several related properties. Finally, we proved that the groupoid $(\mathbb{N}, +)$ cannot be a post groupoid of a B-algebra or an edge d-algebra.

We shall apply the notion of a post groupoid to several algebraic structures, and we will discuss some relations between different post groupoids. Moreover, we shall generalize several theorems and propositions which were discussed in B-algebras to the area of pre-B-algebras with the aid of $\perp$-algebras. Moreover, we shall apply the notion of $Bin(X)$ [16] to post groupoids and provide a ladder to investigate the area of hyper-pre-B-algebras. Since the set of all natural numbers is a concrete example of a semiring, we shall apply these results to the theory of semirings and computer science also.

Author contributions

Siriluk Donganont: Writing-review and editing, investigation; Sun Shin Ahn: Writing-review and editing; Hee Sik Kim: Writing-review and editing, supervision. All authors have read and approved the final version for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Prof. Dr. Hee Sik Kim is a Guest Editor of special issue "General Algebraic Structures and Fuzzy Algebras" for AIMS Mathematics. Prof. Dr. Hee Sik Kim was not involved in the editorial review and the decision to publish this article. The authors hereby declare that there are no conflicts of interest regarding the publication of this paper.

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