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Research article

Optimal quaternary Hermitian self-orthogonal $[n, 5]$ codes of $n \geq 492$

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Abstract: Self-orthogonal (SO) codes, including Hermitian self-orthogonal (HSO) codes, form an important class of linear codes, and such codes have close connections to other mathematical structures such as block designs, lattices, and sphere packings, which can also be used to construct quantum codes. Many scholars try to solve the problem of determining low-dimensional optimal HSO codes over small fields as it is done for optimal linear codes. Let $d_o(n, k)$ be the minimum distance of an optimal quaternary $[n, k]$ linear code, and $d_{so}(n, k)$ be that of an optimal quaternary $[n, k]$ HSO code. In this paper, we try to determine $d_{so}(n, 5)$ for $n \geq 492$ by constructing quaternary $[n, 5]$ HSO codes in detail. Some disjoint HSO blocks have been found from generator matrices of some special optimal HSO codes. These special optimal HSO codes are constructed from quaternary simplex codes and McDonald codes. Then, $[n, 5]$ HSO codes have been constructed for $n \geq 492$, by removing those special blocks from the known optimal HSO codes. As a result, we could show $[n, 5, d_{so}(n, 5)] = [n, 5, 2\lfloor \frac{d_o(n,5)}{2} \rfloor]$ for $n \geq 492$.

Keywords: optimal code; Hermitian self-orthogonal (HSO) code; Griesmer bound; simplex code; McDonald code; HSO block
Mathematics Subject Classification: 94B05, 11T71

1. Introduction

Let F_q^n be the n-dimensional vector space over the Galois field $F_q = GF(q)$. An $[n, k, d]_q$ code is a k-dimensional subspace of F_q^n with minimal distance d. An $[n, k, d]_q$ code is *optimal* if there is no $[n, k, d + 1]_q$ code. Parameters of optimal $[n, k]_q$ codes for the following k and q are solved [1–4]:

1) $q = 2$ and $k \leq 8$;

2) $q = 3$ and $k \leq 5$;

3) $q = 4$ and $k \leq 4$, there are 104 open cases for $k = 5$ and $n \leq 492$.

Self-orthogonal (SO) codes, including self-dual codes, form an important class of linear codes. Such codes have close connections to other mathematical structures such as block designs, lattices, and sphere packings [5–7], which can also be used to construct quantum codes, see [8, 9] and references therein. An $[n, k, d]_q$ SO code is *optimal* if there is no $[n, k, d_{so}]_q$ SO code with $d_{so} > d$. So, people try to solve the problem of determining low-dimensional optimal SO codes over small fields as it is done for optimal linear codes in [1–4].

There are some achievements on low-dimensional optimal SO codes over F_2–F_4. Pless classified certain binary optimal SO codes for $n \leq 20$ in [10]. Bouyukliev et al. determined binary optimal $[n, k]_2$ SO codes for $k \leq 3$, and classified some optimal SO codes with $n \leq 40$ in [11] . Binary optimal $[n, k]_2$ SO codes were solved in [12, 13] for $k = 4$, and in [14–16] for $k = 5, 6$. Shi et al. also determined parameters of most binary optimal SO codes with dimension $k = 7, 8$ in [16]. Bouyukliev et al. classified certain ternary and quaternary optimal SO codes with $n \leq 29$ and $k \leq 6$ in [17]. Guan et al. constructed some ternary optimal SO codes from known self-dual codes in [18]. Li et al. determined parameters of ternary optimal SO codes with dimension $k \leq 5$, except two special $[n, 5]_3$ SO codes [19]. Following [17], Ma et al. established $[n, k]_4$ quaternary optimal HSO codes for $k \leq 3$ in [20] by constructing such codes. In [21], Ren et al. gave exact parameters for $[n, 4]_4$ optimal HSO codes for all but two values of n. For most n with $n \leq 492$, $[n, 5]_4$ optimal HSO codes are determined, and there are also 108 cases remaining open, see [22].

A well-known lower bound on $[n, k, d]_q$ optimal linear codes, is called the Griesmer bound as follows:

$$n \geq g_q(k, d) = \sum_{i=0}^{k-1} \lceil \frac{d}{q^i} \rceil.$$

According to [2, 3], for $q = 4$, and $k = 5$, the Griesmer bound is achieved when $d \geq 369$, hence all optimal $[n, 5]_4$ codes are known for $n \geq 492$. In this paper, we consider optimal $[n, 5]_4$ HSO codes for $n \geq 492$, and our result is the following theorem.

Theorem 1. Let $n \geq 492$. If there is an $[n, 5, d_0(n, 5)]_4$ optimal linear code, then there is an $[n, 5, 2\lfloor d_0(n, 5)/2 \rfloor]_4$ optimal HSO code.

2. Preliminaries

In this section, we prepare some notations and basic results used in this paper.

Let $F_4 = \left\{0, 1, \omega, \omega^2\right\}$ be the field with four elements, where $\omega^2 = 1 + \omega$. For $x \in F_4$, its conjugate is $\overline{x} = x^2$. For $u = (u_1, u_2, \cdots, u_n)$, $v = (v_1, v_2, \cdots, v_n) \in F_4^n$, their Hermitian inner product is $(u, v)_h = \sum_{i=1}^n u_i \cdot \overline{v_i} = \sum_{i=1}^n u_i \cdot v_i^2$. The Hermitian dual code $C^{\perp_h}$ of C is defined as $C^{\perp_h} = \{u \in F_4^n \mid (u, v)_h = 0, \forall v \in C\}$. If $C \subseteq C^{\perp_h}$, then C is called an HSO code. Especially, if $C = C^{\perp_h}$, then C is a Hermitian self-dual code.

For given k, if $n \geq 2k$, there is an $[n, k]$ HSO code over F_4, which is an even code. For the sake of simplicity, we use 2 and 3 to represent ω and ω^2 in the rest of this paper, respectively. Let $\mathbf{1_n}$=$(1, 1, \ldots, 1)_{1\times n}$ and $\mathbf{0_n}$=$(0, 0, \ldots, 0)_{1\times n}$ denote the all-one vector and the all-zero vector of length n, respectively.

For an $m \times n$ matrix M, define $\underline{M}$, $\overline{M}$, $\underline{\underline{M}}$, and $\overline{\overline{M}}$ as the following matrices:

$$\underline{M}=\begin{pmatrix} M \\ \mathbf{0}_n \end{pmatrix}, \overline{M}=\begin{pmatrix} \mathbf{0}_n \\ M \end{pmatrix}, \underline{\underline{M}}=\begin{pmatrix} M \\ \mathbf{0}_n \\ \mathbf{0}_n \end{pmatrix}, \overline{\overline{M}}=\begin{pmatrix} \mathbf{0}_n \\ \mathbf{0}_n \\ M \end{pmatrix},$$

respectively.

Construct

$$S_2=\begin{pmatrix} 10111 \\ 01231 \end{pmatrix}, S_3=\begin{pmatrix} S_2 & \mathbf{0}_{2\times 1} & S_2 & S_2 & S_2 \\ \mathbf{0}_5 & 1 & \mathbf{1}_5 & 2\cdot\mathbf{1}_5 & 3\cdot\mathbf{1}_5 \end{pmatrix},$$

$$S_4=\begin{pmatrix} S_3 & \mathbf{0}_{3\times 1} & S_3 & S_3 & S_3 \\ \mathbf{0}_{21} & 1 & \mathbf{1}_{21} & 2\cdot\mathbf{1}_{21} & 3\cdot\mathbf{1}_{21} \end{pmatrix}=\begin{pmatrix} \underline{S_3} \mid & A_4 \end{pmatrix},$$

$$S_5=\begin{pmatrix} S_4 & \mathbf{0}_{4\times 1} & S_4 & S_4 & S_4 \\ \mathbf{0}_{85} & 1 & \mathbf{1}_{85} & 2\cdot\mathbf{1}_{85} & 3\cdot\mathbf{1}_{85} \end{pmatrix}=\begin{pmatrix} \underline{S_4} \mid & A_5 \end{pmatrix},$$

$$S_4'=\begin{pmatrix} \mathbf{0}_{21} & 1 & \mathbf{1}_{21} & \mathbf{1}_{21} & \mathbf{1}_{21} \\ S_3 & \mathbf{0}_{3\times 1} & S_3 & 2\cdot S_3 & 3\cdot S_3 \end{pmatrix}=\begin{pmatrix} \overline{S_3} \mid & B_4 \end{pmatrix},$$

$$S_5'=\begin{pmatrix} \mathbf{0}_{85} & 1 & \mathbf{1}_{85} & \mathbf{1}_{85} & \mathbf{1}_{85} \\ S_4 & \mathbf{0}_{4\times 1} & S_4 & 2\cdot S_4 & 3\cdot S_4 \end{pmatrix}=\begin{pmatrix} \overline{S_4'} \mid & B_5 \end{pmatrix}=\begin{pmatrix} \overline{\overline{S_3}} \mid & \overline{B_4} \mid & B_5 \end{pmatrix}.$$

It is not difficult to check that both S_4 and S_4' generate $[85,4,64]$ codes, both A_4 and B_4 generate $[64,4,48]$ codes, both S_5 and S_5' generate $[341,5,256]$ codes, both A_5 and B_5 generate $[256,5,192]$ codes, and $(\overline{B_4}\ B_5)$ generates $[320,5,240]$ code. The $[85,4,64]$ code, $[341,5,256]$ code are called as quaternary simplex codes, the $[64,4,48]$ code, $[256,5,192]$ code and $[320,5,240]$ code are called as quaternary McDonald codes. All these codes are HSO codes.

If $A=A_{k,m}$ is a $k\times m$ matrix and the vectors formed by row linear combination of A have largest weight δ, then A is called as an (m,δ) *block* . If $AA^{\dagger}=0$, A is called as an (m,δ) HSO block, where $A^{\dagger}=(A^2)^T$ is the conjugate transpose of A.

Let $G=(\alpha_1,\alpha_2,\cdots,\alpha_n)$ be a $k\times n$ matrix, and $A_j=A_{k,m_j}=(\alpha_{j_1},\cdots,\alpha_{j_{m_j}})$ be an (m_j,δ_j) block of G for $1\le j\le l$. Suppose A_i and A_j have index sets $I_{A_i}=\{i_1,\cdots,i_{m_i}\}$ and $I_{A_j}=\{j_1,\cdots,j_{m_j}\}$, if I_{A_i},I_{A_j} are disjoint. We say that A_i and A_j are disjoint. If any two A_i and A_j are disjoint for $1\le j\le l$, A_1, A_2, $\cdots$, A_l are called disjoint blocks. It naturally follows that l denotes the number of disjoint blocks in G.

Suppose $C=[n,k,d]$ is an HSO code with generator matrix G and G has a $k\times m$ sub-matrix A. If A is an (m,δ) HSO block, then there is an $[n-m,k,d-\delta]$ HSO code, see [22].

Lemma 1. ([22] Lemma 4) Let $N=N_k=\frac{4^k-1}{3}$. If there is an $[n,k,d]$ HSO code with $n\ge N+4$, then there are $[n-j,k,d-2\lceil\frac{j}{2}\rceil]$ HSO codes for $1\le j\le 4$.

Lemma 2. ([22] Lemma 5) If $n=256+m$, $m\ge 170$, and there is an $[m,5,d_{5,m}]$ HSO code, then there are HSO codes with the following parameters: $[n,5,d]=[256+m,5,192+d_{5,m}]$, $[n-5i,5,192+d_{5,m}-4i]$ for $i=1,2,3,4$, and $[n-5i-j,5,192+d_{5,m}-4i-2\lceil\frac{j}{2}\rceil]$ for $i=0,1,2,3$ and $j=1,2,3,4$.

Let $N=N_k=\frac{4^k-1}{3}$. If there is an $[n,k,d]$ HSO code with $n\ge N+4$, then there are $[n-j,k,d-2\lceil\frac{j}{2}\rceil]$ HSO codes for $1\le j\le 4$.

Corollary 1. There are $[512-5i,5,384-4i]$ HSO codes for $i=0,1,2,3,4$, there are $[512-5i-j,5,384-4i-2\lceil\frac{j}{2}\rceil]$ HSO codes for $i=0,1,2,3$ and $j=1,2,3,4$.

Denote the matrix formed by the last 64 columns of S_4 as A_4, and the matrix formed by the last 256 columns of S_5 as A_5.

3. Proof of Theorem 1

In this section, Theorem 1 will be proved by constructing optimal HSO codes for $n \geq 492$, whose generator matrices can be obtained through removing some special HSO blocks from $G_{5,m}$ according to Lemmas 1 and 2, where $G_{5,m}$ means the generator matrix of an $[m, 5]$ code.

For $492 \leq n \leq 1023$, according to the code lengths classification, we will discuss the following six cases, respectively.

Case a. $492 \leq n \leq 597$**.**

Both A_5 and B_5 generate $[256, 5, 192]$ HSO codes, and one can deduce that there are $[512 - 5i, 5, 384 - 4i]$ HSO codes for $0 \leq i \leq 4$, and $[512 - 5i - j, 5, 384 - 4i - 2\lceil \frac{j}{2} \rceil]$ HSO codes for $0 \leq i \leq 4$ and $1 \leq j \leq 4$ according to Lemma 5 in [22].

Let $G_{5,576} = (A_5, \overline{B_4}, B_5)$, and let J be the 2×5 all 2 matrix. We will show that $G_{5,576}$ has two $(21, 16)$ HSO blocks and eight $(5, 4)$ HSO blocks as follows.

Let $x^T = (2, 2)$. Define

$$G_{5,21} = \begin{pmatrix} 0_5 & 1 & 1_5 & 1_5 & 1_5 \\ S_2 & 0 & S_2 & 2S_2 & 3S_2 \\ S_2 & 0 & S_2 & 2S_2 & 3S_2 \end{pmatrix}, \qquad G'_{5,21} = \begin{pmatrix} 0_5 & 1 & 1_5 & 1_5 & 1_5 \\ S_2 & 0 & S_2 & 2S_2 & 3S_2 \\ 2S_2 & x & J + 2S_2 & J + 3S_2 & J + S_2 \end{pmatrix}.$$

$$G_{5,21} = \begin{pmatrix} 00000 & 1111111111111111 \\ 01111 & 0011110222203333 \\ 10123 & 0101232023130312 \\ 01111 & 0011110222203333 \\ 10123 & 0101232023130312 \end{pmatrix}, G'_{5,21} = \begin{pmatrix} 00000 & 1111111111111111 \\ 01111 & 0011110222203333 \\ 10123 & 0101232023130312 \\ 02222 & 2200002111123333 \\ 20231 & 2021301213032301 \end{pmatrix}.$$

The first column of $G_{5,21}$ is chosen from A_5, the second to the fifth columns of $G_{5,21}$ are chosen from $\overline{B_4}$, and the last 16 columns are chosen from B_5. Similarly, the first column of $G'_{5,21}$ is chosen from A_5, the second to the fifth columns of $G'_{5,21}$ are chosen from $\overline{B_4}$, the sixth to the sixteenth columns and the twentieth column are chosen from B_5, the other four columns are chosen from A_5.

Using one of the columns $(00011)^T$, $(00012)^T$, $(00013)^T$, $(01001)^T$ from A_5 and a 4×5 sub-matrix from B_5, one can construct four $(5, 4)$ HSO blocks as

$$G_{a,5} = \begin{pmatrix} 01111 \\ 00000 \\ 00000 \\ 10123 \\ 11032 \end{pmatrix}, G_{b,5} = \begin{pmatrix} 01111 \\ 00000 \\ 01111 \\ 10123 \\ 22013 \end{pmatrix}, G_{c,5} = \begin{pmatrix} 01111 \\ 00000 \\ 02222 \\ 10123 \\ 33021 \end{pmatrix}, G_{d,5} = \begin{pmatrix} 01111 \\ 10123 \\ 03333 \\ 00000 \\ 11032 \end{pmatrix}.$$

Using a 4×5 sub-matrix from A_5 and one of the columns $(01100)^T$, $(01200)^T$, $(01300)^T$, $(01110)^T$ from $\overline{B_4}$, one can construct four $(5, 4)$ HSO blocks as

$$G_{e,5} = \begin{pmatrix} 11110 \\ 01231 \\ 01231 \\ 11110 \\ 11110 \end{pmatrix}, G_{f,5} = \begin{pmatrix} 11110 \\ 01231 \\ 02312 \\ 11110 \\ 22220 \end{pmatrix}, G_{g,5} = \begin{pmatrix} 11110 \\ 01231 \\ 03123 \\ 11110 \\ 33330 \end{pmatrix}, G_{h,5} = \begin{pmatrix} 11110 \\ 01231 \\ 10321 \\ 01231 \\ 11110 \end{pmatrix}.$$

It is not difficult to check that $G_{5,576}$ has two $(21, 16)$ HSO blocks, and eight $(5, 4)$ HSO blocks, and that these 10 blocks are disjoint. Let $G_{5,597} = (A_5, \overline{\overline{B_3}}, \overline{B_4}, B_5)$. Then, $G_{5,576}$ is a sub-matrix of $G_{5,597}$ and $G_{5,597}$ generates a $[597, 5, 448]$ HSO code.

Corollary 2. There are HSO codes with parameters as follows:

(1) $[597-5i, 5, 448-4i]$ HSO codes for $i = 0, 1, 2, 3, 4$, and $[597-5i-j, 5, 448-4i-2\lceil\frac{j}{2}\rceil]$ HSO codes for $i = 0, 1, 2, 3$ and $j = 1, 2, 3, 4$.

(2) $[576-5i, 5, 432-4i]$ HSO codes for $i = 0, 1, 2, 3, 4$, and $[576-5i-j, 5, 432-4i-2\lceil\frac{j}{2}\rceil]$ HSO codes for $i = 0, 1, 2, 3$ and $j = 1, 2, 3, 4$.

(3) $[555-5i, 5, 416-4i]$ HSO codes for $i = 0, 1, 2, 3, 4$, and $[555-5i-j, 5, 416-4i-2\lceil\frac{j}{2}\rceil]$ HSO codes for $i = 0, 1, 2, 3$ and $j = 1, 2, 3, 4$.

(4) $[534-5i, 5, 400-4i]$ HSO codes for $i = 0, 1, 2, 3, 4$, and $[534-5i-j, 5, 400-4i-2\lceil\frac{j}{2}\rceil]$ HSO codes for $i = 0, 1, 2, 3$ and $j = 1, 2, 3, 4$.

(5) $[512-5i, 5, 384-4i]$ HSO codes for $i = 0, 1, 2, 3, 4$, and $[512-5i-j, 5, 384-4i-2\lceil\frac{j}{2}\rceil]$ HSO codes for $i = 0, 1, 2, 3$ and $j = 1, 2, 3, 4$.

Case b. $597 \le n \le 682 = 341 + 341$**.**

Let $G_{5,682} = (S_5\ S'_5) = (\underline{\underline{S_3}}, \underline{A_4}, A_5 \mid \overline{\overline{S_3}}, \overline{B_4}, B_5)$. Then, $G_{5,576}$ is a sub-matrix of $G_{5,682}$. It is not difficult to check that $G_{5,682}$ has four $(21, 16)$ HSO block and eight $(5, 4)$ HSO blocks, and these 12 blocks are disjoint.

Corollary 3. There are HSO codes with parameters as follows:

(1) $[682-5i, 5, 512-4i]$ HSO codes for $i = 0, 1, 2, 3, 4$, and $[682-5i-j, 5, 512-4i-2\lceil\frac{j}{2}\rceil]$ HSO codes for $i = 0, 1, 2, 3$ and $j = 1, 2, 3, 4$.

(2) $[661-5i, 5, 496-4i]$ HSO codes for $i = 0, 1, 2, 3, 4$, and $[661-5i-j, 5, 496-4i-2\lceil\frac{j}{2}\rceil]$ HSO codes for $i = 0, 1, 2, 3$ and $j = 1, 2, 3, 4$.

(3) $[640-5i, 5, 480-4i]$ HSO codes for $i = 0, 1, 2, 3, 4$, and $[640-5i-j, 5, 480-4i-2\lceil\frac{j}{2}\rceil]$ HSO codes for $i = 0, 1, 2, 3$ and $j = 1, 2, 3, 4$.

(4) $[619-5i, 5, 464-4i]$ HSO codes for $i = 0, 1, 2, 3, 4$, and $[619-5i-j, 5, 464-4i-2\lceil\frac{j}{2}\rceil]$ HSO codes for $i = 0, 1, 2, 3$ and $j = 1, 2, 3, 4$.

Case c. $683 \le n \le 705$**.**

Let $G_{5,705} = (S_5 \mid G_{5,364}) = (\underline{S_4}, A_5 \mid G_{5,364})$, then A_5 is a sub-matrix of $G_{5,705}$. It is not difficult to check that A_5 has four disjoint $(5, 4)$ HSO blocks.

Corollary 4. There are HSO codes with parameters as follows:

There are $[705-5i, 5, 528-4i]$ HSO codes for $i = 0, 1, 2, 3, 4$, and $[705-5i-j, 5, 528-4i-2\lceil\frac{j}{2}\rceil]$ HSO codes for $i = 0, 1, 2, 3$ and $j = 1, 2, 3, 4$, and $[685-j, 5, 512]$ HSO codes for $j = 1, 2$.

Case d. $706 \le n \le 768$**.**

Let $\alpha = (0111)$, $\beta = (1023)$, $G_{5,768} = (A_5, D_5, B_5)$, where

$$D_5 = \begin{pmatrix} \alpha & \mathbf{0}_{21} & \mathbf{0}_{21} & \mathbf{0}_{21} & \mathbf{1}_{21} & \mathbf{1}_{21} & \mathbf{1}_{21} & \mathbf{1}_{21} & \mathbf{1}_{21} & \mathbf{1}_{21} & \mathbf{1}_{21} & \mathbf{1}_{21} & \mathbf{1}_{21} \\ \mathbf{0}_{3\times 4} & S_3 & S_3 & S_3 & S_3 & 2S_3 & 3S_3 & S_3 & 2S_3 & 3S_3 & S_3 & 2S_3 & 3S_3 \\ \beta & \mathbf{1}_{21} & \mathbf{2}_{21} & \mathbf{3}_{21} & \mathbf{0}_{21} & \mathbf{0}_{21} & \mathbf{0}_{21} & \mathbf{2}_{21} & \mathbf{2}_{21} & \mathbf{2}_{21} & \mathbf{3}_{21} & \mathbf{3}_{21} & \mathbf{3}_{21} \end{pmatrix}.$$

One can check that D_5 generates a $[256, 5, 192]$ HSO code and $G_{5,768}$ generates a $[3 \times 256, 5, 3 \times 192]$ HSO code. From the construction of D_5, one can see that D_5 has a sub-matrix $\overline{B_4}$, hence $G_{5,576}$ is a sub-matrix of $G_{5,768}$. Thus, $G_{5,768}$ has two $(21, 16)$ HSO blocks and eight $(5, 4)$ HSO blocks, and these 10 blocks are disjoint.

Corollary 5. There are HSO codes with parameters as follows:

(1) $[768-5i, 5, 576-4i]$ HSO codes for $i = 0, 1, 2, 3, 4$, and $[768-5i-j, 5, 576-4i-2\lceil \frac{j}{2} \rceil]$ HSO codes for $i = 0, 1, 2, 3$ and $j = 1, 2, 3, 4$.

(2) $[747-5i, 5, 560-4i]$ HSO codes for $i = 0, 1, 2, 3, 4$, and $[747-5i-j, 5, 560-4i-2\lceil \frac{j}{2} \rceil]$ HSO codes for $i = 0, 1, 2, 3$ and $j = 1, 2, 3, 4$.

(3) $[726-5i, 5, 544-4i]$ HSO codes for $i = 0, 1, 2, 3, 4$, and $[726-5i-j, 5, 544-4i-2\lceil \frac{j}{2} \rceil]$ HSO codes for $i = 0, 1, 2, 3$ and $j = 1, 2, 3, 4$.

Case e. $769 \leq n \leq 832 = 512 + 320$.

Let $G_{5,832} = (A_5, \overline{B_4}, B_5, B_5)$. Then, $G_{5,832}$ generates a $[832, 5, 624]$ HSO code. It easy to see $G_{5,576}$ is a sub-matrix of $G_{5,832}$, hence $G_{5,832}$ has two $(21, 16)$ HSO blocks and eight $(5, 4)$ HSO blocks, and these 10 blocks are disjoint.

Corollary 6. There are HSO codes with parameters as follows:

(1) $[832-5i, 5, 624-4i]$ HSO codes for $i = 0, 1, 2, 3, 4$, and $[832-5i-j, 5, 624-4i-2\lceil \frac{j}{2} \rceil]$ HSO codes for $i = 0, 1, 2, 3$ and $j = 1, 2, 3, 4$.

(2) $[811-5i, 5, 608-4i]$ HSO codes for $i = 0, 1, 2, 3, 4$, and $[811-5i-j, 5, 608-4i-2\lceil \frac{j}{2} \rceil]$ HSO codes for $i = 0, 1, 2, 3$ and $j = 1, 2, 3, 4$.

(3) $[790-5i, 5, 592-4i]$ HSO codes for $i = 0, 1, 2, 3, 4$, and $[790-5i-j, 5, 592-4i-2\lceil \frac{j}{2} \rceil]$ HSO codes for $i = 0, 1, 2, 3$ and $j = 1, 2, 3, 4$.

Case f. $833 \leq n \leq 1023$.

Suppose $n = 341 + n'$. Then, $492 \leq n' \leq 682$. Let $G_{5,n} = (S_5\ G_{5,n'})$, where $G_{5,n'}$ is a generator matrix of an $[n', 5]$ optimal HSO code given in Cases a-e. Then, $G_{5,n} = (S_5\ G_{5,n'})$ generates an $[n = 341 + n', 5]$ optimal HSO code.

Summarizing the above cases, $[n, 5]_4$ optimal HSO codes have been constructed for each $492 \leq n \leq 1023$.

For $n \geq 1024$, denote $s = \lfloor \frac{n}{341} \rfloor$. Then, $s \geq 3$. If $s \geq 3$, let $n = (s-2) \cdot 341 + n''$ and $G_{5,n} = ((s-2) \cdot S_5,\ G_{5,n''})$. Then $G_{5,n}$ generates an $[n = (s-2) \cdot 341 + n'', 5]$ optimal HSO code.

Summarizing the above discussions, we construct $[n, 5]$ HSO codes for each n with $n \geq 492$. It follows that Theorem 1 holds.

4. Conclusions

In this paper, we have verified $[n, 5, d_{so}(n, 5)]_4 = [n, 5, 2\lfloor \frac{d_o(n,5)}{2} \rfloor]_4$ by constructing optimal HSO codes for each $n \geq 492$. When $492 \leq n \leq 1023$, the generator matrices of optimal HSO codes can be obtained by removing disjoint HSO blocks from some $G_{5,n}$. For $n \geq 1024$, optimal HSO codes can be constructed from corresponding ones with short length and some $[341, 5, 256]_4$ quaternary simplex codes.

Author contributions

H. Song: Writing-original draft, writing-review and editing; Y. Ren: Conceptualization, methodology, software; R. Li: Conceptualization, writing-original draft, funding acquisition; Y. Liu: Data curation, writing-review and editing, funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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