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Research article

Existence and uniqueness of solutions for the two-dimensional Euler and Navier-Stokes equations with initial data in H^1

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Abstract: In this paper, we consider the incompressible Euler and Navier-Stokes equations in $\mathbb{R}^2$. It is well known that the Euler and Navier-Stokes equations are globally well-posed for initial data in $H^s(s > 2)$. The main purpose of the present paper is to consider the case $s = 1$. We prove that, for initial data in H^1, the Euler and Navier-Stokes equations both have global solutions, and the solutions are uniformly bounded with respect to time. Moreover, the solution for the Navier-Stokes equations is unique. We also prove that, as the viscosity tends to zero, the solution of the Navier-Stokes equations converges to the one of the Euler equations.

Keywords: Euler equations; Navier-Stokes equations; global existence; weak solutions
Mathematics Subject Classification: 35D30, 35A01

1. Introduction

In this paper, we focus on the two-dimensional incompressible Euler and Navier-Stokes equations. The incompressible Euler equations in $\mathbb{R}^2$ read as

$$\begin{cases} \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = 0, \\ \operatorname{div} \boldsymbol{u} = 0, \end{cases} \tag{1.1}$$

for $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^2$, where the unknowns $\boldsymbol{u} = (u_1, u_2)$ and $p = p(t, x)$ represent the velocity and the pressure of the fluid, respectively.

As is well known, system (1.1) is globally well-posed when the initial velocity lies in $H^s(s > 2)$ (see, for example, [14]), while the existence problem of solutions for the cases $s = 2$ and $s = 0$ are unknown. Indeed, it was shown in [1] that, if the initial data that belongs to H^2-space is perturbed, then the system is ill-posed. Recently, Elgindi and Masmoudi [4] showed ill-posedness in $C^1 \cap L^2$ space. This is partially due to the fact that, by elementary energy methods, we only obtain the following a priori estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_{H^s}^2 \leq \|\nabla \boldsymbol{u}\|_{L^\infty}\|\boldsymbol{u}\|_{H^s}^2, \tag{1.2}$$

which implies that the Sobolev order s should be greater than 2 to close the energy inequality.

To consider the existence problem with initial data of low regularity, we generally resort to the vorticity-stream formulation of the Euler equations, which can be written as

$$\begin{cases} \partial_t w + \boldsymbol{u} \cdot \nabla w = 0, \\ \boldsymbol{u} = K[w], \end{cases} \tag{1.3}$$

where $K := \frac{1}{2\pi}\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right)$ is the kernel of the Biot-Savart law, and w is the vorticity of the fluid. There is a number of literatures about the existence and uniqueness of weak solutions of system (1.3). When the initial vorticity $w_0 \in L^1 \cap L^\infty$, Yudovich [18] proved the global existence and uniqueness of weak solutions. Later, Vishik [17] and Yudovich [19] extended these results into a space slightly larger than L^∞. If w_0 belongs to $L^1 \cap L^p$ with $p > 1$, the global existence result was obtained by Diperna and Majda in [3]. We also refer to [2, 5, 12, 13] for the case that w_0 is a finite Randon measure. It is worth noting that the L^1 condition on w_0 stems from the second subequation of system (1.3), from which we bound the L^∞-norm of $\boldsymbol{u}$ as:

$$\|\boldsymbol{u}\|_{L^\infty} \leq C(\|w\|_{L^1} + \|w\|_{L^\infty}). \tag{1.4}$$

We want to emphasize that the above works are either based on system (1.1) to establish smooth solutions or based on system (1.3) to obtain weak ones. The present paper plans to utilize these two systems together to retrieve the L^1 restriction on the initial vorticity and lower the Sobolev index s to $s = 1$ for the initial velocity. More precisely, let us consider that $\boldsymbol{u}$ is a smooth solution of system (1.1). Then, by multiplying the first subequation with $\boldsymbol{u}$ and integrating over $[0, t) \times \mathbb{R}^2$, we have that

$$\|\boldsymbol{u}\|_{L^2} \leq \|\boldsymbol{u}_0\|_{L^2}. \tag{1.5}$$

On the other hand, as $\boldsymbol{u}$ also satisfies system (1.3), it can be checked that

$$\|\nabla \boldsymbol{u}\|_{L^2} \leq C\|w\|_{L^2} \leq C\|w_0\|_{L^2} \leq C\|u_0\|_{H^1}. \tag{1.6}$$

Collecting (1.5) and (1.6) gives a uniform bound of the H^1-norm of $\boldsymbol{u}$ in time, which brings out the H^1 solution by using approximation and compactness arguments.

We are now in the position to state the following result:

Theorem 1.1. *Suppose that the initial velocity $\boldsymbol{u}_0 \in H^1(\mathbb{R}^2)$ is divergence-free. Then the Euler equations* (1.1) *have a weak solution $\boldsymbol{u} \in L^\infty([0, \infty); H^1(\mathbb{R}^2))$ with the estimate*

$$\sup_{t\in[0,\infty)} \|\boldsymbol{u}\|_{H^1} \leq C\|\boldsymbol{u}_0\|_{H^1}. \tag{1.7}$$

Another interest of the present work is to consider the Cauchy problem of the Navier-Stokes equations, which can be written as:

$$\begin{cases} \partial_t \boldsymbol{u}^\nu + \boldsymbol{u}^\nu \cdot \nabla \boldsymbol{u}^\nu + \nabla p^\nu = \nu \Delta \boldsymbol{u}^\nu, \\ \operatorname{div} \boldsymbol{u}^\nu = 0, \\ \boldsymbol{u}^\nu|_{t=0} = \boldsymbol{u}_0^\nu, \end{cases} \tag{1.8}$$

where $\boldsymbol{u}_0^\nu$ is the given initial velocity and $\nu > 0$ is the viscosity. The unknowns $\boldsymbol{u}^\nu = (u_1^\nu, u_2^\nu)$ and $p^\nu = p^\nu(t, x)$ represent the velocity and the pressure of the fluid, respectively.

With initial data in L^2, Leray [9] proved that the Navier-Stokes equations have a global weak solution in $\mathbb{R}^d$(d=2,3), and Hopf [6] obtained the existence of a global weak solution in domains with boundaries. Since then, many mathematicians studied the uniqueness and regularity of Leray-Hopf solutions. It was proved that for the two-dimensional case, the solution is unique and regular; see [8, 10, 11, 16]. However, the problem of uniqueness of Leray-Hopf solutions for the three-dimensional case is still open.

As is well known, for the case that the initial data lies in $H^s(s > 1)$, Eq (1.8) has a unique solution (see, for example, [7, 14]). Moreover, the solution is global when $s > 2$. We consider in the present work the borderline case: The problem of existence and uniqueness of solutions for (1.8) when the initial data lies in H^1. We find that, when we analyze (1.8) and its vorticitiy-stream formulation together, the obstacle that is caused by the convection term can be bypassed, and for any $s \geq 0$, the following interesting a priori estimate holds:

$$\|\boldsymbol{u}^\nu(t)\|_{H^s}^2 + \nu \int_0^t \|\nabla \boldsymbol{u}^\nu(s)\|_{H^s}^2 \,\mathrm{d}s \leq C(\nu, \boldsymbol{u}_0^\nu), \tag{1.9}$$

where $C \equiv C(\nu, \boldsymbol{u}_0^\nu)$ is a constant that depends on ν and the H^s-norm of the initial data $\boldsymbol{u}_0^\nu$, while does not depend on t. It is worth mentioning that for $s = 1$, the constant C is independent of ν.

With the uniform bound (1.9) in hand, we can establish the global existence and uniqueness of solutions, and we have the following result:

Theorem 1.2. *Let $s \geq 0$ be a given integer. Suppose that the initial velocity $\boldsymbol{u}_0^\nu \in H^s(\mathbb{R}^2)$ is divergence-free. Then the Navier-Stokes equations* (1.8) *have a unique solution $\boldsymbol{u}^\nu \in L^\infty([0, \infty); H^s(\mathbb{R}^2))$ with the estimate* (1.9).

Remark 1.3. *It worth noting that for those $s > 2$ that are not integers, the above conclusion also holds.*

As we know, the vanishing viscosity limit problem is one of the most fundamental problems in fluid mechanics. Masmoudi [15] verified the inviscid limit for initial data in H^s(s¿2). From the above uniform bound (1.9), we may treat the vanishing viscosity limit problem for initial data only in H^1.

Corollary 1.4. *Suppose that $\boldsymbol{u}_0^\nu$ is divergence-free and converges to $\boldsymbol{u}_0$ in $H^1(\mathbb{R}^2)$ as ν tends to zero. Then, as $\nu \to 0$, the solution $\boldsymbol{u}^\nu$ of* (1.8) *converges to a solution $\boldsymbol{u}$ of* (1.1) *with the estimate*

$$\sup_{t\in[0,\infty)} \|\boldsymbol{u}\|_{H^1} \leq C\|\boldsymbol{u}_0\|_{H^1}. \tag{1.10}$$

The remainder of this article is divided into three sections. In Section 2, we establish the global existence of solutions for the Euler equations (1.1). In Section 3, we prove the global well-posedness of the Navier-Stokes equations, i.e., Theorem 1.2. In the last section, we consider the vanishing viscosity limit problem, and prove Corollary 1.4.

2. Global existence of H^1 solutions for the Euler equations

In this section, we prove the global existence of solutions for the Euler equations (1.1) with initial data in H^1, i.e., Theorem 1.1. First, we show that the solutions $\boldsymbol{u}^R$ of some smoothed version of the equations exist and are uniformly bounded in H^1. We then show that, as $R \to \infty$, the limit $\boldsymbol{u}$ of $\boldsymbol{u}^R$ satisfies the original equations.

We first define a Fourier truncation $\mathcal{S}_R$ as

$$\widehat{\mathcal{S}_R f}(\xi) := \mathbf{1}_R \widehat{f}(\xi), \tag{2.1}$$

where $\widehat{f} := \mathscr{F}[f]$ denotes the Fourier transform of f. We construct the approximate Euler equations on the whole plane as:

$$\begin{cases} \partial_t \boldsymbol{u}^R + \mathcal{S}_R \mathbb{P}[\boldsymbol{u}^R \cdot \nabla \boldsymbol{u}^R] = 0, \\ \operatorname{div} \boldsymbol{u}^R = 0, \\ \boldsymbol{u}^R|_{t=0} = \mathcal{S}_R \boldsymbol{u}_0, \end{cases} \tag{2.2}$$

where $\mathbb{P} = \mathbb{I} + \nabla(-\Delta)^{-1} \operatorname{div}$. It can be checked that $\boldsymbol{u}^R$ lies in the space

$$V_R := \{f \in L^2(\mathbb{R}^2) : \widehat{f} \text{ is supported in } B(0, R)\}. \tag{2.3}$$

We then define

$$F(\boldsymbol{u}^R, \boldsymbol{v}^R) := \mathcal{S}_R \mathbb{P}[\boldsymbol{u}^R \cdot \nabla \boldsymbol{v}^R]. \tag{2.4}$$

It can be checked that F is Lipschitz on the space V_R. Indeed, let $\boldsymbol{u}_1^R, \boldsymbol{v}_1^R, \boldsymbol{u}_2^R, \boldsymbol{v}_2^R \in V_R$, by the definition of F, we have

$$\|F(\boldsymbol{u}_1^R, \boldsymbol{v}_1^R) - F(\boldsymbol{u}_2^R, \boldsymbol{v}_2^R)\|_{L^2} = \|\mathbb{P}\mathcal{S}_R[\boldsymbol{u}_1^R \cdot \nabla \boldsymbol{v}_1^R - \boldsymbol{u}_2^R \cdot \nabla \boldsymbol{v}_2^R]\|_{L^2}, \tag{2.5}$$

it follows that

$$\|F(\boldsymbol{u}_1^R, \boldsymbol{v}_1^R) - F(\boldsymbol{u}_2^R, \boldsymbol{v}_2^R)\|_{L^2} \le \|\boldsymbol{u}_1^R \cdot \nabla \boldsymbol{v}_1^R - \boldsymbol{u}_2^R \cdot \nabla \boldsymbol{v}_2^R\|_{L^2}. \tag{2.6}$$

Notice that

$$\begin{aligned} \|\boldsymbol{u}_1^R \cdot \nabla \boldsymbol{v}_1^R - \boldsymbol{u}_2^R \cdot \nabla \boldsymbol{v}_2^R\|_{L^2} &\le \|(\boldsymbol{u}_1^R - \boldsymbol{u}_2^R) \cdot \nabla \boldsymbol{v}_1^R\|_{L^2} + \|\boldsymbol{u}_2^R \cdot \nabla(\boldsymbol{v}_1^R - \boldsymbol{v}_2^R)\|_{L^2} \\ &\le CR^2 \left(\|\boldsymbol{u}_1^R - \boldsymbol{u}_2^R\|_{L^2} \|\boldsymbol{v}_1^R\|_{L^2} + \|\boldsymbol{u}_2\|_{L^2} \|\boldsymbol{v}_1^R - \boldsymbol{v}_2^R\|_{L^2} \right), \end{aligned} \tag{2.7}$$

where we have used the Bernstein's Lemma. From (2.5)–(2.7), we conclude that F is Lipschitz on the space V_R. Hence, by Picard's theorem for infinite-dimensional ordinary differential equations (see, for example, Theorem 3.1 in [14]), there exists a global smooth solution $\boldsymbol{u}^R$ in V_R.

We then show that $\boldsymbol{u}^R$ is uniformly bounded in H^1 with respect to R. Indeed, from Eq (2.2), the L^2 bound of $\boldsymbol{u}^R$ can be established.

Lemma 2.1. *Suppose that $\boldsymbol{u}_0 \in L^2(\mathbb{R}^2)$ is divergence-free. Then, we have*

$$\|\boldsymbol{u}^R\|_{L^2} = \|\boldsymbol{u}_0^R\|_{L^2} \le C\|\boldsymbol{u}_0\|_{L^2}. \tag{2.8}$$

Proof. The L^2 inner product of the first subequation of (2.2) with $\boldsymbol{u}^R$ gives:

$$(\partial_t \boldsymbol{u}^R, \boldsymbol{u}^R) + (\mathcal{S}_R \mathbb{P}[\boldsymbol{u}^R \cdot \nabla \boldsymbol{u}^R], \boldsymbol{u}^R) = 0.$$

Observing that $\boldsymbol{u}^R$ is divergence-free and lies in V_R, we can deduce from the above identity that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}^R\|_{L^2}^2 = 0, \tag{2.9}$$

which implies (2.8) immediately. □

To establish estimates of higher-order derivatives of $\boldsymbol{u}^R$, we resort to the vorticity-stream formulation:

$$\begin{cases} \partial_t w^R + \mathcal{S}_R[\boldsymbol{u}^R \cdot \nabla w^R] = 0, \\ \boldsymbol{u}^R = K[w^R], \end{cases} \tag{2.10}$$

where $K := \frac{1}{2\pi}\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right)$ is the kernel of the Biot-Savart law, and w^R is the approximate vorticity.

Lemma 2.2. *Suppose that $\boldsymbol{u}_0 \in H^1(\mathbb{R}^2)$ is divergence-free. Then, we have*

$$\|\nabla \boldsymbol{u}^R\|_{L^2} \le C\|\nabla \boldsymbol{u}_0\|_{L^2}, \tag{2.11}$$

where C is a constant that is independent of time.

Proof. The L^2 inner product of the first subequation of (2.10) with w^R gives

$$(\partial_t w^R, w^R) + (\mathcal{S}_R[\boldsymbol{u}^R \cdot \nabla w^R], w^R) = 0.$$

Observing that $\boldsymbol{u}^R$ and w^R are divergence-free and lie in V_R, therefore by using integration by parts, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w^R\|_{L^2}^2 = 0, \tag{2.12}$$

which implies

$$\|w^R\|_{L^2} = \|w_0^R\|_{L^2} \le \|\nabla \boldsymbol{u}_0\|_{L^2}. \tag{2.13}$$

On the other hand, by the Calderón-Zygmund theorem, we can deduce from the second subequation of (2.10) that

$$\|\nabla \boldsymbol{u}^R\|_{L^2} \le C\|w^R\|_{L^2}. \tag{2.14}$$

By collecting (2.13) and (2.14), we obtain (2.11). □

Remark 2.3. *We observe that the kernel K of the Biot-Savart law decays at infinity like* $1/r$*, which is not square integrable. This is the cause that we concern only the estimate of* $\nabla \boldsymbol{u}^R$ *from* (2.10).

Collecting the above two lemmas, we immediately have the following result:

Proposition 2.4. *Suppose that $\boldsymbol{u}_0 \in H^1(\mathbb{R}^2)$ is divergence-free. Then, we have*

$$\|\boldsymbol{u}^R\|_{H^1} \le C\|\boldsymbol{u}_0\|_{H^1}. \tag{2.15}$$

Here C is a constant that does not depend on R.

With the uniform bound (2.15), we now proceed with the proof of Theorem 1.1. Indeed, from the Banach-Alaoglu theorem, we find that there exists a subsequence of $\boldsymbol{u}^R$(still denoted by $\boldsymbol{u}^R$) and some $\boldsymbol{u}$ such that

$$\boldsymbol{u}^R \overset{*}{\rightharpoonup} \boldsymbol{u} \text{ in } L^\infty([0,\infty); H^1(\mathbb{R}^2)), \tag{2.16}$$

and $\boldsymbol{u}$ satisfies (1.7).

On the other hand, by the Aubin-Lions lemma and Cantor's diagonal process, there exists a subsequence(still denoted by $\boldsymbol{u}^R$) that, for any bounded open subset K and $T > 0$,

$$\boldsymbol{u}^R \to \boldsymbol{u} \text{ in } L^2(0,T; L^2(K)). \tag{2.17}$$

Furthermore, since $\boldsymbol{u}^R$ satisfies (2.2), it follows that, for any test function $\varphi \in C_0([0,\infty); C^\infty_{0,\sigma}(\mathbb{R}^2))$, we have

$$\int_0^\infty (\boldsymbol{u}^R, \partial_t\varphi) - \int_0^\infty (\mathcal{S}_R\mathbb{P}[\boldsymbol{u}^R \otimes \boldsymbol{u}^R], \nabla\varphi) = (\boldsymbol{u}^R_0, \varphi(0)), \tag{2.18}$$

Collecting (2.16) and (2.17) and letting $R \to \infty$, we conclude that $\boldsymbol{u}$ is a global solution of Eq (1.1). This completes the proof of Theorem 1.1.

3. Global well-posedness of the Navier-Stokes equations

In this section, we first establish some a priori estimates of $\boldsymbol{u}^\nu$. Then, we prove Theorem 1.2 by using approximation and compactness arguments. Finally, we show that the solution $\boldsymbol{u}^\nu$ is unique.

3.1. A priori estimates

Let us denote by Λ^s the fractional derivative operators defined in terms of Fourier transforms as follows:

$$\mathscr{F}[\Lambda^s f](\xi) = |\xi|^s \hat{f}(\xi). \tag{3.1}$$

We now prove the a priori estimate (1.9). Indeed, from the Navier-Stokes equations (1.8), a priori bound of $\|\boldsymbol{u}^\nu\|_{L^2}$ can be established, and we have the following result:

Lemma 3.1. *Suppose that $\boldsymbol{u}^\nu$ is a solution of* (1.8) *with smooth initial data $\boldsymbol{u}^\nu_0$. Then, we have*

$$\|\boldsymbol{u}^\nu\|^2_{L^2} + 2\nu\int_0^t \|\nabla\boldsymbol{u}^\nu\|^2_{L^2} = \|\boldsymbol{u}^\nu_0\|^2_{L^2}. \tag{3.2}$$

Proof. Multiplying the first subequation of (1.8) with $\boldsymbol{u}^\nu$, we have that

$$(\partial_t\boldsymbol{u}^\nu, \boldsymbol{u}^\nu) + (\boldsymbol{u}^\nu \cdot \nabla\boldsymbol{u}^\nu, \boldsymbol{u}^\nu) + (\nabla p, \boldsymbol{u}^\nu) = \nu(\Delta\boldsymbol{u}^\nu, \boldsymbol{u}^\nu).$$

Observing that $\boldsymbol{u}^\nu$ is divergence-free, we use integration by parts to deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}^\nu\|^2_{L^2} + \nu\|\nabla\boldsymbol{u}^\nu\|^2_{L^2} = 0, \tag{3.3}$$

which implies (3.2) immediately. □

To consider the case $s = 1$, we should resort to the vorticity-stream formulation, which can be written as:

$$\begin{cases} \partial_t w^\nu + \boldsymbol{u}^\nu \cdot \nabla w^\nu + \nabla p^\nu = \nu \Delta w^\nu, \\ \boldsymbol{u}^\nu = K * w^\nu, \end{cases} \tag{3.4}$$

where $K(x) := \frac{1}{2\pi}\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right)$ is the kernel of the Biot-Savart law, and $w^\nu := \nabla^\perp \cdot \boldsymbol{u}^\nu$ is called the vorticity of the fluid.

From Eq (3.4), we can establish the following result:

Lemma 3.2. *Suppose that $\boldsymbol{u}^\nu$ is a solution of* (3.4) *with smooth initial data $\boldsymbol{u}_0^\nu$. Then, we have*

$$\|\nabla \boldsymbol{u}^\nu\|_{L^2}^2 + \nu \int_0^t \|D^2 \boldsymbol{u}^\nu\|_{L^2}^2 \leq c\|\nabla \boldsymbol{u}_0^\nu\|_{L^2}^2. \tag{3.5}$$

Proof. By multiplying the first subequation of (3.4) with w^ν, we arrive at

$$(\partial_t w^\nu, w^\nu) + (\boldsymbol{u}^\nu \cdot \nabla w^\nu, w^\nu) = \nu(\Delta w^\nu, w^\nu).$$

Observing that $\boldsymbol{u}^\nu$ is divergence-free, therefore, by using integration by parts, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w^\nu\|_{L^2}^2 = \nu\|\nabla w^\nu\|_{L^2}^2. \tag{3.6}$$

From the above identity, we deduce that

$$\|w^\nu\|_{L^2}^2 + 2\nu \int_0^t \|\nabla w^\nu\|_{L^2}^2 = \|w_0\|_{L^2}^2 \leq C\|\nabla \boldsymbol{u}_0^\nu\|_{L^2}^2. \tag{3.7}$$

On the other hand, from the second subequation of (3.4) and the Calderón-Zygmund theorem, we find that, for $s = 0, 1$,

$$\|\nabla \boldsymbol{u}^\nu\|_{H^s} \leq C\|w^\nu\|_{H^s}. \tag{3.8}$$

By collecting (3.7) and (3.8), we conclude that inequality (3.5) holds. □

With the above two results, we establish the following proposition.

Proposition 3.3. *Suppose that $\boldsymbol{u}^\nu$ is a solution of* (1.8) *with smooth initial data $\boldsymbol{u}_0^\nu$. Then $\boldsymbol{u}^\nu$ satisfies the estimate*

$$\|\boldsymbol{u}^\nu\|_{H^s}^2 + \nu \int_0^t \|\nabla \boldsymbol{u}^\nu\|_{H^s}^2 \,\mathrm{d}s \leq C_1\|\boldsymbol{u}_0^\nu\|_{H^s}^2 \exp^{C_2 t}, \tag{3.9}$$

where $C_2 \equiv C_2(\nu, \boldsymbol{u}_0)$ is a constant that depends on ν and the H^2-norm of $\boldsymbol{u}_0^\nu$, while does not depend on t. Moreover, the Sobolev order s is not necessarily an integer when $s > 2$.

Proof. We first prove for the case $s = 2$. We take the derivative D^α, $|\alpha| = 2$ of the Eq (1.8) and then take L^2 inner product with $D^\alpha \boldsymbol{u}^\nu$:

$$(\partial_t D^\alpha \boldsymbol{u}^\nu, D^\alpha \boldsymbol{u}^\nu) - \nu(D^\alpha \Delta \boldsymbol{u}^\nu, D^\alpha \boldsymbol{u}^\nu) = -(D^\alpha[\boldsymbol{u}^\nu \cdot \nabla \boldsymbol{u}^\nu], D^\alpha \boldsymbol{u}^\nu). \tag{3.10}$$

It is easy to see that

$$(\partial_t D^\alpha \boldsymbol{u}^\nu, D^\alpha \boldsymbol{u}^\nu) - \nu(D^\alpha \Delta \boldsymbol{u}^\nu, D^\alpha \boldsymbol{u}^\nu) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^\alpha \boldsymbol{u}^\nu\|_{L^2} + \nu\|D^\alpha \nabla \boldsymbol{u}^\nu\|_{L^2}. \tag{3.11}$$

To handle the term on the right hand side of (3.10), we rewrite $D^\alpha[\boldsymbol{u}^\nu \cdot \nabla \boldsymbol{u}^\nu]$ as

$$D^\alpha[\boldsymbol{u}^\nu \cdot \nabla \boldsymbol{u}^\nu] = \sum_{\beta \le \alpha, |\beta|>0} D^\beta \boldsymbol{u}^\nu \cdot \nabla[D^{\alpha-\beta}\boldsymbol{u}^\nu] + \boldsymbol{u}^\nu \cdot \nabla D^\alpha \boldsymbol{u}^\nu. \tag{3.12}$$

We use the Gagliardo-Nirenberg inequality to deduce that

$$|(\sum_{\beta \le \alpha, |\beta|>0} D^\beta \boldsymbol{u}^\nu \cdot \nabla[D^{\alpha-\beta}\boldsymbol{u}^\nu], D^\alpha \boldsymbol{u}^\nu)| \le C\|\nabla \boldsymbol{u}^\nu\|_{L^2}\|D^2\boldsymbol{u}^\nu\|_{L^4}^2 \le C\|\nabla \boldsymbol{u}^\nu\|_{L^2}\|D^2\boldsymbol{u}^\nu\|_{L^2}\|D^3\boldsymbol{u}^\nu\|_{L^2}. \tag{3.13}$$

On the other hand, observing that $\boldsymbol{u}$ is divergence-free, it follows

$$(\boldsymbol{u}^\nu \cdot \nabla[D^\alpha \boldsymbol{u}^\nu], D^\alpha \boldsymbol{u}^\nu) = 0. \tag{3.14}$$

Collecting the above inequalities and using Young's inequality gives that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|D^2\boldsymbol{u}^\nu\|_{L^2} + \nu\|D^3\boldsymbol{u}^\nu\|_{L^2} \le \frac{C}{\nu}\|\nabla \boldsymbol{u}^\nu\|_{L^2}^2\|D^2\boldsymbol{u}^\nu\|_{L^2}^2. \tag{3.15}$$

Since $\|\boldsymbol{u}^\nu\|_{H^1}$ is uniformly bounded in time, it follows from the Gronwall's inequality that (3.9) holds for $s = 2$.

We next prove the case $s > 2$, where s is not necessarily an integer. Indeed, we apply the Λ^s operator to the first subequation of (1.8) and then take the L^2 inner product with $\Lambda^s \boldsymbol{u}$:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^s \boldsymbol{u}^\nu\|_{L^2}^2 + \nu\|\Lambda^{s+1}\boldsymbol{u}^\nu\|_{L^2}^2 = -(\Lambda^s[\boldsymbol{u}^\nu \cdot \nabla \boldsymbol{u}^\nu], \Lambda^s \boldsymbol{u}^\nu). \tag{3.16}$$

We observe that

$$\|\Lambda^s[\boldsymbol{u}^\nu \cdot \nabla \boldsymbol{u}^\nu]\|_{L^2} \le \|\Lambda^{s+1}[\boldsymbol{u}^\nu \otimes \boldsymbol{u}^\nu]\|_{L^2}, \tag{3.17}$$

from Lemma 3.4 of [14], the above inequality yields

$$\|\Lambda^s[\boldsymbol{u}^\nu \cdot \nabla \boldsymbol{u}^\nu]\|_{L^2} \le C\|\boldsymbol{u}^\nu\|_{L^\infty}\|\boldsymbol{u}^\nu\|_{H^{s+1}}. \tag{3.18}$$

As $H^2(\mathbb{R}^2)$ is embedded in $L^\infty(\mathbb{R}^2)$, therefore

$$|(\Lambda^s[\boldsymbol{u}^\nu \cdot \nabla \boldsymbol{u}^\nu], \Lambda^s \boldsymbol{u}^\nu)| \le C\|\boldsymbol{u}^\nu\|_{H^2}\|\boldsymbol{u}^\nu\|_{H^s}\|\boldsymbol{u}^\nu\|_{H^{s+1}}. \tag{3.19}$$

Then, by adding (3.2) and (3.16) together, we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}^\nu\|_{H^s}^2 + \nu\|\boldsymbol{u}^\nu\|_{H^{s+1}}^2 \le C\|\boldsymbol{u}^\nu\|_{H^2}\|\boldsymbol{u}^\nu\|_{H^s}\|\boldsymbol{u}^\nu\|_{H^{s+1}}. \tag{3.20}$$

Then, the Holder's inequality implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}^\nu\|_{H^s}^2 + \nu\|\boldsymbol{u}^\nu\|_{H^{s+1}}^2 \le \frac{C}{\nu}\|\boldsymbol{u}^\nu\|_{H^2}^2\|\boldsymbol{u}^\nu\|_{H^s}^2. \tag{3.21}$$

Observing that $\|\boldsymbol{u}^\nu\|_{H^2}$ is uniformly bounded, it follows from the Gronwall's inequality that $\|\boldsymbol{u}^\nu\|_{H^s}$ is also uniformly bounded in time. That is, the inequality (3.9) holds. This completes the proof of Proposition 3.3. □

3.2. Proof of global existence

We are now in the position to prove Theorem 1.2. We first consider the truncated Navier-Stokes equations on the whole plane:

$$\begin{cases}\partial_t \boldsymbol{u}^{\nu,R} + \mathcal{S}_R\mathbb{P}[\boldsymbol{u}^{\nu,R}\cdot\nabla\boldsymbol{u}^{\nu,R}] = \nu\Delta\boldsymbol{u}^{\nu,R},\\ \operatorname{div}\boldsymbol{u}^{\nu,R} = 0,\\ \boldsymbol{u}^{\nu,R}|_{t=0} = \mathcal{S}_R\boldsymbol{u}_0^{\nu},\end{cases} \tag{3.22}$$

where $\mathbb{P} = \mathbb{I} + \nabla(-\Delta)^{-1}\operatorname{div}$. It can be checked that $\boldsymbol{u}^{\nu,R}$ lies in the space V_R. We then define

$$F(\boldsymbol{u}^{\nu,R},\boldsymbol{v}^{\nu,R}) := \mathcal{S}_R\mathbb{P}[\boldsymbol{u}^{\nu,R}\cdot\nabla\boldsymbol{v}^{\nu,R}] - \nu\Delta\boldsymbol{u}^{\nu,R}. \tag{3.23}$$

It can be checked that F is Lipschitz on the space V_R. Indeed, let $\boldsymbol{u}_1^{\nu,R},\boldsymbol{v}_1^{\nu,R},\boldsymbol{u}_2^{\nu,R},\boldsymbol{v}_2^{\nu,R}\in V_R$, by the definition of F, we have

$$\|F(\boldsymbol{u}_1^{\nu,R},\boldsymbol{v}_1^{\nu,R}) - F(\boldsymbol{u}_2^{\nu,R},\boldsymbol{v}_2^{\nu,R})\|_{L^2} = \|\mathbb{P}\mathcal{S}_R[\boldsymbol{u}_1^{\nu,R}\cdot\nabla\boldsymbol{v}_1^{\nu,R} - \boldsymbol{u}_2^{\nu,R}\cdot\nabla\boldsymbol{v}_2^{\nu,R}] - \nu\mathbb{P}\mathcal{S}_R[\Delta\boldsymbol{u}_1^{\nu,R} - \Delta\boldsymbol{u}_2^{\nu,R}]\|_{L^2}, \tag{3.24}$$

it follows that

$$\|F(\boldsymbol{u}_1^{\nu,R},\boldsymbol{v}_1^{\nu,R}) - F(\boldsymbol{u}_2^{\nu,R},\boldsymbol{v}_2^{\nu,R})\|_{L^2} \le \|\boldsymbol{u}_1^{\nu,R}\cdot\nabla\boldsymbol{v}_1^{\nu,R} - \boldsymbol{u}_2^{\nu,R}\cdot\nabla\boldsymbol{v}_2^{\nu,R}\|_{L^2} + \nu\|\Delta\boldsymbol{u}_1^{\nu,R} - \Delta\boldsymbol{u}_2^{\nu,R}\|_{L^2}. \tag{3.25}$$

Notice that the first term on the right hand-side of the above inequality satisfies

$$\begin{aligned}\|\boldsymbol{u}_1^{\nu,R}\cdot\nabla\boldsymbol{v}_1^{\nu,R} - \boldsymbol{u}_2^{\nu,R}\cdot\nabla\boldsymbol{v}_2^{\nu,R}\|_{L^2} &\le \|(\boldsymbol{u}_1^{\nu,R} - \boldsymbol{u}_2^{\nu,R})\cdot\nabla\boldsymbol{v}_1^{\nu,R}\|_{L^2} + \|\boldsymbol{u}_2^{\nu,R}\cdot\nabla(\boldsymbol{v}_1^{\nu,R} - \boldsymbol{v}_2^{\nu,R})\|_{L^2}\\ &\le CR^2\left(\|\boldsymbol{u}_1^{\nu,R} - \boldsymbol{u}_2^{\nu,R}\|_{L^2}\|\boldsymbol{v}_1^{\nu,R}\|_{L^2} + \|\boldsymbol{u}_2^{\nu,R}\|_{L^2}\|\boldsymbol{v}_1^{\nu,R} - \boldsymbol{v}_2^{\nu,R}\|_{L^2}\right),\end{aligned} \tag{3.26}$$

where we have used the Bernstein's Lemma. Similarly, we can obtain that

$$\|\Delta\boldsymbol{u}_1^{\nu,R} - \Delta\boldsymbol{u}_2^{\nu,R}\|_{L^2} \le CR^2\|\boldsymbol{u}_1^{\nu,R} - \boldsymbol{u}_2^{\nu,R}\|_{L^2}. \tag{3.27}$$

From (3.24)–(3.27), we conclude that F is Lipschitz on the space V_R. Hence, by Picard's theorem, there exists a local solution $\boldsymbol{u}$ in V_R. Moreover, it can be checked that $\boldsymbol{u}^{\nu,R}$ satisfies the estimate (1.9), thus the solution exists globally.

By the Banach-Alaoglu theorem, we find that there exists a subsequence of $\boldsymbol{u}^{\nu,R}$(still denoted by $\boldsymbol{u}^{\nu,R}$) and some $\boldsymbol{u}^{\nu}$ such that

$$\boldsymbol{u}^{\nu,R} \overset{*}{\rightharpoonup} \boldsymbol{u}^{\nu} \text{ in } L^\infty([0,\infty);H^s(\mathbb{R}^2)), \tag{3.28}$$

and $\boldsymbol{u}$ satisfies (1.9).

On the other hand, by the Aubin-Lions lemma and Cantor's diagonal process, there exists a subsequence (still denoted by $\boldsymbol{u}^{\nu,R}$) that, for any bounded open subset K and $T > 0$,

$$\boldsymbol{u}^{\nu,R} \to \boldsymbol{u}^{\nu} \text{ in } C([0,T];L^2(K)). \tag{3.29}$$

Collecting (3.28) and (3.29), we conclude that $\boldsymbol{u}^{\nu}$ is a global solution of Eq (1.8).

3.3. Proof of uniqueness

We here prove for the case $s = 1$, as it is well-known that the solution is unique for $s = 0$. Let $\boldsymbol{u}_1^\nu, \boldsymbol{u}_2^\nu$ be two solutions of Eq (1.8). We subtract the first subequation of (1.8) for $\boldsymbol{u}_2^\nu$ from the one for $\boldsymbol{u}_1^\nu$, then multiply with $\boldsymbol{u}_1^\nu - \boldsymbol{u}_2^\nu$ and integrate over $\mathbb{R}^2$ to obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_1^\nu - \boldsymbol{u}_2^\nu\|_{L^2}^2 + \nu\|\nabla(\boldsymbol{u}_1^\nu - \boldsymbol{u}_2^\nu)\|_{L^2}^2 \leq C\|\nabla\boldsymbol{u}_1^\nu\|_{L^2}\|\boldsymbol{u}_1^\nu - \boldsymbol{u}_2^\nu\|_{L^2}\|\nabla(\boldsymbol{u}_1^\nu - \boldsymbol{u}_2^\nu)\|_{L^2}, \tag{3.30}$$

where we have used the Gagliardo–Nirenberg interpolation inequality. Using Young's inequality, we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_1^\nu - \boldsymbol{u}_2^\nu\|_{L^2}^2 \leq \frac{C}{\nu}\|\nabla\boldsymbol{u}_1^\nu\|_{L^2}^2\|\boldsymbol{u}_1^\nu - \boldsymbol{u}_2^\nu\|_{L^2}^2, \tag{3.31}$$

then the Grönwall inequality implies that $\boldsymbol{u}_1^\nu \equiv \boldsymbol{u}_2^\nu$. We thus conclude that the solution is unique.

4. Vanishing viscosity limit

In this section, we prove Corollary 1.4. Let $\boldsymbol{u}_0 \in H^1(\mathbb{R}^2)$ be divergence-free. Assume that $\boldsymbol{u}_0^\nu$ is divergence-free and converges to $\boldsymbol{u}_0$ in H^1 as $\nu \to 0$. We now prove that, as $\nu \to 0$, the solution $\boldsymbol{u}^\nu$ of (1.8) with initial data $\boldsymbol{u}_0^\nu$ converges to a solution $\boldsymbol{u}$ of (1.1) with initial data $\boldsymbol{u}_0$.

We observe that, for arbitrary $\nu > 0$, Eq (1.8) with initial data $\boldsymbol{u}_0^\nu$ has a unique solution $\boldsymbol{u}^\nu$, and $\boldsymbol{u}^\nu$ satisfies that

$$\sup_{t\in[0,\infty)} \|\boldsymbol{u}^\nu(t)\|_{H^1}^2 + \nu\int_0^\infty \|\nabla\boldsymbol{u}^\nu(t)\|_{H^1}^2\,\mathrm{d}t \leq C\|\boldsymbol{u}_0^\nu\|_{H^1}^2 \leq C\|\boldsymbol{u}_0\|_{H^1}^2. \tag{4.1}$$

By the Banach-Alaoglu theorem, we find that there exists a subsequence of $\boldsymbol{u}^\nu$(still denoted by $\boldsymbol{u}^\nu$) and some $\boldsymbol{u}$ such that

$$\boldsymbol{u}^\nu \overset{*}{\rightharpoonup} \boldsymbol{u} \text{ in } L^\infty([0,\infty); H^1(\mathbb{R}^2)), \tag{4.2}$$

and $\boldsymbol{u}$ satisfies

$$\sup_{t\in[0,\infty)} \|\boldsymbol{u}(t)\|_{H^1} \leq C\|\boldsymbol{u}_0\|_{H^1}. \tag{4.3}$$

On the other hand, by the Aubin-Lions lemma and Cantor's diagonal process, there exists a subsequence (still denoted by $\boldsymbol{u}^\nu$) that, for any bounded open subset K and $T > 0$,

$$\boldsymbol{u}^\nu \to \boldsymbol{u} \text{ in } C([0,T]; L^2(K)). \tag{4.4}$$

Collecting (4.2) and (4.4), we conclude that $\boldsymbol{u}$ is a global weak solution of the Euler equations (1.1). This completes the proof of Corollary 1.4.

5. Conclusions

The incompressible Euler and Navier-Stokes equations in $\mathbb{R}^2$ are studied in this paper. We obtain global existence of weak solutions for initial data in H^1. Moreover, it is proved that, as the viscosity tends to zero, the solution of the Navier-Stokes equations converges to the one of the Euler equations.

Author contributions

Shaoliang Yuan: Conceptualization, writing original draft, writing-review and editing; Lin Cheng: Investigation, writing-review and editing; Liangyong Lin: Investigation, writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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